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Problems Related to the Zermelo and Extended Zermelo Model

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PROBLEMS RELATED TO THE ZERMELO AND EXTENDED ZERMELO MODEL

by

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ABSTRACT

PROBLEMS RELATED TO THE ZERMELO
AND EXTENDED ZERMELO MODEL

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In this thesis we consider a few results related to the Zermelo and Extended Zermelo Model as well as outline some partial results and open problems related thereto. First we will analyze a discrete dynamical system considering under what conditions the convergence of this dynamical system predicts the outcome of the Extended Zermelo Model. In the following chapter we will focus on the Zermelo Model by giving a method for simplifying the derivation of Zermelo ratings for tournaments in terms of specific types of strongly connected components. Following this, the idea of stability of a tournament will be discussed and an upper bound will be obtained on the stability of three-team tournaments. Finally, we will conclude with some partial results related to the topics presented in the previous chapters.

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Contents

1 The Zermelo and Extended Zermelo Model

When we encounter a competition between either individuals or teams we will often want to determine who, in the tournament, should be ranked first, second, and so on. Owing to the fact that there is no consensus as to how this should be done many mathematical models have been proposed and devised to answer how one might rank the teams or objects, as we will be callings them, for an arbitrary tournament (e.g. [1] and [5]).

In 1929, Zermelo [8] developed a mathematical model for ranking by paired comparisons under the condition that the outcome matrix for the tournament was irreducible. When this condition held Zermelo was able to show that the ranking generated by his model was unique. However, when the outcome matrix was reducible it was possible for Zermelo's Model to yield only a partial ordering of the objects under consideration. Because of this limitation the Extended Zermelo model was adapted by Grant and Conner [3] in such a way as to naturally extend Zermelo's original model, making it possible to rank any given tournament. This Extended Zermelo Model respected Zermelo's Model in the sense that if a tournament could be ranked by Zermelo's Model, and did not involve ties, then the Extended Zermelo model would provide the same ranking of the tournament.

1.1 Zermelo's Model

To introduce Zermelo's Model consider n objects that are compared against each other in pairs where one of the two objects is determined to be superior to the other in each comparison. If we define the results of such comparisons to be a *tournament* or an *n-team tournament* a useful way of representing such a tournament is by constructing the matrix $A = (a_{ij})$ where a_{ij} or the ijth entry of the matrix is the number of times the ith object is judged superior to the jth. Note that by convention we will assume $a_{ii} = 0$ for all $i = 1, 2, \ldots, n$ or that no object can be judged better than itself. The matrix A or the *outcome matrix* of the tournament will be the way in which we will generally summarize results of tournaments in this thesis. (Note that this definition for a tournament differs from its normal use in which a tournament is defined to be a digraph where, between any two distinct vertices, there is exactly one edge.) Often we will refer to the tournament with matrix representation A simply as the tournament A. Since we will be dealing with such matrices throughout this paper it will be useful to make the following definition.

Definition 1. For $n \geq 2$ let T_n be the set of all $n \times n$ matrices whose diagonal entries are zero and whose off diagonal entries are nonnegative real numbers.

Remark 1. Note that the set T_n is precisely the set of matrices that are positive scalar multiples of matrix representations for the tournaments described above. Also it follows, therefore, that the matrix $A \in T_n$ may not have integer value entries. However, we will still refer to the matrix A as the tournament A.

A second useful way that we may use for depicting the tournament $A \in T_n$ with matrix $A \in T_n$ is that of a weighted digraph $\Gamma(A)$ with n vertices representing the n objects where the directed edge starting from vertex i and ending at vertex j with weight m represents that object i has been determined to be superior to object j m times. For example if we are considering the tournament A in which object 1 is judged better than object 2 once and better than object 2 twice we represent this information as the following matrix or the digraph.

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{1}$$

A

$$\begin{tikzcd} & 1 \arrow[dl, "1"'] \arrow[dr, "2"] & \\ 2 & & 3 \end{tikzcd}$$

In order to rank the objects of a given tournament Zermelo derived the functional

$$P(r) := \prod_{i,j=1}^{n} \left(\frac{r_i}{r_i + r_j} \right)^{a_{ij}} \tag{2}$$

where $r = (r_1, r_2, \ldots, r_n)$, by assuming that object i has a fixed strength or rating r_i and each comparison of objects is an independent binomial trial with the probability of finding object i superior to object j being $\frac{r_i}{r_i+r_j}$. In considering such ratings Zermelo adopted the convention, as we will, that $r_i \geq 0$ for all $1 \leq i \leq n$ or that r lies in the closed positive orthant of $\mathbb{R}^n$.

Definition 2. If the tournament is such that r can be chosen to maximize $P(r)$ the resulting strengths $(r_1, r_2, \ldots, r_n)$ are known as the Zermelo ratings for the tournament.

However, it is not always the case that such a maximizer of (2) exists, for a given tournament. To see this note that fairly simple tournaments exist such that, for r to maximize $P(r)$, r must have two components that are arbitrarily small i.e. two components which are zero. For example tournament (1) has the functional

$$P(r) := \left(\frac{r_1}{r_1+r_2}\right)^1 \left(\frac{r_1}{r_1+r_3}\right)^2 \left(\frac{r_2}{r_2+r_1}\right)^0 \left(\frac{r_2}{r_2+r_3}\right)^0 \left(\frac{r_3}{r_3+r_1}\right)^0 \left(\frac{r_3}{r_3+r_2}\right)^0$$

which, under the condition that no two of $r_1, r_2, r_3 = 0$ can be written as

$$P(r) := \left(\frac{r_1}{r_1+r_2}\right) \left(\frac{r_1}{r_1+r_3}\right)^2.$$

Note that for any $\delta, \epsilon > 0$ both

$$P((r_1, \delta, \epsilon)) < P((r_1, \delta/2, \epsilon))$$

$$P((r_1, \delta, \epsilon)) < P((r_1, \delta, \epsilon/2)).$$

Since $P(r)$ is undefined if $r_2 = r_3 = 0$, a maximizer of P does not exist implying that the Zermelo ratings for this tournament do not exist or that the Zermelo Model fails to rank this tournament.

In such cases Zermelo resorted to limit points $\bar{r}$ of maximizing sequences making the argument that if $P(\bar{r})$ existed then $\bar{r}$ was a maximizer of P. In the event that $\bar{r}$ contained two or more components, which were zero, Zermelo suggested that the objects should be ordered in a way corresponding to values of r close to $\bar{r}$ such that $P(r)$ exists. For this to yield a unique ranking, however, there must be a directed path between every pair of vertices in $\Gamma(A)$ (*see Section 2.2*). Since tournaments exist that do not have this property (for instance tournament (1)) even with this extension of his model, Zermelo was unable to uniquely rank all tournaments.

1.2 The Extended Zermelo Model

A fact proved by Keener [7] states that for the functional P to have a maximizer in the interior of the positive orthant of $\mathbb{R}^n$ the outcome matrix of the tournament must be irreducible.

An condition proved by [6] states that if the matrix A is irreducible then $\Gamma(A)$ is necessarily strongly connected. (That is, $\Gamma(A)$ is a directed graph such that from any vertex to any other vertex in the graph there exists a directed path made up of edges with positive weight from the one to the other.) With this in mind Conner and

Grant [3] have shown it is possible to construct an extension of Zermelo's Model by adding edges to the digraphs of tournaments of arbitrarily small weight. That is, for the tournament represented by the matrix A and a fixed $\epsilon > 0$ we consider the new tournament represented by the matrix $A(\epsilon)$ where we add ϵ to all the off diagonal entries of A. One reason we wish to consider the new tournament $A(\epsilon)$ is that it has a digraph that is necessarily strongly connected. Since this is the case for any $\epsilon > 0$, this is enough to guarantee that the Zermelo rankings for the tournament $A(\epsilon)$ exist. To formally define this idea, let

$$\gamma_{ij} := \begin{cases} 0 & if\ i = j, \\ 1 & otherwise \end{cases}.$$

Then, given an n-team tournament with matrix representation $A = (a_{ij})$, for a given $\epsilon > 0$ we define the tournament $A(\epsilon)$ by $A(\epsilon) := (a_{ij} + \gamma_{ij}\epsilon)$ with a maximizer $r = r(\epsilon)$ corresponding to the functional

$$P(r, \epsilon) := \prod_{i,j=1}^{n} (\frac{r_i}{r_i + r_j})^{a_{ij} + \gamma_{ij}\epsilon}. \tag{3}$$

Naturally, choosing ϵ to be large may cause the ranking of the objects in a given tournament $A(\epsilon)$ to be quite different from the ranking of the objects in the original tournament A. The question that merits consideration then, is under what conditions is this model an extension of the Zermelo Model? Since choosing ϵ to be small makes the difference between the two tournaments A and $A(\epsilon)$ small, the hope is that, first, the ranking vector $r(\epsilon)$ stabilizes in terms of ranking as $\epsilon \to 0^+$. Second, we would hope that for this to be an extension of Zermelo's Model that if a Zermelo ranking for the matrix representation A exists then this ranking is consistent with the ranking generated by $r(\epsilon)$ for small ϵ. To address these concerns we will need a way to distinguish the rankings generated by Zermelo's Model from that of the Extended Model.

Definition 3. The preorder induced by $r(\epsilon)$ denoted $\preceq_\epsilon$, is the relation between the elements of $\{1, \ldots, n\}$ satisfying $i \preceq_\epsilon j \;\Leftrightarrow\; r_i(\epsilon) \leq r_j(\epsilon)$.

If $i \preceq_\epsilon j$ and $j \not\preceq_\epsilon i$, then we write $i \prec_\epsilon j$.

In their article proposing this extension of Zermelo's Model, Conner and Grant [3] prove that there exists an $\epsilon_0 > 0$ such that $\preceq_\epsilon$ is independent of the choice of $\epsilon \in (0, \epsilon_0)$ and that so long as the Zermelo rating vector does not possess any ties then for $\epsilon \in (0, \epsilon_0)$, $r(\epsilon)_i \preceq r(\epsilon)_j$ if and only if $r_i \leq r_j$. That is, so long as there are no ties in the Zermelo rankings then the Extended Zermelo Model is an extension of the Zermelo Model and the rankings stabilize as $\epsilon \rightarrow 0^+$.

2 Dynamical Systems and the Extended Zermelo Model

In this chapter we introduce a discrete dynamical system with the intent of deriving results relating this dynamical system to the Extended Zermelo Model. As no one definition of a dynamical system exists we will develop our ideas of such systems based on generally accepted definitions and terms showing the dynamical system under consideration is consistent with these. Later, we present a conjecture relating the Extended Zermelo Model and our dynamical system motivating the conjecture's conclusion by proving some subcases associated with it.

2.1 A Discrete Dynamical System

As before we will let $r \in \mathbb{R}^n$ written as $r = (r_1, r_2, \ldots, r_n)$. Also, define Ω^n to be the set of points in $\mathbb{R}^n$ whose components are nonnegative and where no two of these components are zero. Then for $n \geq 2$ and $A \in T_n$ we will consider the function $F_A : \Omega^n \to \Omega^n$ defined by

$$F_A(r) := \begin{bmatrix} \frac{a_{12}+a_{13}+\cdots+a_{1n}}{\frac{a_{12}+a_{21}}{r_1+r_2}+\frac{a_{13}+a_{31}}{r_1+r_3}+\cdots+\frac{a_{1n}+a_{n1}}{r_1+r_n}} \\ \frac{a_{21}+a_{23}+\cdots+a_{2n}}{\frac{a_{21}+a_{12}}{r_2+r_1}+\frac{a_{23}+a_{32}}{r_2+r_3}+\cdots+\frac{a_{2n}+a_{n2}}{r_2+r_n}} \\ \vdots \\ \frac{a_{n1}+a_{n2}+\cdots+a_{n,n-1}}{\frac{a_{n1}+a_{1n}}{r_n+r_1}+\frac{a_{n2}+a_{2n}}{r_n+r_2}+\cdots+\frac{a_{n,n-1}+a_{n-1,n}}{r_n+r_{n-1}}} \end{bmatrix} \tag{4}$$

where we denote the ith component of F_A to be

$$F_{A_i}(r) = \frac{a_{i1} + a_{i2} + \cdots + a_{in}}{\frac{a_{i1}+a_{1i}}{r_i+r_1} + \frac{a_{i2}+a_{2i}}{r_i+r_2} + \cdots + \frac{a_{i,n}+a_{n,i}}{r_i+r_n}}. \tag{5}$$

Since the matrix A is fixed we will generally suppress the dependence of F_A on A by writing $F_A(r) = F(r)$ and $F_{A_i}(r) = F_i(r)$. Also, note that requiring F_A to map Ω^n to itself necessarily imposes conditions on the entries of the matrix A.

Definition 4. For $n \geq 2$ let Λ_n be the set of all $n \times n$ matrices $L \in T_n$ satisfying

(a) no two rows of L are zero and

(b) the same column and row of L are not zero.

Proposition 1. If $A \in \Lambda_n$, then $F_A : \Omega^n \to \Omega^n$.

proof: Let $A \in \Lambda_n$, $r \in \Omega^n$ and $i \neq j$. Since $r \in \Omega^n$ the fractions in both the denominators of F_i and F_j have themselves nonzero denominators. Also condition (b) implies that not all of the a_{ij}'s in the denominator of F_i and F_j are zero implying that both denominators must be greater than zero. Similarly, since no two rows of A are zero then the numerator of F_i and F_j must be nonnegative and cannot both be zero. It follows then that both F_i and F_j are nonnegative and that at most one is zero for all $i \neq j$. Therefore, $F_A : \Omega^n \to \Omega^n$.

Proposition 2. If $F_A : \Omega^n \to \Omega^n$ and $A \in T_n$ then $A \in \Lambda_n$.

proof: Suppose for $A \in T_n$ that $F_A : \Omega^n \to \Omega^n$. For $i \neq j$ if both row i and j are zero, equation (5) implies that $F_i = F_j = 0$. Since no two components of any point of Ω_n are zero then F_A is not a function from Ω^n to Ω^n. Since this contradicts the assumption, no two rows of A can be zero. Similarly, if row i and column i are zero, equation (5) is

$$F_{A_i}(r) = \frac{0 + 0 + \cdots + 0}{\frac{0+0}{r_i+r_1} + \frac{0+0}{r_i+r_2} + \cdots + \frac{0+0}{r_i+r_n}} = \frac{0}{0}$$

or undefined, which is a contradiction to $F_A : \Omega^n \to \Omega^n$ so the same column and row cannot both be zero. $\square$

Remark 2. If we let A be an n-team tournament, condition (a) in the previous definition can be interpreted as the condition that no two teams in the tournament have 0 wins. Similarly, condition (b) implies that every team is genuinely involved in the tournament; i.e., every team wins or loses at least one game.

Summarizing the results of Proposition 1 and Proposition 2 with the remark, it follows that the tournaments, where every team is involved and no two teams have 0 wins, are exactly those tournaments with matrices A such that $F_A : \Omega^n \to \Omega^n$. In general, we will associate a given n-team tournament $(a_{ij}) = A$ with the function F_A.

If $A \in \Lambda_n$, it follows from Lemma 1 that the composition $F(F(r_0))$ is defined for all $r_0 \in \Omega^n$. If we then define the inductive identity $F^{\ell+1}(r_0) := F(F^{\ell}(r_0))$ for $\ell \in \{0, 1, 2, \dots\}$ and $r_0 \in \Omega^n$, where $F^0(r_0) = r_0$, the claim is that this generates a dynamical system with phase space Ω^n in which, for every nonnegative integer ℓ, the orbit of r_0 is the set $\{F^{\ell}(r_0) : \ell \in \mathbb{N}\}$.

Definition 5. A dynamical system (X, ϕ) consists of a metric space X and a continuous mapping ϕ that maps elements of the space X to itself.

Under the assumption that $A \in \Lambda_n$, Proposition 1 implies that $F_A : \Omega^n \to \Omega^n$ where Ω^n is a metric space under the standard metric for $\mathbb{R}^n$. To verify the claim that (Ω^n, F) is a dynamical system, note the only condition needing to be satisfied is that F_A is continuous on Ω^n. We therefore prove the following lemma.

Lemma 3. *If $A \in \Lambda_n$ then F_A is a continuous function on Ω^n.*

It follows directly from the proof of Proposition 1 that this is in fact the case. Therefore from Proposition 1 and Lemma 3, (Ω^n, F_A) satisfies the definition for a dynamical system under the condition that $A \in \Lambda_n$.

However, as the function F may not have an inverse, the system generated by iterating F is sometimes referred to as a *semi-dynamical system*. For instance the 2-team tournament in which team one and team two beat the other once has the associated function

$$F(r) = \begin{bmatrix} r_1 + r_2 \\ r_2 + r_1 \end{bmatrix}.$$

Since there exist infinitely many vectors in Ω^2 whose first and second component sum to the same number then F is not a $1-1$ function and is therefore not invertible.

2.2 Weakly Path-Connected Tournaments

In this section we wish to more fully motivate the connection between the n-team tournament with matrix representation $(a_{ij}) = A$ and the dynamical system generated by the function $F_A(r)$. To do this we will limit our focus to a smaller class of tournaments (and therefore matrices) whose properties will allow us to make conclusions about both the dynamical system (Ω^n, F) and the rankings produced by the Extended Zermelo Model.

Definition 6. The digraph $\Gamma(A)$ will be said to be weakly path-connected if for any two distinct vertices in $\Gamma(A)$ there is a directed path from one to the other. We denote the set P_n to be the set of all $n \times n$ matrices in T_n having weakly path-connected digraphs.

Theorem 4. *If $P \in P_n$ then $P \in \Lambda_n$.*

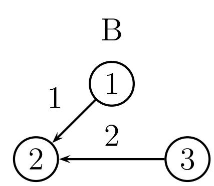
proof: Suppose by way of contradiction that $P \in P_n$ but that the matrix P has two rows i and j that are zero. If this is the case then $\Gamma(P)$ has two vertices i and j that have no directed edges going from them to any other vertex in the graph. Therefore, there can exist no directed path from vertex i to vertex j, which contradicts the supposition that P is weakly path-connected. Therefore, P must fulfil condition (a) of Definition 4.

