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Zeros of a Family of Complex-Valued Harmonic Rational Functions

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ABSTRACT

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The Fundamental Theorem of Algebra is a useful tool in determining the number of zeros of complex-valued polynomials and rational functions. It does not, however, apply to complex-valued harmonic polynomials and rational functions generally. In this thesis, we determine behaviors of the family of complex-valued harmonic functions $f_c(z) = z^n + \frac{c}{\bar{z}^k} - 1$ that defy intuition for analytic polynomials. We first determine the sum of the orders of zeros by using the harmonic analogue of Rouché's Theorem. We then determine useful geometry of the critical curve and its image in order to count winding numbers by applying the harmonic analogue of the Argument Principle. Combining these results, we fully determine the number of zeros of f_c for $c > 0$.

Keywords: complex analysis, complex-valued harmonic function, epicycloid

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Chapter 1. Introduction

Consider the complex rational function

$$r(z) = z^n + \frac{c}{z^k} - 1,$$

where $c \in \mathbb{C} \setminus \{0\}$ and $n, k \in \mathbb{N}$. To determine the total number of zeros of r is a straightforward matter. Writing r as a single fraction,

$$r(z) = z^n + \frac{c}{z^k} - 1 = \frac{z^{n+k} + c - z^k}{z^k},$$

we see that r is zero if and only if $z^{n+k} + c - z^k = 0$ and $z^k \neq 0$. Because $c \neq 0$, we know that $z = 0$ is not a zero of $z^{n+k} + c - z^k$, so we see that r has the same number of zeros as $z^{n+k} + c - z^k$. The Fundamental Theorem of Algebra tells us that $z^{n+k} + c - z^k$, and thus r, has $n + k$ zeros.

We can determine the number of zeros of r in this way, but suppose we now want to determine the number of zeros of the *complex-valued harmonic function*

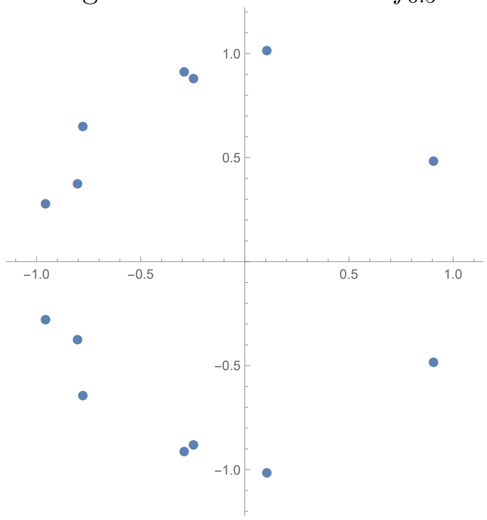
$$f_c(z) = z^n + \frac{c}{\overline{z}^k} - 1$$

(see Chapter 2 for details). Because f_c is not holomorphic, we cannot rely on the Fundamental Theorem of Algebra; instead, we can use computational methods to illustrate the number of zeros in specific cases. As an example, consider

$$f_{0.5}(z) = z^{13} + \frac{0.5}{\overline{z}^7} - 1.$$

The zeros are shown in Figure 1.

Figure 1.1: The zeros of $f_{0.5}$



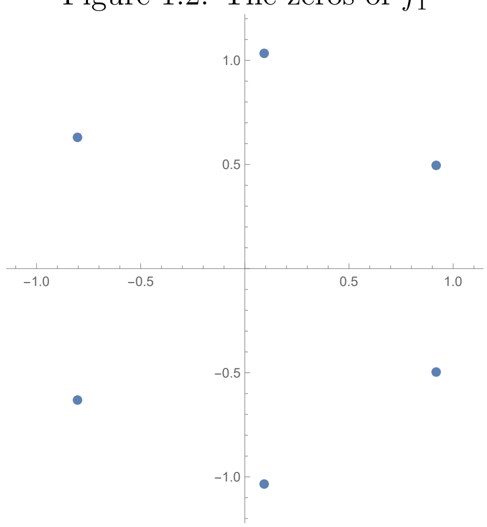
This function has 14 zeros, which bears no immediately clear relation to the degree of the numerator or denominator of $f_{0.5}$.

It is illuminating to further consider the similar function

$$f_1(z) = z^{13} + \frac{1}{\bar{z}^7} - 1,$$

where only the value of c differs from the previous example. The zeros for this function are shown in Figure 2.

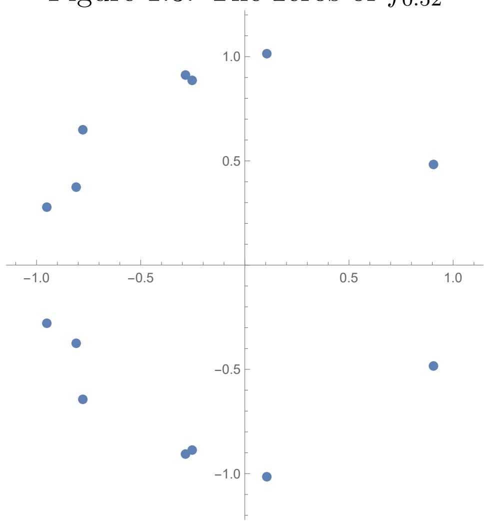
Figure 1.2: The zeros of f_1



Despite having the same degrees as before, this function now has only 6 zeros. These examples provide evidence to demonstrate that the behavior of complex-valued harmonic functions is considerably more complicated than that of holomorphic functions–not only does the number of zeros bear no clear relation to the degrees of the numerator and denominator, but the number of zeros is also dependent on the coefficient c.

