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Variational and Covariational Reasoning of Students with Disabilities

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ABSTRACT

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Mathematics education reform has led to more conceptually focused instruction in the classroom. Yet, students with disabilities are receiving fewer chances than other students to engage in meaningful mathematics. Furthermore, a research divide between mathematics education and special education in mathematics has led to significant gaps in research on the individual and conceptual understanding of students with disabilities. Through task-based interviews and classroom observations, this study begins the process of closing this research gap through an examination of students' understanding of variational and covariational reasoning. Data suggest that the participants, two students with disabilities, increased their conceptual understanding in a reformed learning environment with support from teacher presence and questions. The students were able to increase their understanding of the difference between discrete and continuous functions, demonstrated an ability to self-correct, and improved their ability to choose appropriate levels of reasoning. The results suggest that conceptually oriented instruction with the presence and questioning of a teacher can support students with disabilities in developing a deep and rich understanding of complex mathematics.

Keywords: variational reasoning, covariational reasoning, conceptual understanding, reformed instruction, students with disabilities

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TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1: RATIONALE

During my first three years of teaching, I co-taught a special education math class for 8th graders with mild to moderate disabilities that qualified them for an Individualized Education Program (IEP). My co-teacher, from the special education department, and I were both very new to teaching and were observed many, many times during those three years by district specialists from the Math Department and the Special Education Department. When visited by the Math Specialist, debriefs of any lesson focused on evidence of student thinking. The Math Specialist encouraged and guided us in improving our reformed teaching style. Conversely, after each observation by the Special Education Department, the district leader would inevitably tell us that our reformed style of teaching was inappropriate and too complex for our students. They would encourage us to use the tested and proven method of teaching through step-by-step instructions for our students with disabilities. My co-teacher and I were always baffled. We would sit there and wonder, “Did the special education leader not just see the same lesson we did?” Yes, many of our students had disabilities that affected their processing speed, but given the time, they were capable of rich mathematical thinking. Time and time again, through conceptual tasks and questioning, my students in the co-taught class were thinking deeper about mathematics and making more connections than my honor students.

While I was encouraged by the Math Department to improve my reformed teaching, the Special Education Department told me the complete opposite. My story is not unique. Since the National Council of the Teachers of Mathematics’ (NCTM) publication of *Principles and Standards for School Mathematics* in 2000, mathematics reform has led to more conceptual based, student-centered, and collaboration focused instruction in mathematics classrooms. Many studies since have shown an increase in student learning and achievement through reformed teaching (e.g., Zakaria et al., 2010; Kogan & Laursen, 2014; Boaler, 2001). One such study in mathematics and science classes found that student learning was significantly improved through highly reformed teaching (Sawada et al., 2002).

While students in general mathematics education classes have seen growth in student learning, unfortunately, students in special education have not experienced the same rise in achievement scores (UTPB, 2017; CORE, 2018). According to the 2017 state assessment results from the National Assessment of Educational Progress, there exists a grade level achievement gap on math tests of 32% to 41% between students in special education and those who are not. This achievement gap has changed little in the past ten years for students with disabilities (UTPB, 2017).

Despite efforts to reform mathematics teaching, Jackson and Neel (2006) found that students in special education are receiving fewer chances to engage in meaningful mathematics so they can build conceptual understanding. There could be many reasons why there is a lack of reformed teaching for students with disabilities. My experience has been that the absence may stem from a teacher belief that special education students cannot think conceptually. There is currently little research that addresses the conceptual understanding of students with disabilities or shows the positive or negative effects of reformed mathematics teaching on special education students' understanding or achievement levels. The purpose of this thesis is to help develop this area of research by exploring the conceptual thinking of students with disabilities who are engaged in reformed mathematics teaching.

CHAPTER 2: BACKGROUND

Literature Review

As I began my research, I immediately sought out an explanation for the vastly different feedback I was receiving from the Special Education and Math Education Departments. It was soon clear that a research gap between these two fields may have been the cause of such polarizing advice from my mentors. Additionally, I searched the mathematics education research literature to see what is already known about the conceptual understanding of students with disabilities. I found that the vast majority of the research was focused on students without disabilities. Because of this deficit in the literature, I decided to study disabled students' conceptual understanding of rate of change. I chose the middle school mathematics topic of rate of change because this topic is related to and builds upon many of the central mathematical ideas developed in middle school mathematics (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010) and is pivotal to students' understanding of functions (Byerley & Thompson, 2017; Carlson et al., 2010; Thompson & Carlson, 2017). I further justify my research direction below by first considering definitions for students with disabilities, and then discuss the difference in perspectives used in special education and mathematics education research. Lastly, I examine the current research regarding students' understanding of rate of change.

Students with Disabilities

There are varying definitions of disabilities or what qualifies a person to be considered to have a disability. The United Kingdom's *Equality Act of 2010* defines a disabled person as "someone who has a physical or mental impairment that has a substantial and long-term adverse effect on his or her ability to carry out normal day-to-day activities" (GOV.UK, 2020). Likewise, a student with disabilities is defined by United States Federal Law as, “a student who has a physical or mental impairment that substantially limits one or more major life activities, has a record of such an impairment, or is regarded as having such an impairment” (USLegal, 2019).

These definitions are both legal in nature and focus on the abilities of a person. In contrast, definitions for the term *disability* in the field of Disability Studies refer to disability “...not as an individual defect but as the product of social injustice, one that requires not the cure or elimination of the defective person but significant changes in the social and built environment” (Lambert & Tan, 2016, p. 4). In other words, many scholars consider a disability to be a socially constructed disadvantage that can be altered through changes in the environment and in society.

Other definitions have been created to acknowledge both the medical and societal aspects of disabilities. For example, the World Health Organization’s definition of disabilities states that, “Disabilities is an umbrella term, covering impairments, activity limitations, and participation restrictions.... Disability is not just one health problem. It is a complex phenomenon, reflecting the interaction between features of a person’s body and features of the society in which he or she lives” (WHO, 2020, para. 1-2). Since this definition acknowledges both the medical and societal aspects of disabilities, I adopt this definition for defining disability in my study.

Current Research for Mathematics Special Education

There is a clear research divide between mathematics education and special education in mathematics. This research divide stems from differences both in the content of the research and the theoretical perspective used to interpret the research. In 2017, Lambert and Tan analyzed 149 research articles from 2013 to 2015 for differences in research for students with and without disabilities. They found that mathematical learning for students with disabilities is often understood through behavioral or information processing perspectives in mathematics special education research. In contrast, mathematical learning for students without disabilities is usually approached through constructivist and sociocultural perspectives by general mathematics education researchers. Lambert and Tan (2016, 2017) also determined that while students with disabilities are usually analyzed as a group with little emphasis on the individual or teacher by special education researchers, students without disabilities are studied on an individual level with a focus on the role of the teacher. This difference in theoretical perspectives has led mathematics special education research to have a lack of focus on the meanings and understandings students with disabilities develop as they engage in reasoning and sense making in the classroom. Current general mathematics education research does focus on exploring the individual outcomes of teaching for conceptual understanding through tasks, but the research tends to apply to students in general and excludes students with disabilities. Mathematics special education research is overlooking the individual understanding of students with disabilities and general mathematics education research is overlooking students with disabilities when examining individual understanding of students. This rift in research has led to significant gaps in research on the individual and conceptual understanding of students with disabilities.

Research in mathematics and special education also differ in the type of mathematical instruction they promote. Current research in mathematics education calls for a reformation of teaching styles, from traditional, teacher-centered teaching to reformed, student-centered teaching (NCTM, 2000). A teacher using a reformed instruction style tends to focus more on building the conceptual understanding and procedural fluency of students through student explorations, problem solving, discoveries, and collaboration (NCTM, 2000, 2014; Teach.com, 2020). In contrast, much research for mathematics special education suggests that teacher-focused instruction styles, such as direct instruction, is more beneficial for students with disabilities (e.g., Carnine, 1997; Kroesbergen & Van Luit, 2003). Yet, much of this research focuses on the teaching of basic skills or rote strategies that researchers themselves point out are not generalizable strategies. A teacher using a direct instruction style tends to model procedures to the class and expects them to observe and copy the teacher's process (Teach.com, 2020). A direct instruction style focuses more on procedural competency than conceptual understanding. Boyd and Bargerhuff (2009) found in their research of 25 years of literature that it was difficult to find any reference to the conceptual understanding of students with disabilities. Underlying much of this research seems to be either the assumption that special education students may not be able to understand mathematics and that the best we can hope for special education students is that they learn to perform simple computations, or they believe that mathematics is merely a collection of procedures and that students in special education are learning all there is to learn in mathematics.

Before concluding that direct instruction yields more effective outcomes for students with disabilities than reformed instruction styles, research needs to be conducted to examine whether students with disabilities are able to develop conceptual understanding through reformed teaching. Lambert and Tan (2016) have contributed to this examination by analyzing an article by Bonotto (2013) that highlights the use of artifacts in problem solving activities. They found that the "children with histories as underachievers were able to engage deeply in problem posing when artifacts were relevant to their lives" (p. 1061-1062). This study shows the possibility of underachieving students, like many students with disabilities, learning to think deeply about mathematics through a reformed instruction style. Jo Boaler (2016) echoes this idea in her book, *Mathematical Mindsets:*

> The new evidence from brain research tells us that everyone, with the right teaching and messages, can be successful in math, and everyone can achieve at the highest levels in school. There are a few children who have very particular special educational needs that make math learning difficult, but for the vast majority of children --about 95%-- any levels of school math are within their reach. And the potential of the brain to grow and change is just as strong in children with special needs. (p. 4)

Boaler's claim is that all mathematics is within reach for 95% of students. This implies that the remaining 5% may not be able to access the most advanced mathematics, but it does not imply that the students in the 5% cannot develop rich connections for the mathematics they *can* learn. Research in mathematics education needs to be extended to examine the conceptual understanding that students in special education develop as a result of engaging in reformed mathematics instruction.

Variation and Covariation

In order to better comprehend the conceptual understanding of students with disabilities, I examined the research on students' developing conceptions of the topics of variation and covariation. A conceptualization of variation and covariation supports students' future years of conceptual understanding. The concept of covariation is foundational to understanding rate of change and functions (Thompson & Carlson, 2017) and students, both with disabilities and without, will spend years examining different function types and their rates of change. Thus, the concepts of variation and covariation are pivotal mathematical understandings that students need to develop in middle school in order to build their conceptual understanding of linear, exponential, and quadratic functions in later years.

Previous research has allocated much focus to the conceptualization of variation and covariation. Thompson and Carlson (2017) outline different levels of variational and covariational reasoning that are based, in part, on the “chunky” and “smooth” continuous reasoning described by Castillo-Garsow. According to Castillo-Garsow (2010), chunky continuous covariational reasoning occurs when a student views change as a completed chunk, often happening in the past, and smooth continuous covariational reasoning occurs when a student views change as smooth progress, often happening in the future or not completed yet. The levels of variational and covariational reasoning are described in more detail in the theoretical framework section below.