Similarly, suppose for some $1 \leq i \leq n$ that column and row i of P are 0. This requirement would imply that there are no edges connecting vertex i to any other vertex in $\Gamma(P)$ or that no path exists from vertex i to any other vertex in $\Gamma(P)$ or vice-versa implying that $P \notin P_n$. Since this is a contradiction P must fulfil condition (b) of Definition 4. Since $P \in T_n$ and fulfils conditions (a) and (b) of Definition 4 then $P \in \Lambda_n$. $\square$

Note that the reverse inclusion is not necessarily true. That is, if a matrix is an element of Λ_n it may not be weakly path-connected. The matrix B given by

$$B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix} \tag{6}$$

is an element of Λ_n. However, in the associated digraph $\Gamma(B)$



there is no directed path from the first to the third vertex and vice-versa implying $B \notin P_n$. To summarize these results it follows that for $n \geq 2$ that $P_n \subsetneq \Lambda_n \subsetneq T_n$.

As we will be concerned with the dynamical system generated by (Ω^n, F) for the remainder of this chapter we will concentrate our discussion on the object $F^\ell(r_0)$ for $\ell \in \mathbb{N}$. Specifically, we will consider the convergence or possible divergence of $\lim_{\ell\to\infty} F_A^\ell(r_0)$ as this vector's limiting behavior relates to ranking.

Definition 7. If $r \in \Omega^n$ then define $r^k := F^k(r)$ for $k \in \mathbb{N}$. Similarly, for $1 \leq i \leq n$ define $r_i^k := F_i^k(r)$ for $k \in \mathbb{N}$.

Definition 8. For a fixed $\kappa \in \mathbb{N}$ if $r_i^m > r_j^m$ implies that $r_i^{m+1} > r_j^{m+1}$ for all $m \geq \kappa$ and $1 \leq i, j \leq n$ we will say the vector *stabilizes in order* under iteration.

If a vector $r \in \Omega^n$ does stabilize in order, for some fixed $\kappa \in \mathbb{N}$ in Definition 8 define the vector $r^\infty := r^\kappa$ denoting $r^\infty = (r_1^\infty, r_2^\infty, \ldots, r_n^\infty)$.

Definition 9. The preorder induced by r^∞ denoted $\tilde{\leq}$, is the relation between the elements of $\{1, \ldots, n\}$ satisfying $i \mathrel{\tilde{\leq}} j \Leftrightarrow r_i^\infty \leq r_j^\infty$.

Again we write $i \mathrel{\tilde{<}} j$ or $j \mathrel{\tilde{>}} i$ if $i \mathrel{\tilde{\leq}} j$ and $j \mathrel{\tilde{\nleq}} i$.

We are now in a position to relate the Extended Zermelo Model to the dynamical system (Ω^n, F).

Conjecture : Suppose that $A = (a_{ij})$ is the matrix representation of an n-team tournament where $A \in P_n$ and that the Extended Zermelo rating for the tournament does not contain a tie. Then for all $r \in \Omega^n$

(1) the vector r stabilizes in order under iteration of the dynamical system (Ω^n, F)

(2) and the preorder induced by r^∞ and the ranking given by the extended Zermelo Model of the objects $\{1, \ldots, n\}$ are the same.

Note the condition that $A \in P_n$ is necessary since simple examples exist of tournaments that are not weakly path-connected, which produce different rankings for differing initial conditions. For example the tournament B given by (6) has the associated function

$$F_N(r) = \begin{bmatrix} r_1 + r_2 \\ 0 \\ r_2 + r_3 \end{bmatrix}.$$

For the initial condition $r_0 = (1, 0, 2)$, since $F_N(r_0) = r_0$ it follows that $r_0^\infty = r_0$. Therefore, the preorder induced by the initial condition r_0 is given by $2 \mathrel{\tilde{>}} 1 \mathrel{\tilde{>}} 3$.

Similarly, for the initial condition $s_0 = (2, 0, 1)$ it follows that $s_0^\infty = s_0$, or the ranking induced by the initial condition s_0 is $1 \mathrel{\tilde{>}} 2 \mathrel{\tilde{>}} 3$. Since the rankings induced by r_0^∞ and s_0^∞ are different, the conclusion of the conjecture does not hold for the tournament B. Therefore, for the conjecture to hold it is necessary that $A \in P_n$. However, having this necessary condition that $A \in P_n$ in no way gives us a natural way of showing the hypotheses of the conjecture are sufficient conditions to guarantee the conjecture's conclusion. We leave the proof of the conjecture as an open problem.

Owing to the complexity of the dynamical system (Ω^n, F) we will not prove the conjecture as it is stated above but will prove the result for specific classes of tournaments.

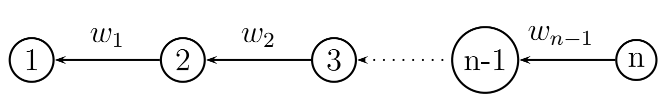
2.4 Some Simple Tournaments

Definition 10. Let W_n be the set of all matrices of the form

$$\begin{bmatrix} 0 & 0 & & & \\ w_1 & 0 & 0 & & \\ 0 & w_2 & \ddots & \ddots & \\ & \ddots & \ddots & 0 & 0 \\ & & 0 & w_{n-1} & 0 \end{bmatrix}$$

where $w_i > 0$ for all $1 \leq i \leq n-1$.

Since the digraph of any matrix $W \in W_n$ has the form



then there is a directed path from team n to team i for $1 \leq i \leq n-1$; this implies that $W_n \subseteq P_n$. It follows from previous results that for any $W \in W_n$ the iterations of F_W of the initial condition $r_0 \in \Omega^n$ generate a dynamical system with phase space Ω^n.

Theorem 5. *If $W \in W_n$ then for all $r \in \Omega^n$*

(1) *the vector r stabilizes in order under iteration and*

(2) *the preorder induced by r^∞ and the ranking given by the extended Zermelo Model of the objects $\{1, \ldots, n\}$ are the same.*

Note that this is the previous conjecture restricted to a smaller class of tournaments, namely W_n, without the hypothesis that the Extended Zermelo Ranking contains no ties.

Remark 3. The reason for this missing hypothesis follows from a result from Conner and Grant. For $W \in W_n$ note that for every $1 \leq j < i \leq n$ there is a directed path from vertex i to vertex j but not in the other direction. Theorem 3.3 of [3] implies that in the Extended Zermelo Ranking that $j \prec i$. Since this is true for every $j < i$ then the Extended Zermelo Model induces the strict ranking $1 \prec 2 \prec \cdots \prec n$.

2.3 Proving Convergence

In order to prove theorem 5 consider the change of coordinates given by the function

$$H_W(r) := \begin{bmatrix} 0 \\ w_2 \frac{r_2}{r_2+r_3} \\ w_3 \frac{r_3}{r_3+r_4} \\ \vdots \\ w_{n-1} \frac{r_{n-1}}{r_{n-1}+r_n} \\ 0 \end{bmatrix} \tag{7}$$

where as before $r = (r_1, r_2, \ldots, r_n)$ is in the set Ω^n and w_i is the corresponding entry in the matrix $W \in W_n$. As before we will suppress the dependence of H_W on W and will simply write H.

Note that the ith component of H can be written as $H_i(r) = w_i \frac{r_i}{r_i + r_{i+1}}$ with the convention that $r_1 = 0$ and $w_n = 0$. The reason $r_1 = 0$ follows from the fact that for any matrix $W \in W_n$, row 1 of W is zero. Therefore, $F_{W_1}(r) = 0$ for any $r \in \Omega^n$. On the other hand letting $w_n = 0$ is for the ease of discussion since it allows the conjugate function H to map a subset of $\mathbb{R}^n$ to another subset in $\mathbb{R}^n$. (We will show later that we lose no infirmation by making this assumption.)

Remark 4. Since only one component of $r \in \Omega^n$ can be zero it follows that $F_i(r) > 0$ for $2 \leq i < n$. This implies in particular that since $H_i(r) = w_i \frac{r_i}{r_i + r_{i+1}}$ and $w_n = 0$ then $H_i(r) < w_i$ for $2 \leq i \leq n-1$.

We now define the set C^n to be the set

$$C^n := \{c = (c_1, c_2, c_3, \ldots, c_n) | 0 \leq c_i \leq w_i,\ c_1 = c_n = 0\}$$

with the added condition that $c_{i+1} \neq 0$ if $c_i = w_i$. Since $0 \leq H_i(r) < w_i$ for $2 \leq i \leq n-1$ then H has the property that $H : \Omega^n \to C^n$ for all $n \geq 2$.

In order to consider the dynamical system (Ω^n, F) under this change of coordinates we seek a function $G : C^n \to C^n$ such that for every $r \in \Omega^n$, $H(F(r)) = G(H(r))$. That is, we need the function G that fits the commutative diagram in Figure 1.

Note the the function H is know as a conjugate function since for any $r \in \Omega^n$ it has the property $H(F^\ell(r)) = G^\ell(H(r))$ for every $\ell \in \mathbb{N}$. (That is, taking an iterate of $r \in \Omega^n$ and mapping the result to C^n by the function H is the same as first mapping r to C^n by H and thereafter iterating with the function G.)

After some calculations the function G satisfying these conditions can be found

$$\begin{tikzcd}
\Omega^n \arrow[r, "H"] \arrow[d, "F"'] & C^n \arrow[d, "G"] \\
\Omega^n \arrow[r, "H"] & C^n
\end{tikzcd}$$

Fig. 1

to be

$$G(c) := \begin{bmatrix} 0 \\ \frac{c_2 w_1 w_2(-c_2+c_3+w_2)}{(-c_1+w_1)w_2^2 - c_2^2(w_1+w_2)+c_2(c_3w_1+w_2(c_1+w_2))} \\ \vdots \\ \frac{c_{n-1}w_{n-1}w_{n-2}(-c_{n-1}+c_n+w_{n-1})}{(-c_{n-2}+w_{n-2})w_{n-1}^2 - c_{n-1}^2(w_{n-2}+w_{n-1})+c_{n-1}(c_nw_{n-2}+w_{n-1}(c_{n-2}+w_{n-1}))} \\ 0 \end{bmatrix}$$

where $c = (c_2, c_3, \dots, c_n) \in C^n$ and the ith component of G is denoted by

$$G_i(c) = \frac{c_i w_i w_{i-1}(-c_i + c_{i+1} + w_i)}{(-c_{i-1} + w_{i-1})w_i^2 - c_i^2(w_{i-1} + w_i) + c_i(c_{i+1}w_{i-1} + w_i(c_{i-1} + w_i))}.$$

Note that G_i can be viewed as the function $G_i : C_i^3 \to C_i$ where $C_i^3 = \{(c_{i-1}, c_i, c_{i+1})\}$ and $C_i = \{c_i\}$(needs to be fixed) under the restrictions that $(c_1, \dots, c_{i-1}, c_i, c_{i+1}, \dots, c_n) \in C^n$. It will be important to note that by our previous calculations that $w_i \geq c_i$ for all $1 \leq i \leq n$. Also for ease of discussion we will commonly use the notation $G(\mathbf{a}) = G(a, a, \dots, a)$ and $G_i(c) = G_i(c_{i-1}, c_i, c_{i+1})$.

The claim will be that, under iteration, the function G produces a dynamical system that has the property $\lim_{\ell\to\infty} G^\ell(c) = \mathbf{0}$ for any $c \in C$. In order to show that this is the case we will use comparison principles that are generally associated with the study of partial differential equations [2]. That is, in the space C^n we will first show that locally the function has a maximum principle that allows us to bound its behavior under iteration. Second, we will consider $\limsup_{\ell\to\infty}\{G^\ell(c)\}$, for $c \in C^n$, showing that for all $c \in C^n$, $\limsup_{\ell\to\infty}\{G^\ell(c)\} = \mathbf{0}$ implying that $\lim_{\ell\to\infty} G^\ell(c) = \mathbf{0}$.

Once this has been established it will follow for all $r \in \Omega^n$ that the vector r stabilizes under iteration of the original dynamical system given by (Ω^n, F), inducing the rankings given by the Extended Zermelo Model for the tournament with matrix W.

In order to establish the claim that G generates the dynamical system (C^n, G) under iteration we will first prove that G is continuous on C^n and second that $G : C^n \to C^n$.

Proposition 6. The function G is continuous on the set C^n.

proof: Since the function $G_i(c)$ is the composition of continuous functions, discontinuity can only result from its denominator being zero at some point in the set C^n. However, given that $c \in C^n$ suppose $c_{i+1} = 0$. By definition then $c_i < w_i$. Since $c_i < w_i$ and $0 \leq c_i$ for $1 \leq i \leq n$ then the following inequality holds.

$$(w_i^2 - c_i w_i) + (w_{i-1}w_i - w_{i-1}c_i) + c_{i+1}w_{i-1} > 0.$$

If $c_{i+1} \neq 0$. then $c_{i+1}w_{i-1} > 0$ and the inequality above similarly holds.

It follows then that

$$(w_i^2 - c_i w_i) + (w_{i-1}w_i - w_{i-1}c_i) + c_{i+1}w_{i-1} > 0 \Rightarrow$$

$$(-c_{i-1}w_i + w_{i-1}w_i) - (c_i w_{i-1} - c_{i+1}w_{i-1} - c_{i-1}w_i + c_i w_i - w_i^2) > 0 \Rightarrow$$

$$-c_{i-1}w_i + w_{i-1}w_i > c_i w_{i-1} - c_{i+1}w_{i-1} - c_{i-1}w_i + c_i w_i - w_i^2.$$

Since $w_i \geq c_i$ and the left hand side of this inequality is positive, then multiplying the left hand side of this by w_i and the right hand side by c_i implies that

$$w_i^2(-c_{i-1} + w_{i-1}) > c_i(-c_{i+1}w_{i-1} - w_i(c_{i-1} + w_i) + c_i(w_{i-1} + w_i)) \Rightarrow$$

$$w_i^2(-c_{i-1}+w_{i-1})-(c_i(-c_{i+1}w_{i-1}-w_i(c_{i-1}+w_i)+c_i(w_{i-1}+w_i)))>0 \Rightarrow$$

$$(-c_{i-1}+w_{i-1})w_i^2-c_i^2(w_{i-1}+w_i)+c_i(c_{i+1}w_{i-1}+w_i(c_{i-1}+w_i))>0.$$

Note that the left hand side of this last inequality is equal to the denominator of G_i implying that G_i is continuous on its domain C_i^3. Since this is true for every $1 \leq i \leq n$ then G must be continuous on C^n. $\square$

To show that $G : C^n \to C^n$ we first prove the following lemma.

Lemma 7. *For the function G_i, given above, (**1**)$\frac{\partial G_i}{\partial c_{i-1}} \geq 0$, (**2**)$\frac{\partial G_i}{\partial c_{i+1}} \geq 0$, and (**3**) $\frac{\partial G_i}{\partial c_i} > 0$ at any point in the domain C^n of G_i.*

proof(**1**): From the definition of G

$$\frac{\partial G_i}{\partial c_{i-1}} = \frac{c_i w_{i-1}(w_i-c_i)(-c_i+c_{i+1}+w_i)w_i^2}{((-c_{i-1}+w_{i-1})w_i^2-c_i^2(w_{i-1}+w_i)+c_i(c_{i+1}w_{i-1}+w_i(c_{i-1}+w_i)))^2}$$

Note that by assumption $0 \leq c_i, w_i$ and $c_i \leq w_i$ for $1 < i < n$. It follows then that $(w_i - c_i) \geq 0$ as well as $(-c_i + c_{i+1} + w_i) > 0$. Since the numerator of the partial derivative is the product of nonnegative numbers it must therefore be nonnegative. Also, by Proposition 6 the denominator is a nonzero number squared so it must be strictly positive. Therefore, $\frac{\partial G_i}{\partial c_{i-1}} \geq 0$. $\square$

proof(**2**): Similarly,

$$\frac{\partial G_i}{\partial c_{i+1}} = \frac{c_i w_{i-1}(w_i-c_i)(-c_{i-1}+c_i+w_{i-1})w_i^2}{(-c_{i-1}+w_{i-1})w_i^2-c_i^2(w_{i-1}+w_i)+c_i(c_{i+1}w_{i-1}+w_i(c_{i-1}+w_i))^2}.$$

Again since $(w_i - c_i) \geq 0$ then the numerator of the derivative is nonnegative. The denominator is the same as in part (**1**) so is strictly positive, implying that $\frac{\partial G_i}{\partial c_{i+1}} \geq 0$ over the set C. $\square$

proof (**3**): The derivative in this case is slightly more complicated, given by

$$\frac{\partial G_i}{\partial c_i} = \frac{w_{i-1}w_i^2(c_i^2(c_{i+1}+w_{i-1}-c_{i-1})+w_i(w_{i-1}-c_{i-1})(w_i-2c_i+c_{i+1}))}{(-c_{i-1}+w_{i-1})w_i^2-c_i^2(w_{i-1}+w_i)+c_i(c_{i+1}w_{i-1}+w_i(c_{i-1}+w_i))^2}.$$

Since the denominator is positive, by Proposition 6, we will focus on the numerator denoting it by $N(c_{i-1}, c_i, c_{i+1})$.

Given that $0 \le c_i \le w_i$ for $1 < i < n$, note that

$$\frac{\partial N}{\partial c_{i+1}} = w_{i-1}w_i^2(c_i^2 + (-c_{i-1}+w_{i-1})w_i).$$

Suppose that $w_{i-1} > c_{i-1}$. In this case since $(-c_{i-1}+w_{i-1}) > 0$ it follows that $\frac{\partial N}{\partial c_{i+1}} > 0$. On the other hand supposing that $w_{i-1} = c_{i-1}$ implies that $c_i \neq 0$ since the point $(c_{i-1}, c_i, c_{i+1}) \in C_i^3$. Therefore, $\frac{\partial N}{\partial c_{i+1}} = w_{i-1}w_i^2(c_i^2 + (-c_{i-1}+w_{i-1})w_i) > 0$.

