2024-04-22

Unique K3 Surfaces with Purely Non-Symplectic Automorphism: Insights from Weighted Projective SpaceUnique K3 Surfaces with Purely Non-Symplectic Automorphism:\\Insights from Weighted Projective Space

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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ABSTRACT

Unique K3 Surfaces with Purely Non-Symplectic Automorphisms: Insights from Weighted Projective Space

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K3 surfaces have garnered attention across various fields, from optics and dynamics to high energy physics, making them a subject of extensive study for many decades. Recent work by mathematicians, including Brandhorst [1], has focused on non-symplectic automorphisms, aiming to categorize K3 surfaces that admit such automorphisms. Brandhorst made a list of unique K3 surfaces with purely non-symplectic automorphisms and established specific criteria for a K3 surface to be isomorphic to one on his list.

This thesis aims to provide an alternative representation of select K3 surfaces from Brandhorst's list. While Brandhorst predominantly characterizes these surfaces as elliptic K3 surfaces, we offer a description of these surfaces as hypersurfaces in weighted projective space. Our approach involves verifying the criteria established by Brandhorst, thereby establishing an isomorphism between the surfaces in question. Through this study, we contribute to the understanding of K3 surfaces and their automorphisms while also demonstrating the correspondence between different spaces and methodologies for analyzing K3 surfaces. This work lays the groundwork for further investigations into K3 surfaces with purely non-symplectic automorphisms, paving the way for deeper insights into their structural properties and geometric intricacies.

Keywords: K3 surfaces, purely non-symplectic automorphisms, weighted projective space, Brandhorst

Acknowledgements

Thanks to my loving amazing dedicated husband for his support. Plus my daughter Aria who helped pull me through mentally hard days and showed me I was ready to face the challenge of a thesis.

Contents

List of Tables

List of Figures

Chapter 1. INTRODUCTION

K3 surfaces boast a long and rich history, dating back to 1957 when Weil coined the term(see e.g [4]). They pique particular interest due to myriad geometric and algebraic properties.

Many studies have focused on analyzing K3 surfaces with symplectic automorphisms, delving into the classification of associated symmetries and structures. Recently, mathematicians such as Brandhorst [1] have turned their attention to non-symplectic automorphisms, aiming to categorize that admit such automorphisms.—- Isomorphism between two K3 surfaces allows interchangeable usage of their descriptions, enabling us to leverage structural advantages from each depiction.

Weighted projective space is as an intriguing framework for studying K3 surfaces, offering significant flexibility in their analysis. The incorporation of weighted coordinates plays a pivotal role in understanding these surfaces. This is the main focus of this thesis.

Our focus centers on a list compiled by Brandhorst in [1] of unique K3 surfaces with purely non-symplectic automorphisms of finite order (see Table 3.1). These automorphisms, denoted as σ satisfy the equation $\sigma \cdot \omega = \zeta_n \omega$, where n is the order, ω is a non-vanishing 2-form, and ζ_n represents a primitive nth root of unity. Brandhorst's theorem establishes specific criteria for a K3 surface to be isomorphic to one on his list.

– Our exploration encompasses automorphisms of orders $13, 26, 11, 22, 10, 5, 14, 7,$ and 12, providing a representative example for each listed n. While other surfaces in weighted projective space may meet the criteria, we present only one instance for each order—though in some cases we make mention of additional examples and possible future research directions.

Our methodology unfolds through the construction of K3 surfaces by resolving singularities, followed by an examination of how the automorphism σ interacts with exceptional curves. We leverage essential lattices like the Picard lattice and multiple invariant lattices to ascertain the eigenvalues of our Picard lattice. An absence of primitive eigenvalues signifies an isomorphic surface.

While many other authors have analyzed K3 surfaces in weighted projective space, to the best of our knowledge, no one has applied this analysis to the surfaces Brandhorst identified—this is what we attempt to do in this work. Through comparative analysis between hypersurfaces in weighted projective space and those on Brandhorst's list, we demonstrate the correspondence between different spaces and methodologies for analyzing K3 surfaces.

In chapter 2, we begin with fundamental background information on lattices, K3 surfaces, weighted projective space, automorphisms, and related tools. In chapter 3, we proceed to integrate prior works, including Brandhorst's contributions and studies focused on weighted projective space, while providing illustrative examples. In chapter 4, we systematically explore each order listed above, detailing the construction of K3 surfaces in weighted projective space and navigating the steps required for Brandhorst's theorem. To conclude, we outline avenues for future research and highlight the significance of our contributions in advancing the understanding of uniquely determined K3 surfaces.

Chapter 2. BACKGROUND

There are a few fundamental concepts to understand, including lattices, K3 surfaces, and weighted projective space. As mentioned earlier, the K3 surfaces we want to study are hypersurfaces in weighted projective space. Thus, they are defined by a polynomial W and a group of symmetries G. In this section we define the conditions for W and G as well as how to construct K3 surfaces. We begin by defining a very useful tool in K3 surfaces, namely integral lattices.

2.1 Lattice

One of the more valuable tools for studying K3 surfaces comes from integral lattices. We define these together with other useful tools.

Definition 2.1. A **lattice** is free abelian group L of finite rank with a non-degenerate symmetric bilinear form $B : L \times L \to \mathbb{Q}$. **Non-degenerate** in this context means that for $x \in L$, $B(x, y) = 0$ for all $y \in L$ if and only if $x = 0$.

We work with **integral** lattices: that is $B : L \times L \to \mathbb{Z}$, or in other words, the bilinear form maps to the integers. Furthermore, an integral lattice is **even** if $B(x, x) \in 2\mathbb{Z}$ for all x.

Definition 2.2. The **Gram matrix** of our lattice is the matrix defined by $[B(x_i, x_j)]_{ij}$ where $\{x_i\}$ is a minimal generating set for the lattice.

We reference both the bilinear form and gram matrix by B.

Definition 2.3. The **signature** of a lattice is the signature of the gram matrix. As our lattices are non-degenerate the signature is an invariant (t_+, t_-), where t_+ is the number of positive and t_- the number of negative eigenvalues of the matrix.

The rank of a lattice is determined by the total number of generators in a minimal generating set. In fact, we have $\operatorname{rk} L = t_+ + t_-$.

There are a few more key tools to describe. We consider the discriminant quadratic form. Let $L^* = \operatorname{Hom}(L, \mathbb{Z})$; then the **discriminant group** H_L is defined by embedding L into L^* via B, and $H_L = L^*/L$. If L is even, then the bilinear form B extends to a quadratic form on H_L, resulting in the **discriminant quadratic form** $q_L : H_L \to \mathbb{Q}/2\mathbb{Z}$. The minimal number of generators of H_L is the **length** of L. When we look at the Gram matrix, the order of H_L is $|\det B|$.

For example, there are two important unimodular lattices, denoted U and E_8. In fact, every unimodular lattice that is not not negative definite or positive definite is a direct sum of these two lattices.

The lattice U, also know as the hyperbolic lattice, is a rank 2 lattice with symmetric bilinear form given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It has signature $(1, 1)$. Since it is unimodular, H_L is trivial for this lattice.

The lattice E_8 is a negative definite lattice of rank 8, whose bilinear form is given by

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}.$$

Finally, for a lattice L, $H_L \equiv (\mathbb{Z}/p\mathbb{Z})^a$ and we say L is **p-elementary**. In this case notice that a is the length of L.

2.2 K3 SURFACES

Next we describe a major focus of our study, the K3 surface. A K3 surface is a complex, compact surface with trivial canonical bundle and dim $H^1(X, \mathcal{O}_X) = 0$ where $\mathcal{O}_X$ is the sheaf of regular functions on X. K3 surfaces have been an important topic of study for many years. In this thesis, we will take a closer look at some specific K3 surfaces and verify that they are isomorphic. Since a K3 surface has a trivial canonical bundle, there exists a 2-form ω that is nowhere vanishing. In fact $H^{2,0}(X)$ is 1-dim and generated by ω. It is widely known that K3 surfaces are determined by a certain integral lattice, called the Picard lattice.

Certain K3 surfaces possess specific automorphisms which we now describe. On the

K3 surface X any automorphism σ induces a map σ^* on $H^{2,0}(X)$. Since $H^{2,0}(X)$ is one-dimension, generated by ω, we have $\sigma^*\omega = c_\sigma \omega$ for some $c_\sigma \in \mathbb{C}^*$. If $c_\sigma = 1$ then σ is called **symplectic**; otherwise, it is **non-symplectic**. In the case σ has order n, then c_σ is an nth root of unity. Furthermore, if $c_\sigma = \zeta_n$, a primitive nth root of unity, then σ is called a **purely non-symplectic automorphism.**

We have a few important lattices associated with our K3 surface: the Picard lattice $\mathrm{Pic}(X)$, the transcendental lattice $T(X)$, and if σ is purely non-symplectic, the invariant lattice of σ denoted by $S_X(\sigma)$.

Definition 2.4. [1] The **transcendental lattice** is the smallest primitive sublattice of $T(X) \subseteq H^2(X, \mathbb{Z})$ whose complexification contains $H^{2,0}(X) \subseteq T(X) \otimes \mathbb{C}$.

Definition 2.5. We define the **Picard lattice** as follows:

$$\mathrm{Pic}(X) = H^2(X, \mathbb{Z}) \cap H^{1,1}(X, \mathbb{C}).$$

In other words, the Picard lattice is the intersection of the second integral cohomology group and the group of $(1, 1)$ forms on X.

For a K3 surface X, the second cohomology group $H^2(X, \mathbb{Z})$ is isomorphic to a unique even unimodular lattice of rank 22 and signature $(3, 19)$. Thus $H^2(X, \mathbb{Z}) \equiv U^3 \oplus (E_8)^2$. The Picard lattice is orthogonal to the transcendental lattice $T(X)$ in $H^2(X, \mathbb{Z})$. Hence, $\mathrm{rk}\,\mathrm{Pic}(X) + \mathrm{rk}\,T(X) = 22$.

Definition 2.6. Given a non-symplectic automorphism σ, the **invariant lattice** $S_X(\sigma)$ is the sublattice of $\mathrm{Pic}(X)$ fixed by σ. In other words,

$$S_X(\sigma) = \{x \in H^2(X, \mathbb{Z}) : \sigma \cdot x = x\}.$$

For any automorphism σ, the invariant lattice is a sublattice of $H^2(X, \mathbb{Z})$ and in the case of a non-symplectic σ it is a sublattice of $\mathrm{Pic}(X)$. Given our focus on non-symplectic automorphisms, we rely on these invariant lattices to help us understand the Picard lattice.

2.3 Weighted projective space

One way of constructing K3 surfaces is via hypersurfaces in weighted projective space, which we will now discuss. In fact, there is a famous list of 95 so-called weight systems that give rise to K3 surfaces (see [5]). Weighted projective space is similar to usual projective space, with coordinates weighted by positive integers. Given $w_1, w_2, \cdots, w_n \in \mathbb{Z}_{>0}$, the **weighted projective space** $\mathbb{P}(w_1, w_2, \cdots, w_n)$ is defined as the quotient space $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. For $\lambda \in \mathbb{C}^*$, the action is given by:

$$\lambda \dot{(}x_1, x_2, \cdots, x_n) = (\lambda^{w_1} x_1, \lambda^{w_2} x_2, \cdots, \lambda^{w_n} x_n).$$

K3 surfaces can be constructed in many ways. In this thesis, we will focus on those defined by a polynomial in weighted projective space. In order to define these, we need a few more definitions.