Even though students start learning about variation and covariation in middle school, most of the research on students’ understanding of variation and covariation has been conducted on college students. Research has shown that conceptualizing covariation is difficult for undergraduate students since they struggle coordinating the variation of two related quantities (Carlson, 1995; Monk & Nemirovsky, 1994; Thompson, 1994a). Carlson et al. (2002) examined the thinking of 20 “academically talented” undergraduate students and found that most were able to determine the general direction of change of one variable with respect to the other and coordinate changes in one variable at the same time as changes in the other variable. These students had difficulty representing a continuously changing instantaneous rate and interpreting why a graph might be a straight or curved line. Stalvey and Vidakovic (2015) found that university students had difficulty with shape, direction, domain and range, and relative rates when graphing two different rates of change. In a literature review of the research on the learning of functions and covariation, Thompson et al. (2017) concluded that it is uncommon for US students to reason covariationally, which is fundamental to understanding rate of change.

While research on students' conceptions of variation and covariation has mostly focused on university students, variation and covariation are first explored and taught in middle school and junior high. According to Byerly and Thompson (2017) and Thompson and Carlson (2017), to conceptualize variation and covariation, a student should have a conceptualization of rate, change, and proportionality. In the United States, the concepts of ratio and rate are introduced in the 6th grade. These concepts are then extended to proportional reasoning in 7th grade, to slope and linear functions in 8th grade, and to exponential functions in 9th grade (NGA Center & CCSSO, 2010). I chose to investigate the understanding of students in a junior high, since this is when students first encounter and begin developing their understanding of concepts that can help them understand chunky and smooth covariation. The difficulty students have had in understanding covariation, as documented in the research, suggests that if students with disabilities can develop an understanding of this topic through reformed instruction, then it is likely students with disabilities can also develop conceptual understanding for other mathematics topics as well.

Theoretical Framework

Framework for Variational and Covariational Reasoning

I have adopted Thompson and Carlson's (2017) variational and covariational reasoning framework for my study. This framework is often used to analyze instances of student thinking to hypothesize what level of reasoning a student might be at. Then the framework can be used to direct teachers in choosing tasks and questions that will push students from a lower to higher level of reasoning. Thus, the framework is predominantly used to classify students. My study uses the framework to classify instances of thinking not for the purpose of labeling what level a student is at, but rather to describe what type of conceptual understanding a student might be expressing in a particular instance. Thus, I am using the framework as a tool for describing in-the-moment understanding rather than classifying students. I am thereby equating students' conceptual understanding to what levels of variational and covariational reasoning are present in instances of their reasoning.

In this framework, a student thinks *variationally* when they envision a quantity's value as changing within a given setting. In their framework, Thompson and Carlson list six levels of variational reasoning: *variable as a symbol, no variation, discrete variation, gross variation, chunky continuous variation*, and *smooth continuous variation*.

To show how this framework might fit with my research, consider how a student might think at each level of variational reasoning for the following problem found in Mathematics Vision Project's (MVP) lesson "Piggies and Pools" in Unit 2 of the Secondary Mathematics One Curriculum (Hendrickson et. al., 2021):

> Our family has a small pool for relaxing in the summer that holds 1500 gallons of water. I decided to fill the pool for the summer. When I had 5 gallons of water in the pool, I decided that I didn't want to stand outside and watch the pool fill, so I had to figure out how long it would take so that I could leave but come back to turn off the water at the right time. I checked the flow on the hose and found that it was filling the pool at a rate of 2 gallons every minute. Use a table, a graph, and an equation to create a mathematical model for the number of gallons of water in the pool at t minutes. (p. 41)

Due to the detailed nature of the framework, Table 1 contains the descriptions and examples for each level of variational reasoning. Levels of reasoning are listed from most to least advanced.

Table 1

Levels of Variational Reasoning

Level	Description	Example
Smooth Continuous Variation	The person thinks of variation of a quantity's or variable's (hereafter, variable's) value as increasing or decreasing (hereafter, changing) by intervals while anticipating that within each interval the variable's value varies smoothly and continuously. The person might think of same-size intervals of variation, but not necessarily.	The student can view the change in time (x-variable) as increasing over an interval of 1 but can still envision an infinite number of ordered values with no unnumbered gaps between the 1-minute intervals, and that time will take on these values in order as it increases.
Chunky Continuous Variation	The person thinks of variation of a variable's value as changing by intervals of a fixed size. The intervals might be the same size, but not necessarily. The person imagines, for example, the variable's value varying from 0 to 1, from 1 to 2, from 2 to 3 (and so on), like laying a ruler. Values between 0 and 1, between 1 and 2, between 2 and 3, and so on, "come along" by virtue of each being part of a chunk – like numbers on a ruler – but the person does not envision that the quantity has these values in the same way it has 0, 1, 2, and so on, as values. Chunky continuous variation is not just a person thinking that changes happen in whole number amounts. Thinking of a variable's value going from -0 to 0.25, 0.25 to 0.5, 0.5 to 0.75, and so on (while thinking that entailed intervals "come along") is just as much thinking in chunky continuous variation as is thinking of increases from 0 to 1, 1 to 2, and so on.	The student can view the change in time as increasing over an interval of 1 since the water is filling the pool at a rate of 2 gallons every minute. For example, the student recognizes there are times between 0 and 1 minutes, 1 and 2 minutes, etc. but does not use these times as possible quantities for the x-variable in the same way as minutes 1, 2, 3, etc. are used.
Gross Variation	The person envisions that the value of a variable increases or decreases, but gives little or no thought that it might have values while changing.	The student can reason that time is always increasing without giving thought to specific time values.

Table 1 Continued

Level	Description	Example
Discrete Variation	The person envisions a variable as taking specific values. The person sees the variable's value changing from *a* to *b* by taking value $a_1, a_2, \ldots, a_n$ but does not envision the variable taking any value between a_i and a_{i+1}.	The student may recognize that the time can change from minute 1 to 2, 2 to 3, etc. without envisioning the possible times between each minute.
No Variation	The person envisions a variable as having a fixed value. It could have a different fixed value, but that would be simply to envision another scenario.	The student may set up an equation, like 1500=2x+5, to represent a single situation where the amount of time does not vary.
Variable as Symbol	The person understands a variable as being just a symbol that has nothing to do with variation.	The student may set up an equation, like y=2x+5, using the procedure of writing the change in the water (taken from a table) in front of an x and adding the starting amount of water. There is no meaning given to the x-variable, nor understanding of what the equation means.

Note. Adapted from "Variational, Covariation, and Functions: Foundational Ways of Thinking Mathematically" by P. W. Thompson and M. P. Carlson, 2017, in *Handbook of Research in Mathematics Education* edited by J. Cai, p. 440. Copyright 2017 by National Council of Teachers of Mathematics.

Thompson and Carlson's (2017) define *covariational* reasoning as occurring when a person "envisions two quantities' value varying and envisions them varying simultaneously" (p. 425). They outline six levels of covariational reasoning: *no coordination, precoordination of values, gross coordination of values, coordination of values, chunky continuous covariation,* and *smooth continuous covariation*. These levels differ from the levels of variational reasoning in that they require the coordination of two varying quantities instead of just one. Thompson and

Carlson's description for each level and an example of student thinking for the same story problem in the previous section is provided in Table 2. Levels of reasoning are listed from most to least advanced.

Table 2

Levels of Covariational Reasoning

Level	Description	Example
Smooth Continuous Covariation	The person envision increases or decreases (hereafter, changes) in one quantity's or variable's value (hereafter, variable) as happening simultaneously with changes in another variable's value, and the person envisions both variables varying smoothly and continuously.	The student can see that the increase in water is always 2 times as large as the increase in time. The student can view the change in time (x-variable) as increasing over an interval of 1 but can still envision an infinite number of ordered values with no unnumbered gaps between the 1-minute intervals, and that time will take on these values in order as it increases. At the same time, the amount of water changes in coordination with time so that at any given time, there is a given amount of water, and the change in water occurs across a range of values that have no gaps and are ordered.
Chunky Continuous Covariation	The person envisions changes in one variable's value as happening simultaneously with changes in another variable's value, and they envision both variables varying with chunky continuous variation.	The student can see the time increasing by 1 minute and the amount of water in the pool increasing by 2 gallons at the same time. The student recognizes there are times between 0 and 1 minutes, 1 and 2 minutes, etc. as well as amounts between 7 and 9 gallons, 9 and 11 gallons, etc., but does not think of how these values for time and amounts of water are coordinated in these intervals of 1 minute and 2 gallons.

Table 2 Continued

Level	Description	Example
Coordination of Values	The person coordinates the values of one variable (*x*) with values of another variable (*y*) with the anticipation of creating a discrete collection or pairs (*x*, *y*).	The student can connect that a time of 1 minute is related to the amount of 7 gallons of water and two minutes is related to the amount of 9 gallons of water. These connections make a discrete collection of pairs, like (1,7), (2,9), (3,11), etc.
Gross Coordination of Values	The person forms a gross image of quantities' values varying together, such as "this quantity increases while that quantity decreases." The person does not envision that individual values of quantities go together. Instead, the person envisions a loose, nonmultiplicative link between the overall changes in two quantities' values.	The student can reason that as the time increases, the amount of water in the pool increases, but does not connect specific values for time with specific values for amount of water.
Precoordination of Values	The person envisions two variables' values varying, but asynchronously – one variable changes, then the second variable changes, then the first, and so on. The person does not anticipate creating pairs of values as multiplicative objects.	The student can envision the time changing by 5 minutes and then compare the amount of water in the pool at each time to find the change in the amount of water. The student does not see these two variables as changing together.
No Coordination	The person has no image of variables varying together. The person focuses on one or another variable's variation with no coordination of values.	The student can think of the time changing or the water changing, but not both simultaneously.

Note. Adapted from "Variational, Covariation, and Functions: Foundational Ways of Thinking Mathematically" by P. W. Thompson and M. P. Carlson, 2017, in *Handbook of Research in Mathematics Education* edited by J. Cai, p. 441. Copyright 2017 by National Council of Teachers of Mathematics.

Research Question

Mathematics special education research focuses on students with disabilities as they learn procedures through direct instruction, and thus lacks attention to conceptual understanding. Mathematics education research focuses on conceptual understanding developed through reformed mathematics instruction, but often fails to extend that research to include students with disabilities, as illustrated in the research regarding students' understanding of variational and covariational reasoning. The learning of variational and covariational reasoning through reformed mathematics instruction needs to be examined more closely for students in special education, which led me to my research question:

- What conceptual thinking about variational and covariational reasoning can surface from students with disabilities in a co-taught classroom using reformed instruction?

Answers to this question can shed light on the suitability of reformed oriented instruction for helping students with disabilities develop conceptual understanding of central middle school mathematical concepts. Additionally, the conceptual understanding surfaced from the participants in this study can provide evidence for an existence proof of what conceptual understanding is possible for students with disabilities.

CHAPTER 3: METHODOLOGY

In order to answer my research question, I completed a series of interviews and classroom observations to ascertain students' understanding of variation and covariation as they participated in reformed instruction on this topic. The next section outlines what types of data were collected, how the data were collected, and how the data were analyzed.

Data Collection

Setting

For the setting of my study, I collected data from students with disabilities in a reformed co-taught special education math class. Students with disabilities are sometimes placed in a co-taught classroom with one regular education mathematics teacher and one special education teacher. Because students are unlikely to develop conceptual understanding if the teachers and the curriculum they use are only focused on procedural fluency, I placed priority on choosing a classroom where the teachers focused on reformed teaching and used a curriculum that promotes building conceptual understanding. This setting is necessary to study what conceptual thinking the students with disabilities are capable of.