In either case since $\frac{\partial N}{\partial c_{i+1}} > 0$ and $N(c_{i-1}, c_i, 0) = w_{i-1}w_i^2$ is strictly positive, then for any $(c_{i-1}, c_i, c_{i+1}) \in C_i^3$ we have $N(c_{i-1}, c_i, c_{i+1}) > N(c_{i-1}, c_i, 0) > 0$, which implies that the N is strictly positive. Therefore, $\frac{\partial G_i}{\partial c_i} > 0$. $\square$

Remark 5. Note that in parts (**1**) and (**2**) of the previous lemma that if c_i is not equal to w_i or 0 that the strict inequalities $\frac{\partial G_i}{\partial c_{i-1}} > 0$, $\frac{\partial G_i}{\partial c_{i+1}} > 0$ hold.

Lemma 8. *The function G has the property that $G : C^n \to C^n$.*

proof: Since the point (w_{i-1}, w_i, w_{i+1}) is in the set C_i^3 and G_i is defined at this point Lemma 7 implies that $G_i(w_{i-1}, w_i, w_{i+1}) \ge G_i(c)$ for all $c \in C_i^3$. A simple computation shows that $G_i(w_{i-1}, w_i, w_{i+1}) = w_i$ implying that $w_i \ge G_i(c_{i-1}, c_i, c_{i+1})$ for all $c \in C^n$.

A similar calculation shows $G(\mathbf{0}) = \mathbf{0}$. Lemma 7 then implies that $G(\mathbf{0}) \le G(c)$ for all $c \in C_i^3$. Therefore, $0 \le G_i(c) \le w_i$ for all $c \in C_i^3$.

To prove that $G_{i+1}(c) \neq 0$ if $G_i(c) = w_i$ note that if $(c_{i-1}, w_i, c_{i+1}) \in C_i^3$ then $G_i(c_{i-1}, w_i, c_{i+1}) = \frac{w_i c_{i+1}}{c_{i+1}}$ which is equal to w_i since, by assumption $c_{i+1} \neq 0$. From the previous remark, given that the partial derivative with respect to c_i is strictly positive on the interval $(0, w_i)$, it follows that $G_i(c_{i-1}, c_i, c_{i+1}) < G_i(c_{i-1}, w_i, c_{i+1}) = w_i$ for $c_i \neq w_i$. Therefore, if $G_i(c) = w_i$ then $c_i = w_i$.

Under the supposition this implies that $c_{i+1} \neq 0$. Therefore, consider the equation

$$G_{i+1}(c_i, c_{i+1}, c_{i+2}) = \frac{c_{i+1} w_i w_{i+1}(w_{i+1} - c_{i+1} + c_{i+2})}{-c_{i+1}^2(w_i + w_{i+1}) + c_{i+1}(c_{i+2} w_i + w_{i+1}(w_i + w_{i+1}))}. \tag{8}$$

Since we have already proved the denominator in (8) is positive note that since $c_{i+1} > 0$ then the only way the numerator is zero is if both $w_{i+1} = c_{i+1}$ and $c_{i+2} = 0$. However, this would violate our assumption that $c \in C^n$ so the numerator in (8) is positive implying that if $G_i(c) = w_i$ then $G_{i+1}(c) \neq 0$.

Since the only conditions placed on the components of points in C^n was that $0 \leq c_i \leq w_i$ and if $c_i = w_i$ then $c_{i+1} \neq 0$ it follows that $G : C^n \to C^n$. $\square$

Theorem 9. *The function (C^n, G) defines a dynamical system.*

proof: To prove (C^n, G) is a dynamical system we need only show, first that G is continuous on its domain C^n, and second that G maps the metric space C^n to itself. That G is continuous on C^n was the result of Proposition 6. Similarly, the condition that $G : C^n \to C^n$ was proved in the previous lemma. $\square$

Now that we have established that (C^n, G) is in fact a dynamical system we wish to consider a local property of this system. That is, for the function $G_i : C_i^3 \to C$ we will show that for any $(c_{i-1}, c_i, c_{i+1}) \in C_i^3$ that $G_i(c_{i-1}, c_i, c_{i+1}) \leq \max\{c_{i-1}, c_i, c_{i+1}\}$. The approach that seems best to prove this will be to consider the statement restricted to different cases, which we will summarize into the following lemmas.

Lemma 10. *Supposing that $c_i \geq c_{i-1}, c_{i+1}$ then $G_i(c_{i-1}, c_i, c_{i+1}) \leq max\{c_{i-1}, c_i, c_{i+1}\}$.*

proof: Assuming $c_i \geq c_{i-1}, c_{i+1}$ implies that

$$(c_i w_{i-1} - c_{i+1} w_{i-1}) + (-c_{i-1} w_i + c_i w_i) \geq 0 \Rightarrow$$

$$-c_{i+1} w_{i-1} - c_{i-1} w_i + c_i (w_{i-1} + w_i) \geq 0 \Rightarrow$$

$$c_i (c_i - w_i)(-c_{i+1} w_{i-1} - c_{i-1} w_i + c_i (w_{i-1} + w_i)) \leq 0 \Rightarrow$$

$$\begin{aligned} &c_i w_{i-1} w_i (-c_i + c_{i+1} + w_i) \\ &\quad - c_i((-c_{i-1} + w_{i-1}) w_i^2 - c_i^2 (w_{i-1} + w_i) + c_i (c_{i+1} w_{i-1} + w_i (c_{i-1} + w_i))) \leq 0. \Rightarrow \end{aligned}$$

$$\begin{aligned} &c_i w_{i-1} w_i (-c_i + c_{i+1} + w_i) \leq \\ &\qquad c_i((-c_{i-1} + w_{i-1}) w_i^2 - c_i^2 (w_{i-1} + w_i) + c_i (c_{i+1} w_{i-1} + w_i (c_{i-1} + w_i))). \end{aligned}$$

The proof of Proposition 6 implies that the right hand side of this inequality is positive. Dividing by the right hand side without the first c_i yields

$$\frac{c_i w_{i-1} w_i (-c_i + c_{i+1} + w_i)}{(-c_{i-1} + w_{i-1}) w_i^2 - (c_i^2 (w_{i-1} + w_i) + c_i (c_{i+1} w_{i-1} + w_i (c_{i-1} + w_i)))} \leq c_i \Rightarrow$$

$$G_i(c_{i-1}, c_i, c_{i+1}) \leq c_i.$$

By assumption $c_i = \max\{c_{i-1}, c_i, c_{i+1}\} \Rightarrow G_i(c_{i-1}, c_i, c_{i+1}) \leq \max\{c_{i-1}, c_i, c_{i+1}\}$. $\square$.

Lemma 11. *Supposing that* $c_{i-1} \geq c_i, c_{i+1}$ *then* $G_i(c_{i-1}, c_i, c_{i+1}) \leq max\{c_{i-1}, c_i, c_{i+1}\}$.

proof: Assuming that $c_{i-1} \geq c_i, c_{i+1}$ consider the following cases.

Case 1: Suppose $w_i \leq c_{i-1}$. Recall that $G_i(c_{i-1}, c_i, c_{i+1}) \leq w_i$ for all $(c_{i-1}, c_i, c_{i+1}) \in C_i^3$. Therefore, the condition that $w_i \leq c_{i-1}$ then implies that $G_i(c_{i-1}, c_i, c_{i+1}) \leq c_{i-1} = \max\{c_{i-1}, c_i, c_{i+1}\}$.

Case 2: Suppose $w_i > c_{i-1}$ and $w_{i+1} > c_{i-1}$. Note that for any point $(\zeta, \zeta, \zeta) \in C_i^3$

that $G_i(\zeta,\zeta,\zeta)=\zeta$. Since $c_{i-1}<w_i,w_{i+1}$ then $(c_{i-1},c_{i-1},c_{i-1})\in C_i^3$ so $G_i(c_{i-1},c_{i-1},c_{i-1})=c_{i-1}$. Given $c_{i-1}\geq c_i,c_{i+1}$, Lemma 7 implies that $G_i(c_{i-1},c_{i-1},c_{i-1})=c_{i-1}\geq G_i(c_{i-1},c_i,c_{i+1})$ or that $G_i(c_{i-1},c_i,c_{i+1})\leq c_{i-1}=\max\{c_{i-1},c_i,c_{i+1}\}$.

Case 3: Suppose that $w_i>c_{i-1}$ and $w_{i+1}\leq c_{i-1}$. Note that under these conditions

$$G_i(c_{i-1},c_{i-1},w_{i+1})=\frac{c_{i-1}w_i(w_{i-1}+w_{i+1}-c_{i-1})}{w_i^2+c_{i-1}w_{i+1}-c_{i-1}^2}$$

which is defined for all c_{i-1} since $w_i^2+c_{i-1}w_{i+1}-c_{i-1}^2>0$. Also, given $w_i>c_{i-1}$ and $w_{i+1}\leq c_{i-1}$ it follows that

$$c_{i-1}(w_i-c_{i-1})(c_{i-1}-w_{i+1})\geq 0\Rightarrow$$

$$c_{i-1}(w_i(c_{i-1}-w_i-w_{i+1})-(c_{i-1}^2-w_i^2-c_{i-1}w_{i+1}))\geq 0\Rightarrow$$

$$c_{i-1}w_i(c_{i-1}-w_i-w_{i+1})\geq c_{i-1}(c_{i-1}^2-w_i^2-c_{i-1}w_{i+1})\Rightarrow$$

$$c_{i-1}w_i(w_i+w_{i+1}-c_{i-1})\leq c_{i-1}(w_i^2+c_{i-1}w_{i+1}-c_{i-1}^2)\Rightarrow$$

$$\frac{c_{i-1}w_i(w_i+w_{i+1}-c_{i-1})}{w_i^2+c_{i-1}w_{i+1}-c_{i-1}^2}\leq c_{i-1}\Rightarrow G_i(c_{i-1},c_{i-1},w_{i+1})\leq c_{i-1}.$$

Since $G_i(c_{i-1},c_{i-1},c_{i+1})\leq G_i(c_{i-1},c_i,w_{i+1})$ by Lemma 7, then $G_i(c_{i-1},c_i,c_{i+1})\leq c_{i-1}$.

Since this is the result of the previous three cases it follows that if $c_{i-1}\geq c_i,c_{i+1}$ then $G_i(c_{i-1},c_i,c_{i+1})\leq\max\{c_{i-1},c_i,c_{i+1}\}$. $\square$

Lemma 12. *Supposing that* $c_{i+1}\geq c_{i-1},c_i$ *then* $G_i(c_{i-1},c_i,c_{i+1})\leq max\{c_{i-1},c_i,c_{i+1}\}$.

proof: Assuming that $c_{i+1}\geq c_{i-1},c_i$ consider the following cases.

Case 1: Suppose $w_i\leq c_{i+1}$. If $c_{i+1}=0$ then $c_{i-1}=c_i=c_{i+1}=0$ or $G_i(c_{i-1},c_i,c_{i+1})=0$ and the result follows. If $c_{i+1}\neq 0$ then Lemma 7 implies that $G_i(c_{i-1},c_i,c_{i+1})\leq G_i(c_{i-1},w_i,c_{i+1})=w_i$. The condition that $w_i\leq c_{i+1}$ in turn implies that $G_i(c_{i-1},c_i,c_{i+1})\leq c_{i+1}$ or $G_i(c_{i-1},c_i,c_{i+1})\leq c_{i+1}=\max\{c_{i-1},c_i,c_{i+1}\}$.

Case 2: Suppose $w_i>c_{i+1}$ and $w_{i-1}>c_{i+1}$. Since $c_{i+1}<w_i,w_{i+1}$ then $(c_{i+1},c_{i+1},c_{i+1})\in$

C_i^3 and $G_i(c_{i+1}, c_{i+1}, c_{i+1}) = c_{i+1}$. Given $c_{i+1} \geq c_{i-1}, c_i$, Lemma 7 implies that $G_i(c_{i+1}, c_{i+1}, c_{i+1}) = c_{i+1} \geq G_i(c_{i-1}, c_i, c_{i+1})$ or that $G_i(c_{i-1}, c_i, c_{i+1}) \leq c_{i+1} = \max\{c_{i-1}, c_i, c_{i+1}\}$.

Case 3: Suppose that $w_i > c_{i+1}$ and $w_{i-1} \leq c_{i+1}$. Under these conditions

$$G_i(w_{i-1}, c_{i+1}, c_{i+1}) = \frac{w_{i-1}w_i}{w_{i-1} + w_i - c_{i+1}}$$

which is defined for all c_{i+1} since $w_{i-1} + w_i - c_{i+1} > 0$. Also, given $w_{i-1} \leq c_{i+1}$ and $w_i > c_{i+1}$ it follows that

$$(w_{i-1} - c_{i+1})(w_i - c_{i+1}) \leq 0 \Rightarrow$$

$$w_{i-1}w_i - w_{i-1}c_{i+1} - c_{i+1}w_i + c_{i+1}^2 \leq 0 \Rightarrow$$

$$w_{i-1}w_i \leq c_{i+1}(w_{i-1} + w_i - c_{i+1}) \Rightarrow$$

$$\frac{w_{i-1}w_i}{w_{i-1} + w_i - c_{i+1}} \leq c_{i+1} \Rightarrow G_i(c_{i-1}, c_{i-1}, w_{i+1}) \leq c_{i+1}.$$

Since $G_i(c_{i-1}, c_i, c_{i+1}) \leq G_i(w_{i-1}, c_{i+1}, c_{i+1})$ by Lemma 7, then $G_i(c_{i-1}, c_i, c_{i+1}) \leq c_{i+1}$.

Since this is the result of the previous three cases it follows that if $c_{i+1} \geq c_{i-1}, c_i$ then $G_i(c_{i-1}, c_i, c_{i+1}) \leq \max\{c_{i-1}, c_i, c_{i+1}\}$. $\square$

Theorem 13. *For every* $(c_{i-1}, c_i, c_{i+1}) \in C_i^3$ $G(c_{i-1}, c_i, c_{i+1}) \leq max\{c_{i-1}, c_i, c_{i+1}\}$.

proof: This is the result of the previous three lemmas.

We will refer to this theorem as the fact that the dynamical system (C^n, G) obeys a local maximum principle. Also useful fact that follows from this theorem is the following corollary

Corollary : For $n \in \mathbb{N}$ let L be the set of consecutive numbers $\{i, i+1, \ldots, j\}$ where $1 < i \leq j < n$. For $c \in C^n$, if $\max\{c_i, c_{i+1}, \ldots, c_j\} = \hat{c}$ where $c_{i-1}, c_{j+1} \leq \hat{c}$ then $\max\{G_i(c), G_{i+1}(c), \ldots, G_j(c)\} \leq \hat{c}$.

proof: Let L be as above and let $\max\{c_i, c_{i+1}, \ldots, c_j\} = \hat{c}$ where $c_{i-1}, c_{j+1} \leq \hat{c}$. If $i \leq k \leq j$ then, by assumption, c_{k-1}, c_k, and c_{k+1} are less than or equal to $\hat{c}$ so Theorem 12 implies that $G_k(c) \leq \hat{c}$ completing the proof. $\square$

Now that we have this result about the local behavior of points in C^n under the iteration of G, we wish to draw conclusions about the eventual convergence or divergence of this dynamical system. Ultimately, we will show that the system does converge for any initial condition $c \in C^n$ to the zero vector. To do so we will show that both $\mathbf{0}$ and $\limsup_{\ell\to\infty}\{G^\ell(c)\}$ are subsolutions for this system. Once this has been done, showing that $\limsup_{\ell\to\infty}\{G^\ell(c)\} \in C^n$ will allow us to prove that $\limsup_{\ell\to\infty}\{G^\ell(c)\} = \mathbf{0}$ completing the proof.

Definition 11. Let $X^m \subseteq \mathbb{R}^m$ where $Q : X^m \to X^m$ is a continuous function with Q_i being its i^{th} component. Then the point $\rho \in X^m$ with i^{th} component ρ_i is a subsolution of the map Q if $Q_i(\rho) \geq \rho_i$ for all i.

Note that any fixed point of a system is a subsolution.

Proposition 14. The point $\mathbf{0} \in C^n$ is a subsolution of the dynamical system (C^n, G).

proof: Simply note that for all $1 < i < n$, $G_i(0,0,0) = \frac{0}{w_{i-1}w_i^2} = 0$. That is $G(\mathbf{0}) = \mathbf{0}$. Since $\mathbf{0}$ is a fixed point of the system then it is a subsolution of G. $\square$

Lemma 15. *For $c \in C^n$ let $\max\{c_1, \ldots, c_n\} = c_j$ where $c_j < w_j$, then c is a subsolution of the map G if and only if $c = \mathbf{0}$.*

proof: Let $c \in C^n$ where $\max\{c_1, \ldots, c_n\} = c_j < w_j$. Suppose that c is a subsolution of the function G, that is $G_i(c) \geq c_i$ for all $1 \leq i \leq n$. Since $c_j = \max\{c_1, \ldots, c_n\}$ Theorem 13 implies that $G_j(c) \leq c_j$ but by assumption $G_j(c) \geq c_j$ implying $G_j(c) = c_j$. Given that $c_j \geq c_{j-1}, c_{j+1}$ and $G_j(c) = c_j$ note that if $c_{j-1} = c_{j+1} = c_j$ the point c fulfils the above conditions since $G_j(c_j, c_j, c_j) = c_j$.

Similarly, note that if $c_j = 0$ then $c = \mathbf{0}$ and we are done. Otherwise, suppose by way of contradiction that $c_j > 0$ and that $G_j(c_{j-1}, c_j, c_{j+1}) = c_j$ but that either c_{j-1} or c_{j+1} is not equal to c_j. Since $c_j = max\{c_1, ..., c_{n-1}\}$, if $c_{j-1} \neq c_j$ then $c_{j-1} < c_j$. By the remark following Lemma 7 it follows that $\frac{\partial G}{\partial c_{j-1}}(c) > 0$. Also, since $c_{j+1} \leq c_j$ then $G_j(c_{j-1}, c_j, c_{j+1}) < G_j(c_j, c_j, c_j) = c_j$. However, this contradicts the supposition implying that $c_{j-1} = c_j$. Similarly, under these conditions it follows that $c_{j+1} = c_j$ or that $c_{j-1}, c_{j+1}, c_j = \max\{c_1, c_2, \ldots, c_{n-1}\}$.