Definition 2.7. A polynomial $W(x_1, x_2, \cdots, x_n)$ is **quasihomogeneous** of degree d with weight system $(w_1, w_2, \cdots, w_n; d)$ if for $\lambda \in \mathbb{C}^*$

$$W(\lambda^{w_1}x_1, \lambda^{w_2}x_2, \cdots, \lambda^{w_n}x_n) = \lambda^d W(x_1, x_2, \cdots, x_n).$$

Definition 2.8. A quasihomogeneous polynomial W is **non-degenerate** when it has an isolated critical point at the origin and the weights w_i can be uniquely determined from the polynomial W.

In the following, we require $\gcd(w_1, \cdots, w_n) = 1$. As we are focusing on K3 surfaces, we set $n = 4$ and consider a quasihomogenous polynomial with coordinates x, y, z, w weighted by w_1, w_2, w_3, w_4.

Example 2.9. Let $W = x^2y + xy^2 + z^4 + w^{12}$. This is a quasihomogeneous polynomial with weight system $(4, 4, 3, 1; 12)$. For $\lambda \in \mathbb{C}^*$,

$$\begin{aligned} W(\lambda^{w_1}x, \lambda^{w_2}y, \lambda^{w_3}z, \lambda^{w_4}w) &= (\lambda^4x)^2\lambda^4y + \lambda^4x(\lambda^4y)^2 + (\lambda^3z)^4 + (\lambda^1x)^{12} \\ &= \lambda^8x^2\lambda^4y + \lambda^4x\lambda^8y^2 + \lambda^{12}z^4 + \lambda^{12}x^{12} \\ &= \lambda^{12}x^2y + \lambda^{12}y^2x + \lambda^{12}z^4 + \lambda^{12}x^{12} \\ &= \lambda^{12}(x^2y + y^2x + z^4 + w^{12}). \end{aligned}$$

Definition 2.10. [6] A non-degenerate and quasihomogeneous polynomial is **invertible** when it has an equal number of variables and monomials. Such polynomials can be represented by an **exponent matrix** $A_W = (a_{ij})$ where a_{ij} is the exponent of the ith variable and jth monomial. As such, the rows represent the monomials of W and the columns represent the variables. When a polynomial is invertible, its associated exponent matrix is also invertible.

We use the exponent matrix to assist in selecting an automorphism. We will discuss this use of the exponent matrix in Sections 2.4 and 2.6.1.

Example 2.11. Let $W = x^2y + xy^2 + z^4 + w^{12}$. We denote the exponent matrix as follows:

$$A_W = \begin{pmatrix} 2 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 12 \end{pmatrix}.$$

As invertible polynomials are an important part of our work, we include the following theorem to assist in recognizing invertible polynomials.

Theorem 2.12. *[7]: A quasi-homogeneous polynomial W is non-degenerate and invertible if and only if it can be written as a direct sum of the three atomic types:*

- $W_{fermat} = x^a$;
- $W_{loop} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_n^{a_n}x_1$;
- $W_{chain} = x_1^{a_1}x_2 + x_2^{a_2}x_3 + \cdots + x_{n-1}^{a_{n-1}}x_n + x_n^{a_n}$.

Here the exponents are all integers greater than 1.

2.4 Automorphisms

For an invertible polynomial W, there are several groups of automorphisms that we need.

Definition 2.13. Given a nondegenerate quasihomogenous polynomial W, the **group of diagonal symmetries** is the group

$$G_W = \{(c_1, c_2, \cdots, c_n) \in (\mathbb{C}^*)^n : W(c_1x_1, c_2x_2, \cdots, c_nx_n) = W(x_1, x_2, \cdots, x_n)\}.$$

If W is invertible, then $|G_W| = \det(A_w)$. It is also well known that G_W is generated by the columns of A_W^{-1}.

For $\lambda = (c_1, c_2, \cdots, c_n) \in G_W$, we have a few notations. The c_i are all roots of unity and thus can be written $c_i = e^{2\pi i g_i}$ for $g_i \in \mathbb{Q}/\mathbb{Z}$. These g_i essentially represent the fractional version of c_i. Instead of writing complex numbers for our automorphisms, we use an additive notation by writing $(g_1, g_2, \cdots, g_n) \in (\mathbb{Q}/\mathbb{Z})^n$ to represent the symmetry $(c_1, c_2, \cdots, c_n) = (e^{2\pi i g_1}, e^{2\pi i g_2}, \cdots, e^{2\pi i g_n})$. We use additive notation unless otherwise stated. Considering $\lambda = (g_1, g_2, \ldots, g_n)$ as representative of the diagonals of a matrix allows us to compute $|\det(\lambda)|$.

When we are in weighted projective space an important automorphism comes from the weights and degree. For a quasihomogenous polynomial with weights $(w_1, w_2, w_3, w_4; d)$, we define $j_W = (\frac{w_1}{d}, \frac{w_2}{d}, \frac{w_3}{d}, \frac{w_4}{d}) \in G_W$. Then j_W has order d. An important group for analyzing the symmetries that arise is $J_W = \langle j_W \rangle$. This is a significant group in mirror symmetry analysis especially. However, its usefulness extends further as we frequently use j_W to gain insight into quotient K3 surfaces where it aids in determining equivalent automorphisms and understanding the transformations performed on our surface.

Indeed, every element $\lambda \in G_W$ induces an automorphism on $\mathbb{P}(w_1, w_2, w_3, w_4)$. Since G_W fixes W, λ will also define an automorphism on the hypersurface defined by W. However, the automorphism j_W acts trivially on $\mathbb{P}(w_1, w_2, w_3, w_4)$, so any two automorphism in G_W are equal if they differ by an element of J_W(see Example 2.15). Another group of automorphisms that we use is the following.

Definition 2.14. The group SL_W is defined by

$$\mathrm{SL}_W = \{\lambda = (g_1, g_2, \cdots, g_n) \in G_W : \sum_i g_i \in \mathbb{Z}.$$

Note that when $\sum_i w_i = d$, the group J_W is in SL_W. Thus in this case,

$$J_W \subseteq \mathrm{SL}_W \subseteq G_W.$$

2.5 Constructing K3 surfaces

We consider a nondegenerate quasihomogeneous polynomial W with weight system $(w_1, w_2, w_3, w_4; d)$. In weighted projective space $\mathbb{P}(w_1, w_2, w_3, w_4)$, W defines a hypersurface Y_W. If W is quasihomogeneous with respect to one of the 95 weight systems, then the minimal resolution X_W of Y_W is a K3 surface.

A surface defined by a nondegenerate quasihomogenous polynomial is quasismooth. Thus when W is nondegenerate, all singularities on Y_W occur on coordinate curves $x = 0, y = 0, z = 0$, and $w = 0$, or as we will refer to them, C_x, C_y, C_z, and C_w. When we resolve singularities we obtain a diagram of curves on our K3 surface. We illustrate the creation of these diagrams further in Chapter 3.

We can view elements of G_W as automorphisms on Y_W, and these extend as expected to automorphisms on X_W. Weighted projective space is an equivalence relation based on the weight system and since J_W is defined by the weights of the polynomial, J_W acts trivially on Y_W. Because J_W acts trivially on Y_W, it is necessary for any subgroup acting on Y_W to contain J_W. Thus the non-symplectic automorphism σ on Y_W extends to a non-symplectic automorphism on X_W. We also denote this as σ. Sometimes two elements of G_W induce the same automorphism, as mentioned previously.

Example 2.15. We consider $x^2y + xy^2 + z^4 + w^{12}$. This polynomial is invertible, nondegenerate and quasihomogenous with weights $(4, 4, 3, 1; 12)$. The corresponding $j_W = (\frac{1}{3}, \frac{1}{3}, \frac{1}{4}, \frac{1}{12})$ helps us determine equivalent automorphisms. Let $\sigma = (0, 0, \frac{1}{4}, 0)$ and $\sigma' = (0, 0, 0, \frac{1}{4})$. Using $j_W^3 = (0, 0, \frac{3}{4}, \frac{1}{4})$, we can see that $\sigma = \sigma' + j_W^3$. Thus $\sigma = (0, 0, \frac{1}{4}, 0) \equiv (0, 0, 0, \frac{1}{4}) = \sigma'$. The fixed locus of σ also includes things fixed by σ'.

When we construct X_W as a K3 surface, in some cases we can construct another K3 surface by considering the quotient by a group of symmetries. We demonstrate what this looks like. As our quotient must preserve our canonical bundle, we must quotient by a subgroup of SL_W. Let G be a group of symmetries such that $J_W \subseteq G \subseteq \mathrm{SL}_W$. Then $\tilde{G} = G/J_W$ preserves the canonical bundle and the polynomial W now defines a hypersurface in the quotient weighted projective space $\mathbb{P}(w_1, w_2, w_3, w_4)/\tilde{G}$. We call this $Y_{W,G}$. The minimal resolution $X_{W,G}$ of $Y_{W,G}$ is a K3 surface. You may also construct $X_{W,G}$ by taking the quotient $X_W/\tilde{G}$. Similarly to X_W, a non-symplectic automorphism σ acting on X_W extends to $X_W/\tilde{G}$. Thus we obtain a non-symplectic automorphism σ of $X_{W,G}$. We denote each version of this as σ.

The diagram for a quotient K3 surface is related to the original K3 surface. By looking at the fixed points on the old diagram, you can determine the singularities for the quotient surface $X_W/\tilde{G}$. The automorphism σ on X_W or $X_{W,G}$ leaves the coordinate curves invariant and induces an action on the set of exceptional curves. We will leverage this fact to determine the action of σ on $\mathrm{Pic}(X_{W,G})$

2.6 Tools

There are some extra tools that we will use in our analysis, including dual spaces, quotient spaces, and the genus of curves. However, before introducing these tools, we introduce the following lemma to assist in Chapter 4.

Lemma 2.16. *If a primitive nth root of unity is an eigenvalue of $\sigma|_{\mathrm{Pic}\otimes\mathbb{C}}$, then each primitive nth root of unity is an eigenvalue of $\sigma|_{\mathrm{Pic}\otimes\mathbb{C}}$.*

Proof. Suppose we have a K3 surface with a purely non-symplectic automorphism σ of order n. We consider the action of σ on $\mathrm{Pic}(X)\otimes\mathbb{C}$ as a integer matrix. There exists a polynomial $p(x) \in \mathbb{Z}[x]$ that is considered the characteristic polynomial of the action. We wish to show that if ζ_n, a primitive root of unity, satisfies $p(\zeta_n) = 0$, then all other primitive nth roots of unity are also roots of $p(x)$.