I purposefully selected a junior high (grades 7-9) in a suburban community located in the Western United States. The chosen junior high serves around 2,000 students, with minority enrollment making up 15% of the student body. The district's and school's mathematics departments promote reformed teaching styles, thus promoting conceptual thinking. I chose a 9th grade co-taught math class taught by two teachers, Mrs. Daley (general education math teacher) and Mr. Wright (special education teacher). Mrs. Daley has taught for 16 years, and Mr. Wright has taught for 17 years. These teachers work together to build the conceptual understanding of the students in their co-taught class. The 9th grade team at the school uses the MVP Curriculum for Secondary Mathematics I (Hendrickson et. al., 2021), a curriculum that focuses on building procedural fluency through a solid conceptual understanding. Mrs. Daley and Mr. Wright use the same curriculum for their 9th grade co-taught class and taught using tasks from MVP's Module 2, a unit specifically designed to help students develop an understanding about the differences between linear/exponential and discrete/continuous functions. This unit was chosen for the study because pre- and post-assessments could be used to show what covariational understanding the students developed through their Module 2 instruction.

Participants

I introduced the study to the students of Mrs. Daley and Mr. Wright's co-taught class to see if any students were interested in participating. If they were, I provided them with a parental permission form to take home and decide with their parents/guardians if they would participate. The bottom of this form also asked the parent to identify which disability(s) the student has. While I had several students ask for these forms, three students were particularly motivated to participate and were the only three to return signed permission forms. Luckily, all three students were proficient in sharing their thinking, an essential characteristic for participation in the study. I had originally planned to select three students with the anticipation that one of the three students might drop out of the study. So, when one of my participants did drop out of the study half-way through the unit, I knew I would still have enough data from the other two participants to see what conceptual understanding can surface from the students.

The first participant in my study, whose pseudonym is Adam, has Moebius Syndrome, high-functioning autism, ADHD, and anxiety. Moebius Syndrome is "a rare congenital (present at birth) condition that results from underdevelopment of the facial nerves that control some of the eye movements and facial expressions. The condition can also affect the nerves responsible for speech, chewing and swallowing" (John Hopkins Medicine, 2022). Adam said that his parents always tell him that his autism makes him think a bit differently than his peers, and he believes that it helps him to find easier ways to do math problems. He also said that his ADHD makes it hard to remember to do his homework and to concentrate in class, especially math class. Adam acknowledged that even though math can be boring sometimes, but that it is necessary in order to do "basic things" in his current and future life. The thing that he loves most about his current math class and what helps him learn math the best is when he is given chances to work with his classmates and ask each other questions.

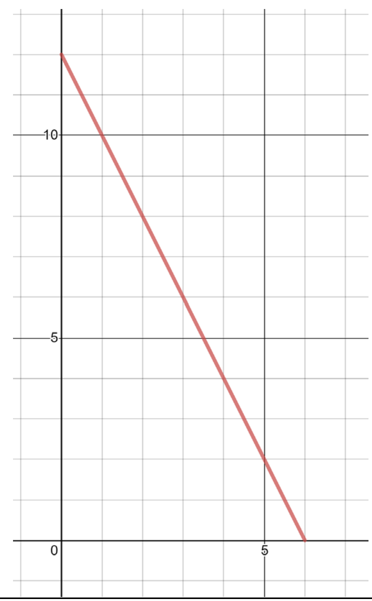
The second participant, whose pseudonym is Cohen, has math and reading specific learning disabilities. When I asked Cohen whether he felt his disability affected how he learned math, he said, "Not really. Cuz I feel like it's been going really easily. I feel like I'm actually getting stuff done. I've been not being a lot of distraction or anything." Cohen had a drive for doing better in school this year since his 9th grade grades mattered more and he said his increased attention has helped him get good grades, like B- or above, in math class. Cohen's teachers seemed to be a big help to him in understanding the mathematics taught in class. He liked that his teachers this year did the problem with them instead of just having them take notes and not explain things. He feels that Mrs. Daley (a pseudonym), the general education teacher, just makes math easier to understand.

Data Types

Each student was interviewed for approximately 30 minutes a total of three times: once before, during, and after the unit. Before Mrs. Daley and Mr. Wright began teaching from MVP's Module 2, I used the first of the three interviews to administer a written pre-assessment that was comprised of three questions to each student (see Figure 1). These questions were designed to give students an opportunity to make different representations for discrete and continuous functions to reveal how they coordinated the changing quantities in each situation. The last question in the interview gave them a chance to explicitly state what similarities and differences they noticed between a discrete linear situation and a continuous exponential situation (see Figure 1). When I gave the participants the tasks for the pre-assessment during the first interview, I prompted them to engage in a think-aloud, meaning that they were asked to explain how they were thinking about the problem as they solved it. Then I briefly interviewed each student to push for any needed explanations of their reasoning and understanding for each question. Preliminary interview questions can be found in Appendix A. Follow-up questions were asked during the pre-interview to clarify student thinking with the purpose of helping the researcher determine what level of reasoning the participant was using. In order to not affect the data from the rest of the unit, I did not discuss possible solutions with any of the interviewees during the first interview.

Figure 1

Interview Task Titles and Prompts

<table>
<tr><th>Interview</th><th>Task Title</th><th>Task Prompt</th></tr>
<tr><td rowspan="2">Pre-Unit</td><td>Buying Soda Task</td><td>Mike was given a gift card with $20 on it. He used it to buy a $1.25 soda each day. Use a table, graph, and equation to represent this situation.</td></tr>
<tr><td>Book Set Task</td><td>Carrie bought a rare book set worth $250 in 2000. The book set has doubled in value every year since 2000. Use a table, graph, and equation to represent this situation.</td></tr>
<tr><td rowspan="2">Mid-Unit</td><td>Create a Story Task</td><td>Create a story to match the graph below. Make sure to label each axis and connect the story to the graph.


</td></tr>
<tr><td>Practice Time Task</td><td>Deven plans to practice his trombone 30 minutes every day this week. He has already practiced 140 minutes before this week started. Sketch a graph to represent how many total minutes Deven has practiced his trombone.</td></tr>
<tr><td rowspan="2">Post-Unit</td><td>Candle Burning Task</td><td>Jared lit a candle with a height of 8-inches and let it burn until it was gone. The candle burns at a rate of ½ inch per hour. Use a table, graph, and equation to represent this situation.</td></tr>
<tr><td>Growing Tree Task</td><td>Stephanie planted a new tree in her yard and wanted to measure its growth. The tree was already 0.8 feet tall when Stephanie planted it. She measured the height of the tree each month and noticed it grew about 0.2 feet per month. Use a table, graph, and equation to represent this situation.</td></tr>
</table>

After the pre-assessment interview, I began my classroom observations to make sure that future interview tasks and questions were aligned with the conceptual focus and vernacular of the classroom teachers. For the classroom observations, I looked for how the teachers posed questions about and talked about the ideas of rate of change, variation, and covariation. Specifically, I used a form (see Appendix B) to make note of what types of questions the teachers were asking and how they referenced the main concepts. I used the data collected to phrase interview questions in a way that would be more familiar to the students. I also used the information from the observations to know what levels of reasoning had been used, modeled, or discussed in the classroom and developed interview questions to see if each student could reason at those levels. Additionally, I made a note of various levels of variational and covariational reasoning that each participant showed in class and for possible gaps in the students' understanding. I used this information to help adjust questions asked in later interviews to better explore the depth of thinking these students were capable of. I found that the levels they showed in class were on par with the thinking they shared in their interviews.

The tasks given during the interviews prompted the students to examine different representations of linear, exponential, discrete, and continuous functions. Participants were questioned about how they viewed the quantities in the representations changing, how they viewed the intervals of change, and how they coordinated the change in the quantities. Additionally, I verbally checked with the participants in the interviews to see if I was correctly interpreting their statements and made adjustments to my interpretation as needed. Each interview with the students was video recorded with the camera pointed at the student's workspace. The video recordings captured what mathematics the students were doing and allowed me to coordinate what the students said to what they wrote or pointed to. All interviews were transcribed and used for the data analysis.

Data Analysis

Since one of the purposes of this study is to construct an existence proof of what conceptual understanding is possible for students with disabilities, I chose to fully analyze each interview from the two participants that completed the study. This decision was made to help me see the depth and breadth in their variational and covariational reasoning to better answer the research question.

I performed a cursory analysis of the data as it was collected. An ongoing analysis of the data was especially important so that I could plan specific interview questions based on my classroom observations and experiences with the previous interviews. After each interview, I took notes of any questions that I wanted to explore in the next interview, key instances when the student showed different levels of variational or covariational reasoning, and ideas for what to look for in the next classroom observation. For example, I realized after my pre-interviews that I was not asking enough questions that really let the students articulate whether they were envisioning values between their benchmark times and benchmark heights. So, I added more questions to provide the students more chances to share their level of reasoning.

When all the interviews were completed, I began to transcribe each interview and separated out any off-topic portions of the interviews from the data since those moments would not have shed light on the mathematical understanding of the participants. For example, I did not think that Adam's story about the café where he likes to buy pretzels would be helpful to understanding his mathematical reasoning and it was separated from the rest of the data. With the remaining data, I used the student's approach for solving a problem as the unit of analysis and did a preliminary coding for what levels of reasoning were shown for each approach. For example, if a student used three different approaches in their attempt to make a graph, each approach was coded for what level of variational or covariational reasoning was demonstrated. To code the different approaches, I used the framework for the levels of variational and covariational reasoning to match explanations given by the student to the descriptions of the levels in the framework. I also recorded my reasons for choosing each specific level. For example, when Adam was making a graph for the Practice Time Task in the post-interview, he explained his thinking by saying, "So, on the first week, on the zero day, technically he would have 30 because technically it is his first day [makes a coordinate point at (0,30)]." His approach to creating the graph was coded as showing *coordination of values* for his level of reasoning, and I noted that I chose this level because Adam had connected a practice time to a number of days to make a distinct coordinate point. After the initial coding, I shared my results and reasons with my advisor, who agreed with my coding. Additionally, I went back a few days (up to a week) later and re-coded the interview and compared my results. For the most part, I agreed with my initial coding. There were a few instances where I changed the coding for the student's approach and made note of why I changed my mind in my listed reasons.

In addition to coding which levels of reasoning were demonstrated by the student during the interviews, I also coded approaches made by the student where the student was definitely *not* using a specific level of reasoning that would have been more appropriate for that item. For example, when Cohen was making a graph for the Practice Time Task in the post-interview, he explained his thinking by saying:

> So, on day 1, he did 140 [points to the coordinate point with his pencil]. Then on day 2, he jumped up to 130 [uses pencil to go from 2 on x-axis and jumps up to where

coordinate point (2, 170) was drawn]. I meant 170. …At the end of day 3 [pencil at coordinate point] he's jumped up to 190.

Cohen's description that the values were jumping to the next pair of coordinated values did demonstrate that he was using *coordination of values* for his level of reasoning and was coded as such. But, his description also clearly indicates that Cohen was definitely *not* using *chunky continuous covariation* since his "jumping" explanation negates any use of values in between benchmark values in his approach. As such, this approach was also coded as *negating* a *chunky continuous covariational* level of reasoning. The coded, negated levels of reasoning were used as further evidence that the student was not using a particular level of reasoning, especially when the question provided an opportunity for the student to use a more advanced level of reasoning. Negated codes helped me confirm that the type of reasoning I had attributed to a solution fit the data.