Since the only condition needed to show $c_{j-1}, c_{j+1} = \max\{c_1, ..., c_{n-1}\}$ was that $c_j = \max\{c_1, ..., c_{n-1}\}$ this implies that $c_{j-2}, c_{j+2} = \max\{c_1, ..., c_{n-1}\}$ by the same argument. If we continue this argument inductively then each of $c_1, c_2, \ldots, c_n = \max\{c_1, \ldots, c_n\}$. Since $c_1 = c_n = 0$ then for all $1 \leq i \leq n$, $c_i = 0$. Hence, $c = \mathbf{0}$. (Note that the reverse implication is the result of Proposition 14.) $\square$

Since we will be considering the function G under iteration the following definition will be useful.

Definition 12. If $c \in C^n$ then define $G^k(c) := c^k$ for $k \in \mathbb{N}$ where $G^0(c) := c$. Similarly, define $G_i^k(c) := c_i^k$ where $G_i^0(c) := c_i$.

Lemma 16. *For $c \in C^n$ let $\tilde{c} = limsup_{k\to\infty}\{c^k\}$ then $\tilde{c} \in C^n$.*

proof: Since the interval $[0, w_i]$ is compact it follows for $1 \leq i \leq n-1$ that $\tilde{c}_i \in [0, w_i]$. To show that $\tilde{c} \in C^n$ the only condition that remains to be checked is whether or not it is possible that both $\tilde{c}_i = w_i$ and $\tilde{c}_{i+1} = 0$ for some $1 \leq i \leq n$.

Case 1: Suppose that for some $1 \leq i \leq n$ that $c_i = w_i$. Previously, we showed that if this was the case then $G_i(c) = w_i$, that is, c is a fixed point of G_i or $\tilde{c}_i = w_i$. If we now consider the function G_{i+1} at the point c it follows that $G_{i+1}(c) = G_{i+1}(w_i, c_{i+1}, c_{i+2})$. Since the partial derivative of G_{i+1} with respect to c_{i+1} is nonnegative then intuitively, the smaller the input the smaller the output. Since G_{i+1} is undefined when both $c_i = w_i$ and $c_{i+1} = 0$ we will consider the output of G_{i+1} for arbitrarily small c_{i+1}. Note that in the limit

$$\lim_{c_{i+1} \to 0} G_{i+1}(c) = \frac{w_i w_{i+1}(c_{i+2} + w_{i+1})}{c_{i+2} w_i + w_{i+1}(w_i + w_{i+1})} > 0.$$

Since this is the limit of the possible smallest output of G_{i+1} then for all c_{i+1} and c_{i+2}

$$G_{i+1}(w_i, c_{i+1}, c_{i+2}) > \frac{w_i w_{i+1}(c_{i+2} + w_{i+1})}{c_{i+2} w_i + w_{i+1}(w_i + w_{i+1})}.$$

Since the output of G_{i+1} is bounded away from zero then $\limsup_{k \to \infty}\{c_{i+1}^k\}$ is also bounded away from zero or $\tilde{c}_{i+1} \neq 0$.

Case 2: Suppose that $c \in C^n$ where $0 < c_i < w_i$. Having shown for $c_i \in (0, w_i)$ that $\frac{\partial G_i}{\partial c_i}(c_i) > 0$ the following holds. If $c_i < w_i$ then $G_i(c) < w_i$ since $G_i(c_{i-1}, w_i, c_{i+1}) = w_i$. Given that $c_i \neq w_i$ by assumption, then for all $k \in \mathbb{N}$ it follows that $c_i^k \neq w_i$. With this in mind suppose by way of contradiction that $\tilde{c}_i = w_i$ and that $\tilde{c}_{i+1} = 0$. Since $\limsup_{k \to \infty}\{c_{i+1}^k\} = 0$ where $c_{i+1}^k \in [0, w_{i+1}]$ then it must be the case that $\lim_{k \to \infty} c_{i+1}^k = 0$. Therefore, there exists $N \in \mathbb{N}$ such that $w_i^2 - c_{i+1}^N w_{i-1}$ is positive for all $m \geq N$. Given that $c_i^k \to 0$ there must exist an $m_0 \geq N$ such that

$$\frac{w_i^2 - c_{i+1}^{m_0} w_{i-1}}{w_{i-1} + w_i} \leq \frac{w_i^2 - c_{i+1}^{m} w_{i-1}}{w_{i-1} + w_i}$$

for all $m \geq N$. Note that this implies that $c_{i+1}^{m_0} \geq c_{i+1}^m$ where $m \geq N$.

Choose $\epsilon < \min\{\frac{w_i^2 - c_{i+1}^{m_0} w_{i-1}}{w_{i-1} + w_i}, w_i - c_i^{m_0}\}$. Then

$$\epsilon(w_{i-1} + w_i) < w_i^2 - c_{i+1}^{m_0} w_{i-1} \Rightarrow$$

$$\epsilon(w_{i-1}+w_i) - w_i^2 + c_{i+1}^{m_0} w_{i-1} < 0 \Rightarrow$$

$$\epsilon(w_i - \epsilon)(c_{i+1}^{m_0} w_{i-1} - w_i^2 + \epsilon(w_{i-1}+w_i)) < 0.$$

Note that this last inequality holds since $\epsilon < w_i - c_i^{m_0}$ implying that $\epsilon < w_i$. Therefore,

$$\begin{aligned}&(c_{i+1}^{m_0}+\epsilon)w_{i-1}w_i(w_i-\epsilon)- \\ &\quad (w_i-\epsilon)(-(w_i-\epsilon)^2(w_{i-1}+w_i)+(w_i-\epsilon)(c_{i+1}^{m_0}w_{i-1}+w_i(w_{i-1}+w_i))) < 0 \Rightarrow\end{aligned}$$

$$\frac{(c_{i+1}^{m_0}+\epsilon)w_{i-1}w_i(w_i-\epsilon)}{-(w_i-\epsilon)^2(w_{i-1}+w_i)+(w_i-\epsilon)(c_{i+1}^{m_0}w_{i-1}+w_i(w_{i-1}+w_i))} < w_i - \epsilon.$$

Since the left hand side of the previous inequality is equal to $G_i(w_{i-1}, w_i - \epsilon, c_{i+1}^{m_0})$ then

$$G_i(w_{i-1}, w_i - \epsilon, c_{i+1}^{m_0}) < w_i - \epsilon.$$

By Lemma 7 it follows that since $c_{i-1}^{m_0} \leq w_{i-1}$ then $G_i(c_{i-1}^{m_0}, w_i - \epsilon, c_{i+1}^{m_0}) \leq G_i(w_{i-1}, w_i - \epsilon, c_{i+1}^{m_0})$. Similarly, since we chose $\epsilon < w_i - c_i^{m_0}$ then $c_i^{m_0} < w_i - \epsilon$, which implies so $G_i(w_{i-1}, c_i^{m_0}, c_{i+1}^{m_0}) \leq G_i(w_{i-1}, w_i - \epsilon, c_{i+1}^{m_0})$. Together these inequalities imply that $G_i(c_{i-1}^{m_0}, c_i^{m_0}, c_{i+1}^{m_0}) \leq w_i - \epsilon$. Therefore, $c_i^{m_0+1} < w_i - \epsilon$.

Since both $c_{i-1}^k \leq w_{i-1}$ for all $0 \leq m$ and $c_{i+1}^k \leq c_{i+1}^{m_0}$ for all $m_0 \leq m$ then continuing inductively, it follows that for any $m \geq m_0$ that

$$G_i(c_{i-1}^m, c_i^m, c_{i+1}^m) \leq w_i - \epsilon.$$

This implies that $\limsup_{k\to\infty}\{c_i^k\} \neq w_i$, contradicting the supposition.

Case 3: Suppose that $c_i = 0$. Since $G_i(c) = 0$ then c is a fixed point of G_i implying that $\tilde{c}_i = 0$ or $\tilde{c}_i \neq w_i$. Therefore, since it is not possible in any of the above cases for both $\tilde{c}_i = w_i$ and $\tilde{c}_{i+1} = 0$ then for all $c \in C^n$ it follows that $\tilde{c} \in C^n$. $\square$

Lemma 17. *For the function G and any $c \in C^n$, if $\tilde{c} = \limsup_{k\to\infty}\{c^k\}$ then $\tilde{c}$ is a subsolution of G.*

proof: Let $\tilde{c} = \limsup_{k\to\infty}(c^k)$ where $c \in C^n$ and suppose by way of contradiction that $G_i(\tilde{c}) < \tilde{c}_i$ for some $1 \leq i \leq n$. By the previous lemma $G(\tilde{c}) \in C^n$ so $0 \leq G_i(\tilde{c}) \leq w_i$. If $\tilde{c}_i = 0$ then as we have already shown $G_i(\tilde{c}) = 0$ and we are done. If, on the other hand, $\tilde{c}_i = w_i$ then $G_i(\tilde{c}) = w_i$, which contradicts the supposition so we can assume that $0 < \tilde{c}_i < w_i$. It follows from this inequality, for some $\epsilon_0 > 0$, that $G_i(\tilde{c}) = \tilde{c}_i - 2\epsilon_0$.

Let $l_i \in C_{3_i}$ be the point $(\tilde{c}_{i-1}, \tilde{c}_i, \tilde{c}_{i+1})$ and let p_i be any arbitrary point in C_{3_i}. Since G_i is continuous at the point l_i then for every $\epsilon > 0$ there is a $\delta > 0$ such that $|G_i(l_i) - G_i(p_i)| < \epsilon$ if $||l_i - p_i|| < \delta$. Since $\epsilon_0 > 0$ there is a corresponding δ_0 such that $|G_i(l_i) - G(p_i)| < \epsilon_0$ if $||l_i - p_i|| < \delta_0$.

Given that $\limsup_{k\to\infty}\{c^k_{i-1}\} = \tilde{c}_{i-1}$, $\limsup_{k\to\infty}\{c^k_i\} = \tilde{c}_i$, and $\limsup_{k\to\infty}\{c^k_{i+1}\} = \tilde{c}_{i+1}$ then for only finitely many k can

$$c^k_{i-1} \geq \tilde{c}_{i-1} + \frac{\delta_0}{2\sqrt{3}},\ c^k_i \geq \tilde{c}_i + \frac{\delta_0}{2\sqrt{3}},\ c^k_{i+1} \geq \tilde{c}_{i+1} + \frac{\delta_0}{2\sqrt{3}}.$$

Therefore, there exist a natural number M such that for all $m \geq M$

$$c^m_{i-1} < \tilde{c}_{i-1} + \frac{\delta_0}{2\sqrt{3}},\ c^m_i < \tilde{c}_i + \frac{\delta_0}{2\sqrt{3}},\ c^m_{i+1} < \tilde{c}_{i+1} + \frac{\delta_0}{2\sqrt{3}}.$$

Note that, Lemma 7 then implies for all $m \geq M$ that

$$G_i(\tilde{c}_{i-1} + \delta_0/2\sqrt{3}, \tilde{c}_i + \delta_0/2\sqrt{3}, \tilde{c}_{i+1} + \delta_0/2\sqrt{3}) > G_i(c^m_{i-1}, c^m_i, c^m_{i+1}).$$

Since $||(\tilde{c}_{i-1} + \delta_0/2\sqrt{3}, \tilde{c}_i + \delta_0/2\sqrt{3}, \tilde{c}_{i+1} + \delta_0/2\sqrt{3}) - l_i|| = \frac{\delta_0}{2} < \delta_0$ it follows that $G_i(\tilde{c}_{i-1} + \delta_0/2\sqrt{3}, \tilde{c}_i + \delta_0/2\sqrt{3}, \tilde{c}_{i+1} + \delta_0/2\sqrt{3}) \in (G_i(l_i) - \epsilon_0, G_i(l_i) + \epsilon_0)$. Since $G_i(l_i) = \tilde{c}_i - 2\epsilon_0$ then $G_i(l_i) + \epsilon_0 = \tilde{c}_i - \epsilon_0$ which implies that

$$G_i(\tilde{c}_{i-1} + \delta_0/2\sqrt{3}, \tilde{c}_i + \delta_0/2\sqrt{3}, \tilde{c}_{i+1} + \delta_0/2\sqrt{3}) \in (G_i(\tilde{c}) - \epsilon, \tilde{c}_i - \epsilon_0).$$

Therefore, $G_i(\tilde{c}_{i-1} + \delta_0/2\sqrt{3}, \tilde{c}_i + \delta_0/2\sqrt{3}, \tilde{c}_{i+1} + \delta_0/2\sqrt{3}) < \tilde{c}_i - \epsilon_0$.

By the observation above it follows for any $m \geq M$ that $G_i(c_{i-1}^m, c_i^m, c_{c+1}^m) < \tilde{c}_i - \epsilon_0$. However, this contradicts the assumption that $\limsup_{k\to\infty}\{c_i^k\} = \tilde{c}_i$ implying that $G_i(\tilde{c}) \geq \tilde{c}_i$. Since i was arbitrary it must the case that $G_i(\tilde{c}) \geq \tilde{c}_i$ for all $1 \leq i \leq n$. By definition it follows that $\tilde{c}$ is a subsolution of the function G. $\square$

We are now in a position to prove the convergence of the dynamical system (C^n, G) for any $c \in C^n$. To do so we will show that for any $c \in C^n$ that $\tilde{c}$ is not only a subsolution of the function G but that it must be the vector $\mathbf{0}$. This result will then be applied to the original dynamical system (Ω^n, F) to prove Theorem 5.

Theorem 18. *If $c \in C^n$ where $c_i < w_i$ for all $1 \leq i \leq n$ then $\tilde{c} = \mathbf{0}$.*

Suppose by way of contradiction that $c \in C_n$ satisfies $c_i < w_i$ for $i = 2, \ldots, n-1$, but $\tilde{c} \neq \mathbf{0}$. Let

$$\overline{c} = \max\{\tilde{c}_1, \tilde{c}_2, \ldots, \tilde{c}_n\},$$

and let $\mathcal{I}$ be a maximal set of consecutive indices such that $c_i = \overline{c}$ for every $i \in \mathcal{I}$. By Lemma 17, $\tilde{c}$ is a subsolution, Lemma 15 implies that $\tilde{c}_i = w_i$ for every $i \in \mathcal{I}$. Note also that $\{1, n\} \cap \mathcal{I} = \emptyset$, because $\tilde{c}_1 = \tilde{c}_n = 0 < \overline{c}$. Thus, $a := (\min \mathcal{I}) - 1$ and $b := (\max \mathcal{I}) + 1$ are indices in $\{1, 2, \ldots, n\}$ satisfying $\tilde{c}_a, \tilde{c}_b < \overline{c}$. Let $\epsilon = \min\{\overline{c} - \tilde{c}_a, \overline{c} - \tilde{c}_b\}/2$, then we can pick $K \in \mathbb{N}$ such that $\max\{c_a^k, c_b^k\} \leq \overline{c} - \epsilon$ for every $k \geq K$.

Let

$$\hat{c} = \max\left\{\overline{c} - \epsilon, \max\left\{c_i^K \;\middle|\; i \in \mathcal{I}\right\}\right\}.$$

Since c_a^k, $b_b^k \leq \hat{c}$ for all $k \geq K$ and $c^i \leq \hat{c}$ for all $i \in \mathcal{I}$ the corollary to Theorem 12 implies that $\max_{i\in\mathcal{I}}\{c_i^{K+1}\} \leq \hat{c}$. Since c_a^k, $b_b^k \leq \hat{c}$ for all $k \geq K$, continuing inductively it follows that $c_i^k \leq \hat{c}$ for all $k \geq K$. This contradicts the definition of $\tilde{c}$, and this contradiction proves the theorem. $\square$

We are now at the point where it is possible to give a proof for Theorem 5.

proof (Theorem 5): Recall from Remark 4 that for any $r \in \Omega^n$, that $H(r) \in C^n$ where the i^{th} component $H_i(r)$ is strictly less than w_i for all $1 \leq i \leq n-1$. Under this condition the previous theorem implies that $\limsup_{k\to\infty}\{G^k(H(r))\} = \mathbf{0}$. Note that since $0 \leq x$ for all $x \in [0, w_i]$ and $1 \leq i \leq n-1$ then it must in fact be the case that $\lim_{k\to\infty} G^k(H(r)) = \mathbf{0}$.

Since $G^\ell(H(r)) = H(F^\ell(r))$ for all $\ell \in \mathbb{N}$ then in the limit it follows that $\lim_{\ell\to\infty} H(F^\ell(r)) = \mathbf{0}$ for all $r \in \Omega^n$. From the formula for H_i this implies that

$$\lim_{\ell\to\infty} w_i \frac{r_i^\ell}{r_i^\ell + r_{i+1}^\ell} = 0.$$

By dividing by w_i and using the definition of a limit this equality implies that for every $\epsilon > 0$ there exists a ℓ_i such that if $\ell > \ell_i$ then $\frac{r_i^\ell}{r_i^\ell + r_{i+1}^\ell} < \epsilon$. Since this implies that $r_i^\ell < \epsilon r_i^\ell + \epsilon r_{i+1}^\ell$, choosing $\epsilon < \frac{1}{2}$ in turn implies that $r_i^\ell < r_{i+1}^\ell$ for all $\ell > \ell_i$. Given that $1 \leq i \leq n-1$ then under iteration the vector r stabilizes implying that r^∞ exists and has the property that $r_1^\infty < r_2^\infty < \cdots < r_n^\infty$ inducing the ranking $1 \mathrel{\tilde{<}} 2 \mathrel{\tilde{<}} \ldots \mathrel{\tilde{<}} n$. Since this is the ranking generated by the Extended Zermelo model for any tournament in W_n (*see Remark 3*) this completes the proof of Theorem 5. $\square$

2.4 Other Tournaments

Since we have proved that the dynamical system (Ω^n, F) generates the same rankings for tournaments in W_n as the Extended Zermelo model does, it may seem likely that we can now prove the same for the set of lower diagonal matrices $L_n \subset T_n$ of the form

$$L_n = \begin{bmatrix} 0 & 0 & & & & \\ w_{21} & 0 & 0 & & & \\ w_{31} & w_{32} & 0 & 0 & & \\ w_{41} & w_{42} & w_{43} & 0 & \ddots & \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ w_{n1} & w_{n2} & \dots & w_{n,n-2} & w_{n,n-1} & 0 \end{bmatrix}$$

where $w_{ij} \geq 0$ and $w_{i-1,i} > 0$ for $1 \leq i, j \leq n$.