Since ζ_m is of order m, then there are $\phi(m)$ other primitive roots of unity. Now the minimal polynomial of ζ_n over $\mathbb{Q}$ is the m-cyclotomic polynomial $f_m(x)$. The zeros of $f_n(x)$ are all the nth primitive roots. Since ζ_m is a root of both $p(x)$ and $f_m(x)$ and $f_m(x)$ is minimal, we have that $f_m(x)$ divides $p(x)$. This means that $p(x) = f_m(x)^t h(x)$, where t is the multiplicity of the primitive root of unity. Note that all primitive roots of unity have the same multiplicity. The zeros of $f_m(x)$ are all the mth primitive roots of unity and thus are also roots of $p(x)$, as desired. $\square$

2.6.1 Dual spaces.

One use for K3 surfaces is in mirror symmetry. In [6], Comparin and Priddis discuss a particular form of mirror symmetry called BHK mirror symmetry. Dual spaces are an integral part of studying mirror symmetry. We won't use the full implications, but we will briefly discuss the dual polynomial and dual group. The dual group is defined by

$$G_W^T = \{g \in G_{W^T} |\ gA_W h^T \in \mathbb{Z} \text{ for all } h \in G\}.$$

The dual group is not strictly necessary for our analysis; however, we need to understand the dual polynomial W^T. This is as follows: after we form A_W, the exponent matrix of W, we can create the dual matrix by taking the transpose of A_W. Then W^T is the unique invertible polynomial determined by A_{W^T}, i.e. $A_{W^T} = A_W^T$. A polynomial is **self dual** when A_W is a diagonal matrix. We denote the degree of W^T by d^T.

The following theorem will be used frequently in determining the rank of the Picard lattice. In [8], Lyons and Olcken use W as an invertible quasihomogenous polynomial and G as a group of symmetries containing J_W. As stated before, we will not use the construction of the dual group but will use the dual degree d_T to determine the rank of the Picard lattice.

Theorem 2.17. *[8, Theorem 1.2] Let $X_{W,G}$ be a K3 surface of BHK type that are BHK mirrors. Let d and d_T denote the degree of the quasihomogeneous polynomials: W and W^T. The Picard number of $X_{W,G}$ is:*

$$\text{rk}(\text{Pic}(X_{W,G})) = 22 - \phi(d_T)$$

where ϕ denotes Euler's totient function.

Using this, we can compute $\operatorname{rk}\operatorname{Pic}(X) = 22 - \phi(n) = 22 - \phi(d^T)$. As we are seeking to identify a basis for the Picard lattice, the rank tells us how many elements we are looking for. Understanding these elements is essential for analyzing K3 surfaces and determining isomorphisms.

2.6.2 Quotient spaces. Having discussed the construction of a quotient space, we now introduce the calculations to determine when a quotient is possible. As our quotient G must fall between J_W and SL_W, that narrows the possibilities for G. The first step is to calculate $|\mathrm{SL}_W/J_W|$ and determine how many options for G exist. One can find $|\mathrm{SL}_W/J_W|$ by hand. Alternatively, there is an easy way to determine $|\mathrm{SL}_W/J_W|$, and that is by the dual group G_W^T. In the following paragraphs, we use the dual group to assist our calculations.

We use a couple of facts along with this diagram:

$$\{0\} \hookrightarrow \ J_W \hookrightarrow \mathrm{SL}_W \hookrightarrow G_W.$$

Since $|\det(A_W)| = |G_W|$, we can use the determinant of A_W to determine $|\mathrm{SL}_W/J_W|$. The natural inclusions above illustrate that $|G_W|$ can be factored into pieces:

$$|G_W| = |J_W| \cdot |\mathrm{SL}_W/J_W| \cdot |G_W/\mathrm{SL}_W|.$$

Because $|G_W/\mathrm{SL}_W| = |\mathrm{SL}_W^T/G_W^T| = |J_W^T/\{0\}|$ (see [9]), we can simplify our equation:

$$|G_W| = |J_W| \cdot |\mathrm{SL}_W/J_W| \cdot |J_W^T|.$$

#	N	c	configuration	G
1	2	8	$8A_1$	C_2
2	3	12	$6A_2$	C_3
3	4	12	$12A_1$	C_2^2
4	4	14	$4A_3 + 2A_1$	C_4
5	5	16	$4A_4$	C_5
6	6	14	$3A_2 + 8A_1$	D_6
7	6	16	$2A_5 + 2A_2 + 2A_1$	C_6
8	7	18	$3A_6$	C_7
9	8	14	$14A_1$	C_2^3
10	8	15	$2A_3 + 9A_1$	D_8

Table 2.1: Classification of types quotients and their fixed points, reproduced from [2].

Thus, we can divide $|\det(A_W)|$ by $|J_W^T|$ and $|J_W|$ to receive $|\mathrm{SL}_W/J_W|$. If the value is greater than 1, we have a quotient. Note that if $|\mathrm{SL}_W/J_W| = 1$, then we only have one choice for G, namely J_W. In fact, J_W acts trivially on X_W, so $X_W = X_{W,G}$.

When resolving the singularities of the quotient surface $X_{W,G}$, it is important to start with a list of singularities and fixed points of the original surface X. From there you can determine how many are in an orbit together, and which ones are missing partners. Those not manifesting the quotient relationship are new singularities on the quotient surface. [2] provides a table describing how many new singularities are added based on the quotient taken. We have reproduced a portion of that list in Table 2.1. We use this to check our work and verify we have found all singularities. These singularities are resolved and give us our $X_{W,G}$ diagram.

2.6.3 Genus. The genus of a curve in $\mathbb{P}(w_1, w_2, w_3; d)$ is calculated using the following formula (found, e.g., in [10]):

$$g(C) = \frac{1}{2}\Big(\frac{d^2}{w_1 w_2 w_3} - d \sum_{1 \le j < i} \frac{\gcd(w_i, w_j)}{w_i w_j} + \sum_{i=1}^{3} \frac{\gcd(w_i, d)}{w_i} - 1\Big).$$

The genus of a curve aids in the classification of curves on a K3 surface and is used in determining the rank of a lattice. An example calculation demonstrates the application of this formula.

Example 2.18. In $\mathbb{P}(5,2,2,1)$, we look at the hypersurface defined by $x^2+y^5+z^5+w^{10}=0$ of degree 10. A curve C_x can be defined on the hypersurface Y_W by setting $x=0$. We can think of this curve as living in $\mathbb{P}(2,2,1)$ defined by $W|_{x=0}$, so we are looking at a polynomial of degree 10 in $\mathbb{P}(2,2,1)$. We then do similarly for each variable and obtain curves C_x, C_y, C_z, and C_w in $\mathbb{P}(2,2,1)$, $\mathbb{P}(5,2,1)$, $\mathbb{P}(5,2,1)$, and $\mathbb{P}(5,2,2)$. We can compute the genus as follows:

$$\begin{aligned}
g(C_x) &= \frac{1}{2}\left(\frac{10^2}{4} - 10\left(\frac{2}{4}+\frac{1}{2}+\frac{1}{2}\right) + \left(\frac{2}{2}+\frac{2}{2}+\frac{1}{1}\right) - 1\right)\\
&= \frac{1}{2}\left(25 - 5\left(3\right) + \left(3\right) - 1\right) = 6;\\
g(C_y) &= \frac{1}{2}\left(\frac{10^2}{10} - 10\left(\frac{1}{10}+\frac{1}{5}+\frac{1}{2}\right) + \left(\frac{5}{5}+\frac{2}{2}+\frac{1}{1}\right) - 1\right)\\
&= \frac{1}{2}\left(10 - 8 + \left(3\right) - 1\right) = 2;\\
g(C_z) &= \frac{1}{2}\left(\frac{10^2}{10} - 10\left(\frac{1}{10}+\frac{1}{5}+\frac{1}{2}\right) + \left(\frac{5}{5}+\frac{2}{2}+\frac{1}{1}\right) - 1\right)\\
&= \frac{1}{2}\left(10 - 8 + \left(3\right) - 1\right) = 2;\\
g(C_w) &= \frac{1}{2}\left(\frac{10^2}{20} - 10\left(\frac{1}{10}+\frac{1}{10}+\frac{2}{4}\right) + \left(\frac{5}{5}+\frac{2}{2}+\frac{2}{2}\right) - 1\right)\\
&= \frac{1}{2}\left(5 - 7 + \left(3\right) - 1\right) = 0.
\end{aligned}$$

Knowing the genus of a curve is essential for our analysis of the fixed locus of a non-symplectic automorphism σ.

Finally, the genus of C_x, C_y, C_z, and C_w helps us find the genus of C'_x, C'_y, C'_z, and C'_w when we work in a quotient K3 surface. We use the curve's old genus, the order of quotient N, and a count of points with ramification e_ρ to calculate the new genus with the Reimann-Hurwitz formula:

$$2 - 2g_{old} = N(2 - 2g_{new}) - \sum_{\rho \in S'} (e_\rho - 1).$$

Example 2.19. Continuing from above example, we calculate the new curve genus. Here $N = 5$. We use Figure 2.1 and Section 4.5 to determine ramification points. For C_x, we have $g_{old} = 6$ with no ramification points. Thus:

$$\begin{aligned} 2 - 2g_{old} &= 5(2 - 2g_{new}) - \sum_{\rho \in S'} (e_\rho - 1) \\ 2 - 2(6) &= 5(2 - 2g_{new}) - 0 \\ 2 &= g_{new}. \end{aligned}$$

For C_y, we have $g_{old} = 2$ with three ramification points index 5.

$$\begin{aligned} 2 - 2g_{old} &= 5(2 - 2g_{new}) - \sum_{\rho \in S'} (e_\rho - 1) \\ 2 - 2(2) &= 5(2 - 2g_{new}) - (5 - 1)(5 - 1)(5 - 1) \\ -2 &= 10 - 10g_{new} - 12 \\ 0 &= g_{new}. \end{aligned}$$

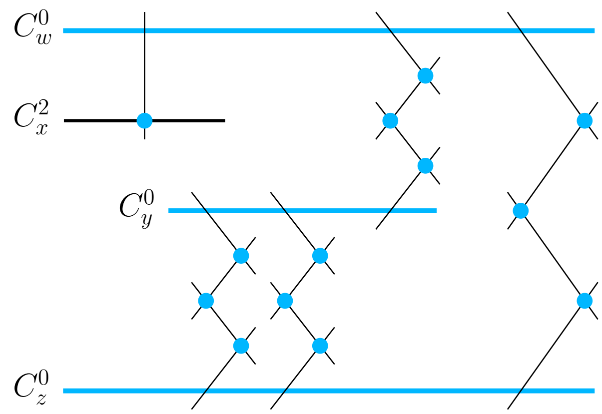

Figure 2.1: Quotient Curve configuration for X_{W,SL_W}, $n = 10$

For C_z, we have $g_{old} = 2$ with three ramification points index 5. Thus:

$$2 - 2g_{old} = 5(2 - 2g_{new}) - \sum_{\rho \in S'} (e_\rho - 1)$$

$$2 - 2(2) = 5(2 - 2g_{new}) - (5 - 1)(5 - 1)(5 - 1)$$

$$-2 = 10 - 10g_{new} - 12$$

$$0 = g_{new}.$$

For C_w, we have $g_{old} = 0$ with two ramification points index 5. Thus:

$$2 - 2g_{old} = 5(2 - 2g_{new}) - \sum_{\rho \in S'} (e_\rho - 1)$$

$$2 - 2(0) = 5(2 - 2g_{new}) - (5 - 1)(5 - 1)$$

$$2 = 10 - 10g_{new} - 8$$

$$0 = g_{new}.$$

Chapter 3. METHODS

3.1 Brandhorst

In [1], Brandhorst proves that under some conditions, certain K3 surfaces are uniquely determined by a purely non-symplectic automorphism, as in the following theorem.