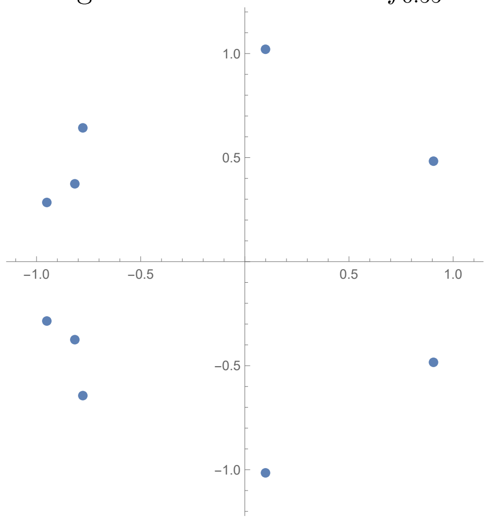
In order to gain a more full intuition for this problem, we examine f_c for various values of c and see if we can observe any notable behavior of the zeros. One may notice from the Figure 1 that there are two pairs of zeros, one pair in the second quadrant, the other in the third quadrant, which consist of zeros very close to each other. If we increase c to be 0.52, then the zeros get closer:

Figure 1.3: The zeros of $f_{0.52}$



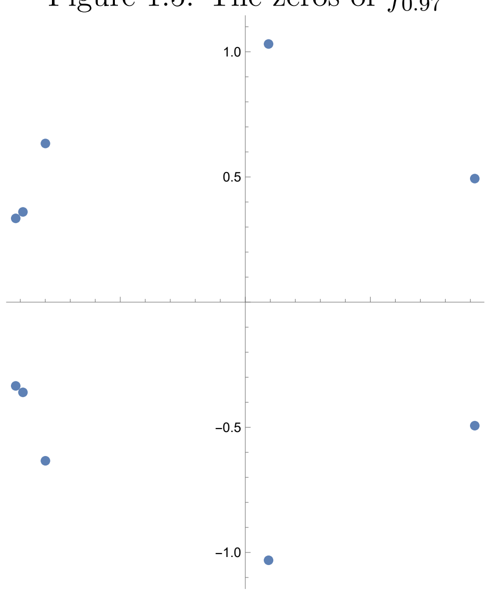
If we again increase c to be 0.55, then both pairs of zeros vanish entirely:

Figure 1.4: The zeros of $f_{0.55}$



These examples demonstrate more characteristics of these complex-valued harmonic functions: certain pairs of zeros appear to converge to a single point and then vanish as c increases. It seems fair to suggest that this type of behavior occurs again when we notice that we have ten zeros for $c = 0.55$ and only six zeros for $c = 1$. We can investigate this behavior by considering the case where $c = 0.97$, which gives rise to the following set of zeros.

Figure 1.5: The zeros of $f_{0.97}$



We again have two pairs of zeros converging to a single point (they are seen as the farthest left objects in the figure); by the time $c = 1$, they have disappeared, giving us six zeros total.

We call the c-values at which the number of zeros changes *critical values.*

Reasonable questions to ask based on this new information are things such as the following:

- Is the number of zeros a decreasing function in c?
- Are there infinitely many critical values?
- When zeros disappear, do they always disappear in two pairs of two?

Similar questions have been asked by Brilleslyper et al. [2] about the family of complex-valued harmonic functions

$$p_c(z) = z^n + c\bar{z}^k - 1,$$

where, using the harmonic generalizations of the Argument Principle and Rouché's Theorem, they determined exactly how many zeros this function has for various values of c:

Theorem 1.1 (Brilleslyper et al. Main Theorem)**.** *Let $p_c(z)$ be as above and let $N = \lceil k/2 \rceil + 1$. There exist N critical values c_j, with $0 < c_1 < c_2 < \cdots < c_N$, such that*

(a) if $0 \leq c < c_1$, then $p_c(z)$ has n distinct zeros,

(b) if $c_j < c < c_{j-1}$ for some $1 \leq j \leq N-1$, then $p_c(z)$ has $n + 4j - 2$ distinct zeros, and

(c) if $c > c_N$, then $p_c(z)$ has $n + 2k$ distinct zeros.

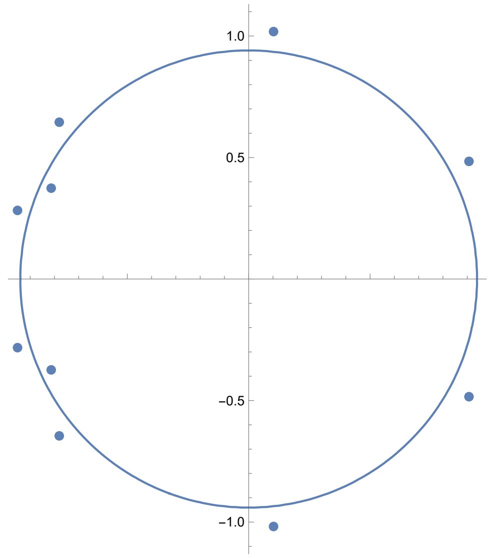
In their paper, they utilized the concept of the *critical curve*, which separates the complex plane into sense-preserving and sense-reversing regions. Dividing the plane in this way allows us to make sense of the orders of zeros; in the sense-preserving region, we count zeros with positive order, while in the sense-reversing region, we count zeros with negative order (see Chapter 2 for details).

Example 1.2. Consider again the function

$$f_{0.55}(z) = z^{13} + \frac{0.55}{\bar{z}^7} - 1.$$

In Figure 1.6, we again depict the zeros of f, now with the function's critical curve, which is the circle shown. For this specific family, the region within the critical curve is sense-reversing, and the region outside the critical curve is sense-preserving. Then we count the two zeros within the critical curve with negative order, and we count the eight zeros outside the critical curve with positive order.