One reason we might conclude that this is possible is that in the tournament $L \in L_n$ if $i < j$ then team j has has been judged superior to team i w_{ji} times for all $i < j$. Since this was only true for $i = j - 1$ for the tournaments in W_n we may suspect that for $L \in L_n$ that the convergence of the dynamical system (Ω^n, F_L) will be bounded above by the convergence of the system (Ω^n, F_W) for some suitable $W \in W_n$.

However, it turns out that even for only small differences between the matrices $W \in W_n$ and $L \in L_n$ that the behaviors of the dynamical system (Ω^n, F_W) and (Ω^n, F_L) are quite different. That is, under a change of coordinates, similar to those used in the proof of Theorem 5, the system does not posses a local maximum principle. (Note that since the matrix $\mathcal{U}$ in the dynamical system $(\Omega^n, F_{\mathcal{U}})$ can be thought of as a parameter then this behavior mimics to some degree the phenomena generally studied in bifircation theory.) As it turns out the best we can hope for then is to find another method to prove the convergence of the dynamical system (Ω^n, F_L) other than the one previously used for the set of matrices W_n.

However, owing to the fact that no such method has been found we will motivate the claim that the system (Ω^n, F_L) generates the same ranking as the Extended Zermelo Model on the tournament L by proving the following subcases.

Proposition 19. If the matrix $D \in L_3$ has the form

$$D = \begin{bmatrix} 0 & 0 & 0 \\ w_{21} & 0 & 0 \\ w_{31} & w_{32} & 0 \end{bmatrix}$$

then the dynamical system (Ω^n, F_D) generates the same rankings as the Extended Zermelo Model does for the tournament D.

proof: By Theorem 3.3 of [3] it follows that since there is a directed path from team 3 to team 1 and team 2 and also one from team 2 to team 1 but not in the other direction then the Extended Zermelo rankings are $1 \prec 2 \prec 3$.

As before we will consider the dynamical system (Ω^n, F_D) under a change of coordinates generated by the function

$$X(r_1, r_2, r_3) := \begin{bmatrix} 0 \\ (w_{31} + w_{32})\frac{r_2}{r_3} \\ 0 \end{bmatrix}.$$

Note that the ith component of X can be written as

$$X_i(r_1, r_2, r_3) = (w_{i+1,i-1} + w_{i+2,i})\frac{r_i}{r_{i+1}}$$

with the convention that $r_1 = 0$ and $w_{42} + w_{43} = 0$. The reason $r_1 = 0$ again follows from the fact that for any matrix $L \in W_n$, row 1 of L is zero implying, $F_{L_1}(r) = 0$ for any $r \in \Omega^3$. On the other hand letting $w_{42} + w_{43} = 0$ simply allows the function X to map a subset of $\mathbb{R}^n$ to another subset in $\mathbb{R}^n$. If we then define

$$\Psi^n := \{(0, p_2, p_3, \dots, p_n) | p_i > 0, 1 < i < n\}$$

it follows that $X : \Psi^3 \to \Psi^3$.

As before we desire the function $\mathcal{G} : \Psi^3 \to \Psi^3$ such that $X(F(r)) = \mathcal{G}(X(r))$ for all $r \in \Omega^3$.

After some straightforward calculations we find the formula for $\mathcal{G}$ to be

$$\mathcal{G}(p_1, p_2, p_3) = \begin{bmatrix} 0 \\ \frac{p_2 w_{21}(p_2 w_{31} + (w_{31}+w_{32})^2)}{(w_{31}+w_{32})(p_2(w_{21}+w_{32}+w_{21}(w_{31}+w_{32})))} \\ 0 \end{bmatrix}.$$

Since w_{21}, w_{32}, and $r_2 > 0$ and the rest of the the constants in the denominator of $\mathcal{G}$ are nonnegative then the denominator is greater than $w_{32}(p_2(w_{21}+w_{32}+w_{21}w_{32})) > 0$. It follows then that $\mathcal{G}$ is a continuous function mapping Ψ^3 to itself or by definition $(\Psi^3, \mathcal{G})$ is a dynamical system. Therefore, for any $\ell \in \mathbb{N}$ the equation $X(F^{\ell}(r)) = \mathcal{G}^{\ell}(X(r))$ holds.

Since we are only concerned with the limiting behavior of the vector $p \in \Psi^3$ under iteration as it relates to ranking we will first show that $\mathcal{G}_2^{\ell}(p) < 1$ for ℓ larger than some natural number N. This will then imply that in the original dynamical system (Ω^n, F_D) that the vector r stabilizes under iteration for all $r \in \Omega^n$.

To do this first note that we have reduced the dynamical system (Ω^n, F_D) through our change of coordinates into a one dimensional system that depends only on the component p_2. Therefore, we may consider the function $\mathcal{G}$ to be the scalar valued function

$$\mathcal{G}_2(p_2) = \frac{p_2 w_{21}(p_2 w_{31} + (w_{31}+w_{32})^2}{(w_{31}+w_{32})(p_2(w_{21}+w_{32}+w_{21}(w_{31}+w_{32})))}.$$

To prove convergence first note that

$$p_2 - \mathcal{G}_i(p_2) = \frac{p_2^2 w_{32}(w_{21}+w_{31}+w_{32})}{(w_{31}+w_{32})(p_2(w_{21}+w_{32}) + w_{21}(w_{31}+w_{32}))}.$$

Supposing that $p_2 > w_{31} + w_{32}$ then

$$\frac{p_2^2 w_{32}(w_{21}+w_{31}+w_{32})}{(w_{31}+w_{32})(p_2(w_{21}+w_{32}) + w_{21}(w_{31}+w_{32}))} =$$

$$\frac{w_{32}(w_{21}+w_{31}+w_{32})}{\frac{(w_{31}+w_{32})}{p_2}\left(\frac{p_2(w_{21}+w_{32})}{p_2} + \frac{w_{21}(w_{31}+w_{32})}{p_2}\right)} > \frac{w_{32}(w_{21}+w_{31}+w_{32})}{w_{21}+w_{32}+w_{21}}.$$

Letting

$$c = \frac{w_{32}(w_{21} + w_{31} + w_{32})}{w_{21} + w_{32} + w_{21}}$$

then $p_2 - \mathcal{G}_2(p_2) > c$. Therefore, under iteration it follows that $\mathcal{G}_2^\ell(p_2) < p_2 - \ell \cdot c$. Then for some $N \in \mathbb{N}$ it must be the case that $\mathcal{G}_2^N(r_2) \leq w_{31} + w_{32}$.

Note that, at this point we have no guarantee that $\mathcal{G}_2^{\ell+1}(p_2) \leq w_{31} + w_{32}$ since our assumption was that $p_2 > w_{31} + w_{32}$. However, consider the following. Since

$$-w_{32}(w_{21} + w_{31} + w_{32}) < 0 \Rightarrow$$

$$w_{21}(2w_{31} + w_32) - (w_{31} + w_{32})(2w_{21} + w_{32}) < 0 \Rightarrow$$

$$w_{21}(2w_{31} + w_32) < (w_{31} + w_{32})(2w_{21} + w_{32}) \Rightarrow$$

$$\frac{w_{21}(2w_{31} + w_32)}{(2w_{21} + w_{32})} < w_{31} + w_{32} \Rightarrow$$

$$\mathcal{G}_2({}_{31}{+}w_{32}) < w_{31} + w_{32}.$$

Also

$$\frac{\partial \mathcal{G}_2}{\partial p_2} = \frac{p_2 w_{32}(w_{21} + w_{31} + w_{32})(p_2(w_{21} + w_{32}) + 2w_{21}(w_{31} + w_{32}))}{((w_{31} + w_{32})(p_2(w_{32} + w_{21}) + w_{21}(w_{31} + w_{32})))^2}.$$

Since we have already proved the denominator to be nonzero then because both the numerator and denominator are the sum and products of positive quantities then for all $p_2 \geq 0$ we have that $\frac{\partial \mathcal{G}_2}{\partial p_2} \geq 0$. Therefore, from the previous result that $\mathcal{G}_2(w_{31} + w_{32}) < w_{31} + w_{32}$ it follows that $\mathcal{G}_2(p_2) < w_{31} + w_{32}$ for all $p_2 \in [0, w_{31} + w_{32}]$ (since the derivative would have to be negative on some interval otherwise). Since $\mathcal{G}_2^N(p_2) \leq w_{31} + w_{32}$ then it follows that for all $n > N$ that $\mathcal{G}_2^n(p_2) < w_{31} + w_{32}$.

Recall that for any $\ell \in \mathbb{N}$ the equation $X(F^\ell(r)) = \mathcal{G}^\ell(X(r))$ holds. Then for every $r \in \Omega^3$ there exists an $M \in \mathbb{N}$ such that if $m > M$ then $X(F_2^\ell(r)) < w_{31} + w_{32}$. Since $X_2(r_1, r_2, r_3) = (w_{31} + w_{32})\frac{r_2}{r_3}$ this implies that for all $m > M$ that

$$(w_{31} + w_{32})\frac{F_2^m(r_1, r_2, r_3)}{F_3^m(r_1, r_2, r_3)} < w_{31} + w_{32} \Rightarrow \frac{F_2^m(r_1, r_2, r_3)}{F_3^m(r_1, r_2, r_3)} < 1 \Rightarrow$$

$$F_2^m(r_1, r_2, r_3) < F_3^m(r_1, r_2, r_3).$$

Recall that since only one component of $r \in \Omega^n$ can be zero and $F_1(r_1, r_2, r_3) = 0$ then for all $m > M$ it follows that $F_1^m(r_1, r_2, r_3) < F_2^m(r_1, r_2, r_3) < F_2^m(r_1, r_2, r_3)$. Therefore, for any $r \in \Omega^3$, r^∞ exists inducing the ranking $1 \tilde{<} 2 \tilde{<} 3$. Note that this is the same as the ranking induced by the Extended Zermelo Model for the tournament D. $\square$

As it turns out the method used to prove the previous proposition does not lend itself to proving the same is true for the tournaments in L_n where $n \geq 4$. The best we can do in this case is the single weighted tournament $E \in L_4$ given by

$$E = \begin{bmatrix} 0 & 0 & 0 & 0 \\ w_1 & 0 & 0 & 0 \\ w_1 & w_1 & 0 & 0 \\ w_1 & w_1 & w_1 & 0 \end{bmatrix}$$

where $w_1 > 0$.

Proposition 20. The rankings induced by the dynamical system (Ω^4, F_E) for any $r \in \Omega^4$ are the same as the Extended Zermelo rankings for the tournament E.

proof: Again from [3] the ranking given to the objects in the tournament E by the Extended Zermelo Model is given by $1 \prec 2 \prec 3 \prec 4$. From the matrix E note that it follows that the function F_E is defined as

$$F_E(r) = \begin{bmatrix} \frac{1}{\frac{1}{r_1+r_2}+\frac{1}{r_1+r_3}+\frac{1}{r_1+r_4}} \\ \frac{1}{\frac{1}{r_2+r_1}+\frac{1}{r_2+r_3}+\frac{1}{r_2+r_4}} \\ \frac{1}{\frac{1}{r_3+r_1}+\frac{1}{r_3+r_2}+\frac{1}{r_3+r_4}} \\ \frac{1}{\frac{1}{r_4+r_1}+\frac{1}{r_4+r_2}+\frac{1}{r_4+r_3}} \end{bmatrix}$$

where $r \in \Omega^4$. Another result that is lengthy, although not complicated to derive is that if $r_1 < r_2$ then $F_1(r) < F_2(r)$. Similarly, if $r_2 < r_3$ then $F_2(r) < F_3(r)$ and finally if $r_3 < r_4$ then $F_3(r) < F_4(r)$. Since these calculations involve simple manipulation of inequalities we will omit then.

Therefore, consider the change of coordinates given by the function

$$Y(r) = \begin{bmatrix} 0 \\ \frac{r_2}{r_3} \\ \frac{r_3}{r_4} \\ 0 \end{bmatrix}.$$

Note that $Y_i(r) = \frac{r_i}{r_{i+1}}$ where $r_1 = 0$ and r_5 is arbitrarily large. As before we assume $r_1 = 0$ since the top row of the matrix E is zero implying that $F_1^k(r) = 0$ for all $k \geq 1$. The assumption that r_5 is arbitrarily large is simply to ensure that $Y : \Omega^4 \to \Psi^4$. Again we seek the function $\mathcal{J} : \Psi^4 \to \Psi^4$ such that $Y(F(r)) = \mathcal{J}(Y(r))$ for all $r \in \Omega^4$.

After some simple calculation we find that

$$\mathcal{J}(p) = \begin{bmatrix} 0 \\ \frac{p_2(1+p_2p_3)(2+p_2+3p_3+2p_2p_3)}{2(1+p_3)(1+2p_2+p_2(2+3p_2)p_3)} \\ \frac{2(1+p_2)p_3(3+p_3(2+p_2(2+p_3)))}{3(1+p_2p_3)(2+p_2+3p_3+2p_2p_3)} \\ 0 \end{bmatrix}.$$

Since both $\mathcal{J}_2$ and $\mathcal{J}_3$ have numerator and denominator which are the product and sum of strictly positive quantities then both must be strictly positive for all $p \in \Psi^4$. It follows then that $\mathcal{J}$ is a map from Ψ^4 to itself and is a continuous function on this set. Therefore, by definition $(\Psi^4, \mathcal{J})$ defines a dynamical system. Again it follows for all $\ell \in \mathbb{N}$ that the equation $Y(F^\ell(r)) = \mathcal{J}^\ell(Y(r))$ holds.

To show for any $p \in \Psi_4$ that $\mathcal{J}^\ell(p)$ converges to some subset of Ψ^4 as $\ell \to \infty$ we will first consider the function $\mathcal{J}$ a component at a time. Since, for all $\ell \in \mathbb{N}$ and $p \in \Psi^4$, the functions $\mathcal{J}_1^\ell(p) = \mathcal{J}_4^\ell(p) = 0$ we will concern ourselves first with $\mathcal{J}_2$ then with $\mathcal{J}_3$.

For the function $\mathcal{J}_2$ note that for all $(p_1, p_2, p_3, p_4) \in \Psi^4$ where $p_2 \geq 1$ it follows that

$$(\frac{p_2}{2} - \frac{p_2^2}{2}) + (p_2p_3 - p_2^3p_3) + (\frac{p_2^2p_3^2}{2} - \frac{p_2^3p_3^2}{2}) < 0 \Rightarrow$$

$$\frac{p_2}{2}(1 + p_2p_3)(2 + p_2 + 3p_3 + 2p_2p_3) - \frac{p_2}{2}(1 + p_3)(1 + 2p_2 + p_2(2 + 3p_2)p_3) < 0 \Rightarrow$$

$$\frac{p_2(1 + p_2p_3)(2 + p_2 + 3p_3 + 2p_2p_3)}{2(1 + p_3)(1 + 2p_2 + p_2(2 + 3p_2)p_3)} < \frac{p_2}{2} \Rightarrow$$

$$\mathcal{J}_2(p) < \frac{p_2}{2}.$$

Repeated use of this last inequality implies that $\mathcal{J}_2^\ell(p) < \left(\frac{1}{2}\right)^\ell p_2$. Therefore, for any $p \in \Psi^4$ there exists some $\ell_2 \in \mathbb{N}$ such that $\mathcal{J}_2^{\ell_2}(p) < 1$. Note that this implies that for every $r \in \Omega^4$ that there exist a $k_2 \in \mathbb{N}$ such that $\mathcal{J}_2^{k_2}(Y(r)) < 1$. Since $Y_2(F^\ell(r)) = \mathcal{J}_2^\ell(Y(r))$ it follows that $Y_2(F^{k_2}(r)) < 1$ which implies that $\frac{F_2^{k_2}(r)}{F_3^{k_2}(r)} < 1$ or that $F_2^{k_2}(r) < F_3^{k_2}(r)$. From the result that if $r_2 < r_3$ then $F_2(r) < F_3(r)$ it follows that for all $\kappa > k_2$ that $F_2^\kappa(r) < F_3^\kappa(r)$.

If we now consider the function $\mathcal{J}_3$ then for any $(p_1, p_2, p_3, p_4) \in \Psi^4$ where $p_3 \geq 1$ it follows that

$$(2p_3 - 2p_3^2) + (4p_3p_2 - 4p_3^3p_2) + (2p_3^2p_2^2 - 2p_3^3p_2^2) < 0 \Rightarrow$$

$$2p_3(1 + p_2)(3 + p_3(2 + (2 + p_3)p_2)) - \frac{2}{3}p_3 3(1 + p_3p_2)(2 + 3p_3 + p_2 + 2p_3p_2) < 0 \Rightarrow$$

$$\frac{2p_3(1 + p_2)(3 + p_3(2 + (2 + p_3)p_2))}{3(1 + p_3p_2)(2 + 3p_3 + p_2 + 2p_3p_2)} < \frac{2}{3}p_3 \Rightarrow$$

$$\mathcal{J}_3(p) < \frac{2}{3}p_3.$$

Again use of this inequality implies that $\mathcal{J}_3^\ell(p) < \left(\frac{2}{3}\right)^\ell p_3$. From this it follows that for any $p \in \Psi^4$ there exists some $\ell_3 \in \mathbb{N}$ such that $\mathcal{J}_3^{\ell_3}(p) < 1$.