Theorem 3.1. *[1, Theorem 5.9] Let X be a K3 surface and f a purely non-symplectic automorphism of order n such that $\operatorname{rk} T = \phi(n)$ and ζ_n is not an eigenvalue of $\sigma|\operatorname{Pic}(X) \otimes \mathbb{C}$.*

Set $d = |\det \operatorname{Pic}(X)|$, then X is determined up to isomorphism by the pair (n, d). Conversely, all possible pairs (n, d) and equations for X and (some) f are given in Table 3.1.

As stated in Theorem 3.1, we see that each of these K3 surface is uniquely determined by two things, the order of the automorphism and the determinant of the Picard lattice.

As we can see from Table 3.1, each of the K3 surfaces in Brandhorst's list is given as either an elliptic surface or a double cover of $\mathbb{P}^2$. Our goal is to use Theorem 3.1 and show that each K3 surface on Brandhorst's list can be represented as (the resolution of) a hypersurface in weighted projective space. Brandhorst's theorem demands three steps. First, that we have a purely non-symplectic automorphism σ of order n on our surface X. Second, that $\operatorname{rk} T(X) = \phi(n)$. Finally, that ζ_n is not an eigenvalue of $\sigma|_{\operatorname{Pic}(X)}$.

Recall, $T(X)$ is the transcendental lattice, which is orthogonal to the Picard lattice in $H^2(X, \mathbb{Z})$. Thus, $\operatorname{rk} T(X) + \operatorname{rk} \operatorname{Pic}(X) = 22 = \operatorname{rk} H^2(X, \mathbb{Z})$. So we can show either $\operatorname{rk} T(X) = \phi(n)$ or $\operatorname{rk} \operatorname{Pic}(X) = 22 - \phi(n)$.

n	$\det \mathrm{Pic}(X)$	X	f
$3, 6$	3	$y^2 = x^3 - t^5(t-1)^5(t+1)^2$	$(\zeta_3 x, \pm y, t)$
4	2^2	$y^2 = x^3 + 3t^4 x + t^5(t^2-1)$	$(-x, \zeta_4 y, -t)$
$5, 10$	5	$y^2 = x^3 + t^3 x + t^7$	$(\zeta_5^3 x, \pm\zeta_5^2 y, \zeta_5^2 t)$
8	2^2	$y^2 = x^3 + tx^2 + t^7$	$(\zeta_8^6 x, \zeta_8 y, \zeta_8^6)$
	2^4	$t^4 = (x_0^2 - x_1^2)(x_0^2 + x_1^2 + x_2^2)$	$(\zeta_8 t; x_1 : x_0 : x_2)$
12	1	$y^2 = x^3 + t^5(t^2-1)$	$(-\zeta_3 x, \zeta_4 y, -t)$
	$2^2 3^3$	$y^2 = x^3 + t^5(t^2-1)^2$	$(-\zeta_3 x, \zeta_4 y, t)$
	2^4	$y^2 = x^3 + t^5(t^2-1)^3$	$(-\zeta_3 x, \zeta_4 y, -t)$
$7, 14$	7	$y^2 = x^3 + t^3 x + t^8$	$(\zeta_3^7 x, 7 \pm \zeta_7 y, \zeta_7^2 t)$
$9, 18$	3	$y^2 = x^3 + t^5(t^3-1)$	$(\zeta_9^2 x, \pm\zeta_9^3 y, \zeta_9^3 t)$
	3^3	$y^2 = x^3 + t^5(t^3-1)^2$	$(\zeta_9^2, \pm y, \zeta_9^3 t)$
16	2^2	$y^2 = x^3 + t^2 x + t^7$	$(\zeta_{16}^2 x, \zeta_{16}^{11} y, \zeta_{16}^{10} t)$
	2^4	$y^2 = x^3 + t^3(t^4-1)x$	$(\zeta_{16}^6 x, \zeta_{16}^9 y, \zeta_{16}^4 t)$
	2^6	$y^2 = x^3 + x + t^8$	$(-x, iy, \zeta_{16} t)$
20	2^4	$y^2 = x^3 + (t^5-1)x$	$(-x, \zeta_4 y, \zeta_5 t)$
	$2^4 5^2$	$y^2 = x^3 + 4t^2(t^5+1)x$	$(-x, \zeta_4 y, \zeta_5 t)$
24	2^2	$y^2 = x^3 + t^5(t^4+1)$	$(\zeta_3\zeta_8^6 x, \zeta_8 y, \zeta_8^2 t)$
	2^6	$y^2 = x^3 + (t^8+1)$	$(\zeta_3 x, y, \zeta_8 t)$
	$2^2 3^4$	$y^2 = x^3 + t^3(t^4+1)^2$	$(\zeta_3\zeta_8^6 x, \zeta_8 y, \zeta_8^6 t)$
	$2^6 3^4$	$y^2 = x^3 + x + t^{12}$	$(-x, \zeta_{24}^6 y, \zeta_{24} t)$
$15, 30$	5^2	$y^2 = x^3 + 4t^5(t^5+1)$	$(\zeta_3 x, \pm y, \zeta_5 t)$
	3^4	$y^2 = x^3 + t^5 x + 1$	$(\zeta_{15}^{10} x, \pm y, \zeta_{15} t)$
$11, 22$	11	$y^2 = x^3 + t^5 x + t^2$	$(\zeta_{11}^5 x, \pm\zeta_{11}^2 y, \zeta_{11}^2 t)$
$13, 26$	13	$y^2 = x^3 + t^5 x + t$	$(\zeta_{13}^5 x, \pm\zeta_{13} y, \zeta_{13}^2 t)$
26	13	$y^2 = x^3 + t^7 x + t^4$	$(\zeta_{13}^{10} x, -\zeta_{13}^2 y, \zeta_{13} t)$
26	13	$w^2 = x_0^4 y_0^4 + x_1^4 y_1^3 y_2 + x_0 x_1^3 y_1^4$	$((\zeta_{13} x_0 : x_1), (\zeta_{13}^9 y_0 : y_2), -\zeta_{13}^7 w)$
$21, 42$	1	$y^2 = x^3 + t^5(t^7-1)$	$(\zeta_{42}^2 x, \zeta_{42}^3 y, \zeta_{42}^{18} t)$
$21, 42$	7^2	$y^2 = x^3 + 4t^4(t^7-1)$	$(\zeta_3\zeta_7^6 x, \pm\zeta_7^2 y, \zeta_7 t)$
$21, 42$	7^2	$y^2 = x^3 + t^3(t^7+1)$	$(\zeta_3\zeta_7^3 x, \pm\zeta_7 y, \zeta_7^3 t)$
21	7^2	$x_0^3 x_1 + x_1^3 x_2 + x_0 x_2^3 - x_0 x_3^3$	$(\zeta_7 x_0 : \zeta_7 x_1 : x_2, \zeta_3 x_3)$
28	1	$y^2 = x^3 + x + t^7$	$(-x, \zeta_4 y, -\zeta_7 t)$
	2^6	$y^2 = x^3 + (t^7+1)x$	$(-x, \zeta_4 y, \zeta_7 t)$
	2^6	$y^2 = x^3 + (t^7+1)x$	$(x - (y/x)^2, \zeta_4(y - (y/x)^3), \zeta_7 t)$
$17, 34$	17	$y^2 = x^3 + t^7 x + t^2$	$(\zeta_{17}^7 x, \pm\zeta_{17} y, \zeta_{17}^2 t)$
34	17	$x_0 x_1^5 + x_0^5 x_2 + x_1^2 x_2^4 = y^2$	$(-y; x_0 : \zeta_{17} x_1, \zeta_{17}^5 x_2)$
32	2^2	$y^2 = x^3 + t^2 x + t^{11}$	$(\zeta_{32}^{18} x, \zeta_{32}^{11} y, \zeta_{32}^2 t)$
	2^4	$y^2 = x_0(x_1^5 + x_0^4 x_2 + x_1 x_2^4)$	$(\zeta_{32} y; \zeta_{32}^2 x_0 : x_1 : \zeta_{32}^{24} x_2)$
36	1	$y^2 = x^3 - t^5(t^6-1)$	$(\zeta_{36}^2 x, \zeta_{36}^3 y, \zeta_{36}^{30} t)$
	3^4	$y^2 = x^3 + x + t^9$	$(-x, \zeta_4 y, -\zeta_9 t)$
	$2^6 3^2$	$x_0 x_3^3 + x_0^3 x_1 + x_1^4 + x_2^4$	$(x_0 : \zeta_9^3 x_1 : \zeta_4\zeta_9^3 x_2 : \zeta_9 x_3)$
40	2^4	$z^2 = x_0(x_0^4 x_2 + x_1^5 - x_2^5)$	$(x_0 : \zeta_{20} x_1 : \zeta_4 x_2; \zeta_8 z)$
48	2^2	$y^2 = x^3 + t(t^8-1)$	$(\zeta_{48}^2 x, \zeta_{48}^3 y, \zeta_{48}^6 t)$
$19, 38$	19	$y^2 = x^3 + t^7 x + t$	$(\zeta_{19}^7 t, \pm\zeta_{19} y, \zeta_{19}^2 t)$
38	19	$y^2 = x_0^5 x_1 + x_0 x_1^4 x_2 + x_2^6$	$(x_0 : \zeta_{19} x_1 : \zeta_{19}^{16} x_2; -\zeta_{19}^{10} y)$
$27, 54$	3	$y^2 = x^3 + t(t^9-1)$	$(\zeta_{27}^2 x, \zeta_{27}^3 y, \zeta_{27}^6 t)$
27	3^3	$x_0 x_3^3 + x_0^3 x_1 + x_2(x_1^3 - x_2^3)$	$(x_0 : \zeta_{27}^3 x_1 : \zeta_{27}^{21} x_2 : \zeta_{27} x_3)$
$25, 50$	5	$z^2 = (x_0^6 + x_0 x_1^5 + x_1 x_2^5)$	$(z; x_0 : \zeta_{25}^5 x_1 : \zeta_{25}^4 x_2)$
$33, 66$	1	$y^2 = x^3 + t(t^{11}-1)$	$(\zeta_{66}^2 x, \zeta_{66}^3 y, \zeta_{66}^6 t)$
44	1	$y^2 = x^3 + x + t^{11}$	$(-x, \zeta_4 y, \zeta_{11}, t)$

Table 3.1: Brandhorst's [1] list of K3 surfaces with purely non-symplectic automorphism.