Figure 1.6: The zeros and critical curve of $f_{0.55}$



Using similar concepts to those used by Brilleslyper et al., we show in Chapter 3 that

Theorem 1.3. *Let $n, k \in \mathbb{N}$ and $c > 0$. For*

$$f_c(z) = z^n + \frac{c}{\bar{z}^k} - 1,$$

the sum of the orders of the zeros is always $n - k$.

With this result in hand, we compute the winding number associated with the critical curve, allowing us to count certain zeros. Together with the above result, we ultimately prove our main theorem:

Theorem 1.4. *Consider the family of functions*

$$f_c(z) = z^n + \frac{c}{\bar{z}^k} - 1,$$

where $n, k \in \mathbb{N}$ with $gcd(n, k) = 1$ and $n > k$, and $c > 0$. There exist $N = \lceil \frac{k+1}{2} \rceil$ critical values $0 < c_1 < c_2 < \cdots < c_N$. Then *(a) if* $0 < c < c_1$*,* f_c *has* $n + k$ *zeros.*

(b) if $c_{j-1} < c < c_j$*, where* $2 \leq j \leq N$*,* f_c *has* $n + k + 6 - 4j$ *zeros.*

(c) if $c > c_N$*,* f_c *has* $n - k$ *zeros.*

We proceed as follows:

In Chapter 2, we introduce the necessary definitions and preliminary results for our arguments. Two notable results included are harmonic analogues to the Argument Principle and Rouché's Theorem.

In Chapter 3, we first prove Theorem 1.3; the proof is found in Section 3.1. Throughout Section 3.2, we prove geometric facts about the image of the critical curve to help us apply the Harmonic Argument Principle. We prove Theorem 1.4 at the end of Section 3.2.

We briefly discuss possible generalizations of our problem in Chapter 4.

A function f is *complex differentiable* at all points $z \in \mathbb{C}$ for which the complex-theoretic limit

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

exists. We say f is *analytic* on a domain $D \subseteq \mathbb{C}$ if this limit exists for every point in D. If every point of f in D is either analytic or a pole, we say that f is *meromorphic* [5].

Recall that a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if it is twice continuously differentiable and satisfies Laplace's equation $\phi_{xx} + \phi_{yy} = 0$. A complex-valued function $f = u + iv : \mathbb{C} \to \mathbb{C}$ is harmonic if both u and v are. Because an analytic function satisfies the Cauchy-Riemann equations, all analytic functions are harmonic [5]. It is well-known that, on a simply connected domain $D \subseteq \mathbb{C}$, any complex-valued harmonic function f can be written in the form $f = h + \overline{g}$, where both h and g are analytic functions [4]. We call h the *analytic part of f* and g the *co-analytic part of f.*

Example 2.1. Consider the complex-valued function

$$f_1(z) = z + \frac{1}{\overline{z}} - 1$$

We show that f_1 is harmonic. Letting $z = x + iy$, we have

$$f_1(x, y) = z + \frac{z}{|z|^2} - 1 = \Big(x + \frac{x}{x^2 + y^2} - 1\Big) + i\Big(y + \frac{y}{x^2 + y^2}\Big).$$

Then we need to show that each of

$$u(x, y) = x + \frac{x}{x^2 + y^2} - 1 \quad \text{and} \quad v(x, y) = y + \frac{y}{x^2 + y^2}$$

are harmonic. We compute

$$u_{xx} = \frac{2x^3 - 6xy^2}{(x^2 + y^2)^3},$$
$$u_{yy} = \frac{-2x^3 + 6xy^2}{(x^2 + y^2)^3},$$

so $u_{xx} + u_{yy} = 0$. Similarly, $v_{xx} + v_{yy} = 0$; then f_1 is harmonic.

It is notable that we can clearly represent f_1 as $f = h + \overline{g}$, where h is analytic and g is meromorphic.

An analytic function is *sense-preserving* everywhere, in the following sense: if we take a curve with positive (resp. negative) orientation on a simply connected domain in the complex plane and pass it through an analytic function, its image will also have positive (resp. negative) orientation. This behavior does not hold generally for harmonic functions: in certain cases, harmonic functions are *sense-reversing*, i.e., if one passes a curve with positive (resp. negative) orientation on a simply connected domain through a harmonic function, it is possible that the image of the curve has negative (resp. positive) orientation.

In order to determine when a complex-valued harmonic function $f = u + iv$ is sense-preserving or sense-reversing, consider its *Jacobian*

$$J_f(z) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix}.$$

Harmonic mappings are sense-preserving when $J_f(z) > 0$ and sense-reversing when $J_f(z) < 0$ [4]. However, computing with the Jacobian is generally not as easy as one might wish; to make computations more manageable, we define the *complex dilatation* of $f = h + \overline{g}$ to be the function

$$\varphi(z) = \frac{g'(z)}{h'(z)}.$$

By considering f under the differential operators

$$\frac{\partial}{\partial z} = \frac{1}{2}\Big(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\Big)$$
$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\Big(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\Big),$$

one can computationally show the following [6]:

Proposition 2.2. *When* $|\varphi(z)| < 1$*, the function is sense-preserving; when* $|\varphi(z)| > 1$*, the function is sense-reversing.*

When determining where a function f is sense-preserving or sense-reversing, it will be useful to consider the subset of $\mathbb{C}$ on which f is neither:

Definition 2.3 (Critical Curve)**.** The set of $\mathbb{Z} \in \mathbb{C}$ for which $|\varphi(z)| = 1$ will be called the *critical curve.*

The order of a zero is defined differently depending on what region it is in. In the analytic case, there are several equivalent ways of defining the order of a zero; the one we will focus on is the following. Let f be an analytic function with a zero z_0. Around z_0, we can determine the Taylor series of f:

$$f(z) = \sum_{j=0}^{\infty} \frac{f^{(j)}(z_0)}{j!}(z - z_0)^j.$$

We define the order of the zero z_0 to be the smallest value j for which $f^{(j)}(z_0) \neq 0$.