Once all interviews went through two rounds of coding, I created a summary table of the results and used both the table and the student responses to help me look for patterns of understanding for the students individually and collectively. In particular, I looked for evidence that students were using increasingly advanced levels of thinking. When I noticed that students continued to use lower levels of reasoning in later interviews, I examined the students' approaches closer and found that not all uses of lower levels of reasoning in the later interviews were inappropriate. While coding the data, I also took note of how many times the students changed up their approach to a problem or self-adjusted their thinking. These self-corrections seemed to play a significant role in the students' solutions. To examine this pattern closer, it became necessary to code the data for different student corrections.. I coded a self-correction in the data when the student verbally acknowledged having made an error and adjusted their solution or when the student erased at least part of their work and adjusted their solution. Once all instances of self-correction were identified in the data, I coded each self-correction for what seemed to prompt the self-correction. Each of these themes in the students' understanding became more solidified as the writing process for the results section commenced. Writing up the results with feedback from my advisor pushed me to be more detailed in my analysis, discover more nuanced findings within the broader themes mentioned above, and have more confidence in my results.

CHAPTER 4: RESULTS

After analyzing the data from the students' interviews, it was clear that both students showed growth in their understanding of the mathematical concepts. One participant, Adam, began the unit with a more established understanding of rate of change than the other participant, Cohen. As a result, Adam ended the unit with a greater ability to use variational and covariational reasoning. Despite this difference, both participants had similarities in the understanding they demonstrated. The students, to a varying degree, improved in their understanding of discrete and continuous functions, increased their accuracy in and range of levels of variational and covariational reasoning used, and displayed their ability to self-correct their mistakes. This section will outline the results for both participants in each area of understanding.

Discrete and Continuous Variation and Covariation

During the unit on discrete and continuous variation and covariation, the co-teachers encouraged students to use three methods for determining if a situation was discrete or continuous. First, decide if a situation has breaks (discrete) or no breaks (continuous) in the variation of each variable. Second, use values like half a day or half a book to see if values between benchmark values make sense in the given situation. And third, identify whether a graph has just coordinate points (representing discrete covariation) or coordinate points connected with a line (representing continuous covariation). In order to reason using *chunky and smooth variational and covariational* levels of reasoning, students must first understand how to determine whether the covariation is continuous or discrete. As such, this section will investigate each students' development of their ability to distinguish between discrete and continuous variation and covariation, which will in turn expose their ability to reasoning using *chunky and smooth variational and covariational* levels of reasoning.

Since the aforementioned methods were the methods used in their class, one indicator of the participants' understanding of discrete and continuous variation and covariation was their usage (or lack) of a line on their graph and their reasoning for that decision. Both students had little to no meaning ascribed to the line on a graph before the unit. They viewed the line as showing a pattern or indicating the variational or covariational change in a situation was continuing forever. Their understanding of the line on a graph grew throughout the unit. By the end of the study, both students used the idea of breaks or no breaks in a situation's covariation to decide if a graph was continuous or discrete, and then used a line to connect their points if the covariation was continuous and no line if it was discrete.

A second indicator of their understanding of discrete and continuous variation and covariation was their ability to envision all the values in between their chosen benchmarks for a situation and use them as possible values for the variables. Even though the co-teachers highlighted using points in between to make sense of a situation's possible inputs and outputs, the participants did not naturally use these values while making their tables, graphs, and equations. However, when pushed, they could usually reason successfully whether or not values between benchmarks (e.g., 1 and 2) made sense in the situation. Adam was much more successful in this than Cohen, who was evidently still in the midst of developing his ability to reason using *smooth and chunky covariational reasoning*.

To address both indicators of the participants' understanding of discrete and continuous variation and covariation, this section is divided into the participant's understanding of the line of a graph and their usage of values between their benchmarks.

Adam

Meaning of the Line

During the pre-interview it was apparent that Adam did not know that the line of a graph could indicate anything more than a linear or exponential pattern. He recognized the Buying Soda Task as linear because it had a straight line and the Book Set Task as exponential because the line was curved. Adam connected the points on his graph for each task with a line. To explain what the lines meant, he said:

> So, I guess it is like symbolizing the thing…[laughs]? Give me a sec to think about that. Well, I guess, just so that it's connecting the dots so that you know that it's like, I guess, the same thing. And the dots are there so that you can like go like this [traces pencil from y-axis and x-axis to a coordinate point] and figure out when and how much.

Adam used the line in both graphs to show the coordinated values he had plotted on the graph and to show the pattern of the coordinate points. The line did not indicate whether the variation and covariation in a situation was happening continuously or discretely. Thus, Adam did not show evidence in this interview that he could distinguish between *discrete variation* and *chunky or smooth continuous variation.*

In the second interview, Adam created the following story for the Create a Story Task: "Jessica likes to buy a pretzel every other day for dinner at 6 pm. They cost $5." He made up a discrete situation for the provided continuous graph, limiting the levels of variational and covariational reasoning that would be appropriate for this task from *smooth levels of reasoning* to *chunky levels of reasoning*. Despite this limitation, he later said in the interview:

> So, technically it could be a discrete pattern, because what we're doing is, she's not continuously like boom, pretzel, boom, pretzel, boom, pretzel. It's every other, it's like every other day. So, technically it makes sense if we just had dots.

Adam was able to reflect on his work, assess whether the line was appropriate for representing the situation he had created, and correctly adjust the graph. This shows that, at the half-way mark of the unit, Adam was developing a conceptual understanding for the differences between discrete and continuous covariation and how a line (or lack thereof) on a graph can better represent a situation. For the second problem in the mid-unit interview, Adam decided the situation in the Practice Time Task was discrete because there were breaks in the practice time, and he decided he did not need to connect his points with a line. Adam was able to make sense of the situation, identify whether the variation occurred every second or whether Deven had breaks in his practice time, and determine the situation was discrete. This is tremendous progress from the beginning of the unit since he has now shown an ability to use *discrete variational* and *chunky continuous variational* levels of reasoning to determine whether the covariation should be represented as discrete or continuous on a graph.

He continued to show this growth in the final interview on both questions. For the Growing Plant Task, Adam made a graph with discrete coordinate points. When asked about whether his graph should have a line or just the coordinate points, he explained, as he connected his points with a line, that the plant was consistently growing and so the covariation was continuous. He used his understanding of the growth of a plant to decide that there would be no jumps or breaks in the covariation. This led him to connect his points with a line. He clearly was not drawing the line out of habit because he only added it after determining that the situation involved continuous covariation. Adam's progress in understanding discrete and continuous variation and covariation and how that affects the meaning and usage of a line was considerable. He went from having no understanding of how the line of a graph could indicate discrete or continuous variation and covariation before the unit to being able to using levels of *chunky and smooth continuous variational and covariational* reasoning to match his graph to how he interpreted the covariation of a situation by the end of the unit.

Using Values Between Benchmark Values

Before the unit began, Adam used benchmark values (like 1, 2, 3, etc.) for time and the other variable in the situation to make his tables and graphs. He also showed in the pre-interview that he was able to envision time passing between benchmark values for x, but did not use the intermediate values in the same way as benchmark values. For example, in the first interview I had asked him if a point half-way between his point for the year 2000 and the year 2001 would make sense for the Book Set Task. He responded, "Yeah. It could still be 2000. It'd just be like…it'd be June." He did not initially use June as an input for his table or graph, but he was able to reason that it would be possible to have part of a year in this situation. As he said, "A lot can happen in a year!" he was not envisioning or using June of 2000 in the same way he viewed the year 2001 (such as to help make his tables and graphs, to produce outputs, etc.), but he was confident that the worth of the book set could be calculated at 1.5 years into the pattern. Adam did not show that he thought about these values in between his benchmark values until I questioned him about it. Yet, when given the chance to reason with the intermediate values between his benchmarks, Adam showed that he was able to reason with these values, but not in the same way as his benchmark values. This shows that, for the variable of time, he can reason at a *chunky continuous variational level* but not necessarily at a *smooth continuous variational level*.

In the mid-unit interview, Adam showed an ability to split both variables into smaller units, even if such splitting was not appropriate to the given situation or his interpretation of the situation. Initially, Adam had trouble envisioning the values between the benchmarks for a variable that was not time if it did not make sense to him to use in the given (or his created) situation. For example, when I asked him whether the coordinate point (5,2) would work for his pretzel story in the Create a Story Task, he said:

> So, technically it could if she decided to be a little off on her pattern of doing it like exactly at 6. … So, like 6:30, maybe? Cuz that would still make sense, because you can do that. That's how time works. But you can't do that with money, because, ya know. I mean, you can! But umm… don't expect to get anything.

Adam showed once again that he envisioned smaller units of time than the ones he used for his benchmark values. He also showed that he can envision splitting money up into smaller increments, even if he determined it did not make sense for his pretzel situation. This was the first instance of Adam using *chunky continuous variational reasoning* for a variable other than time. For the Practice Time Task, Adam seemed to have no problem envisioning values between his benchmarks when asked, even though he decided the situation was discrete. For example, I drew a coordinate point at about (1.5, 75) and asked him if it had meaning for Deven's practice time. Adam said:

> Ooh. Okay. Well, that would be 1 and a half days, because it is in the middle [pencil between 1 and 2 on x-axis] and it would also be at half here [pencil between 60 and 90 on y-axis], which would be 75? I think? I'm like 99% sure.

He had not used the values 1.5 or 75 in his table or connected his points with a line, but Adam was able to put the new coordinated values that were not part of his data set into his context and make sense of them. Adam could imagine points between the benchmark points that made sense in the situation even when he determined that the covariation was discrete, using *chunky continuous variational and covariational* reasoning.

By the final interview, Adam demonstrated an ability to envision even more values between the benchmarks by repeatedly splitting up intervals of units for both variables. His ability to reason using *chunky and smooth continuous variation and covariation* was evident when asked about the points between his benchmark points, as is shown in the following excerpt that is taken from our discussion of his work for the Burning Candle Task.

L: So, would that point [(4.5, 5.75)] work for your story?

A: Yes.

L: How tall would the candle be about?

A: Uh oh.

L: You can just estimate.

A: It would be a half… let's see. It would be a fourth of a half of a half of an inch.

L: Why do you think that point works?

A: Because you can, because it's constantly burning, so technically you could have half of a half of a fourth of a third of it [laughs] feet tall, which is very, very specific.

It seemed like Adam was able to use his understanding of the constant burning of the candle to reason about the "very specific" decimal possibilities in between his chosen benchmark values. Besides determining the covariation of this situation as continuous in this interview, he still did not mention or refer to these intermediate values unless questioned, perhaps indicating that Adam was not using intermediate values in the same way as his benchmark values. In turn, this difference in use of intermediate and benchmark values suggests that Adam may have been using *chunky continuous variational and covariational reasoning* rather than *smooth continuous variational and covariational reasoning*. Overall, Adam showed growth from the first interview in his ability to reason and envision intermediate values for *both* variables in a situation. The combination of his improved ability to envision intermediate values and his understanding of the meaning of a line shows that Adam was forming a conceptual understanding of discrete and continuous variation and covariation throughout the unit.