In order to show that all eigenvalues of σ are non-primitive, we consider the invariant lattices $S_X(\sigma^p)$ of powers of σ which are sublattices of the Picard lattice. With these, we can count the eigenvalues associated with each pth root of unity. For example, $S_X(\sigma)$ is the invariant lattice of σ and $|\operatorname{rk} S_X(\sigma)|$ counts how many eigenvalues are equal to 1. Then σ^2 contributes eigenvalues that are equal to -1, and σ^3 gives eigenvalues that are third roots of unity, etc. In this way, we can account for all the eigenvalues.

3.2 Methods for K3 surfaces in weighted projective space.

Many mathematicians study K3 Surfaces and categorize the invariant lattices and Picard lattices. Recall that we are interested in K3 surfaces with a purely non-symplectic automorphism σ of order n. A few papers are of particular importance. Comparin et al., in [9], find the rank of lattices in the case when n is prime and not equal to 2. In [6], Comparin and Priddis consider different methods for analyzing a surface when n is not prime except for $n = 4, 8, 12$. Bott et al., in [11], analyze the case when $n = 4, n = 8$, and $n = 12$. In each paper, the invariant lattice is computed by finding the rank and the discriminant quadratic form.

Some relevant results are as follows:

Theorem 3.2. *[6, Theorem 3.1] Given a K3 surface with a non-symplectic automorphism σ of prime order p, the invariant lattice $S_X(\sigma)$ is p-elementary.*

Theorem 3.3. *[6, Theorem 3.2] For a prime $p \neq 2$, a hyperbolic, p-elementary lattice L with rank $r \geq 2$ is completely determined by the invariants (r, a) where a is the length. An indefinite 2-elementary lattice is determined by the invariants (r, a, δ), where $\delta \in 0, 1$ and*

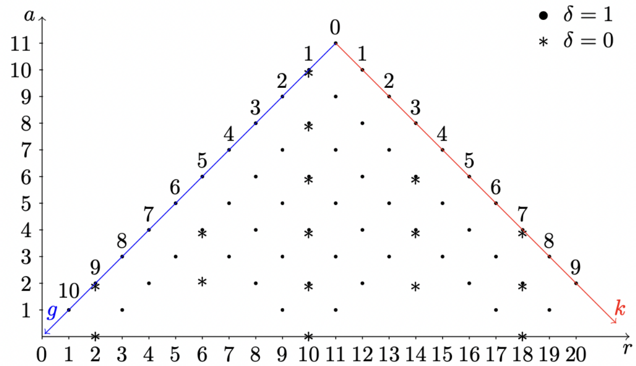

Figure 3.1: Diagram for $p = 2$ and $|\sigma| = 2$. Reproduced from [3]. In this case there are no isolated fixed points.

$\delta = 0$ *if the discriminant quadratic form takes values* 0, *or* 1 *only and* $\delta = 1$ *otherwise.*

Thus if n is prime, we can easily find the rank of the invariant lattice using the following theorem. Similarly, if n is not prime we can find powers of σ that are of prime order and use their invariant lattices to understand the Picard lattice.

In what follows, given a lattice L, we will generally denote the rank of L by r. Artebani et al., in [3], provide equations that relate the fixed locus of σ to the rank of the invariant lattice. The fixed locus for σ includes one curve with genus $g \geq 0$, l isolated fixed points, and k additional genus 0 curves. Then (g, l, k) and (r, a) are two sets of surface invariants. There is a direct relationship between the two sets of invariants, so knowing one allows us to calculate the other (see [9]). For a σ of order $p = 2$, see Figure 3.1.

Theorem 3.4. *[9, 2.4] Let X be a K3 surface with a non-symplectic automorphism σ of prime order $p \neq 2$. Then the fixed focus X^σ is nonempty, and consists of either isolated points or a disjoint union of smooth curves and isolated points of the following form:*

$$X^\sigma = C \cup R_1 \cup \cdots \cup R_k \cup \{p_1, \cdots, p_l\}.$$

Here C is a curve of genus $g \geq 0$, R_i are rational curves and p_i are isolated points.

Furthermore, if X^σ contains a curve and $S(\sigma)$ has invariants (r, a) then the following hold:

- $m = 2g + a$;
- *if* $p = 3$ *then* $1 - g + k = (r - 8)/2$ *and* $l = 10 - m$;
- *if* $p = 5$ *then* $1 - g + k = (r - 6)/4$ *and* $l = 16 - 3m$;
- *if* $p = 7$ *then* $1 - g + k = (r - 4)/6$ *and* $l = 18 - 5m$;
- *if* $p = 13$ *then* $(g, l, k) = (0, 9, 0)$ *and* $S(\sigma) = H_13 \oplus E_8$.

Here the invariant k does not include the curve C.

3.3 Process

This process is described in [6]; we provide an example here.

Consider $n = 12$. In Table 3.1(see also [1]), there are three unique surfaces with a purely non-symplectic automorphism of order 12. In weighted projective space $\mathbb{P}(4, 4, 3, 1)$, we can look at the surface defined by

$$W = x^2y + xy^2 + z^4 + w^{12}.$$

Our polynomial is invertible with weight system $(4, 4, 3, 1; 12)$. The surface X_W has a purely non-symplectic automorphism defined by $\sigma = (0, 0, 0, \frac{1}{12})$.

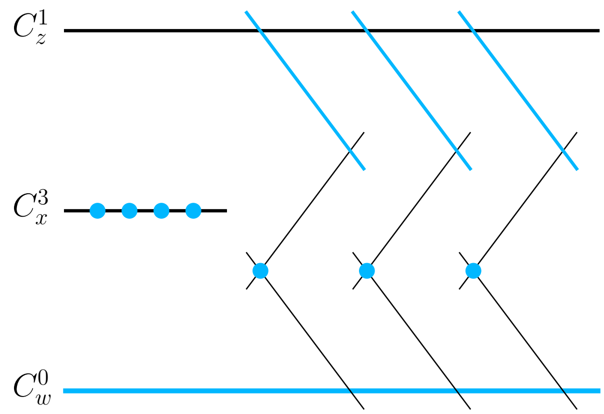

Figure 3.2: Curve configuration, $n = 12$

We will be demonstrating that X_W is actually one of 3 surfaces in Table 3.1. According to Theorem 3.1, we need to verify three conditions: Firstly, that σ is purely non-symplectic of order 12; secondly, that $\operatorname{rk} T(X) = \phi(12)$; and thirdly, that ζ_{12} is not an eigenvalue of $\sigma|_{\operatorname{Pic}(X)\otimes\mathbb{C}}$.

We start by finding all the singularities. Notice if $z = w = 0$ then $\lambda = i$ acts trivially on $(x, u, 0, 0)$. Then all points satisfying $w = 0$ satisfy $x^2y + xy^2 = xy(x + y) = 0$, giving us $3A_3$ singularities. There are no other singularities. Figure 3.2 is a diagram illustrating the resolution of our singularities. Each horizontal line represents the curves on X_W defined by $x = 0, y = 0, z = 0$ and $w = 0$, respectively. We will denote these by C_x^a, where the superscript a represents the genus. The vertical trees map out the singularities. We should note that each of these curves has genus 0.

Step 1. First we verify that σ is purely non-symplectic. Since $\frac{1}{12}$ represents a primitive 12th root of unity, we have a purely non-symplectic automorphism with order 12.

Step 2. Since our surface is self dual, i.e., $W^T = W$, it follows that $d^T = 12$. So $\phi(d^T) = \phi(n) = 4$ and we find that our Picard lattice has rank $22 - \phi(d^T) = 22 - \phi(n) = 22 - \phi(12) = 22 - 4 = 18$ by Theorem 2.17. This verifies the second condition.

Step 3. This step is to find the eigenvalues of $\sigma|_{\mathrm{Pic}(X)\otimes\mathbb{C}}$. Because the $\operatorname{rk}\operatorname{Pic}(X) = 18$, we need to find 18 eigenvalues and verify none are primitive $12th$ root of unity.

Figure 3.2 is the blow-up of exceptional curves on our surface. Applying σ, we see each curve is in its own orbit. When W is of the form $w^n + f(x, y, z)$, then we can use this helpful theorem to count the rank of our invariant lattice.

Theorem 3.5. *[6] The rank r_X of $S(\sigma)$is equal to 1 plus the number of orbits of exceptional curves in the blow-up $X_{W,G} \to Y_{W,G}$.*

For this specific example, we see that there are 9 exceptional curves. Since each curve is its own orbit, $\operatorname{rk} s_X(\sigma) = 1 + 9 = 10$. These represent ten eigenvalues equal to 1. The rank of the Picard lattice is 18, so we are still looking for eight. We look at powers of σ and their associated invariant lattices to find the remaining eigenvalues.

First we consider $\sigma^4 = (0, 0, 0, \frac{1}{3})$. Note that it is purely non-symplectic of order 3. We start with finding the fixed locus and then use Theorem 3.4 to find the rank of its invariant lattice.

The blue lines and points in figure represent the fixed locus of σ^4. C_W has genus 0 and is fixed and each third line in the trees is also fixed. The three intersection points are fixed and isolated. σ can also be written as $(\frac{2}{3}, \frac{2}{3,0,0})$, from which we see that when $x = y = 0$ we have $z^4 + w^{12} = (z + \zeta_4 w)(z + \zeta_4^2 w)(z + \zeta_4^3 w)(z + w)$, thus 4 more intersection points. So we have $(g, l, k) = (0, 7, 3)$. Then by Theorem 3.4, $(r, a) = (16, 3)$. Thus $|\operatorname{rk}(S_X(\sigma^4)| = 16$ and we have 16 of our 18 eigenvalues needed.

To review the eigenvalues we have checked, we have 10 equal to 1, and six more equal to either -1 or $\pm i$. Note that we are still missing two eigenvalues.

Looking at the fixed locus of $\sigma^3 = (0, 0, 0, \frac{1}{4})$, we have C_W genus 0 and 13 points. Each curve has its own orbit so we have $|\operatorname{rk} S_X(\sigma^3)| = 10$, the same amount as $S_X(\sigma)$. This means that we have no third root of unity as an eigenvalue. This is significant as we look at σ^6.

We seek to show that the final two eigenvalues are not primitive $12th$ roots of unity.

Proof. We proceed by Contradiction. Assume that one of the remaining two eigenvalues is ζ_{12}. Then by Lemma 2.16, all 4 primitive 12th roots of unity must be eigenvalues as well. This is too many eigenvalues, thus contradiction. $\square$

Since there are no third roots of unity the only other choice for eigenvalues are primitive sixth roots of unity.

This verifies the third condition, thus we have no primitive 12th roots of unity and this surface is isomorphic to one on Brandhorst's list. On Table 3.1, we see that there are three K3 surfaces with a non-symplectic automorphism of order 12. A future work will be determining which of Brandhorst's surfaces is isomorphic to our X_W with by finding the determinant d.

3.4 Purpose of Thesis

In [1], Brandhorst has a complete list of K3 surfaces with purely non-symplectic automorphisms. As we can see from the preceding example, working with automorphisms on K3 surfaces in weighted projective space is rather straightforward. Our goal is to use Theorem

3.1 and show that each K3 surface on Brandhorst's list is isomorphic to a hypersurface in weighted projective space.