Example 2.4. Consider the complex-valued function

$$f(z) = z^3 - z^2 + z - 1,$$

which has a zero at $z = 1$. The Taylor series expansion of f about $z = 1$ is

$$f(z) = 2(z - 1) + 2(z - 1)^2 + (z - 1)^3.$$

From this expansion, we see that $f'(1) = 2$ is the first $f^{(j)}(1) \neq 0$, and so f has a zero of order 1 at $z = 1$.

This definition implicitly depends on the zero being in a sense-preserving region. Because a harmonic function is not always sense-preserving, we extend our definition to all cases [4]:

Definition 2.5 (Order of a Zero)**.** Let $f = h + \overline{g}$ be a complex-valued harmonic function. Because h and g are analytic at z_0, they have Taylor expansions about z_0 and we may write

$$f(z) = \sum_{j=0}^{\infty} \frac{h^{(j)}(z_0)}{j!}(z - z_0)^j + \overline{\sum_{j=0}^{\infty} \frac{g^{(j)}(z_0)}{j!}(z - z_0)^j}.$$

If z_0 is in a sense-preserving region, we define the order of z_0 to be the smallest value $j \geq 1$ for which $h^{(j)}(z_0) \neq 0$. If z_0 is in a sense-reversing region, we define the order of z_0 to be $-j$ for the smallest value $j \geq 1$ for which $g^{(j)}(z_0) \neq 0$. If z_0 is not in a sense-preserving region or a sense-reversing region, then the order of z_0 is undefined, and we call z_0 a *singular zero.*

The following definition of a pole and its order in the harmonic case comes from Suffridge and Thompson [8]:

Definition 2.6 (Order of a Pole)**.** Assume f is harmonic in $\{z \mid 0 < |z - z_0| < r\}$ for some $r > 0$. Define z_0 to be a pole of f provided $\lim_{z \to z_0} |f(z)| = \infty$. If z_0 is a pole of the harmonic function f on $\{z \mid |z - z_0| < r\}$, then the order of the pole is

$$-\frac{1}{2\pi}\Delta_\gamma \arg(f(z)),$$

where γ is the circle $|z - z_0| = \delta$ for δ sufficiently small.

Computing the order of a pole in this way can be somewhat cumbersome. Suffridge and Thompson [8] have provided the following result to ease computation, which also helps determine whether the region around a pole is sense-preserving or sense-reversing:

Lemma 2.7. *Let f be a harmonic mapping on a domain $D \subseteq \mathbb{C}$. Suppose that the local representation of f around a pole z_0 is*

$$f(z) = \sum_{j=-\ell}^{\infty} a_j (z - z_0)^j + \overline{\sum_{j=-m}^{\infty} b_j (z - z_0)^j} + 2A \log |z - z_0|,$$

for some constant A and where ℓ and m are finite.

- *If $a_{-\ell} \neq 0$ for some $\ell > 0$ and $\ell > m$, or $\ell = m$ with $|a_{-\ell}| > |b_{-\ell}|$, then f is sense-preserving near z_0 and f has a pole at z_0 of order ℓ.*

- *If $b_{-m} \neq 0$ for some $m > 0$ and $\ell < m$, or $\ell = m$ with $|a_{-m}| < |b_{-m}|$, then f is sense-reversing near z_0 and f has a pole at z_0 of order $-m$.*

Example 2.8. Consider the complex-valued harmonic function

$$f_1(z) = z^3 + \frac{1}{\overline{z}^2} - 1.$$

Considering $\lim_{z \to 0} |f_1(z)| = \infty$, we see that f_1 has a pole at $z = 0$. To compute the order of this pole, note that we have a representation as in Lemma 2.7 with $A = 0$ and

$$h(z) = z^3 - 1 \quad \text{and} \quad g(z) = \frac{1}{z}^2.$$

Then, using the notation given in the lemma, we have $b_{-2} = 1 \neq 0$ and $a_{-2} = 0$. Hence, f_1 is sense-reversing near the origin and has a pole at the origin of order -2.

Two very useful tools for determining the number of zeros of a meromorphic function are the Argument Principle and Rouché's Theorem. The statement of the Argument Principle comes from Saff and Snider [5].

Theorem 2.9 (Meromorphic Argument Principle)**.** *If f is analytic on a neighborhood of and nonzero at each point of a simple closed positively oriented contour C and is meromorphic inside C, then*

$$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}dz = N_0(f) - N_p(f),$$

where $N_0(f)$ and $N_p(f)$ are, respectively, the number of zeros and poles of f inside C (multiplicity included).

The statement of the analytic Rouché's Theorem comes from Stein and Shakarchi [7]:

Theorem 2.10 (Analytic Rouché's Theorem)**.** *Suppose that f and g are analytic in an open set containing a simple closed contour C and its interior. If*

$$|f(z)| > |g(z)| \quad \textit{for all}\, z \in C,$$

then f and $f + g$ have the same number of zeros inside C.