Cohen

Meaning of the Line

In the pre-interview, Cohen did not demonstrate that he understood that the line of a graph could provide information about the way quantities change in a situation (e.g., linear, exponential, discrete, continuous, etc.). When asked why he had a line on both graphs in the first interview, he said, "I just…I got taught that and followed it... But it's an arrow, cuz it's happening every single day." He did not describe how the line connects to the covariation of the situation, but he did indicate that there should be an arrow on the line. This could suggest that Cohen recognizes that a line could be used to indicate that the covariation will continue. There was no indication in this interview that Cohen could distinguish between *discrete variation* and *chunky continuous variation.*

By the mid-unit interview, Cohen had developed some idea of why some graphs might have a line and others might not. For example, he had connected his points on the graph for the Practice Time Task. When asked whether it mattered if we have the line or just coordinate points, he responded:

> Umm. Yeah, it matters. I just drew the line, because, well, to see if they were lined up. But it wouldn't be a line, because he's doing it 30 minutes a day. It doesn't matter what

hours he does it, like he can do 20 minutes at like 2 and then another 20-uh other minutes at 12 or something. So, there's breaks in between.

He seemed to have initially drawn the line to see if there was a pattern in the coordinate points. However, after further reflection, Cohen was able to use a way of reasoning that was discussed by his teachers in class to decide that the situation was discrete. His teachers often talked about the difference in discrete and continuous variation and covariation in terms of a situation having breaks or no breaks in the pattern, respectively. Since Cohen determined that Deven's practice time could happen any time during the day and would have breaks, he chose to erase the line connecting his points. This shows that Cohen now has some understanding of what the line on a graph means and has grown in his understanding of the difference between *discrete variation and chunky continuous variation.* Cohen's decision to change his graph from a continuous model to a discrete model also shows that, at this point of the unit, he needed prompting to reason about discrete and continuous variation and covariation.

By the end of the unit, Cohen, like Adam, was able to match his interpretation of the given situation to whether his graph should be discrete or continuous. In the post-interview, he determined that the Burning Candle Task was continuous and should have a line because the candle was always burning. The line for this graph was intentionally drawn and justified without prompting, suggesting that Cohen had developed a more solid connection between the line (or lack thereof) on a graph and what this indicates about the variation and covariation in the situation. He showed this same understanding for the Growing Plant Task in the same interview. Cohen decided that the graph should be discrete with no line, because "… she's doing it [checking the tree] every month. So, she can do it any [sweeping hand motion] time that month. And then she has to wait 'til next month. Discrete." His interpretation of the situation was to represent when the growth of the tree was checked rather than the growth of the tree itself. As such, he determined that the situation would be discrete because there were breaks between each check on the growth of the tree. Without needing to be prompted, Cohen had already matched his graph to this discrete interpretation. For both problems, Cohen clearly thought about whether he should connect the points or not. In the first interview, this was not a consideration at all. By the end of the unit, Cohen had clearly grown in his conceptual understanding of why a situation involves discrete *coordination of values* or *chunky and smooth continuous covariation* and how that covariation would be represented on a graph.

Using Values Between Benchmark Values

During the first half of the unit on discrete and continuous variation and covariation, Cohen was on the verge of his ability to use *chunky continuous variational and covariational reasoning*. In the pre-interview, he struggled with thinking about the effect that a half-unit of time would have on the other variable in a situation. This struggle may have stemmed from his interpretation of the situation, precluding his ability to conceptualize intermediate values for the other variable besides time. This would limit Cohen's covariational reasoning to coordinating benchmark values. For example, in the Book Set Task, he had trouble thinking about how a year and a half would affect the worth of the books, as is shown in the passage below.

L: For Carrie's situation, would it make sense to have a year and a half?

C: … a half? …Mm I don't think so.

L: Why not?

C: Because it's already 2-, we're 250 and we're multiplying by 2. But, if it's a year and a half you have to like …do something with 250. Or something like that.

L: What do you mean "do something with the 250"?

C: Cuz it's always like, it's rising, the money, every year. So, it's every year and a half… Umm…

L: So, would the book set be worth stuff at a year and a half? Would we be able to figure that out?

C: Like… I don't know… [Long pause]. Cuz if you like do a year and half that means…you have to double it again and the more money she needs to spend.

Cohen does show that he recognizes that the worth of the book set should change at a year and a half, but he seems unable to determine what the worth of the books would be. His struggle is reasonable, as he is dealing with exponential growth defined in time units of 1 year, not 1/2 years. Consequently, while it is possible that Cohen cannot identify intermediate values between the benchmark money values of $250 and $500, it is more likely that he does not name an intermediate value because he does not know how to calculate that intermediate value. Similarly, Cohen had an easier time thinking about the values in between his benchmark times compared to the benchmark money amounts in the Soda Buying Task. There was no evidence in this initial interview that he could consider all the possible times in between each year/day as possible inputs for the x-variable in the same way as his benchmark values (i.e., as being values of time with corresponding monetary values), but he did show he was able to envision some intermediate values between benchmark times. At this point in the unit, Cohen mainly used *discrete variation and coordination of values* levels of reasoning.

As mentioned above, Cohen had trouble in the mid-unit interview with creating a story that matched the given continuous graph in the Create a Story Task. This severely limited his ability to show his thinking with intermediate values. However, he did show improved reasoning with the intermediate values for the second problem in this interview, as shown in the following excerpt from the Practice Time Task.

> L: What would this coordinate [makes a coordinate point between 2 and 3 days on the line created by the student] mean for your story, for Deven's story? [Long pause from student] You can estimate like you did in class what you think that coordinate point is.
>
> C: That coordinate point [puts pencil on it] would be… [uses pencil to trace faint line from coordinate point over to the y-axis] would be exactly 170.
>
> L: So, what does that mean for the story?
>
> C: That means if he did another half a day… no, you can't have half a day. Yeah, you can. That he would get to 170.
>
> L: K. You just said you can't have half a day. Why do you think that?
>
> C: Well, you can, because each day is 24 hours. And you can half an hour.
>
> L: Would half a day make sense for Deven's story?
>
> C: No, it would not make sense. Because he's doing it 30 minutes every day, not half a day.

There are several great moments of reasoning for Cohen in this passage. First, he was able to think about a value between his benchmark in the context of the story. Second, he was able to determine the value for the y-variable, something he did not show an ability to do in the pre-interview. Third, he demonstrated that he was able to envision more times in a day than just his benchmark values. And finally, he was able to use the context to decide if half a day made sense for Deven's situation. Combined, these new skills show Cohen's vast improvement in his ability to now use *chunky continuous variational and covariational reasoning*.

By the final interview, Cohen had grown in his ability to envision the intermediate values for both variables and use intermediate values as possible inputs, but he was inconsistent in his implementation. The following excerpt from our discussion of the Burning Candle Task shows this inconsistency in his ability to envision values between benchmarks.

> L: Would that coordinate point that you just made [at 6.5 hours], does that make sense in the story?
>
> C: Mmm… yeah!
>
> L: Why?
>
> C: Cuz the candle is melting. So, it can melt to 6 and a half, too.
>
> L: Okay. Could it be melting at 6 ¼?
>
> C: Yeah.
>
> L: In class you've talked a lot about when to put a line and when not to put a line. Does it make sense to have a line for this situation?
>
> C: Yeah, because there's no breaks. It's always melting.
>
> L: So, what kind of numbers could we choose for the hours?
>
> C: Like 8, 8 and a half, 7, 7 and a half, 6 and a half, that many.

In this passage, Cohen is able to reason that the candle is burning at 6.25 hours (and later in the interview at 7.5265 hours) when asked to think about the possibility. But he makes no connection in these times to the height of the candle, and I did not question him further to see if he could. I was more focused on his ability to truly envision all the possible times in between his hourly benchmarks since this was a level of reasoning he had previously struggled with in earlier interviews. Fascinatingly, Cohen agreed that specific times could make sense for the given situation, but when asked later for possible inputs, he only listed his benchmark times as possible values for the hours. So, Cohen showed again that even if he was truly envisioning all of the values between benchmarks, he was not using these intermediate in the same way as his benchmark values.

Cohen performed similarly on the Growing Plant Task from the same interview. When I asked if it was possible to have a tree height between 1 and 1.2, he said:

> 1 and 1.2? Yeah, you could do in between. It would just change up the graph a little. Cuz if you do, if you subtract, if you put it like between [pointing to between 1 and 1.2 on the table] then you're gonna have to add more numbers and more dots.

He was able to reason that it was possible to examine values between the heights of 1 and 1.2 feet, but he felt the need to add these coordinated values to his table in order to add them to the graph. This indicates to me that he was on the verge of understanding the continuous variation that he saw in some of these situations, and thus be able to use *smooth continuous variational reasoning*. The previous example also suggests that Cohen has continued in his ability (as was shown in the mid-unit interview) of envisioning intermediate values for a quantity besides time. He could be pushed to reason about values in between the benchmark heights. However, there was insufficient evidence to suggest that he envisioned intermediate values on his own or simultaneously considered all the possible values for time. Furthermore, Cohen seemed to treat newly named intermediate values as additional benchmarks, adding them to his table of benchmarks instead of being able to think of the values without recording them.

Compared to the first interview, Cohen has grown from using no *smooth continuous* levels of reasoning in the pre-interview to one instance of using *smooth continuous variational reasoning* in the post-interview. By the end of the final interview, Cohen showed that he was not fully able to conceptualize the idea of continuous variation for either task. Yet, Cohen showed great improvement in his ability to use *chunky continuous variation and covariation* and conceptualize the intermediate values in the given stories.

Choosing Appropriate Levels of Reasoning

In the beginning, both students regularly used *discrete variation* and *coordination of values* for their level of reasoning regardless of the question asked. Specifically, these two levels of reasoning comprised 15 instances out of the 25 instances of reasoning for Adam in the pre-interview and 15 instances out of the 28 instances of reasoning for Cohen. As discussed above, both students increased in their ability to distinguish between and use *chunky and smooth continuous reasoning*, enabling them to better answer the questions posed in later interviews. Even so, the participants would sometimes conserve cognitive resources by using lower levels of variational and covariational reasoning when problems did not require advanced levels. The students' ability to choose levels strategically to conserve cognitive resources shows a greater understanding of variation and covariation than always using advanced levels for all problems. Strategic selection of levels demonstrates a more nuanced understanding of variation and covariation levels and when each is appropriate and useful. Their strategic use of levels suggests that rather than replace lower levels of reasoning with higher levels, they instead modified their original conceptions of variation and covariation to create a coherent, integrated understanding that included both lower and higher levels of reasoning. This section provides an example for each participant that demonstrates the students' growing ability to answer interview questions using appropriate, resource-saving levels of reasoning.

Adam

The results found in the previous section and Table 3 show that, by the end of the unit, Adam could use *chunky and smooth continuous covariational* reasoning but did not always choose to do so. Adam had 5 instances of *chunky and smooth continuous reasoning* in the pre-interview and increased to 12 instances by the post-interview. Even with this increase in advanced levels of reasoning, Adam still chose to use lower levels of reasoning 22 times in the post-interview. Adam's use of lower levels of reasoning in the post-interview was appropriate 86.4% of the time. Comparing this percentage of appropriate lower levels to Adam's percentage in the pre-interview (65%) shows that Adam grew significantly more accurate in his appropriate use of lower levels of reasoning throughout the unit.