This implies that for every $r \in \Omega^4$ that there exist a $k_3 \in \mathbb{N}$ such that $\mathcal{J}^{k_3}(Y(r)) < 1$.Again, since $Y_3(F^\ell(r)) = \mathcal{J}_3^\ell(Y(r))$ then $Y_3(F^{k_3}(r)) < 1$ which implies that $\frac{F_3^{k_3}(r)}{F_3^{k_3}(r)} <$

1 or that $F_3^{k_3}(r) < F_4^{k_3}(r)$. It follows from the result that if $r_3 < r_4$ then $F_3(r) < F_4(r)$ that for all $\kappa > k_2$, $F_3^{\kappa}(r) < F_4^{\kappa}(r)$.

Therefore, if $k_0 = \max\{k_2, k_3\}$ then for all $\kappa > k_0$, $F_1^{\kappa}(r) < F_2^{\kappa}(r) < F_3^{\kappa}(r) < F_4^{\kappa}(r)$ implying that for any $r \in \Omega^4$ that r^{∞} exists and induces that ranking $1 \tilde{<} 2 \tilde{<} 3 \tilde{<} 4$. Since this is the same as the order induced by the Extended Zermelo model this completes the proof. $\square$

To conclude this chapter it should be reiterated that the only tournaments that we have proved the conjecture holds for are those in the tournament class W_n as well as the previous two tournament classes represented by the matrix D and the matrix E. As these are relatively simple tournaments in P_n it seems unlikely that this small fraction of possible tournaments will be enough to convince the reader that the conjecture is true, nor should it. However, it is worth noting that the original goal of finding a proof to the conjecture was motivated by numerical evidence gained by studying many tournament classes for a large number of initial conditions.

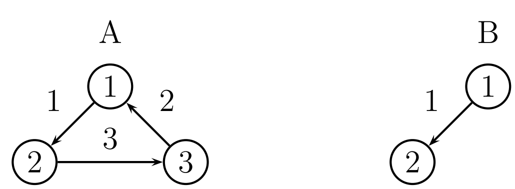
As of yet no counterexample has been found to disprove the conjecture. Still, as noted the proof is incomplete but it does seem likely that if a proof does exist then it is not trivial.

3 Uncoupling Strongly Connected Tournaments

In this chapter we wish to consider under what circumstances we may uncouple tournaments in order to gain insight into their Zermelo Rankings. By uncoupling we mean, finding the Zermelo Rankings for a specific subset of a tournament and relating this subset's rankings to that of the whole. One reason we wish to do so is that computing Zermelo rankings for an n-team tournament requires that we maximize the equation (2) in n variables. Computationally, we reduce the problem of maximizing this equation significantly by reducing the number of variables or teams we are considering.

Definition 13. If $m \leq n$ then a subtournament $B \in T_m$ of the tournament $A \in T_n$ is a collection of objects from the set $\{1, 2, \ldots, n\}$ as well as the outcome of the paired comparisons between these objects from the tournament A.

Since we will developing the theory of how to uncouple tournaments, it is necessary to note that it is not always possible, as far as is known, to do so for any given tournament. That is, knowing the Zermelo Rankings for a particular subtournament does not always lead us to know anything useful about the rankings of the tournament as a whole. For instance, in the tournament A below, knowing that in its subtournament B, that team 1 is ranked better than team 2 does not imply that this is true for the entire tournament.



In fact, the ranking generated for the tournament A by Zermelo's Model is given by $1 \prec 3 \prec 2$. Note that, as above, we will often refer to the digraph $\Gamma(A)$ as the tournament A when the meaning is clear.

In this chapter we will, therefore, consider when it possible to uncouple tournament in a way that preserves the order of a given subtournament if it is reintroduced back into the original tournament. After doing so we will then relate some consequences of these ideas to a specific class of tournaments.

3.1 Revisiting the Zermelo Model

Recall from chapter 1 that the Zermelo rankings for a tournament are guaranteed to exist if the matrix representation of the tournament is irreducible or equivalently if the tournament's corresponding digraph is strongly connected. Under the assumption that this is the case then the functional (2) given in chapter 1

$$P(r) := \prod_{i,j=1}^{n} (\frac{r_i}{r_i + r_j})^{a_{ij}}$$

has a maximizer r in the positive orthant of $\mathbb{R}^n$. From this equation it is possible then to derive an equivalent condition under which $r = (r_1, r_2, \ldots, r_n)$ is a maximizer of the functional P. After deriving this equivalent condition we will then give conditions under which it is possible to uncouple a tournament.

One way to find a maximizer of the functional (2) is to take its partial derivative with respect to r_i for all $1 \leq i \leq n$ setting each derivative equal to zero and solving the resulting system of equations. However, if we do the same except we take the logarithmic partial derivatives with respect to r_i then for the tournament $A \in T_n$ where $A = (a_{ij})$ we have the following. Since

$$\log[P(r)] = a_{12} \log(\frac{r_1}{r_1 + r_2}) + a_{13} \log(\frac{r_1}{r_1 + r_3}) + \cdots + a_{n,n-1} \log(\frac{r_n}{r_n + r_{n-1}}) \Rightarrow$$

$$\log[P(r)] = a_{12} \log(r_1) - a_{12} \log(r_1 + r_2) + \cdots + a_{n,n-1} \log(r_n) - a_{n,n-1} \log(r_n + r_{n-1})$$

Then for all $1 \leq i \leq n$

$$\frac{\partial}{\partial r_i}(\log P(r)) = \frac{a_{i1}}{r_i} - \frac{a_{i1}}{r_i + r_1} - \frac{a_{1i}}{r_1 + r_i} + \cdots + \frac{a_{i,n}}{r_i} - \frac{a_{i,n}}{r_i + r_n} - \frac{a_{n,i}}{r_n + r_i}.$$

Setting this equation equal to zero and moving all the negative fractions to the right hand side gives us the equation

$$\frac{a_{i1}}{r_i} + \cdots + \frac{a_{i,n}}{r_i} = \frac{a_{i1}}{r_i + r_1} + \frac{a_{1i}}{r_1 + r_i} + \cdots + \frac{a_{i,n}}{r_i + r_n} + \frac{a_{n,i}}{r_n + r_i}$$

which if we multiply both sides by r_i can be written as

$$\sum_{j=1}^{n} a_{ij} = \sum_{j=1}^{n} (a_{ij} + a_{ji}) \frac{r_i}{r_i + r_j}.$$

Note that this last equation can be thought of as implying that for $r = (r_1, \ldots, r_n)$ to maximize $P(r)$ then the actual wins for team i must equal its expected wins.

If we repeat this process for all $1 \leq i \leq n$ we have the resulting nonlinear system of equations

$$\begin{aligned}
\sum_{j=1}^{n} a_{1,j} &= \sum_{j=1}^{n} (a_{1,j} + a_{j,1}) \frac{r_1}{r_1 + r_j} \\
\sum_{j=1}^{n} a_{2,j} &= \sum_{j=1}^{n} (a_{2,j} + a_{j,2}) \frac{r_2}{r_2 + r_j} \\
&\vdots \\
\sum_{j=1}^{n} a_{n,j} &= \sum_{j=1}^{n} (a_{n,j} + a_{j,n}) \frac{r_n}{r_n + r_j}.
\end{aligned} \tag{9}$$

Definition 14. For $n \geq 2$, define the the set S_n to be the set of all strongly connected matrices in T_n.

Definition 15. For the matrix $A \in S_n$ define the the set $\mathbb{Z}(A)$ to be the set of all positive solutions $(r_1, r_2, ..., r_n)$ of the system of equation (9).

It follows from the previous derivation that the set $\mathbb{Z}(A)$ is the set of Zermelo Ratings for the tournament A. A fact proved by [7] states that such solutions are unique up to some positive constant i.e. if $(r_1, r_2, ..., r_n)$ is a solution to the above system then for $c > 0$ the vector $(cr_1, cr_2, ..., cr_n)$ is also a solution to (9). However, note that the ranking is unique since for any $c > 0$ if $r_i > r_j$ then $c \cdot r_i > c \cdot r_j$.

3.2 Method of Uncoupling

Theorem 21. *For $n, m \geq 2$ let $(s_1, s_2, ..., s_n)$ be a solution to the system of equations (9) for the tournament with matrix*

$$C = \begin{bmatrix} 0 & c_{1,2} & c_{1,3} & \cdots & c_{1,n} \\ c_{2,1} & 0 & c_{2,3} & \cdots & c_{2,n} \\ c_{3,1} & c_{3,2} & 0 & & \\ \vdots & \vdots & & \ddots & c_{n-1,n} \\ c_{n,1} & c_{n,2} & & c_{n,n-1} & 0 \end{bmatrix}$$

where $C \in S_n$. Similarly, let $(t_1, t_2, \ldots, t_m)$ be a solution to the system of equations (9) for the tournament with matrix

$$D = \begin{bmatrix} 0 & d_{1,2} & d_{1,3} & \cdots & d_{1,m} \\ d_{2,1} & 0 & d_{2,3} & \cdots & d_{2,m} \\ d_{3,1} & d_{3,2} & 0 & & \\ \vdots & \vdots & & \ddots & d_{m-1,m} \\ d_{m,1} & d_{m,2} & & d_{m,m-1} & 0 \end{bmatrix}$$

where $D \in S_m$. Then the solution to (9) for the matrix $(e_{i,j}) = E$ where

$$E = \begin{bmatrix} 0 & c_{1,2} & c_{1,3} & \dots & c_{1,n} & 0 & \dots & \dots & 0 \\ c_{2,1} & 0 & c_{2,3} & \dots & c_{2,n} & 0 & \dots & \dots & 0 \\ c_{3,1} & c_{3,2} & 0 & & & & & & \\ \vdots & \vdots & & \ddots & c_{n-1,n} & 0 & \dots & \dots & 0 \\ c_{n,1} & c_{n,2} & & c_{n,n-1} & 0 & d_{1,2} & d_{1,3} & \dots & d_{1,m} \\ 0 & 0 & & 0 & d_{2,1} & 0 & d_{2,3} & \dots & d_{2,m} \\ \vdots & \vdots & & \vdots & d_{3,1} & d_{3,2} & 0 & & \\ \vdots & \vdots & & \vdots & \vdots & \vdots & & \ddots & d_{m-1,m} \\ 0 & 0 & & 0 & d_{m,1} & d_{m,2} & & d_{m,m-1} & 0 \end{bmatrix}$$

is given by $(r_1, r_2, \dots, r_{n+m-1})$ *where* $E \in S_{n+m-1}$ *and*

$$\begin{aligned} r_1 &= s_1 s_n \\ r_2 &= s_2 s_n \\ &\vdots \\ r_n &= s_n t_1 \\ r_{n+1} &= s_n t_2 \\ &\vdots \\ r_{n+m-2} &= s_n t_{m-1} \\ r_{n+m-1} &= s_n t_m. \end{aligned}$$

proof: Since Zermelo ratings are guaranteed to exist if the tournament is strongly connected then the ratings $(s_1, s_2, \dots, s_n)$ for tournament C and the ratings $(t_1, t_2, \dots, t_m)$ for tournament D exist and are unique up to a constant. To see that $(r_1, \dots, r_{n+m-1})$ is a solution to the system of equations (9) it must be shown that the equations

$$\begin{aligned} \sum_{j=1}^{n+m-1} e_{1,j} &= \sum_{j=1}^{n+m-1} (e_{1,j} + e_{j,1}) \frac{r_1}{r_1 + r_j} \\ \sum_{j=1}^{n+m-1} e_{2,j} &= \sum_{j=1}^{m+n-1} (e_{2,j} + e_{j,2}) \frac{r_2}{r_2 + r_j} \\ &\vdots \\ \sum_{j=1}^{n+m-1} e_{n+m-1,j} &= \sum_{j=1}^{n+m-1} (e_{n+m-1,j} + e_{j,n+m-1}) \frac{r_{n+m-1}}{r_{n+m-1} + r_j} \end{aligned}$$

hold true.

Designating the k^{th} equation above to be

$$\sum_{j=1}^{n+m-1} e_{k,j} = \sum_{j=1}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j} \tag{10}$$

we will proceed by assuming $(r_1, \ldots, r_{n+m-1})$ are as given above, then show that the righthand side of (10) is equal to the left hand side of (10) for $1 \leq k \leq n + m - 1$. This will be done in three steps, first showing equation (10) holds for $1 \leq k \leq n - 1$, second the case $k = n$ and finally the case $n + 1 \leq k \leq n + m - 1$.

For the case $1 \leq k \leq n - 1$ since $(e_{k,j} + e_{j,k}) = 0$ for $j > n$ it follows that the righthand side of (10) is

$$\sum_{j=1}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j} = \sum_{j=1}^{n} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j}.$$

By assumption, for $1 \leq k \leq n$, $r_k = s_k t_1$, which implies that

$$\sum_{j=1}^{n} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j} = \sum_{j=1}^{n} (e_{k,j} + e_{j,k}) \frac{s_k t_1}{s_k t_1 + s_j t_1} =$$

$$= \sum_{j=1}^{n} (e_{k,j} + e_{j,k}) \frac{t_1(s_k)}{t_1(s_k + s_j)} = \sum_{j=1}^{n} (e_{k,j} + e_{j,k}) \frac{s_k}{s_k + s_j}.$$

Given that $(s_1, \ldots, s_n)$ are the Zermelo ratings for tournament C then the following system of equations hold

$$\begin{aligned}
\sum_{j=1}^{n} c_{1,j} &= \sum_{j=1}^{n} (c_{1,j} + c_{j,1}) \frac{s_1}{s_1 + s_j} \\
\sum_{j=1}^{n} c_{2,j} &= \sum_{j=1}^{n} (c_{2,j} + c_{j,2}) \frac{s_2}{s_2 + s_j} \\
&\vdots \\
\sum_{j=1}^{n} c_{n,j} &= \sum_{j=1}^{n} (c_{n,j} + c_{j,n}) \frac{s_n}{s_n + s_j}.
\end{aligned} \tag{11}$$

Since $e_{i,j} = c_{i,j}$ for $1 \leq i, j \leq n$ and $e_{i,j} = 0$ for $i, j > n$ then this, together with (11), implies that

$$\sum_{j=1}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{s_k}{s_k + s_j} = \sum_{j=1}^{n} (c_{k,j} + c_{j,k}) \frac{s_k}{s_k + s_j} = \sum_{j=1}^{n} c_{k,j} = \sum_{j=1}^{n+m-1} e_{k,j} \Rightarrow$$

$$\sum_{j=1}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j} = \sum_{j=1}^{n+m-1} e_{k,j}.$$

That is, $(r_1, \ldots, r_{n+m+l-2})$ is a solution to equation (10) for $1 \leq k \leq n-1$.

Consider the case where $k = n$ for equation (10). Note that in this case the righthand side of equation (10), can be rewritten as follows

$$\sum_{j=1}^{n+m-1} (e_{n,j} + e_{j,n}) \frac{r_n}{r_n + r_j} =$$

$$\sum_{j=1}^{n} (e_{n,j} + e_{j,n}) \frac{r_n}{r_n + r_j} + \sum_{j=n+1}^{n+m-1} (e_{n,j} + e_{j,n}) \frac{r_n}{r_n + r_j}. \tag{12}$$

From the equations for $(r_1, \ldots, r_{n+m-1})$ if $k < n$ then $r_k = s_k t_1$. Similarly, for $k > n$ then $r_k = s_n t_{k-n}$. Therefore, given that $r_n = s_n t_1$ then (12) can be written as

$$\sum_{j=1}^{n} (e_{n,j} + e_{j,n}) \frac{s_n t_1}{s_n t_1 + s_j t_1} + \sum_{j=n+1}^{n+m-1} (e_{n,j} + e_{j,n}) \frac{s_n t_1}{s_n t_1 + s_n t_{j-n+1}} =$$

$$\sum_{j=1}^{n} (e_{n,j} + e_{j,n}) \frac{t_1(s_n)}{t_1(s_n + s_j)} + \sum_{j=n+1}^{n+m-1} (e_{n,j} + e_{j,n}) \frac{s_n(t_1)}{s_n(t_1 + t_{j-n+1})} =$$

$$\sum_{j=1}^{n} (e_{n,j} + e_{j,n}) \frac{s_n}{s_n + s_j} + \sum_{j=n+1}^{n+m-1} (e_{n,j} + e_{j,n}) \frac{t_1}{t_1 + t_{j-n+1}}. \tag{13}$$

Since $e_{n,j} = c_{n,j}$ and $e_{j,n} = c_{j,n}$ for $1 \leq j \leq n$ as well as $e_{n,j} = d_{1,j-n+1}$ and $e_{j,n} = d_{j-n+1,1}$ for $n+1 \leq j \leq n+m-1$ it follows that (13) can be rewritten as

$$\sum_{j=1}^{n} (c_{n,j} + c_{j,n}) \frac{s_n}{s_n + s_j} + \sum_{j=n+1}^{n+m-1} (d_{1,j-n+1} + d_{j-n+1,1}) \frac{t_1}{t_1 + t_{j-n+1}}. \tag{14}$$

Since $d_{1,1} = 0$ the righthand side of (14) is equal to

$$\sum_{j=n}^{n+m-1} (d_{1,j-n+1} + d_{j-n+1,1}) \frac{t_1}{t_1 + t_{j-n}}$$

which, under the change of index $j \to j + n - 1$, is

$$\sum_{j=1}^{m} (d_{1,j} + d_{j,1}) \frac{t_1}{t_1 + t_{j-n}}.$$

From the assumption that $(t_1, \ldots, t_n)$ are the Zermelo ratings for the tournament represented by the matrix D the following equations hold

$$\begin{aligned}
\sum_{j=1}^{m} d_{1,j} &= \sum_{j=1}^{n} (d_{1,j} + d_{j,1}) \frac{t_1}{t_1 + t_j} \\
\sum_{j=1}^{m} d_{2,j} &= \sum_{j=1}^{m} (d_{2,j} + d_{j,2}) \frac{t_2}{t_2 + t_j} \\
&\vdots \\
\sum_{j=1}^{m} d_{m,j} &= \sum_{j=1}^{m} (d_{m,j} + d_{j,m}) \frac{t_m}{t_m + t_j}.
\end{aligned} \tag{15}$$

In particular, the top most equation of (15) implies that the righthand side of (14) is equal to $\sum_{j=1}^{m} d_{1,j}$. Similarly, the bottom most equation of (11) implies that the left hand side of (14) is equal to $\sum_{j=1}^{n} c_{n,j}$. Since

$$\sum_{j=1}^{n} c_{n,j} + \sum_{j=1}^{m} d_{1,j} = \sum_{j=1}^{n+m-1} e_{n,j}$$

then it follow from the previous results that

$$\sum_{j=1}^{n} (c_{n,j} + c_{j,n}) \frac{s_n}{s_n + s_j} + \sum_{j=n+1}^{n+m-1} (d_{1,j-n+1} + d_{j-n+1,1}) \frac{t_1}{t_1 + t_{j-n+1}} = \sum_{j=1}^{n+m-1} e_{n,j} \Rightarrow$$

$$\sum_{j=1}^{n+m-1} (e_{n,j} + e_{j,n}) \frac{s_k}{s_k + s_j} = \sum_{j=1}^{n+m-1} e_{n,j}.$$

This in turn implies that $(r_1, \ldots, r_{n+m-1})$ satisfies equation (10) for $k = n$.