As stated, these statements apply to analytic (or meromorphic) functions, but they do not apply to complex-valued harmonic functions generally. Fortunately, there is a harmonic analogue to the Argument Principle, proven in Suffridge and Thompson [8]:

Theorem 2.11 (Harmonic Argument Principle)**.** *Let f be harmonic, except for a finite number of poles, in a simply connected domain $D \subseteq \mathbb{C}$. Let C be a simple closed curve contained in D not passing through a pole or a zero, and let Ω be the open bounded region created by C. Suppose that f has no singular zeros in D and let $Z_{f,C}$ be the sum of the orders of the zeros of f in Ω (counting multiplicity). Let $P_{f,C}$ be the sum of the orders of the poles of f in Ω (counting multiplicity). Then $\Delta_C arg f(z) = 2\pi(Z_{f,C} - P_{f,C})$.*

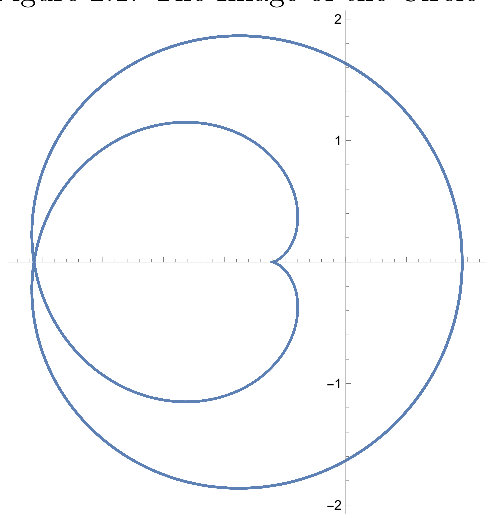
Throughout the rest of our arguments, we use the notation $Z_{f,C}$ and $P_{f,C}$ as given in Theorem 2.11. Using this theorem, we can count winding numbers and sums of zeros.

Example 2.12. Consider again the complex-valued harmonic function

$$f_1(z) = z^3 + \frac{1}{\overline{z}^2} - 1,$$

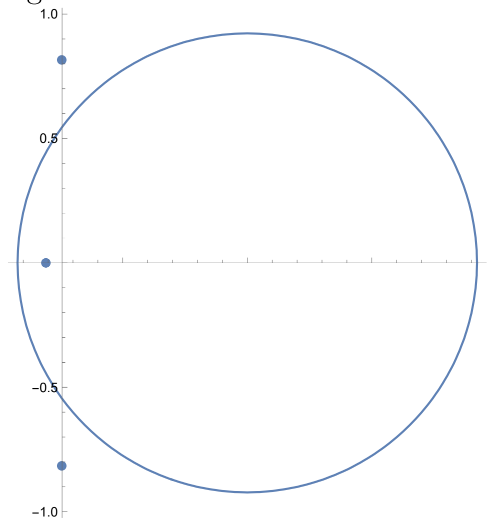
and suppose that we wish to count the number of zeros within the circle C with radius $\left(\frac{3}{2}\right)^{\frac{1}{5}}$ (to understand why we might pick this circle, see Chapter 3). The image of this circle is given in Figure 2.1.

Figure 2.1: The Image of the Circle C



By Lemma 3.9, $\Delta_C \arg f_1(z) = 2\pi$; then, by the Harmonic Argument Principle, $Z_{f_1,C} - P_{f_1,C} = 1$. From Example 2.8, we know that $P_{f_1,C} = -2$; then it must be that $Z_{f_1,C} = -1$. This fact is corroborated by Figure 2.2; we show in Lemmas 3.1 and 3.2 that the zero within C does indeed have order -1.

Figure 2.2: Zeros within the Circle C



In this instance, the image of the critical curve is a special curve known as an *epicycloid*. We show in Chapter 3 that this behavior is not coincidental.

Using the Harmonic Argument Principle, we can also prove a harmonic analogue of Rouché's Theorem. The following proof is adapted from a proof found in Brown and Churchill [3].

Theorem 2.13 (Harmonic Rouché's Theorem)**.** *Suppose that f and g both satisfy the hypotheses for the Harmonic Argument Principle. If f and g are both harmonic functions in and on the closed contour C, if $|f(z)| > |g(z)|$ at each point on C, and if f and g have no poles on C and no singular zeros in C, then $Z_{f,C} - P_{f,C} = Z_{f+g,C} - P_{f+g,C}$.*

Proof. Without loss of generality, assume that the orientation of C is positive. Neither f nor $f + g$ has a zero on C, since

$$|f(z)| > |g(z)| \geq 0 \quad \text{and} \quad |f(z) + g(z)| \geq ||f(z)| - |g(z)|| > 0$$

when z is on C.

From the Harmonic Argument Principle, we know that

$$\frac{1}{2\pi}\Delta_C \arg(f(z)) = Z_{f,C} - P_{f,C} \quad \text{and} \quad \frac{1}{2\pi}\Delta_C \arg(f(z) + g(z)) = Z_{f+g,C} - P_{f+g,C}.$$

Rewrite

$$\begin{aligned}\Delta_C \arg(f(z)+g(z)) &= \Delta_C \arg\Big[f(z)\Big(1+\frac{g(z)}{f(z)}\Big)\Big] \\ &= \Delta_C \arg(f(z)) + \Delta_C \arg\Big(1+\frac{g(z)}{f(z)}\Big).\end{aligned}$$

Then

$$Z_{f+g,C} - P_{f+g,C} = Z_{f,C} - P_{f,C} + \frac{1}{2\pi}\Delta_C \arg(F(z)),$$

where

$$F(z) = 1 + \frac{g(z)}{f(z)}.$$

We know, however, that

$$|F(z)-1| = \frac{|g(z)|}{|f(z)|} < 1,$$

meaning that under the transformation $w = F(z)$, the image of C lies in the open disk $|w-1| < 1$. Then the image of C does not enclose the origin. Hence, $\Delta_C(F(z)) = 0$, implying that $Z_{f+g,C} - P_{f+g,C} = Z_{f,C} - P_{f,C}$, as desired. □

These are the tools that we use to prove Theorem 1.4.