Table 3

Appropriate Levels of Reasoning for Adam

Adam	Pre-Interview	Mid-Interview	Post-Interview
Number of Different Levels of Reasoning Used	6	6	10
Instances of Variational and Covariational Reasoning	25	31	34
Instances of Reasoning at Levels Below Chunky Continuous	20	26	22
Appropriate Use of Levels Below Chunky Continuous	13 (65%)	22 (84.6%)	19 (86.4%)

For example, in the final interview, Adam was asked how the height of the tree and the number of months were related in the Growing Plant Task. He could have responded using *smooth continuous covariational* reasoning by describing the simultaneous, incremental, and continuous increase in both the time and the height of the tree, noting that every possible value between benchmarks was used for both quantities. Instead, Adam appropriately responded to the question using *gross coordination of values*, saying, "Okay, so every month it grows 0.2 feet. So, it's always growing every month." A higher level of reasoning, like *smooth continuous covariation*, would have been unnecessary to sufficiently answer the question. Adam's strategic choice in reasoning shows that he can choose to use lower levels of reasoning when appropriate.

Cohen

Similar to Adam, Cohen appropriately used lower levels of reasoning for some questions in the post-interview even when he had more advanced levels of reasoning at his disposal. The findings detailed in the Discrete and Continuous Variation and Covariation section and Table 4 show that, by the end of the unit, Cohen could reason using *chunky continuous covariation* and *smooth continuous variation.* This improvement in reasoning led Cohen to increase the number of instances of advanced levels of reasoning from 2 in the pre-interview to 7 in the post-interview. Despite this, he appropriately chose to use *discrete variation* and *coordination of values* to make his tables and graphs in the final interview. Choices like this helped Cohen to significantly increase his ability to appropriately use lower levels of reasoning from 58% in the pre-interview to 71% in the mid-interview and post-interview.

Table 4

Appropriate Levels of Reasoning for Cohen

Cohen	Pre-Interview	Mid-Interview	Post-Interview
Number of Different Levels of Reasoning Used	8	7	10
Instances of Variational and Covariational Reasoning	28	25	38
Instances of Reasoning at Levels Below Chunky Continuous	26	21	31
Appropriate Use of Levels Below Chunky Continuous	15 (57.7%)	15 (71.4%)	22 (71%)

For example, in the Growing Plant Task, Cohen used *discrete variation* to list possible inputs for the months in the table. Then he used *coordination of values to* connect 0 months with a height of 0.8 feet for the tree. He continued to make his table using *discrete variation* to find the next benchmark heights for the tree. When Cohen turned to make his graph, he used *discrete variation* to mark the scales on the axes and *coordination of values* to take the benchmark values for each variable in the table and graph distinct pairs of coordinate points. These two levels of reasoning were sufficient for creating a basic table and graph. Consequently, Cohen's choice of levels of reasoning was appropriate for this part of the task. While Cohen might have been envisioning values between his benchmarks while he made the table and graph, he did not need to use more advanced levels of reasoning to make the representations. This example shows that Cohen could use lower levels of reasoning appropriately and that he did not replace his lower levels of reasoning when he developed his ability to reason using more advanced levels.

Summary

Comparatively, Adam was more proficient on a wider variety of levels by the end of the unit than Cohen and consequently had more instances of advanced levels of reasoning in the final interview. Despite this discrepancy, both students used more levels of *chunky continuous reasoning* and *smooth continuous reasoning* in later interviews than they used in the pre-interview. They also both significantly improved the appropriateness of their chosen levels of reasoning throughout the unit. Adam's number of appropriate lower levels of reasoning improved by 21% and Cohen's improved by 13% by the end of the unit (see Appendix C for more detailed tables describing appropriate levels of reasoning used for both students). Since the appropriateness of the lower levels of reasoning improved over time, it follows that the participants grew in their ability to choose a level of reasoning that sufficiently answered an interview question instead of blindly using advanced levels of reasoning regardless of the depth of thinking required by the question. This shows that Adam and Cohen did not replace the lower levels of reasoning with the newly learned advanced levels but instead learned to strategically use these varied levels of reasoning to conserve cognitive resources during the interviews.

Self-Correction

While analyzing the data, it became apparent that the students often adjusted their thinking (for better or worse) as they participated in each interview. The participants were frequently able to improve or correct their mathematical thinking when given sufficient time to progress through a task. Participants' ability to self-correct suggests the following: first, students with disabilities are able to engage in mathematical reasoning without constant correction and handholding; and second, that the mistakes they make can be poor indicators of whether or not they are capable of engaging in deep mathematical thinking. Thus, even though the students' moments of self-correction do not directly answer the research question as to students' ability to engage in variational and covariational reasoning, these moments still shed light on the conceptual understanding and mathematical abilities of students with disabilities and should be included in the results of the study.

Both Adam and Cohen had several instances where they corrected what they perceived as mistakes they had previously made while working on a given task. These mistakes included mislabeling axes on the graph, incorrectly recording numbers on a table, making incorrect calculations, and misinterpreting the situation. After analyzing all corrections made by the participants during the interviews, participants seemed prompted to make a correction in three different ways: self-detections, routine questions, and facilitating questions. This section will outline each type of prompt and then summarize individual trends for each participant.

Self-Detections

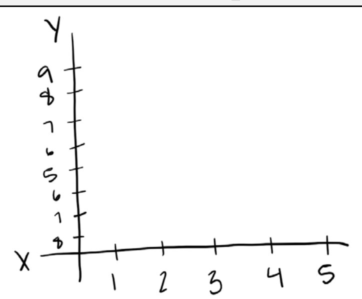
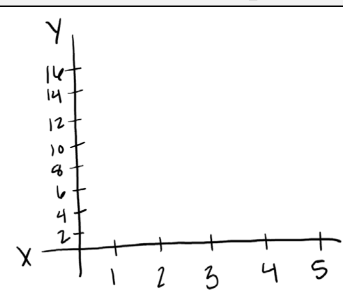
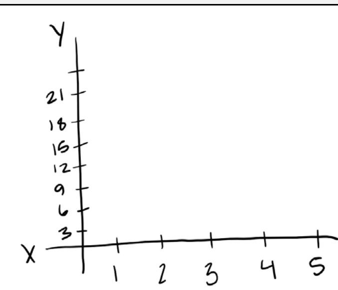
A self-detection prompt occurred when the participant adjusted their mathematical thinking or representations with no feedback or input from the researcher. The students were prompted by self-detections when they noticed errors or contradictions between representations as they continued to work on the task. Most of the modifications made by the participants after self-detection led to accurate mathematical thinking. When modifications following self-detection did not result in a correct answer, they nonetheless frequently yielded improved mathematical thinking. Despite this improvement, I coded these self-corrections as being incorrect.

The following example includes three instances of self-detection by Cohen that helped him create an accurate graphical representation of the covariation in the Buying Soda Task. The modifications resulting from self-detection in this example initially led to incorrect (but improved) mathematical thinking, and eventually to correct mathematical thinking.

When making a graph for the Buying Soda Task in the first interview, Cohen started to use a mathematically incorrect scale for the y-axis. He had accidentally recorded 18.75 as 8.75 in his table for Day 1, so Cohen began his scale with 8 units and decreased his scale by 1 unit as he ascended up the y-axis, stopping at 5 units. As he plotted his coordinated values from the table as coordinate points on the graph, he accidentally read the 17.5 for Day 2 in his table as 7.5. Then for Day 3, he seemed to have looked at his coordinated values for Day 3 and realized that his scale on the y-axis needed to reach 16.25. So, he resumed making his scale starting at his mark for 5 units but began *increasing* by 1 unit instead of continuing the previous decreasing pattern (see Figure 2). When he reached 9 units, Cohen decided to erase his first attempt at the scale and try again.

Figure 2

Cohen's Attempted Scales

1st Attempt	2nd Attempt	3rd Attempt
		

Cohen's second attempt at the scale for the y-axis of his graph represents another moment of modification prompted by self-detection. By plotting the coordinate values from his table, Cohen recognized, with no prompt from the researcher, that his original scale would not work for his data. His new attempt for the y-axis scale ascended from 0 units to 16 units, with each tick mark increasing by equal increments of 2 units. His correction led him to an improved but still incorrect graph.

Cohen was prompted by another self-detection to make a third attempt at modifying the scale for the y-axis when he realized the coordinate points still did not fit on the improved graph. As he started plotting points, Cohen looked back and forth between the table and graph, and then exclaimed, "Second day… wait a minute. Oh! It's 17!" Cohen's actions suggest that he used a comparison between the two representations to identify his mistake of misreading the table. Cohen was then able to self-correct this mistake and create a third scale for the y-axis. His final graph for this task had a scale with equal increments of 3 units ascending from 0 units to 21 units, which fit all of the coordinated values from the table.

If someone had looked at Cohen's original graph, it would have been easy to assume that Cohen had no idea what he was doing. But Cohen showed his conceptual understanding of decreasing linear patterns when he recognized that his graph did not match what he expected it to look like and caught his procedural mistakes. The modifications that resulted from his self-detection varied in their correctness, but he was able to figure out this problem on his own and ended up with a mathematically sound graphical representation of the covariation.

Routine Questions

The students were also prompted to make modifications after the researcher asked a routine question that gave them an opportunity to reflect on their work. These routine questions were asked for each task in every interview regardless of the correctness of the student's work. Examples of these questions include, "How did you make your table?" and "How are these two problems different?" The types of errors noticed by participants after routine questions were similar to the errors they noticed during self-detection, and changes made after routine questions varied in correctness similarly to changes made from self-detection.

A typical example for changes following routine questions is Adam's response to a question asked in the post-interview about the Candle Burning Task. When asked whether having a line on his graph or just points mattered for the situation, Adam responded, "Well, the candle is constantly burning. It doesn't just go [gestures a jump down with his hands] oh, I've burned. So, it would be constantly burning." He then connected the points on his graph with a line. Adam initially created a graph with discrete points, but he was able to reflect on his work in response to the routine question asked and correctly modified his thinking. The reflection opportunity given to him by the routine question allowed Adam to show his conceptual understanding of why a situation might have continuous covariation.

Facilitating Questions

The students were also prompted to adjust their thinking after the researcher asked a facilitating question in response to a student error (i.e., not labeling what is measured on each axis or not understanding the prompt). Facilitating questions like, "What are you measuring on your x- and y-axis?" and "How tall is the candle at first?" were asked with the purpose of moving the student forward in their thinking after a mistake was made by the student. It seemed that the facilitating question itself is what prompted students to modify their work since all four facilitating questions asked throughout the interviews led to direct changes by the participants. According to the data, there were no incorrect modifications after this type of prompt.

Sometimes Cohen had trouble interpreting the given situations and needed a series of facilitating questions to help him engage in the task. These series of facilitating questions were counted as one instance of a facilitating question prompting the student to modify their work, since I was often unable to determine which facilitating question in the series prompted the subsequent self-correction. For example, in the Buying Soda Task, Cohen got stuck while making his table and the researcher asked a series of facilitating questions to help him make sense of the situation.

C: I'm so confused.

L: What's confusing you?

C: I dunno. I just haven't done these in forever.

L: Okay. So, tell me about how you are starting to make the table.

C: I started with 20 bucks, and I subtracted 1.25 and I got 18.75, but I kinda forgot where to write that.