Chapter 4. WEIGHTED PROJECTIVE SPACE K3 SURFACES

Our goal is to show that each of the K3 surfaces in Table 3.1 can be represented by a hypersurface in weighted projective space. In this thesis, we will look at $n = 13, 26, 11, 22, 10, 5, 14, 7,$ and 12. We proceed by choosing a value n and giving an example of a weighted projective K3 surface that is isomorphic to those on Table 3.1. We conjecture that there are more surfaces and automorphisms than those studied here that are isomorphic to a surface on 3.1.

Having chosen W, we demonstrate that the surface X_W meets the 3 conditions listed in Theorem 3.1. First, we can consider each $\lambda \in G_W$ as a diagonal matrix. As mentioned previously, $\lambda\omega = (\det \lambda)\omega)$. We will find σ in G_W, and we can see that σ is a purely non-symplectic automorphism of order n by showing the determinant is a primitive root of unity.

Second, we will use theorem 2.17 to show that that $\operatorname{rk} T = \phi(n)$ or $\operatorname{rk} \operatorname{Pic}(X) = 22 - \phi(n) = 22 - \phi(d^T)$.

Third, we demonstrate that all eigenvalues of σ^* on $\operatorname{Pic}(X) \otimes \mathbb{C}$ are not primitive nth roots of unity. This last step is the most difficult, and we describe the details in each case.

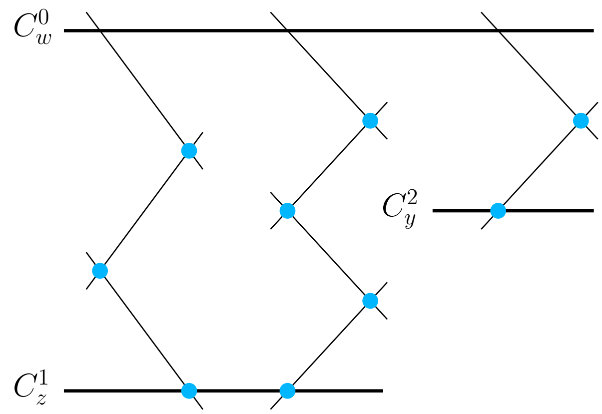

Figure 4.1: Curve configuration, $n = 13$

4.1 Order 13

We consider Brandhorst's surface $(13, 13)$. We find an isomorphic K3 surfaces with corresponding automorphism in weighted projective space $\mathbb{P}(5, 4, 3, 1)$. Let

$$W = x^2z + xy^2 + yz^3 + w^{13}, \text{ and } \sigma = \left(0, 0, 0, \frac{1}{13}\right).$$

Let us begin by describing the process of resolving the singularities of Y_W. Since W is nondegenerate, Y_W is quasismooth and all singularities are inherited from weighted projective space.

The point $(1, 0, 0, 0)$ has a $\mathbb{Z}_5$ isotropy group or A_4 singularity. The point $(0, 1, 0, 0)$ has a $\mathbb{Z}_4$ isotropy group or A_3 singularity. The point $(0, 0, 1, 0)$ has a $\mathbb{Z}_3$ isotropy group or A_2 singularity. When we blow up each of these points we get the configuration pictured in Figure 4.1.

Step 1. $n = 13$ and $|\sigma| = 13$. Since $\det(\sigma) = \frac{1}{13}$, the determinant is a primitive root of unity and so σ is a purely non-symplectic automorphism.

Step 2. Our surface is self dual. Thus $d^T = 13$ so $\phi(n) = \phi(d^T) = \phi(13) = 12$. Our Picard lattice has rank $22 - \phi(13) = 10$ by Theorem 2.17, as desired.

Step 3. Next we check that each of the ten eigenvalues are non-primitive. If we consider the action of σ on the exceptional curves, we see that each is invariant under the action as σ fixes the curve C_w. Thus by Theorem 3.5, we see that $|\operatorname{rk} s_X(\sigma)| = 10$. This means the invariant lattice of σ contains all the eigenvalues of the Picard lattice and all are non-primitive roots of unity. This means that this example corresponds with the Brandhorst surface.

The last part is finding the determinant of the Picard lattice. This case is easy since there is only one option for determinants when $n = 13$. So our Picard lattice has determinant 13. Hence, X_W is isomorphic to the $(13, 13)$ Brandhorst surface. In particular, $\mathrm{Pic}(X_W)$ is generated by the curves in the invariant lattice.

4.2 Order 26

We consider Brandhorst's surface $(26, 13)$. This appears 3 times in Table 3.1. This means that there is one surface with invariants $(26, 13)$ but three different automorphisms of order 26. We leave for a future work distinguishing the three different automorphisms. In weighted projective space $\mathbb{P}(9, 5, 3, 1)$, we find an isomorphic surface with purely non-symplectic au-

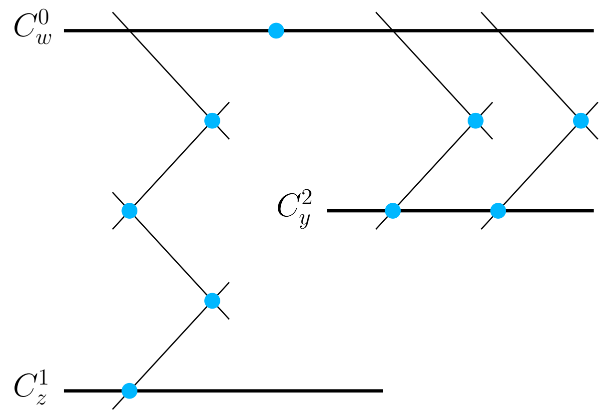

Figure 4.2: Curve configuration, $n = 26$

tomorphism. Let

$$W = x^2 + y^3 z + z^6 + yw^{13}, \text{ and } ,\sigma = \left(\frac{1}{2}, 0, 0, \frac{1}{13}\right).$$

We begin by describing the resolution of singularities on Y_W. The point $(0,1,0,0)$ has a $\mathbb{Z}_5$ isotropy group and so an A_4 singularity. When $y = w = 0$, $W = x^2 + z^6 = (x + iz^3)(x - iz^3)$. This gives us 2 points with $\mathbb{Z}_3$ isotropy. So we have two A_2 singularities. When we blow up these points we get the curve configuration in Figure 4.2.

Step 1. $n = 26$ and $|\sigma| = 26$. We check that

$$\det(\sigma) = \frac{1}{2} + \frac{1}{13} = \frac{13}{26} + \frac{2}{26} = \frac{15}{26}.$$

Since this is a primitive 26th root of unity we have a purely non-symplectic automorphism.

Step 2. Our surface is not self dual, so we check the dual. The dual surface is $W^T = x^2 + y^3w + yz^6 + w^{13}$ with weight system $(13, 8, 3, 2; 26)$. Thus $d^T = 26$. Thus $22 - \phi(d^T) = 22 - \phi(n) = 22 - \phi(26) = 22 - 12 = 10$ and our Picard lattice has rank 10, as desired.

Step 3. We check all the eigenvalues. If we consider the action of σ on the exceptional curves, we see that each is invariant as σ fixes C_w. Thus by Theorem 3.5, we see that $|\operatorname{rk} s_X(\sigma)| = 9$. Since $|\operatorname{rk} s_X(\sigma)| = 9$, we have nine eigenvalues equal to 1.

Next we look at powers of σ for the final eigenvalue, specifically $\sigma^2 = (0, 0, 0, \frac{1}{13})$. The fixed locus of σ^2 contains C_w of genus 0 and 9 fixed points, highlighted blue in Figure 4.2. This gives us the invariant $(0, 9, 0)$. Now comparing with Theorem 3.4 in [9], we have $(r, a) = (10, 1)$. This means the final eigenvalue is -1 and all ten eigenvalues are non-primitive eigenvalues. This example corresponds with the Brandhorst surface.

Since there is only one determinant on Brandhorst's list for $n = 26$ determinants, our lattice has determinant 13 and X_W is isomorphic to the $(26, 13)$ on Brandhorst list.

Remark: While it is not necessary to calculate the determinant for this example, we include the calculation here as we already have all the elements needed. Above we identified $a = 1$ which means our determinant is $13^a = 13$ by [9]. Similarly, this defines a sublattice corresponding to $(0, 0, 0, \frac{1}{13})$ with determinant 13. This is an additional example of $(13, 13)$ Brandhorst surface, i.e. this surface and the previous surfaces are isomorphic.

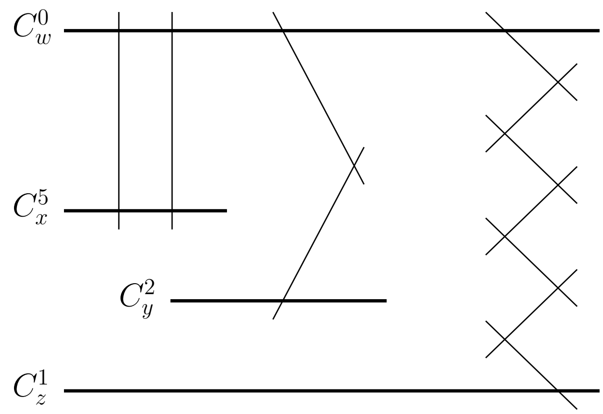

Figure 4.3: Curve configuration, $n = 11$

4.3 Order 11

We consider Brandhorst's surface $(11, 11)$. In weighted projective space $\mathbb{P}(15, 8, 6, 1)$, we find an isomorphic surface defined by

$$W = x^2 + y^3z + z^5 + yw^{22}, \text{ and } \sigma = \left(0, 0, 0, \frac{1}{11}\right).$$

We start by resolving the singularities on Y_W. When $y = w = 0$, we have $W = x^2 + z^5$ so we have one A_2 singularity. The point $(0, 1, 0, 0)$ has a A_7 singularity. Finally, when $x = w = 0$, $W = y^3z + z^5 = z(y^3 + z^4)$. This gives us two A_1 singularities. The curve configuration in Figure 4.3 shows the blow up of the singularities.

Step 1. $n = 11$ and $|\sigma| = 11$. Since $\frac{1}{11}$ is primitive 11th root we have a purely non-symplectic automorphism.

Step 2. Since our surface is not self dual, we check the dual. The dual polynomial is $W^T = x^2 + y^3w + yz^5 + w^{22}$ with weight system $(11, 7, 3, 1; 11)$, so $d^T = 22$. Thus

$\phi(d^T) = \phi(22) = 10 = \phi(11)$ and $22 - \phi(d^T) = 22 - \phi(n) = 22 - \phi(22) = 22 - 10 = 12$. This shows that our Picard lattice has rank 12, as desired.

Step 3. When we consider the action of σ on our exceptional curves, we see that each is invariant under our action. Thus by Theorem 3.5, we find that $|\operatorname{rk} s_X(\sigma)| = 12$. All eigenvalues are 1 and so are non-primitive roots of unity. There is only one determinant choice for $n = 11$, so the determinant is 11. Hence, X_W is another Brandhorst surface.