In order to apply the Harmonic Argument Principle and Rouché's Theorem, we need to determine the orders of the zeros and of the pole at the origin. To determine the orders, we first determine the sense-preserving and sense-reversing regions of our function:

Lemma 3.1. *The critical curve of*

$$f_c(z) = z^n + \frac{c}{\bar{z}^k} - 1$$

is the circle $|z| = \left(\frac{kc}{n}\right)^{\frac{1}{n+k}}$*, which we denote by* Γ_c*. The region inside this circle is sense-reversing, and the region outside this circle is sense-preserving.*

Proof. Here, the analytic part of f_c is $h(z) = z^n - 1$, and the co-analytic part is $g(z) = \frac{c}{z^k} = cz^{-k}$. Then

$$\omega(z) = -\frac{kcz^{-k-1}}{nz^{n-1}} = -\frac{kc}{nz^{n+k}}.$$

Hence,

$$|\omega(z)| = \frac{kc}{n|z|^{n+k}} = 1 \quad \text{if and only if} \quad |z| = \left(\frac{kc}{n}\right)^{\frac{1}{n+k}},$$

which exactly describes the critical curve. We also see that, if $|z| < \left(\frac{kc}{n}\right)^{\frac{1}{n+k}}$, $|\omega(z)| > 1$, and when $|z| > \left(\frac{kc}{n}\right)^{\frac{1}{n+k}}$, $|\omega(z)| < 1$, determining the sense-reversing and sense-preserving regions, respectively. □

Throughout the rest of our arguments, we denote the radius of Γ_c by $R_c := \left(\frac{kc}{n}\right)^{\frac{1}{n+k}}$.

3.1 The Sum of the Orders of the Zeros

Lemma 3.1 shows that the zeros and the pole within the critical curve have negative order, while zeros outside the critical curve have positive order. We now determine the exact values:

Lemma 3.2. *All non-singular zeros of*

$$f_c(z) = z^n + \frac{c}{\bar{z}^k} - 1$$

are simple. The pole at the origin has order $-k$.

Proof. Observe that $z = 0$ cannot be a zero of f_c, so suppose that $z_0 \neq 0$ is a zero of f_c. Noting that both $h'(z_0) = nz_0^{n-1}$ and $g'(z_0) = -kcz_0^{-k-1}$ are nonzero, we see that any zero of f_c must have order 1 or -1; that is, any zero of f_c with defined order is simple.

For the second claim, first observe that $\lim_{z\to 0} |f_c(z)| = \infty$, so f_c has a pole at the origin. We know that

$$g(z) = \frac{c}{z^k}$$

is its own series expansion about the origin, with the lowest term corresponding to $j = -k$. Hence, by Lemma 2.7 the order of the pole at the origin is $-k$. $\square$

We use these facts to apply the generalized form of Rouché's Theorem to our family:

Theorem 3.3. *For the family*

$$f_c(z) = z^n + \frac{c}{z^k} - 1,$$

with $c > 0$, the sum of the orders of the zeros (excluding instances in which there are singular zeros) is $n - k$.

Proof. Let $R > \sqrt[n]{c+1}$, and let $h(z) = z^n$ and $g(z) = \frac{c}{z^k} - 1$. Because $c > 0$, $R > 1$. Then on the circle $|z| = R$, we have

$$\begin{aligned}\left|\frac{c}{z^k} - 1\right| &\leq \frac{c}{|z|^k} + 1 \\ &= \frac{c}{R^k} + 1 \\ &\leq c + 1 \\ &< R^n = |z|^n.\end{aligned}$$

By Rouché's Theorem, then,

$$Z_{h,C} - P_{h,C} = Z_{h+g,C} - P_{h+g,C},$$

where the contour C we take is the circle $|z| = R$. We know that $Z_{h,C} = n$ and $P_{h,C} = 0$, so $Z_{h+g,C} - P_{h+g,C} = n$. By Lemma 3.2, we know that $P_{h+g,C} = -k$. Hence, $Z_{h+g,C} = n - k$, as desired. □

In the analytic case, where all zeros have positive order, these results would provide a complete solution. In the harmonic case, however, we have zeros with negative order, which means that the sum of the orders of the zeros does not directly give the total number of zeros. The next section focuses on determining the number of zeros in the sense-reversing region, or within the circle Γ_c, to resolve this issue.

3.2 The Geometry of the Image of the Critical Curve

To fully determine the number of zeros in the sense-reversing region, we compute the winding number of the image of the critical curve. Parametrize Γ_c by

$$z(\theta) = R_c e^{i\theta}, \quad \text{where} \quad \theta \in [0, 2\pi].$$

Then

$$\begin{aligned} f_c(z(\theta)) &= \left(\left(\frac{kc}{n}\right)^{\frac{1}{n+k}} e^{i\theta}\right)^n + c\left(\left(\frac{kc}{n}\right)^{\frac{1}{n+k}} e^{-i\theta}\right)^{-k} - 1 \\ &= \left(\frac{kc}{n}\right)^{\frac{n}{n+k}} e^{in\theta} + c\left(\frac{kc}{n}\right)^{-\frac{k}{n+k}} e^{ik\theta} - 1 \\ &= \left(\frac{kc}{n}\right)^{-\frac{k}{n+k}} \left(\frac{kc}{n} e^{in\theta} + ce^{ik\theta}\right) - 1. \end{aligned}$$

Then we parameterize the image of the critical curve by

$$z(\theta) = R_c^{-k}\left(\frac{kc}{n} e^{in\theta} + ce^{ik\theta}\right) - 1. \tag{3.1}$$

The simplest instance of this parametrization occurs when $n = k$; in this instance, we have

$$f_c(z(\theta)) = 2\sqrt{c}e^{in\theta} - 1. \tag{3.2}$$

We investigate this case fully; the general case $n > k$ will follow similar logic.