L: So, one thing that will be helpful to keep track of is like how many times you've bought soda. Right? So, when you have $20, how many times have you bought soda? (facilitating question)

C: 20. Wait [reaches for calculator].

L: So, you still have $20, but have you bought any soda yet? (facilitating question)

C: No.

L: And then if you have $18.75…? (facilitating question)

C: You've boughten one soda.

L: Yeah.

C: So… [long pause] Ah crap. Umm. [Grabs calculator] So, I know you can buy 16 sodas.

This series of facilitating questions helped Cohen reason about the covariation in the Buying Soda Task. Once he had a better understanding, Cohen was able to progress in his work in the task by determining that only 16 sodas can be bought with the $20 and then making mathematical representations of the covariation. Because the questions seemed to support Cohen in making sense of the problem, he was able to progress further in his solution and had more opportunities to show what he did understand about discrete and continuous variation and covariation

Individual Trends

While both students made a lot of modifications, the types of prompts and answers varied drastically across students. Table 5 and Table 6 represent the number of occurrences for each category of modifications for Adam and Cohen, respectively.

Table 5

Self-Correcting Prompt Counts for Adam

Correctness	Self-Detection	Routine Question	Facilitating Question
Correct	10	3	1
Incorrect	3	1	0

Table 6

Self-Correcting Prompt Counts for Cohen

Correctness	Self-Detection	Routine Question	Facilitating Question
Correct	8	3	3
Incorrect	5	5	0

Adam was more likely to make modifications as a result of self-detection and commonly used the routine questions in the interviews to explain his thinking rather than change his thinking. He tended to be successful in correctly adjusting his thinking in response to any of the three prompts. Cohen made changes more frequently after self-detections than after being asked a question, but he changed his thinking after a routine question more often than Adam did. Additionally, his adjustments after a routine question were more often incorrect variations of his previously given incorrect answer rather than a correct answer. This could indicate that Cohen thought the researcher insinuated he was wrong and should make an adjustment, that he was in doubt of his answers, or that he was still developing his understanding of the concepts from the unit.

Despite their differences, both students were frequently able to improve or fully correct their mathematical thinking and representations without constant coddling. Their ability to correct their thinking was likely a result of time with the task, multiple representations, and reflection opportunities created by routine and facilitating questions. Furthermore, Adam's and Cohen's ability to improve their thinking while working on a task shows that the initial mistakes that students with disabilities make on a problem are a poor indicator of the conceptual understanding they have or can construct about the task and the underlying mathematics.

Discussion

As aforementioned, most research surrounding covariation is typically done with college students. Covariation is not an easy concept for undergraduate students to understand, because they find it difficult to coordinate the variation of two related quantities (Carlson, 1995; Monk & Nemirovsky, 1994; Thompson, 1994a; Stalvey & Vidakovic, 2015). Adam rarely had moments when he had trouble coordinating values for both variables and continually improved his ability to coordinate continuous covariation through the unit. On the other hand, Cohen frequently struggled with coordinating values for both variables in the interview tasks, similar to the struggles of college students. Remarkably, when Cohen figured out how to coordinate the values in the interviews (whether on his own or with guidance from the researcher), he was able to progress in the tasks and in his understanding of variation and even covariation. The progress that Adam and Cohen made in their understanding of this difficult topic, a topic they were just barely introduced to a few weeks before the post-interview, is a significant achievement for ordinary students and particularly impressive for students with disabilities.

The progress that Adam and Cohen experienced in this unit was due in part to the instruction they received from their co-taught class. It is evident that the students' instruction affected how the students reasoned about the interview tasks by how they used specific strategies and vocabulary from their class discussions during the interviews. Additionally, the tremendous progress the participants showed in reasoning about a topic that is hard for students much further along in their mathematical understanding shows there is no way that Adam and Cohen's instruction did not contribute to their understanding. This demonstrates that reformed instruction can support the mathematical learning of students with disabilities. This finding is supported by general mathematics education research on the learning achievements of students without disabilities in reformed classrooms (e.g., Zakaria et al., 2010; Kogan & Laursen, 2013; Boaler, 2001) and contradicts mathematics special education research that labels direct instruction as the preferred learning environment for students with disabilities (e.g., Carnine, 1997; Kroesbergen & Van Luit, 2003). Since direct instruction commonly emphasizes procedural fluency over or instead of conceptual understanding, direct instruction would not have produced the deep and rich conceptual understanding that Adam and Cohen showed in this study. Additionally, direct instruction would not have provided the participants the experiences they needed to increase their sophistication in choosing appropriate levels of reasoning since direct instruction usually focuses on a single solution method that is designed to be applicable to all problems of a certain type. Thus, the understanding and reasoning that Adam and Cohen showed in this study would not have been possible if they had only received direct instruction.

In addition to reformed instruction, it is likely that the interviews, particularly the think-aloud tasks, also contributed to student learning. It is very likely that my presence as a researcher in the interviews was beneficial for Adam and Cohen. While I did not give feedback on how accurate their mathematical thinking was, I did provide an attentive adult presence that probably helped both students to focus on and persevere in solving the tasks. Additionally, the routine questions, facilitating questions, and expectation that the students would explain their thinking set by the researcher provided ample opportunities for the students to reflect on their thinking. Asking Adam and Cohen to think aloud while working on the interview tasks might have helped them to process their reasoning and the mathematics more easily. It is important to note that, in this study, facilitating questions were asked by the researcher when a student was stuck and unable to progress forward in their thinking on their own, and helped open up opportunities for the student to choose and develop other levels of reasoning. From the results of the selfadjustments the students made during the interviews, it is apparent that the presence of and interactions with the researcher most likely were helpful for the students by increasing focus, reflection, and effort.

It is also important to note that Adam and Cohen did experience different levels of success in their conceptual understanding of covariation by the end of the unit. It is a natural reality that reformed instruction may yield different results for different students. Adam ended the unit with a workable knowledge of *chunky and smooth continuous variation and covariation*; in contrast, Cohen was just beginning to use *smooth continuous variational* reasoning. Cohen also tended to make more mistakes in his work with the tasks than Adam did. It was apparent in all three interviews that Cohen struggled with this unit on covariation more than Adam did. Despite this, the aforementioned results show that Cohen still significantly improved his conceptual understanding throughout the unit. In the end, regardless of their differences in the understanding they developed, both Adam and Cohen showed the capacity to develop conceptual understanding.

This study shows that Adam and Cohen, two students with disabilities, were able to increase their conceptual understanding of covariation in a reformed, co-taught classroom that focused on building the conceptual understanding of their students through task-based instruction. As such, this study is an existence proof that students with disabilities can successfully engage in task-based instruction and develop understanding of complex mathematical ideas. Their increase in conceptual understanding in this learning environment suggests that students with disabilities should not be limited to receiving only direct instruction about procedures. The participants' progress suggests that the type of support they received in this study—conceptually oriented instruction and the presence and questioning of a teacher—can support the development of deep and rich understanding.

CHAPTER 5: CONCLUSION

The purpose of this thesis was to explore the conceptual understanding of variational and covariational reasoning for two students with disabilities and examine how their conceptual understanding changed as a result of reformed instruction. From my own experiences, I hypothesized that students with disabilities can improve their conceptual understanding of mathematics when given conceptually focused learning opportunities. Both participants were observed during their reformed, co-taught class for each class period in the three-week unit on discrete and continuous functions. The students also participated in three task-based interviews conducted before, during, and after their unit on covariation. Each interview was approximately 30 minutes long. The students' thinking was analyzed using Thompson and Carlson's (2017) framework for levels of variational and covariational reasoning. The results showed that both students were able to reason using levels of *chunky continuous* reasoning (with Adam also able to use *smooth continuous* reasoning) by the end of the unit, improved their ability to choose appropriate levels of reasoning, and demonstrated an ability to self-correct their work. This study showed evidence that these two students with disabilities were able improve their mathematical understanding in a reformed classroom environment.

Contributions

First, this study adds an existence proof that students with disabilities can increase their conceptual understanding and can learn in a reformed classroom environment. Very few studies have investigated students with disabilities in a reformed classroom or the conceptual understanding of students with disabilities. This study provides a valuable case study of two students with disabilities who were able to learn difficult mathematics, both procedurally and conceptually, without direct instruction. This case study contributes evidence that some students with disabilities can benefit from reformed instruction on important mathematical concepts and ideas.

Second, this study adds to the list of supports that may help students with disabilities meaningfully engage in mathematics. The interviews in this study suggest other ways that students with disabilities can be supported besides receiving reformed instruction, namely, by having a teacher present who occasionally asks questions and keeps them focused. Part of the reason the participants were able to engage deeply with the tasks in the interviews was their ability to self-correct. The researcher's presence in the interviews and the multitude of questions asked gave plenty of opportunities for Adam and Cohen to reflect on their work and choose to adjust their thinking. More often than not, these self-modifications led to correct or improved mathematical thinking. As such, other students with disabilities might benefit greatly from the presence of a teacher to help them focus and provide questions that give them an opportunity to reflect on and change their thinking.

Third, this study extends the research on students with disabilities by using a mathematics education lens to examine the learning of students with disabilities. The mathematics education lens is essential to examining the conceptual understanding of students with disabilities. This lens allows the researcher to go beyond whether or not the students can perform computations or the behavior of the students to examine what understanding the students are constructing of important mathematical concepts.

Implications

First, reformed instruction should not be ruled out for students with disabilities. The existence proof inherent in this study shows that attending to the conceptual understanding of students with disabilities in their instruction can help them learn difficult mathematics. Students with disabilities should be given opportunities in the classroom to experience problem solving and explore mathematics to build their conceptual understanding of mathematics. In fact, limiting students with disabilities to direct instruction inhibits their ability to develop conceptual understanding and is inequitable in light of the findings of this study.

Second, co-teachers should use teacher presence and questioning to support their students in their learning. Specifically for reformed classrooms, an increase in teacher presence should be a focus for one or both co-teachers in the exploration phase of a task and questions can be used by both co-teachers during the exploration and discussion phases. Routine questions can be used by the teacher to help individuals or groups of students to explain and reflect on their thinking. These questions should be used consistently in the classroom as support for students with disabilities—regardless of whether a student's thinking is mathematically correct or not—to help provide opportunities to reflect and make conceptual connections. Facilitating questions can also be used to support students with disabilities, but these questions should be used sparingly as they may limit reasoning opportunities for the student. The use of routine and facilitating questions will likely also serve the purpose of providing a teacher presence to help the students focus on the task at hand. Since teacher presence alone seemed to encourage the students to persevere in their problem solving, teachers should refrain from constantly interrupt the students' work with routine or facilitating questions. Because students in a co-taught class are sometimes pulled out for instruction time with the special education teacher, it is important to note that both co-teachers should be consistent in giving the students time to work on the problems and find mistakes on their own. Then the teachers can use the different questions to help the students further reflect on their work.

Another support that co-teachers can use with their students with disabilities is to give the students opportunities to think out loud while they solve a problem. Verbalizing their solutions, with or without a teacher present, might help the students to process their reasoning and the mathematics more easily. However, if every student is thinking aloud, then the classroom could quickly become overwhelming for some of the students. This suggests the need for smaller class sizes for co-taught classes, so that students can have the opportunity to process their solution strategy verbally without the noise in the classroom becoming overwhelming.