4.4 Order 22

We consider Brandhorst's surface $(22, 11)$. This has one representation in Brandhorst's paper. In weighted projective space $\mathbb{P}(11, 6, 4, 1)$, we find an isomorphic surface with automorphism defined by

$$W = x^2 + y^3 z + yz^4 + w^{22}, \text{ and } \sigma = \left(0, 0, 0, \frac{1}{22}\right).$$

We start by resolving the singularities on Y_W. The point $(0, 1, 0, 0)$ has a A_5 singularity. The point $(0, 0, 1, 0)$ has a A_3 singularity. When $x = w = 0$, we have $W = y^3 z + yz^4 = yz(y^2 + z^3)$, so we get three A_1 singularities. We blow each singularity up to get the configuration in Figure 4.4.

Step 1. $n = 22$ and $|\sigma| = 22$. Since $\frac{1}{22}$ is primitive 22th root we have a purely non-symplectic automorphism.

Step 2. Since our surface is self dual, $d^T = 22$. So $22 - \phi(d^T) = 22 - \phi(n) = 22 - \phi(22) = 22 - 10 = 12$. Our Picard lattice has rank 12.

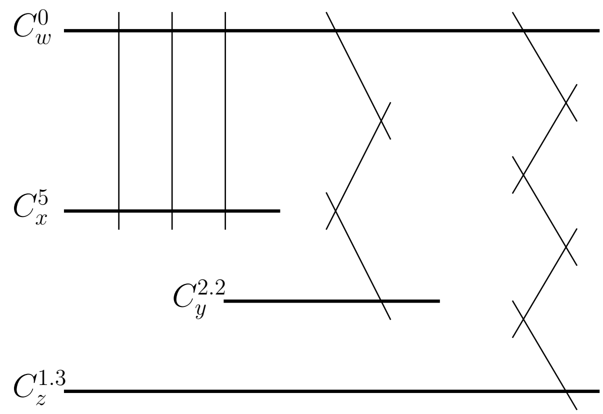

Figure 4.4: Curve configuration, $n = 22$.

Step 3. Next we check that each of the twelve eigenvalues are non-primitive. The action of σ leaves all exceptional curves invariant, so $|\operatorname{rk} s_X(\sigma)| = 12$. This means we have all twelve eigenvalues as non primitive eigenvalues. There is only one determinant choice for the $n = 22$ Brandhorst's surface so the Picard lattice has determinant 11. Thus X_W corresponds with the $(22, 11)$ Brandhorst surface.

Remark: If we consider $\sigma = (0, 0, 0, \frac{1}{11})$, we get another representation of the $(11, 11)$ surface on Brandhorst's list.

4.5 Order 10

We consider Brandhorst's surface $(10, 5)$. This has only one representation of surface and automorphism (see Table 3.1). In weighted projective space $\mathbb{P}(5, 2, 2, 1)$, we define the K3 surface with

$$W = x^2 + y^5 + z^5 + w^{10}, \text{ and } \sigma = \left(0, 0, 0, \frac{1}{10}\right).$$

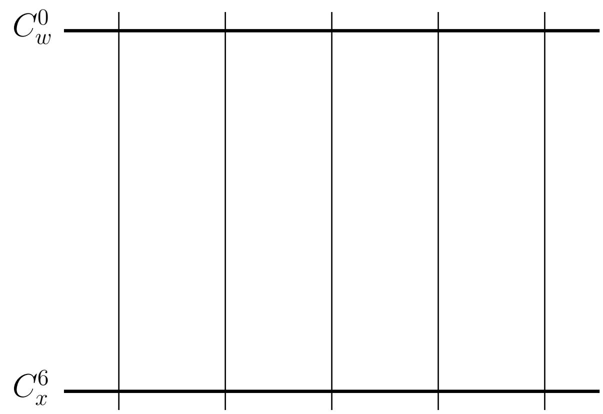

Figure 4.5: Curve configuration for X_W, $n = 10$

We start by finding all the singularities. When $x = w = 0$, then $W = y^5 + z^5 = (y + \zeta_5 z)(y + \zeta_5^2 z)(y + \zeta_5^3 z)(y + \zeta_5^4 z)(y + z)$ and so this gives us five A_1 singularities.

The surface X_W is not on Brandhorst's list, but we will consider a further quotient.

$$\det(A_W) = \det \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 10 \end{pmatrix} = 500 = 10 * 10 * 5.$$

Because $\det(A_W) = |G_W| = |J_W| \cdot |\mathrm{SL}_W/J_W| \cdot |J_{W^T}|$, and we have $|J_W| = 10$ and $|J_{W^T}| = 10$, we find $|\mathrm{SL}_W/J_W| = 5$. We let $G = \mathrm{SL}_W$ and so $\tilde{G} = G/J_W = \mathrm{SL}_W/J_W \cong \mathbb{Z}_5$. We quotient X_W by $\tilde{G}$.

SL_W/J_W is generated by $(0, \frac{1}{5}, \frac{4}{5}, 0)$, which is equivalent (via J_W) to $(0, 0, \frac{3}{5}, \frac{2}{5})$, and also to $(0, \frac{2}{5}, 0, \frac{3}{5})$. Hence the fixed points can be found where $y = z = 0$, where $y = w = 0$ and where $z = w = 0$. The first conditions ($y = z = 0$) means that $W = x^2 + w^{10} = (x + iw^5)(x - iw^5)$,

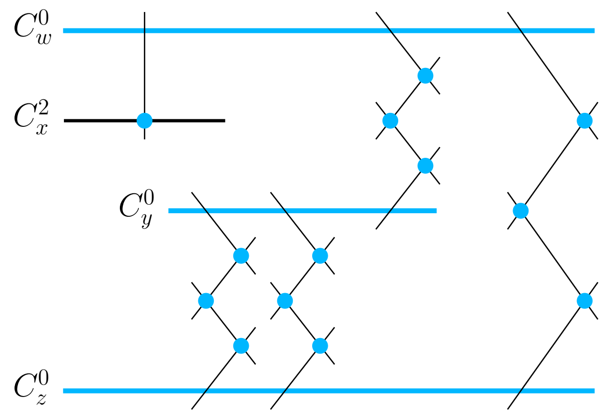

Figure 4.6: Quotient Curve configuration for X_{W,SL_W}, $n = 10$

which contributes two A_4 singularities. The other sets of conditions each contribute another A_4 singularity. Furthermore, the five A_1 curves in Figure 4.5 are permuted by SL_W. Thus we arrive at the configuration of curves described in Figure 4.6.

Step 1. Since $\frac{1}{10}$ is primitive 10th root we have a purely non-symplectic automorphism.

Step 2. Since our surface is self dual, $d^T = 10$ and $\phi(h_T) = \phi(n) = 4$. So our Picard Lattice has rank $22 - \phi(d^T) = 22 - \phi(n) = 22 - 4 = 18$.

Step 3. Now consider the action of σ on these exceptional curves. Each is invariant as σ fixes the curve C_w. Thus by Theorem 3.5, $|\operatorname{rk} s_{\tilde{X}}(\sigma)| = 18$ and we have eighteen eigenvalues equal to 1. Thus X_{W,SL_W} corresponds with the Brandhorst surface $(10, 5)$.

Remark: As there is only one determinant choice for $n = 10$, we don't need to find the determinant for the Picard lattice. However, it can be calculated by looking at the fixed locus of $\sigma^2 = (0, 0, 0, \frac{1}{5})$, as in the next example.

4.6 Order 5

We see from Table 3.1 that the surface with invariants $(5, 5)$ is the one we just analyzed. However, we did not give an automorphism of order 5. But the automorphism $(0, 0, 0, \frac{1}{10})$ from the previous surfaces gives us also a purely non-symplectic automorphism of order 5 on this surface as well, namely $\sigma^2 = \left(0, 0, 0, \frac{1}{5}\right)$.

In the previous calculations, we saw that $S_X(\sigma) = \mathrm{Pic}(X_{W,\mathrm{SL}_W})$, so in fact, we also have $S_X(\sigma^2) = \mathrm{Pic}(X_{W,\mathrm{SL}_W})$, and so again all 18 eigenvalues are equal to one for this automorphism as well. Since all 18 eigenvalues are non-primitive, by Theorem 3.1 X_{W,SL_W} is isomorphic to the K3 surface $(5, 5)$ on Brandhorst's list.

Remark: As there is only one $n = 5$, it is not necessary to find the determinant. However we can verify it by looking at the fixed locus, which contains a curve C_w of genus 0 and 13 fixed points (marked in blue in Figure 4.6). As we are really considering the class of automorphisms equivalent to σ we use J_W to check if there are alternate versions that yield extra information.

Note that $J_W = (\frac{1}{2}, \frac{1}{5}, \frac{1}{5}, \frac{1}{10})$ and so $J_W^8 = (0, \frac{3}{5}, \frac{3}{5}, \frac{4}{5})$. Since $\sigma \equiv \sigma + J_W^8$ then $(0, 0, 0, \frac{1}{5}) \equiv (0, \frac{3}{5}, \frac{3}{5}, 0)$.

Furthermore, to construct this surface, we took a quotient by $\tilde{G}$, which is generated by $(0, \frac{1}{5}, \frac{4}{5}, 0)$. Thus we can further see that σ can be represented by $(0, \frac{1}{5}, 0, 0)$ and $(0, 0, \frac{1}{5}, 0)$. Thus the fixed locus also contains the curves C_y and C_z, both of which have genus 0.

So $g = 0$, $k = 2$ and $l = 13$. The invariant $(0, 13, 2)$ using Theorem 3.4 gives $(r, a) = (18, 1)$. Thus the determinant of our Picard lattice is $5^a = 5$.

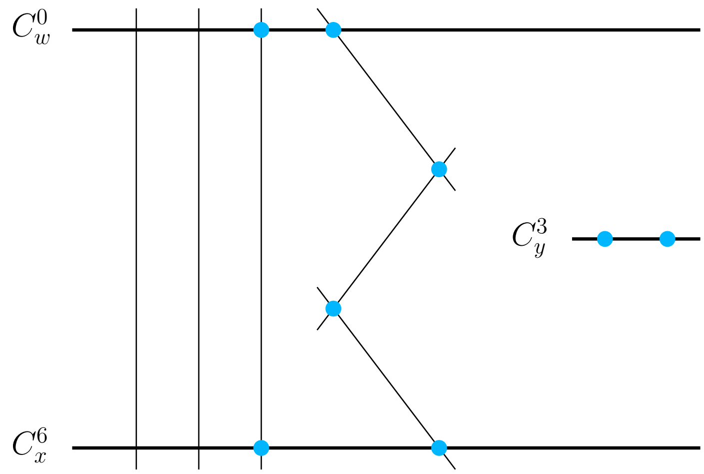

Figure 4.7: Curve configuration for X_W, $n = 14$

4.7 Order 14

We consider Brandhorst's surface $(14, 7)$. This has one representation on Brandhorst's paper. In weighted projective space $\mathbb{P}(7, 4, 2, 1)$, we find an isomorphic surface with automorphism, given by

$$W = x^2 + y^3z + yz^5 + w^{14}, \text{ and } \sigma = \left(0, 0, 0, \frac{1}{14}\right).$$

We start by describing the process of resolving singularities of Y_W. The point $(0, 1, 0, 0)$ has an A_3 singularity. When $x = w = 0$, we have $W = y^3z + yz^5 = yz(y^2 + z^4) = yz(y + iz^2)(y - iz^2)$. When $y = 0$, we see that the point $(0, 0, 1, 0)$, which we just described, gives us an A_3 singularity. The other three points satisfying $W = 0$ each give us an A_1 singularity. Once we blow up these points, we get the curve configuration in Figure 4.7.