3.2.1 The Case $n = k$. In our arguments throughout this section, we will write n and k as such, rather than utilizing the fact that $n = k$, until we reach the end of the argument.

Writing it in this way will help in showing the connection to the more general case.

Equation 3.2 describes a circle with radius $r = 2\sqrt{c}$ and center $z = -1$, which is traversed n times as θ ranges from 0 to 2π. We compute the winding number in all cases.

If $c < \frac{1}{4}$, the radius r is less than 1. Then the circle does not contain the origin, and so the winding number of the critical curve is zero. By the Argument Principle, then,

$$Z_{f_c,C} - P_{f_c,C} = 0,$$

where the contour C taken in applying the Argument Principle is Γ_c. We know that $P_{f_c,C} = -k$, which implies that $Z_{f_c,C} = -k$. Since the order of the zeros is simple and all zeros inside the critical curve are in the sense-reversing region, it follows that f_c has k total zeros inside the critical curve. Now, from our corollary to Rouché's Theorem, the sum of the orders of the zeros must be $n - k$; since there are $-k$ zeros in the sense-reversing region, there must then be n zeros in the sense-preserving region, giving us a total of $n + k$ zeros. Since $n = k$, we have $2n$ zeros.

When $c = \frac{1}{4}$, the image of the critical curve contains the origin. In this case, there is a zero on the critical curve, i.e., there is a singular zero. Singular zeros have undefined order, and so there is no statement to make.

Finally, consider $c > \frac{1}{4}$. In this case, $r > 1$, indicating that the image of the critical curve does contain the origin. Since this image is traversed n times, the winding number of the critical curve is n. Hence,

$$Z_{f_c,C} = n + P_{f_c,C} = n - k.$$

This exactly equals the sum of the orders of the zeros, so there are $n - k$ zeros in the sense-reversing region and no zeros in the sense-preserving region. This gives us $n - k$ zeros total. Because $n = k$, we see that f_c has no zeros in this case.

Putting these cases together, we arrive at the following result:

Lemma 3.4. *The function*

$$f_c(z) = z^n + \frac{c}{\bar{z}^n} - 1$$

- *has* $2n$ *zeros when* $c < \frac{1}{4}$.

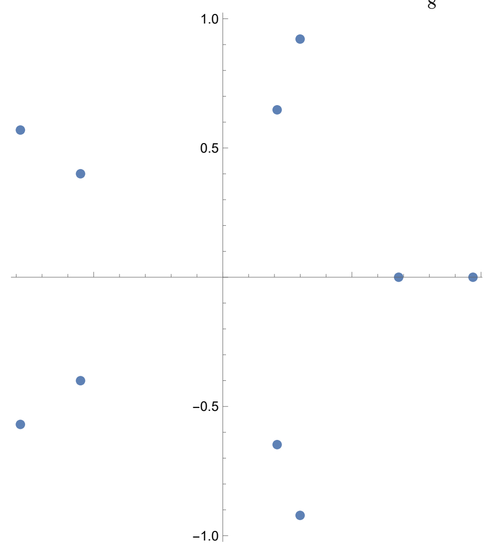
- *has no zeros when* $c > \frac{1}{4}$.

Example 3.5. Consider the family of functions

$$f_c(z) = z^5 + \frac{c}{\bar{z}^5} - 1.$$

When $c < \frac{1}{4}$, Lemma 3.4 tells us that f_c 10 zeros. This is demonstrated for the case $c = \frac{1}{8}$ in Figure 3.1. On the other hand, for any value $c > \frac{1}{4}$, f_c has no zeros.

Figure 3.1: The zeros of $f_{\frac{1}{8}}$



Notice that the statement of Lemma 3.4 fits the statement of Theorem 1.4 when $n = k$ (albeit, we clearly do not have the condition that n and k are relatively prime).

3.2.2 The Case $n > k$. From here, we assume that $n > k$ and that $\gcd(n, k) = 1$. Before moving on to the geometric arguments, we show that we are justified in making this assumption. Let $d = \gcd(n, k)$. If $d > 1$, then $n = dn'$ and $k = dk'$ where n' and k' are relatively prime. Then

$$\begin{aligned} f_c(z) &= z^n + \frac{c}{\overline{z}^k} - 1 \\ &= (z^d)^{n'} + \frac{c}{\overline{(z^d)}^{k'}} - 1 \\ &= w^{n'} + \frac{c}{\overline{w}^{k'}} - 1 = g_c(w), \end{aligned}$$

where $w = z^d$. The zeros of f_c then directly correspond to the zeros of g_c. For fixed $w \neq 0$, the function $z^d - w$ has d distinct solutions; using this, if we find the number of zeros of $g_c(w)$, and multiply by d, we have then found the number of zeros of f_c.

So, assume that $\gcd(n, k) = 1$, and recall that a parameterization of the image of Γ_c is given by

$$\left(\frac{kc}{n}\right)^{-\frac{k}{n+k}} \left(\frac{kc}{n} e^{in\theta} + ce^{ik\theta}\right) - 1.$$

In Example 2.12, we saw the curve parameterized by this equation when $n = 3$, $k = 2$, and $c = 1$, displayed in Figure 2.1. This curve is known as an *epicycloid*:

Definition 3.6. An epicycloid is the plane curve produced by tracing the path of a fixed point on a circle which rolls smoothly around a fixed circle. It is parametrized by the complex equation

$$z(\theta) = re^{i(s+1)\phi} - r(s+1)e^{i\phi}, \tag{3.3}$$

where θ ranges between 0 and 2π, the fixed circle has radius sr, and the moving circle has radius r. We refer to the radius of the fixed circle as the *inner radius* and to the radius of the moving circle as the *outer radius.*

If we can write the ratio of the inner radius to the outer radius as a reduced fraction $\frac{p}{q}$, then the epicycloid has p cusps.