Third, a mathematics education lens should be considered when studying the mathematical learning of students with disabilities. Since this case study showed that two students with disabilities were able to develop conceptual understanding, future studies need to adopt a perspective toward mathematics and its learning and teaching that is not limited to the learning and teaching of procedures only. Researchers, especially those hoping to extend research for students with disabilities, need to conceptualize learning in a way that allows them to notice and study students' understanding and not just their competency with procedures.

Limitations and Direction for Future Research

One critical limitation of this study is the small sample size. The results of this study are limited to the thinking of only two students with disabilities and cannot be generalized to *all* students with disabilities. Additionally, this study only looked at one mathematics topic. Further research should expand to include more students with disabilities and focus on different areas of mathematics to see if similar growth in conceptual understanding is found.

Another limitation of this study is that the participants did not have disabilities that affected their processing speed or IQ. The priority placed on choosing a co-taught classroom that focused on conceptual understanding paired with the full-time class schedule of the researcher narrowed the pool of possible participants. Future research should focus on the conceptual understanding of students with other disabilities, particularly those that affect processing speed or IQ.

A third limitation of this study is that it is unclear from this study how much of the participants' learning was due to instruction and how much was due to the interaction with the researcher during the interviews. This may be an unavoidable limitation in studying the development of conceptual understanding by students with disabilities. Relying on data collected without the presence of a researcher might well be a poor reflection of the participants' understanding, because they may not know how to express their thinking clearly, particularly in writing. Thus, data collection that does not allow opportunities to ask follow-up questions may lead to severe underestimations of students' understanding.

Conclusion

I began this thesis with the hope that my personal experiences with the deep, rich thinking of students with disabilities was not an anomaly. This study shows that it is possible for at least these two students with disabilities to reason conceptually about difficult mathematical concepts. Furthermore, this study shows that Adam and Cohen were able to find success in increasing their understanding in a reformed mathematics classroom. Since Adam and Cohen are two very different students, it is likely that other students with disabilities could find success in a similar learning environment. Students with disabilities should be given more opportunities to develop their understanding of mathematics and not just computational proficiency. Future research should be focused on the conceptual understanding of students with disabilities and how to build it. I hope to continue this line of study to help give *all* students an opportunity to engage in meaningful mathematics.

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APPENDIX A

Pre-Assessment Tasks and Questions

Task 1:
Mike was given a gift card with $20 on it. He used it to buy a $1.25 soda each day. Use a table, graph, and equation to represent this situation.

Task 2:
Carrie bought a rare book set worth $250 in 2000. The book set has doubled in value every year since 2000. Use a table, graph, and equation to represent this situation.

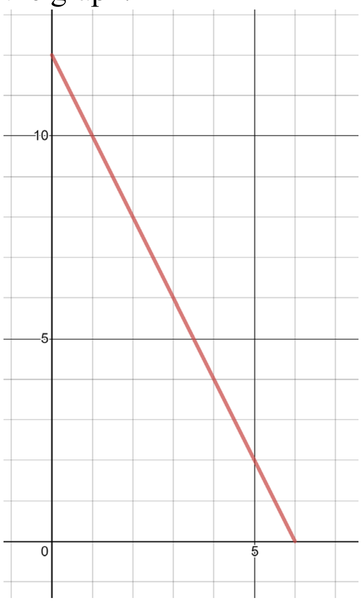
Task 3:
Compare the last two questions. What is similar between the two situations? What is different?

Possible Follow-up Questions:

1. What were the similarities you saw between the two situations? How do you know it is the same for both?
2. What were the differences you saw between the two situations? How do you know it is different?
3. What is changing in Mike's situation? How often does ______________ change?
4. Can you explain what you were thinking as you made the table for Mike's situation?
5. Why did/didn't you use a line to represent Mike's situation on the graph?
6. Can you provide some examples of possible inputs (or values for x) I could use in your equation for Mike's situation? Why would those work?
7. Would it make sense if Mike's gift card had $9.75 on it on the 8^{th} day? Why or why not?
8. What is changing in Carrie's situation? How often does ______________ change?
9. Was making the table for Carrie's situation different than making a table for Mike's? If so, how? If not, why?
10. Why did/didn't you use a line to represent Carrie's situation on the graph?
11. Can you provide some examples of possible inputs (or values for x) I could use in your equation for Carrie's situation? Why would those work?
12. Would it make sense if Carrie's book set was at some point worth $1250? If so, how long until it would be worth that much? If not, why?
13. Can you think of an example that would be similar to Carrie's situation? If so, why would you say the situations are similar? If not, why?

Mid-Unit Interview Tasks and Questions

Task 1:
Create a story to match the graph below. Make sure to label each axis and connect the story to the graph.



Backup Situation A:
The temperature outside started at 12 degrees F and then steadily decreased until it was 0 degrees on the 6th day.

I had a bucket with 12 gallons of water. It drains water at a rate of 2 gallons per minute and was empty by the 6th minute.
If trouble coming up with story: Does this situation fit? Why?

Possible Task 1 Follow-Up Questions:

1. What story did you come up with to match the graph?
2. What is changing in your story? How does _______________ change?
3. How do you see your story in the graph?
4. How would you need to label the x-axis and y-axis in order for the graph to match your story?
5. Can you see the change in (y-axis) from Day 1 to Day 2 in your graph?
6. Can you see the change in (y-axis) from Day 1 to Day 4 in your graph?
7. Can you see the change in (y-axis) from Day 1 to Day 1 ½ in your graph?
8. Pick a coordinate point and tell me what it means in your story.
    a. On grid
    b. Decimal
    c. Does it matter whether you have a line or just points on the graph for your situation?

9. Would the coordinate point (0.5, 11) make sense for your story? Why or why not?
10. How are your (x-axis) and (y-axis) related?
11. Suppose someone said that as (x-axis) increases the (y-axis) increases. Are they right or wrong? Why?
12. If I tell you what time it is, does that help you know (y-axis)?

Task 2:
Deven plans to practice his trombone 30 minutes every day this week. He has already practiced 140 minutes before this week started. Sketch a graph to represent how many total minutes Deven has practiced his trombone.

Possible Task 2 Follow-Up Questions:
1. If trouble with graph: Is there different way that you can represent the situation?
2. Can you see the change in practice time from Day 1 to Day 2 in your graph?
3. Can you see the change in practice time from Day 1 to Day 4 in your graph?
4. Can you see the change in practice time from Day 1 to Day 1 ½ in your graph?
5. How did you label each axis? If you didn't, do so now. Why did you label the x-axis as ______________? Why did you label the y-axis as ______________?
6. Pick a coordinate point and tell me what it means in your story.
    a. On grid
    b. Decimal
    c. Does it matter whether you have a line or just points on the graph for your situation?
7. How is Deven's practice time and the number of days related?
8. Suppose someone said that as the number of days increases Deven's practice time increases. Are they right or wrong? Why?
9. If I tell you how many days have passed, does that help you know how long Deven has practiced?

Post-Assessment Tasks and Questions

Task 1:
Jared lit a candle with a height of 8-inches and let it burn until it was gone. The candle burns at a rate of ½ inch per hour. Use a table, graph, and equation to represent this situation.

Task 2:
Stephanie planted a new tree in her yard and wanted to measure its growth. The tree was already 0.8 feet tall when Stephanie planted it. She measured the height of the tree each month and noticed it grew about 0.2 feet per month. Use a table, graph, and equation to represent this situation.

Task 3:
Compare the last two questions. What is similar between the two situations? What is different?

Possible Follow-up Questions:

1. What were the similarities you saw between the two situations? How do you know it is the same for both?
2. What were the differences you saw between the two situations? How do you know it is different?
3. What is changing in Jared's situation? How often does ______________ change?
4. Can you explain what you were thinking as you made the table for Jared's situation?
5. Can you provide some examples of possible inputs (or values for x) I could use in your equation for Jared's situation? Why would those work? What does x represent?
6. Can you see the change in the height of the candle from Hour 1 to Hour 2 in your graph?
7. Can you see the change in the height of the candle from Hour 1 to Hour 4 in your graph?
8. Can you see the change in the height of the candle from Hour 1 to Hour 1.5 in your graph?
9. Pick a coordinate point and tell me what it means in your story.
    a. On grid
    b. Decimal
10. Does it matter whether you have a line or just points on the graph for Jared's situation?
11. How is the candle's height and the number of hours related?
12. Suppose someone said that as the number of hours increases the height of the candle increases. Are they right or wrong? Why?
13. If I tell you how many hours have passed, does that help you know how tall the candle is?

14. What is changing in Stephanie's situation? How often does ______________ change?
15. Was making the table for Stephanie's situation different than making a table for Jared's? If so, how? If not, why?

16. Can you provide some examples of possible inputs (or values for x) I could use in your equation for Stephanie's situation? Why would those work? What does x represent?
17. Can you see the change in the height of the tree from Month 1 to Month 2 in your graph?
18. Can you see the change in the height of the tree from Month 1 to Month 4 in your graph?
19. Can you see the change in the height of the tree from Month 1 to Month 1.5 in your graph?
20. Pick a coordinate point and tell me what it means in your story.
    a. On grid
    b. Decimal
21. Does it matter whether you have a line or just points on the graph for Stephanie's situation?
22. How is the height of the tree and the number of months related?
23. Suppose someone said that as the number of months increases the height of the tree increases. Are they right or wrong? Why?
24. If I tell you how many months have passed, does that help you know how tall the tree is?

APPENDIX B

Classroom Observation Form

Date:	Lesson:	
Teacher Questions	**Teacher Vocabulary/Concepts**	**Levels Modeled by Class**

Student Levels	Student Gaps
#1	#1
#2	#2
#3	#3

APPENDIX C

Table 7

Detailed Appropriate Levels of Reasoning for Adam

Adam	Pre-Interview		Mid-Interview		Post-Interview	
	Appropriate	Not	Appropriate	Not	Appropriate	Not
Variable as a Symbol	0	1	0	0	0	1
No Variation	0	0	0	0	0	0
Discrete Variation	6	1	6	1	7	2
Gross Variation	1	3	0	0	1	0
Chunky Continuous Variation	2	1	1	1	4	0
Smooth Continuous Variation	0	0	0	0	2	0
No Coordination	0	0	0	0	0	0
Precoordination of Values	0	0	2	2	2	0
Gross Coordination of Values	0	0	8	0	4	0
Coordination of Values	6	2	5	2	5	0
Chunky Continuous Covariation	1	1	0	3	2	0
Smooth Continuous Covariation	0	0	0	0	4	0

Table 8

Detailed Appropriate Levels of Reasoning for Cohen

Cohen	Pre-Interview		Mid-Interview		Post-Interview	
	Appropriate	Not	Appropriate	Not	Appropriate	Not
Variable as a Symbol	0	4	0	0	0	0
No Variation	0	1	0	1	0	2
Discrete Variation	8	3	4	3	8	5
Gross Variation	0	2	0	0	4	1
Chunky Continuous Variation	0	0	2	1	3	1
Smooth Continuous Variation	0	0	0	0	1	0
No Coordination	0	0	0	0	0	1
Precoordination of Values	1	0	1	0	2	0
Gross Coordination of Values	2	1	6	1	1	0
Coordination of Values	4	0	4	1	7	0
Chunky Continuous Covariation	2	0	1	0	2	0
Smooth Continuous Covariation	0	0	0	0	0	0