The final case to consider is when $n+1 \leq k \leq m+n-1$ for equation (10). Since $(e_{k,j} + e_{j,k}) = 0$ for $j < n$ and $n+1 \leq k \leq m+n-1$ it follows that

$$\sum_{j=1}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j} = \sum_{j=n}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j}.$$

Also for $n+1 \leq k \leq m+n-1$, $r_k = s_n t_{k-n+1}$, which in turn implies that

$$\sum_{j=n}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j} = \sum_{j=n}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{s_n t_{k-n+1}}{s_n t_{k-n+1} + s_n t_{j-n+1}} =$$

$$\sum_{j=n}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{s_n (t_{k-n+1})}{s_n (t_{k-n+1} + t_{j-n+1})} = \sum_{j=n}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{t_{k-n+1}}{t_{k-n+1} + t_{j-n+1}}.$$

Since $e_{k,j} = d_{k-n+1,j-n+1}$ and $e_{j,k} = d_{j-n+1,k-n+1}$ for $n+1 \leq j \leq n+m-1$ then

$$\sum_{j=n}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{t_{k-n+1}}{t_{k-n+1} + t_{j-n+1}} = \sum_{j=n}^{n+m-1} (d_{k-n+1,j-n+1} + d_{j-n+1,k-n+1}) \frac{t_{k-n+1}}{t_{k-n+1} + t_{j-n+1}}.$$

Again, using the change of index $j \to j + n - 1$ it follows that

$$\sum_{j=n}^{n+m-1} (d_{k-n+1,j-n+1} + d_{j-n+1,k-n+1}) \frac{t_{k-n+1}}{t_{k-n+1} + t_{j-n+1}} = \sum_{j=1}^{m} (d_{k-n+1,j} + d_{j,k-n+1}) \frac{t_{k-n+1}}{t_{k-n+1} + t_j}.$$

Since $2 \leq k - n + 1 \leq m$ if $p = k - n + 1$ then

$$\sum_{j=1}^{m} (d_{k-n+1,j} + d_{j,k-n+1}) \frac{t_{k-n+1}}{t_{k-n+1} + t_j} = \sum_{j=1}^{m} (d_{p,j} + d_{j,p}) \frac{t_p}{t_p + t_j}$$

and therefore (15) implies that

$$\sum_{j=1}^{m} (d_{p,j} + d_{j,p}) \frac{t_p}{t_p + t_j} = \sum_{j=1}^{m} d_{p,j}.$$

From the above it follows that

$$\sum_{j=1}^{m} d_{p,j} = \sum_{j=1}^{m} d_{k-n+1,j} = \sum_{j=n}^{n+m-1} d_{k-n+1,j-n+1} = \sum_{j=n}^{n+m-1} e_{k,j}$$

and since $e_{k,j} = 0$ for $j < n$ then

$$\sum_{j=n}^{n+m-1} e_{k,j} = \sum_{j=1}^{n+m-1} e_{k,j}.$$

Combining this with the previous results implies that

$$\sum_{j=1}^{n+m-1} e_{k,j} = \sum_{j=1}^{n+m-1} (e_{k,j} + e_{j,k}) \frac{r_k}{r_k + r_j}.$$

Therefore, the solution set $(r_1, \ldots, r_{n+m+l-2})$ satisfies equation (10) for $n+1 \leq k \leq n+m-1$. From the previous results $(r_1, \ldots, r_{n+m-1})$ satisfies (10) for every $1 \leq k \leq n+m-1$ so $(r_1, \ldots, r_{n+m-1}) \in \mathbb{Z}(E)$ or is a unique Zermelo ratings for the tournament E up to a constant. □

Since we only loosely defined uncoupling previously, we now give a more formal definition as it relates to the previous theorem.

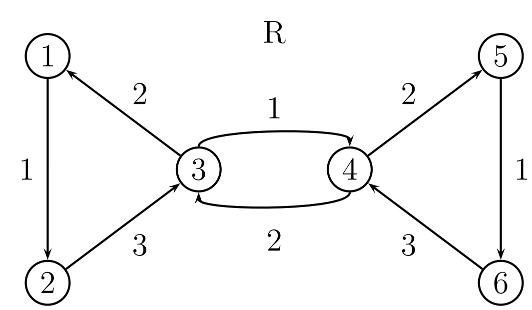
Definition 16. We say that the tournament $A \in S_n$ can be uncoupled into two subtournaments when A_1 and A_2 if both $A_1 \in S_m$ and $A_2 \in S_k$ where $m + k - 1 = n$. That is, the digraphs $\Gamma(A_1)$ and $\Gamma(A_2)$ are both strongly connected and have one vertex in common.

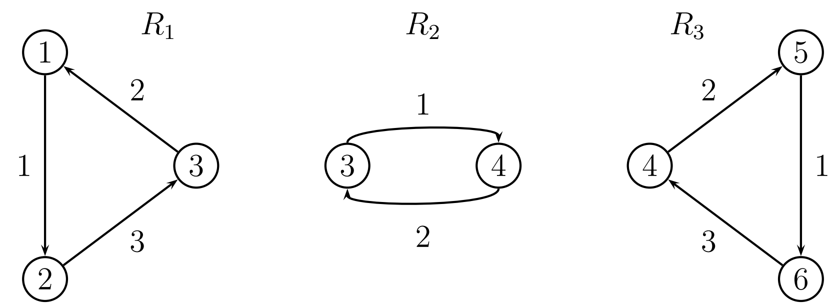
If the tournament A can be uncoupled into the the two tournaments A_1 and A_2, then we will adopt the notation that $A = A_1 \cup A_2$. Note that since the structure of these two subtournaments is given to us, from the underlying structure A, there is no ambiguity in this notation.

Remark 6. Suppose that the tournament A can be uncoupled into two strongly connected subtournaments A_1 and A_2. Theorem 19 then implies that for the subtournament A_1, where $\Gamma(A_1)$ contains the vertices i and j, if in the Zermelo rankings for the tournament A_1 it happens that $i \leq j$ then in the Zermelo rankings for the tournament A the same will be true. Also let the common vertex of these two tournaments represent the team h where the Zermelo Model for the tournament A_1 gives team h the rating h_1 and the Zermelo Model for the tournament A_2 give team h the rating h_2. If $h_1 = h_2$ then the ratings for the tournament A are exactly those given by the combination of the Zermelo ratings tournaments A_1 and A_2 independently. On the other hand if $h_1 \neq h_2$ then to gain the Zermelo ratings for the tournament A we scale all the ratings for the tournament A_1 by h_2 and all the ratings for the tournament A_2 by h_1. (Note that since all the rating vectors in $\mathbb{Z}(A)$ are equal up to a constant then we could have just as easily scaled the ratings for the tournament A_1 by $\frac{h_2}{h_1}$ and been done.)

Since the subtournaments A_1 and A_2 in the remark above may themselves have two strongly connected subtournament whose intersection is a single vertex it may be possible to uncouple the matrix A further into three or more possible subtournaments. That is if the sub-tournament A_1 has two subtournaments B_1 and B_2 where both are strongly connected and share a single vertex then Theorem 19 can be used to generate the Zermelo ratings for the tournament A given the individual ratings for the subtournaments B_1,B_2, and A_2.

For instance the tournament R with digraph given below can be uncoupled into the three subtournaments R_1, R_2, and R_3 (where $R_1 \cup R_2 \cup R_3 = R$).





Computing the Zermelo ratings in R_1 yields $r_1 = 1, r_2 \approx 4.07, r_3 \approx 1.49$ and similarly the ratings in the subtournament R_2 are given by $r_3 = 1$ and $r_4 = 2$. Since R_1 and R_2 are strongly connected and meet in one vertex then Theorem 19 implies for the tournament $R_1 \cup R_2$ that $r_1 = 1, r_2 \approx 4.07, r_3 \approx 1.49$ and $r_4 \approx 2.98$. However, since the sub-tournament $R_1 \cup R_2$ is strongly connected and intersects R_3 only at vertex 4 then given that $r_4 \approx 1.49, r_5 = 1$ and $r_6 \approx 4.07$ in R_3, Theorem 19 implies that $r_1 \approx 1.49, r_2 \approx 6.06, r_3 \approx 2.22, r_4 \approx 4.44, r_5 \approx 2.98$ and $r_6 \approx 12.12$ for the tournament R inducing the ranking $6 > 2 > 4 > 5 > 3 > 1$.

Since there is no reason to assume for a given tournament that it can only uncouple at most twice, we prove the following proposition.

Proposition 22. Let $A \in S_m$ have the subtournaments $A_1, A_2, \ldots, A_n$ where every vertex in $\Gamma(A)$ is in the digraph of some subtournament $\Gamma(A_i)$. Suppose then for every $i \in \{1, 2, \ldots, n\}$ that the subtournaments A_i have the following properties.
(**a**) There exists some $j \neq i$ such that $\Gamma(A_i)$ and $\Gamma(A_j)$ intersect at one vertex
(**b**) Every path in $\Gamma(A)$ from a vertex in $\Gamma(A_i)$ to any other vertex in $\Gamma(A_i)$ must contain an edge in $\Gamma(A_i)$.
Then $A = A_1 \cup A_2, \cup \cdots \cup A_n$ and knowing the Zermelo ratings for the every subtournament A_i allows us to find the Zermelo ratings for A by repeated use of Theorem 19. (Note that one way to express condition (b) is to say that outside the subtournaments A_i there are no cycles.)

Before we prove Proposition 22 we will first prove the following lemma.

Lemma 23. *Suppose that $A \in S_m$ and has the subtournaments $A_1, A_2, \ldots, A_n$. If for all $i \in \{1, 2, \ldots, n\}$ the subtournament A_i fulfils condition (b) of Proposition 22 then $A_i \in S_l$ for some $l \leq m$.*

proof: Let $A \in S_m$ having the subtournaments $A_1, A_2, \ldots, A_n$. For every $i \in \{1, 2, \ldots, n\}$ suppose that A_i fulfils condition (b) above. Then suppose by way of contradiction that for some $j \in \{1, 2, \ldots, n\}$ that $A_j \notin S_l$ for some $l > m$. It follows then that in the digraph of A_j there exist two vertices x_1 and x_k such that there is no directed path from the vertex x_1 to the vertex x_k. However, since $A \in S_n$ then the digraph of A contains a directed path $[x_1, x_2 \ldots, x_{k-1}, x_k]$ from the vertex x_1 to the vertex x_k. Note that this path must leave A_j since there is no directed path in $\Gamma(A_j)$ from x_1 to x_k. Therefore, there exists a smallest number $s \in \{2, 3, \ldots, k-1\}$ such that x_s is not a vertex in $\Gamma(A_j)$. Also, since the vertex x_1 is in $\Gamma(A_j)$ there must be a smallest number $t \in \{s+1, s+2, \ldots, k\}$ such that x_t is a vertex in $\Gamma(A_j)$.

Note that by construction the directed path $[x_{s-1}, \ldots, x_t]$ is a path containing no edges in $\Gamma(A_j)$. However, both x_{s-1} and x_t are vertices in $\Gamma(A_j)$ a result which is a violation of condition (b). Therefore, it must be the case that for all $j \in \{1, 2, \ldots, n\}$ that $A_j \in S_l$ for some $l \leq m$. $\square$

It should be noted that requiring the subtournaments $A_1, A_2, \ldots, A_n$ of A to meet condition (b) is a strong requirement and will not generally be the case for a given collection of subtournaments.

proof (Proposition 22): Let $A \in S_m$ have the subtournaments $A_1, A_2, \ldots, A_n$ where for every $p \in \{1, 2, \ldots, n\}$ every vertex in $\Gamma(A)$ is in the digraph of some subtournament $\Gamma(A_p)$ and every subtournament A_p fulfils conditions (a) and (b) of

Proposition 22.

Since every A_p has the property (b) then by the previous lemma it follows that $A_p \in S_l$ for some $l \leq m$. Therefore, choose any sub-tournament A_i of A. By condition (a) there exists j such that $\Gamma(A_i)$ and $\Gamma(A_j)$ intersect at a single vertex v_1. Given that both $\Gamma(A_i)$ and $\Gamma(A_j)$ are strongly connected then by definition A contains the subtournament $A_i \cup A_j$. Note, it follows that $\Gamma(A_i \cup A_j)$ is strongly connected since both of its components are strongly connected and intersect at a vertex.

Therefore, if $n = 2$ then $A_i \cup A_j = A$ and we are done. Otherwise for $n > 2$ there must exist a $k \neq i, j$ such that $\Gamma(A_k)$ intersects either $\Gamma(A_i)$ or $\Gamma(A_j)$ since A is itself connected. Note that if $\Gamma(A_k)$ intersects both $\Gamma(A_i)$ and $\Gamma(A_j)$ at vertices v_2 and v_3 respectively then $v_1 = v_2 = v_3$ since there would otherwise be a path from v_1 to v_2 through $\Gamma(A_j)$ and $\Gamma(A_k)$ containing no edges in A_i. Since this violates condition (b) this is not possible. Likewise, if $\Gamma(A_k)$ shared two vertices with either $\Gamma(A_i)$ or $\Gamma(A_j)$ then there would be a path from these two vertices outside of $\Gamma(A_i)$ or $\Gamma(A_j)$ from one to the other. Since this is not the case then $\Gamma(A_k)$ must intersect $\Gamma(A_i \cup A_j)$ at exactly one vertex. Therefore, since both $A_i \cup A_j$ and A_k are strongly connected then A contains the subtournament $A_i \cup A_j \cup A_k$. Continuing this argument inductively it follows that $A_1 \cup A_2 \cup \cdots \cup A_n = A$.

Since for all $p \in \{1, 2, \ldots, n\}$ A_p is strongly connected then, as we have mentioned before, the Zermelo ratings for A_p exists. If it is the case that we know the Zermelo ratings for the every subtournament A_p then starting as we did with A_i and A_j using Theorem 19 we can compute the Zermelo ratings for $A_i \cup A_j$. Since $A_i \cup A_j$ has known Zermelo ratings and its digraph intersects A_k at one vertex where A_k has known Zermelo ratings then Theorem 19 can again be used to compute the Zermelo ratings for the subtournament $A_i \cup A_j \cup A_k$. If we continue in the like manner it follows that after $n-1$ applications of Theorem 19 we can derive the Zermelo ratings for the tournament A=$A_2 \cup A_2 \cup \cdots \cup A_n$. $\square$

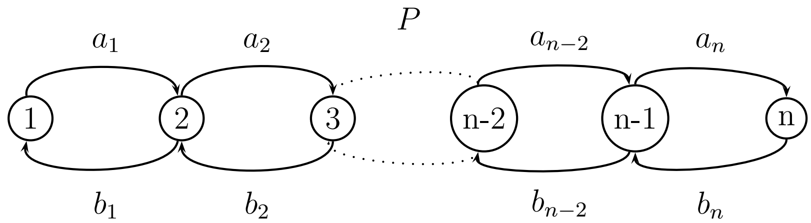
In summary, Proposition 22 implies that if any tournament in S_m has n subtournament where (1) there exist no cycles outside these subtournaments, (2) every subtournament has a corresponding subtournament that it intersects it at one vertex, and (3) every vertex of the tournament is in some subtournament, then Theorem 19 is a useful tool in reducing the computations required to find the Zermelo ratings of the given tournament.

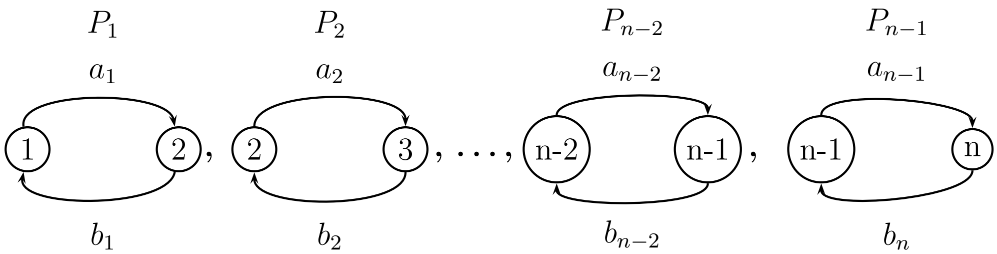
3.3 An Application of Uncoupling

Given the algorithm for computing Zermelo ratings proposed in Proposition 22 it follows that if we know the Zermelo ratings for the subtournaments $T_1, T_2, \ldots, T_n$ of the tournament T where $T = T_1 \cup T_2 \cdots \cup T_n$ then without much trouble we should be able to write down the Zermelo ratings for the tournament T. In practice, however, computing Zermelo ratings for even small tournament can be quite complicated. With this in mind we will consider the class of tournaments represented by the matrix

$$\mathcal{P}_n = \begin{bmatrix} 0 & a_1 & 0 & \ldots & 0 \\ b_1 & 0 & a_2 & \ddots & \vdots \\ 0 & b_2 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{n-1} \\ 0 & \ldots & 0 & b_{n-1} & 0 \end{bmatrix}$$

where $a_i, b_i > 0$ for $1 \leq i \leq n-1$ and $n \geq 2$. The reason we wish to do so is that any tournament $P \in \mathcal{P}_n$ uncouples into $n-1$ subtournaments $P_1, P_2, \ldots, P_{n-1}$ where for all $i \in \{1, 2, \ldots, n-1\}$ the subtournament P_i has Zermelo ratings that are fairly easy to calculate. To see this we claim that the tournament $P \in \mathcal{P}_n$ with digraph given below can be uncoupled into the $n-1$ subtournaments $P_1, P_2, \ldots, P_{n-1}$.