This surface is not in Table 3.1. To find a surface that is in Table 3.1, we take a further quotient. We proceed by checking if there is a quotient surface by looking at our A_W. We find that

$$\det(A_W) = \det \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 14 \end{pmatrix} = 392 = 14 * 14 * 2.$$

Because $\det(A_W) = |G_W| = |J_W| \cdot |\mathrm{SL}_W/J_W| \cdot |J_{W^T}|$ and $|J_W| = |J_{W^T}| = 14$, we calculate $|\mathrm{SL}_W/J_W| = 2$. As discussed in Sections 2.5 and 2.6.2, we have two choices for G: J_W and SL_W. As J_W acts trivially on Y_W, our initial analysis corresponds to J_W. We let $G = \mathrm{SL}_W$ and so $\tilde{G} = G/J_W = \mathrm{SL}_W/J_W$. We quotient X_W by $\mathbb{Z}_2$.

Table 2.1 says we have $8A_1$ singularities in the quotient. We look at the singularities and fixed points of Y_W to find isotropies. The blue points are fixed points, we proceed to discuss each one.

If we look at the original $3A_1$, they come from $x = w = 0$, so W becomes $y^3z + yz^5 = yz(y + iz^2)(y - iz^2) = 0$. The $(y + iz^2)(y - iz^2)$ have the Z_2 structure and do not contribute any new A_1 to our quotient but become equivalent under our quotient operation. Finally, the yz represent the final A_1 and A_3. We discuss each one. The A_1 has two fixed points occurring on either end. These points do not preserve the structure under the quotient operation so they are singularities on our quotient surface. These are two new A_1.

Looking at our original A_3 singularity, there are four fixed points occurring at the intersection points. These points do not preserve the structure under the quotient operation so they are singularities on our quotient surface. They yields $4A_1$ making our A_3 look like an A_7. The original A_3 in Figure 4.8 are drawn as wavy lines, while the four new A_1 are straight.

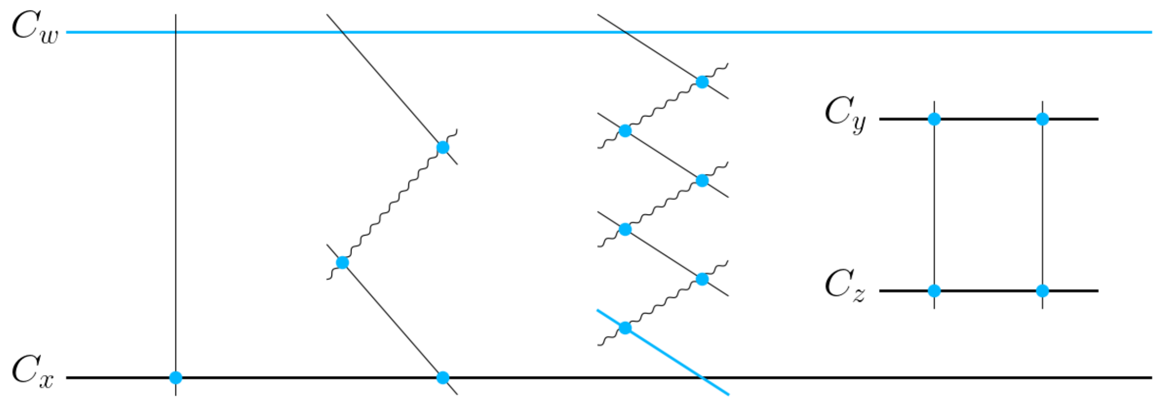

Figure 4.8: Quotient Curve configuration for X_{W,SL_W}, $n = 14$

Finally, SL_W/J_W is generated by $(0, \frac{1}{2}, \frac{1}{2}, 0)$. Thus, all points of the form $y = z = 0$. There are two of these $(x + w^7)(x - w^7)$. So SL_W/J_W also fixes the points $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ which we blew up. So we get fixed points as depicted on C_y. These blow up into two new A_1 singularities. Figure 4.8 shows all eight of the new $8A_1$ singularities and depicts the quotient curve configuration.

Step 1. $n = 10$ and $|\sigma| = 10$. Since $\det(\sigma) = \frac{1}{14}$ is primitive $14th$ root, we have a purely non-symplectic automorphism.

Step 2. Since our surface is self dual, $d^T = 14$ and $\phi(h_T) = \phi(n) = 6$. So our Picard Lattice has rank $22 - \phi(d^T) = 22 - \phi(14) = 22 - 6 = 16$.

Step 3. The trees of curves off of C_w are all invariant under the action of σ. But the other 2 exceptional curves are permuted. Hence by Theorem 3.5, $|\operatorname{rk} s_{\tilde{X}(\sigma)}| = 13$. That leaves us with three eigenvalues. Now if $\mathrm{Pic}(X_{W,\mathrm{SL}_W})$ has a primitive 14th root of unity as an eigenvalue, then it has all six. This is a contradiction so the remaining three eigenvalues are -1. This means that X_{W,SL_W} corresponds with the Brandhorst surface.

As there is only one option for $n = 14$ determinant, our Picard lattice has determinant

7. Hence, X_{W,SL_W} is isomorphic to the $(14,7)$ Brandhorst surface.

4.8 Order 7

We see from Table 3.1 that the surface with invariants $(7,7)$ is the one we just analyzed. However, we did not give an automorphism of order 7. But the automorphism $(0,0,0,\frac{1}{14})$ from the previous surface gives us also a purely non-symplectic automorphism of order 7 on this surface as well, namely $\sigma^2 = (0,0,0,\frac{1}{7})$.

We look at the fixed locus σ^2. The blue lines and points in Figure 4.8 assist in pointing out the fixed locus. C_W has genus 0 and is fixed; furthermore the seventh curve in the A_7 tree is fixed. We count five fixed intersection points on the A_7 tree, three fixed intersection points on the A_3 tree, and one fixed intersection point on A_1. The C_y and C_z hold four more fixed intersection points. This gives us a total of 13 fixed points with two curves of genus 0. Thus, we have $(g,l,k) = (0,13,1)$ and by Theorem 3.4, $(r,a) = (16,1)$.

Furthermore, we have $S_X(\sigma^2) = \mathrm{Pic}(X_{W,\mathrm{SL}_W})$, so in fact, all 16 eigenvalues are non primitive. Since all 16 eigenvalues are non-primitive, by Theorem 3.1 X_{W,SL_W} is isomorphic to the K3 surface $(7,7)$ on Brandhorst's list.

4.9 Order 12

Finally for the sake of completeness, we return to the surface described in Section 3.3. We give an abbreviated version here. We have $n = 12$. Recall, in weighted projective space

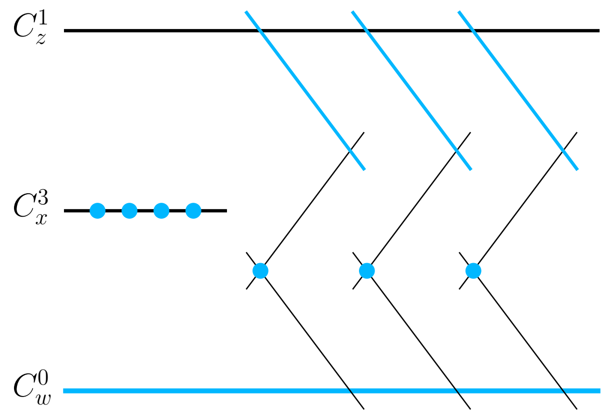

Figure 4.9: Curve configuration, $n = 12$

$\mathbb{P}(4,4,3,1)$, we look at the surface defined by

$$W = x^2y + xy^2 + z^4 + w^{12}, \text{ and } \sigma = \left(0, 0, 0, \frac{1}{12}\right).$$

We begin by resolving the singularities of Y_W. When $z = w = 0$, we have three A_3 singularities. We blow up these singularities to get Figure 3.2

Step 1. Since $\frac{1}{12}$ is primitive 12th root we have a purely non-symplectic automorphism.

Step 2. Since our surface is self dual, $d^T = 12$. So our Picard lattice has rank $22 - \phi(d^T) = 22 - \phi(n) = 22 - \phi(12) = 22 - 4 = 18$.

Step 3. Next we check that each of the 18 eigenvalues are non-primitive. The action of σ on our exceptional curves we see that each is invariant. Thus by Theorem 3.5 we see that $|\operatorname{rk} s_X(\sigma)| = 10$. This means that we have ten eigenvalues equal to 1.

We look at σ^4. The blue lines and points in Figure 3.2 represent the fixed locus of $\sigma^4 = (0, 0, 0, 1/3)$. C_W has genus 0 and is fixed. Furthermore each third curve in each of the trees, as well as the three intersection points. Recall, σ is equivalent to $(\frac{2}{3}, \frac{2}{3}, 0, 0)$ so when $x = y = 0$, we have 4 more fixed intersection points. So we have $(g, l, k) = (0, 7, 3)$ and therefore $(r, a) = (16, 3)$. Thus $|\operatorname{rk}(S_X(\sigma^4)| = 16$.

In summary, we have ten eigenvalues equal to 1 and six equal -1 or $\pm i$ eigenvalues. We are still missing two. Further analysis of σ^3 helps us place the last two eigenvalues.

The fixed locus of σ^3 has C_W genus 0 and 13 points. So σ^3 leaves the curves invariant and we have $|\operatorname{rk} s_X(\sigma^3)| = 10$. Hence we have no third root of unity.

If one of the remaining two eigenvalues are ζ_{12}, then all four primitive 12th roots of unity are also eigenvalues by Lemma 2.16. This would be too many eigenvalues. Since there are no third roots of unity the only other choice for eigenvalues are primitive sixth roots of unity.

Thus we have no primitive 12th roots of unity and this surface is isomorphic to one on Brandhorst's list.

Chapter 5. FUTURE WORK

We have examined K3 surfaces in weighted projective space and determined which ones are isomorphic to the (n, d) K3 surfaces listed by Brandhorst. This list includes many n values that we have not discussed in detail. In future work, we can explore other values of n.

In analyzing the Picard lattice and invariant lattice of σ, we have gathered theorems that relate (g, l, k) invariants to (r, a) invariants for σ of order $2, 3, 5, 7$, and 13. This should allow analysis for surfaces with n having factors of $2, 3, 5, 7$, and 13.

Some of these surfaces will require additional tools not discussed here. For example, with $n = 12$, we may be able to determine a surface is on the list without being able to point to which one. A future work would gather and apply tools needed to find the determinant of the Picard lattice and thus complete the identification.

Similarly, with surface $n = 26$, there are three different purely non-symplectic automorphisms identified. Another future work involves distinguishing these automorphisms on the K3 surfaces in weighted projective space.

In conclusion, our analysis not only identifies isomorphic K3 surfaces within Brandhorst's list, but also lays the groundwork for future investigations into a broader spectrum of automorphisms and the differentiation of K3 surfaces with purely non-symplectic automorphisms in weighted projective space.

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