Since the digraph of every subtournament P_i intersects the digraph of P_{i+1} or P_{i_1} at one vertex then the set of subtournament $P_1, \ldots P_{n-1}$ fulfils condition (a) of Proposition 22. Also by observation, any path from one vertex to another in the subtournament P_i must contain an edge in P_i, that is P_i fulfils condition (b) of Proposition 22. Since every vertex in $\Gamma(P)$ is in at least one of the subtournament then Proposition 22 verifies the claim that $P = P_1 \cup P_2 \cup \cdots \cup P_{n-1}$.

However, the fact that the tournament P uncouples is not as interesting as what it uncouples into. In this case note that for all $1 \leq i \leq n-1$ the subtournament P_i has the matrix representation given by

$$P_i = \begin{bmatrix} 0 & a_i \\ b_i & 0 \end{bmatrix} \tag{16}$$

Using the system of equations (9) for the tournament P_i, the vector $(r_i, r_{i+1}) \in \Omega^2$ that maximizes the associated functional $P((r_i, r_{i+1}))$ must satisfy the equations

$$\begin{aligned} r_i &= (a_i + b_i)\frac{r_i}{r_i + r_{i+1}} \\ r_{i+1} &= (a_i + b_i)\frac{r_{i+1}}{r_i + r_{i+1}}. \end{aligned}$$

Since $r_i = a_i$ and $r_{i+1} = b_i$ is a solution to this system then all solutions to this system are scalar multiples of the vector (a_i, b_i). Using this result and the method described in Proposition 22 we will prove the following lemma.

Proposition 24. The matrix $P \in \mathcal{P}_n$ has Zermelo ratings given by some scalar multiple of

$$\begin{aligned}
r_1 &= a_1 \dots a_{n-1} \\
r_2 &= b_1 a_2 \dots a_{n-1} \\
r_3 &= b_1 b_2 a_3 \dots a_{n-1} \\
&\vdots \\
r_n &= b_1 b_2 \dots b_{n-1}.
\end{aligned} \tag{17}$$

where $r_i = b_1 \dots b_{i-1} a_i \dots a_{n-1}$ for all $i \in \{1, 2, \dots, n\}$.

proof: Note that we have already shown that $P = P_1 \cup P_2 \cup \dots \cup P_{n-1}$ where P_i has matrix representation given by (16). Also by the above the Zermelo ratings for the subtournament P_i are given by $r_i = a_i$ and $r_{i+1} = b_i$. Therefore, since $\Gamma(P_1)$ and $\Gamma(P_2)$ meet at a single vertex and are strongly connected, Theorem 19 implies that the Zermelo ratings for the subtournament $P_1 \cup P_2$ are given by $r_1 = a_1 a_2, r_2 = a_1 b_2$, and $r_3 = b_1 b_2$. Note that if n=2 these are the exactly the ratings given by (16).

Assume now for $k < n - 1$ that (16) holds when we replace n by k under the condition that these equations give us the Zermelo ratings for the subournament $A_1 \cup A_2 \cup \dots \cup A_k$ where $A_1 \cup A_2 \cup \dots \cup A_k$ is strongly connected. Under this assumption since $A_1 \cup A_2 \cup \dots \cup A_k$ and A_{k+1} are strongly connected and meet at one vertex then Theorem 19 implies that $A_1 \cup A_2 \cup \dots \cup A_k \cup A_{k+1}$ is a subtournament of A with Zermelo ratings given by

$$\begin{aligned} r_1 &= a_1 \ldots a_k \\ r_2 &= b_1 a_2 \ldots a_k \\ r_3 &= b_1 b_2 a_3 \ldots a_k \\ &\vdots \\ r_{k+1} &= b_1 b_2 \ldots b_k. \end{aligned}$$

Since these are the ratings given by (16) if $n = k+1$, then by the induction hypothesis it follows that (16) gives the Zermelo ratings for any $n \geq 2$. $\square$

Therefore, Proposition 24 effectively gives us a way to calculate the Zermelo ratings for any tournament in $\mathcal{C}_n$ by simple multiplication rather than maximizing the n simultaneous equations given by (9) of the form

$$a_i + b_{i-1} = (a_{i-1} + b_{i-1})\frac{r_i}{r_{i-1} + r_i} + (a_i + b_i)\frac{r_i}{r_i + r_{i+1}}, \ for\ i \in \{1, 2, \ldots, n\}.$$

It is interesting to note that the idea behind uncoupling tournaments resembles the uncoupling of linear systems in the study of ordinary differential equations. That is, to solve the system of equations $\dot{\mathbf{x}} = A\mathbf{x}$ we normally seek a change of coordinates such that in the transformed space the associated system $\dot{\mathbf{x}} = A'\mathbf{x}$ has the property that A' is a diagonal matrix. The fact that A' is diagonal implies that the system $\dot{\mathbf{x}} = A'\mathbf{x}$ can be solved an equation at a time. Similarly, when we uncouple a tournament we are in essence doing the same thing. We can think of uncoupling a tournament as breaking our tournament up into smaller sets of equations in which we can calculate the Zermelo ratings independent of the rest of the tournament.

However, as most systems of ordinary differential equations are nonlinear, the majority of tournaments that are strongly connected, cannot be uncoupled into two or more subtournaments. But when it is the case that a tournament can be uncoupled the calculations are in general, significantly reduced by this method of uncoupling.

4 Some Results and Open Problems

In this chapter we wish to discuss the concept of stability of a tournament. Also we will consider a few partial results pertaining to the Zermelo and Extended Zermelo and the open problems related thereto.

4.1 Stability of a Tournament

For any ongoing tournament a natural question to ask is to what degree can future paired comparisons effect the present ranking of the tournament? Such questions arise for instance when we are considering the best and worse case scenarios at a certain point in a tournament. With this in mind we make the following definition.

Definition 17. If $A \in T_n$ let L be the set of nonnegative real numbers that when added to any off diagonal entrie of the matrix A, change the Extended Zermelo ranking of the tournament. Then let $Stab_+(A)$ to be the $\inf\{L\}$.

Note that the reason we add and do not subtract from the off diagonal entries of a matrix is due to the fact that future comparisons between the objects in the tournament A do not negate previous wins or losses.

Lemma 25. *If $A \in T_n$ and c is a positive constant then both tournaments $c \cdot A$ and A have the same ranking in the Extended Zermelo Model.*

proof: Let $A \in T_3$ where $A = (a_{ij})$. Since the tournament $A(\epsilon)$ is strongly connected for any $\epsilon > 0$ it follows that the equations given by (9) hold for the tournament $A(\epsilon)$. Then for the positive constant c consider the tournaments given by $A(\epsilon)$ and $c \cdot A(\epsilon)$. Note that for the tournament $c \cdot A(\epsilon)$ that the i^{th} equation of (9) is given by

$$\sum_{j=1}^{n} c \cdot (a_{i,j} + \epsilon) = \sum_{j=1}^{n} (c \cdot (a_{i,j} + \epsilon) + c \cdot (a_{j,i} + \epsilon)) \frac{r_n}{r_n + r_j}.$$

Since the constant c can be factored out of both side of the equation and is nonzero the the previous equation can be rewritten as

$$\sum_{j=1}^{n}(a_{i,j}+\epsilon)=\sum_{j=1}^{n}((a_{i,j}+\epsilon)+(a_{j,i}+\epsilon))\frac{r_n}{r_n+r_j}.$$

However this is the i^{th} equation for the tournament $A(\epsilon)$ implying that scaling the tournament by a positive constant does not change the rankings generated by the Extended Zermelo Model since these equations hold for all $\epsilon > 0$. □

Proposition 26. Let $A \in T_3$ where $A = (a_{ij})$. If $m = \max\{a_{ij}\}$ then $Stab_+(A) \leq m$.

Note that since a rigorous proof of this proposition requires that we repeatedly use large and rather cumbersome inequalities we will simply give a sketch the proof in that we will omit some of these details.

With this in mind, since lemma 25 implies that we can scale any tournament A and not impact the Extend Zermelo rankings we will consider the tournament A' with matrix representation $\frac{1}{m}A$. In the case that $m = 0$ note that A is the zero matrix and all teams are tied under the ranking generated by the Extended Zermelo model. Theorem 3.1 in [3] therefore implies that $Stab_+(A) = 0$, since any positive number added to any of the off diagonal entries of A must break the tie and the infimum of all such numbers is 0. Therefore, if $m = 0$ the proposition holds.

If $m \neq 0$ then under the normalization $\frac{1}{m}A = A'$ where $A' = (a'_{ij})$ it follows that $max\{a'_{ij}\} = 1$.

For ease of discussion let

$$(a'_{ij}) = \begin{bmatrix} 0 & a & b \\ c & 0 & d \\ e & f & 0 \end{bmatrix}$$

and define

$$
\begin{aligned}
p_1 &:= ad + af + bd + bf - cd - cf - ed - ef \\
p_2 &:= -ab - ae + bc + ce + bd + ed - bf - ef \\
p_3 &:= -ab - cb - ad - cd + ae + ce + af + cf
\end{aligned} \tag{18}
$$

For the three team tournament (a_{ij}) a result from Grant [4] states that the ranking induced by the vector $p = (p_1, p_2, p_3)$ is the same as the Extended Zermelo ranking for the tournament (a'_{ij}) under the condition that no ties exist in the ranking induced by the vector p.

Since any tournament in T_3 has an associated ranking induced by the Zermelo Model, we will assume that for the matrix above that $p_1 \geq p_2 \geq p_3$. If this is not the case then a renumbering of the objects in the tournament will give us this ranking. Then if we let $n_1 = d + f, n_2 = a + c, n_3 = b + e$ we have the following cases. (Since the proofs of case 1 and case 2 involve large inequalities they will be omitted.)

Case 1: If $p_2 + n_1 + n_2 \geq p_1$ then adding m to the entry a_{21} of A causes $p_2 > p_1$.

Case 2: If $p_3 + n_2 + n_3 \geq p_2$ then adding m to the entry a_{32} of A causes $p_3 > p_2$.

Case 3: If $p_2 + n_1 + n_2 < p_1$ and $p_3 + n_2 + n_3 < p_2$ then it is the case that $p_1 - p_2 - n_1 - n_2 > 0$ and $p_2 - p_3 - n_2 - n_3 > 0$. Therefore define the functions

$$S(a, b, c, d, e, f) := p_1 - p_2 - n_1 - n_2$$

$$T(a, b, c, d, e, f) := p_2 - p_3 - n_2 - n_3$$

where $a, b, c, d, e, f \in [0, 1]$. Note that $\frac{\partial S}{\partial e} = -2d + a - c - 1 < 0$. Therefore assuming that there are values $a_1, b_1, c_1, d_1, e_1, f_1 \in [0, 1]$ such that $S(a_1, b_1, c_1, d_1, e_1, f_1) > 0$ it follows that $S(a_1, b_1, c_1, d_1, 0, f_1) > 0$ by the above conditions. Similarly, since $\frac{\partial T}{\partial e} = -2a + d - f - 1 < 0$ by the above. Then assuming there are values $a_2, b_2, c_2, d_2, e_2, f_2 \in [0, 1]$ such that $T(a_2, b_2, c_2, d_2, e_2, f_2) > 0$ then, as before, $T(a_2, b_2, c_2, d_2, 0, f_2) > 0$.

Similarly, after some work with large inequalities, since $\frac{\partial S}{\partial e}, \frac{\partial T}{\partial e} > 0$ then from the previous results, it follows that both $S(a_1, 1, c_1, d_1, 0, f_1) > 0$ and $T(a_2, 1, c_2, d_2, 0, f_2) > 0$. Furthermore, some straightforward calculations show that

$$\begin{aligned}
\frac{\partial S}{\partial c} &:= -1 - b - d - e - f < 0 \\
\frac{\partial S}{\partial d} &:= -1 + a - c - 2e \leq 0 \\
\frac{\partial T}{\partial a} &:= -1 + d - 2e - f \leq 0 \\
\frac{\partial T}{\partial f} &:= -a - b - c - e \leq 0.
\end{aligned}$$

Therefore, choosing $a_3 = min\{a_1, a_2\}$, $c_3 = min\{c_1, c_2\}$, $d_3 = min\{d_1, d_2\}$, and $f_3 = min\{f_1, f_2\}$ it follows that $S(a_3, 1, c_3, d_3, 0, f_3) > 0$, $T(a_3, 1, c_3, d_3, 0, f_3) > 0$.

Define the function

$$U(a, b, c, d, e, f) := S(a, b, c, d, e, f) + T(a, b, c, d, e, f).$$

By the above $U(a_3, 1, c_3, d_3, 0, f_3) > 0$. Also note that the partial derivatives of the function U with respect to a, c, d, and f are

$$\frac{\partial U}{\partial a} = 2d \geq 0, \frac{\partial U}{\partial c} = -2f \leq 0, \frac{\partial U}{\partial d} = 2a \geq 0, \frac{\partial U}{\partial f} = -2c \leq 0.$$

It follows then that $U(a_3, 1, c_3, d_3, 0, f_3) < U(1, 1, 0, 1, 0, 0)$. However, $U(1, 1, 0, 1, 0, 0) = 0$ contradicting the fact that $U(a_3, 1, c_3, d_3, 0, f_3) > 0$ for some $a_3, c_3, d_3, f_3 \in [0, 1]$. It follows then that at least one of the previous cases must be true. Since both case 1 and case 2 implying that the ranking generated by the Extended Zermelo Model changes by adding m to some off diagonal entry of the matrix A this completes the proof. □

Note that the hypothesis in Proposition 26 that $Stab_+(A)$ is possibly equal to m is in fact a necessary condition since there exist tournaments whose maximal entry is equal to their stability. For example, it can be shown that the tournament A with matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

has $stab_+(A) = 1$.

One way to summarize the result of the previous proposition is to say that for a given three team tournament the largest number of games needed to change the tournament's Extended Zermelo ranking is less than or equal to the largest number of wins of all the teams in the tournament. On the other hand, given that the results of [4] apply only to three team tournaments it is still an open problem as to weather or not this result holds for higher dimensional tournaments.

4.2 Simple 4-team Tournaments

Since it is desirable to be able to compute the Extended Zermelo rankings for tournaments in T_n without referencing the functional $P(r)$ we give the following partial result, which is comparable to the set of equations (18) for a subset of T_4.

Conjecture: Let Z_n be the set of matrices in T_4 having the form

$$Z_n = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & i \\ j & 0 & 0 & 0 \end{bmatrix}$$

where $a, e, i, j \geq 0$. Then in the Extended Zermelo ranking induced by (r_1, r_2, r_3, r_4)

(1) $r_1 \geq r_4 \Leftrightarrow 2eij + j^3 + 2aej + 2aij \geq ej^2 + ij^2 + 4aei + j^2 i.$

(2) $r_1 \geq r_2 \Leftrightarrow 2aij + a^3 + 2aie + 2aij \geq ia^3 + ja^2 + 4eij + a^2 e.$

$$(3)\ r_2 \geq r_3 \Leftrightarrow 2eaj + e^3 + 2iej + 2aie \geq je^2 + ae^2 + 4aji + e^2 i.$$

$$(4)\ r_3 \geq r_4 \Leftrightarrow 2aei + i^3 + 2aij + 2eij \geq ai^2 + ei^2 + 4aej + i^2 j.$$

The reason this is a partial result is that the $\Rightarrow$ implication in (1) through (4) can be proven to hold. However, the reverse implication given by $\Leftarrow$ has does not as yet have a proof. That is we have the necessary but not the sufficient conditions to prove the conjecture.

However, from a numerical point of view these equations seem to catch even the most complicated rankings given by the Extended Zermelo Model for any tournament in Z_n For example, the tournament with matrix

$$\begin{bmatrix} 0 & 1.5 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 2 & 0 & 0 & 0 \end{bmatrix}$$

has the Extended Zermelo ranking given by $4 = 1 \prec 3 = 2$. Note, for $a = 1.5, e = 2, i = 3$ and $j = 2$, that the conjecture implies that since

$$(1)\ 2eij + j^3 + 2aej + 2aij = ej^2 + ij^2 + 4aei + j^2 i \Rightarrow r_1 = r_4.$$

$$(2)\ 2aij + a^3 + 2aie + 2aij < ia^3 + ja^2 + 4eij + a^2 e \Rightarrow r_1 < r_2.$$

$$(3)\ 2eaj + e^3 + 2iej + 2aie = je^2 + ae^2 + 4aji + e^2 i \Rightarrow r_2 = r_3.$$

$$(4)\ 2aei + i^3 + 2aij + 2eij > ai^2 + ei^2 + 4aej + i^2 j \Rightarrow r_3 > r_4.$$

or the conjecture gives the ranking $4 = 1 \prec 3 = 2$.

Note that from the complexity of the equations it seems likely that it is more than chance that we arrived at the same ranking. Also another point that can be made is that if in fact the conjecture is true then the complexity for 4-team tournaments is far greater than that of 3-team tournament, although there are some similarities between the above equations and those given in [4]. For instance every term in the above equations is a cubic whereas for the related equations of a three team tournament every term is quadratic.

Another interesting fact is that the method used to derive these equations was able to independently generate the equations for the 3-team tournament (18). Therefore, is seems likely that the conjecture is in fact correct. However, since this method does not work for arbitrary 4-team tournaments it may be the case that the equations that predict the Extended Zermelo rankings for a 4-team tournament are very different from that of the 3-team tournaments. On the other hand it seems likely from numerical calculations that there is some underlying structure behind even these higher dimensional tournaments.

Still, most of the results that have been proved in this paper are much like the conjecture above. That is we seem to have only partial results related to some open problem or some conjecture. However, even these partial results are interesting in their own right since they point to the inherent complexity of the Zermelo Model.

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