Motivated by this, we claim the following:

Theorem 3.7. *The curve parameterized as in Equation 3.1 is an epicycloid.*

Proof. Observe that multiplying by $\left(\frac{kc}{n}\right)^{-\frac{k}{n+k}}$ simply scales the curve parametrized by $\theta \mapsto \frac{kc}{n} e^{in\theta} + ce^{ik\theta}$, and subtracting by 1 shifts the curve left by 1. Keeping these in mind, we mainly focus on the expression

$$\frac{kc}{n}e^{in\theta} + ce^{ik\theta}.$$

There are some immediate similarities to the parameterization of an epicycloid and the parameterization of our curve. We have two exponential terms, and a common factor in each term is c.

We show that our curve is an epicycloid rotated by $-\frac{n\pi}{n-k}$ radians. So, multiply our parameterization by $e^{i(\frac{n\pi}{n-k})}$:

$$e^{i(\frac{n\pi}{n-k})}\left(\frac{kc}{n}e^{in\theta} + ce^{ik\theta}\right) = \frac{kc}{n}e^{i(n\theta+\frac{n\pi}{n-k})} + ce^{i(k\theta+\frac{n\pi}{n-k})}.$$

Let $\varphi = n\theta + \frac{n\pi}{n-k}$; solving for θ, we see that

$$\theta = \frac{\varphi}{n} - \frac{\pi}{n-k}.$$

Hence,

$$\begin{aligned}\frac{kc}{n}e^{i(n\theta+\frac{n\pi}{n-k})} + ce^{i(k\theta+\frac{n\pi}{n-k})} &= \frac{kc}{n}e^{i\varphi} + ce^{i(\frac{k\varphi}{n}-\frac{k\pi}{n-k}+\frac{n\pi}{n-k})} \\ &= \frac{kc}{n}e^{i\varphi} + ce^{i(\frac{k\varphi}{n})+i\pi} \\ &= \frac{kc}{n}e^{i\varphi} + ce^{i\pi}e^{i(\frac{k\varphi}{n})} \\ &= \frac{kc}{n}e^{i\varphi} - ce^{i(\frac{k\varphi}{n})}.\end{aligned}$$

To get our parameterization to fully match the form of Equation 3.3, we need the term $ce^{i(\frac{k\varphi}{n})}$ to be positive and the term $\frac{kc}{n}e^{i\varphi}$ to be negative. To facilitate this, make the further substitution $\tau = \frac{k}{n}\varphi$ to get the parametrization

$$\begin{aligned}\frac{kc}{n}e^{i\varphi} - ce^{i(\frac{k\varphi}{n})} &= \frac{kc}{n}e^{i(\frac{n\tau}{k})} - ce^{i\tau} \\ &= -\frac{k}{n}\left(\frac{nc}{k}e^{i\tau} - ce^{i(\frac{n\tau}{k})}\right),\end{aligned}$$

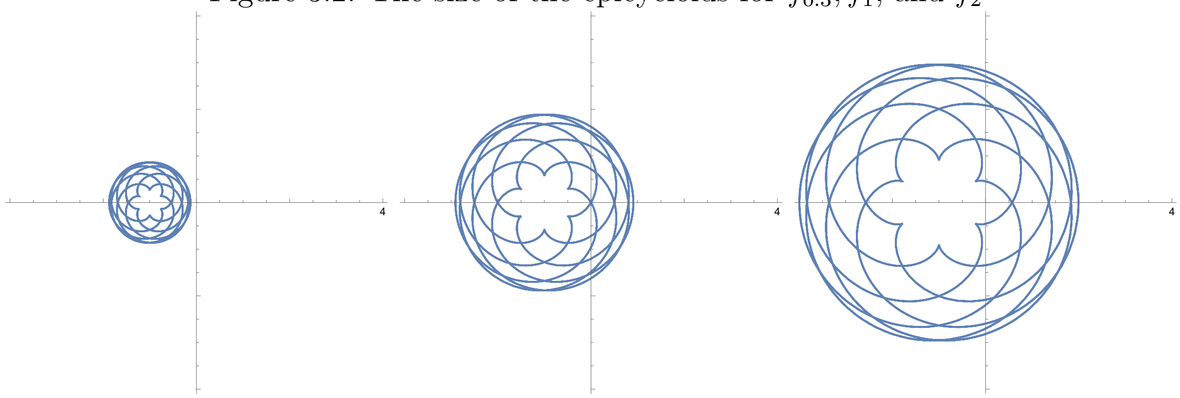
which gives an epicycloid of inner radius $\left(\frac{n}{k}-1\right)c = \left(\frac{n-k}{k}\right)c$ and outer radius c, dilated by a factor of $-\frac{k}{n}$. □

Thus, the image of our critical curve is an epicycloid. As in the case with $n = k$, we would like to count the winding number of our given epicycloids as their size increases, although epicycloids are of course more complicated than the circles given in the other case. In light of this increased complexity, we prove various facts about these epicycloids that will be essential in calculating the winding numbers for various values of c. First, note that as c increases, so does the size of the epicycloid. This is illustrated in Figure 3.2, where we consider the family of functions

$$f_c(z) = z^{13} + \frac{c}{\overline{z}^7} - 1,$$

with $c = 0.5$, $c = 1$, and $c = 2$ in the left, middle, and right figures respectively.

Figure 3.2: The size of the epicycloids for $f_{0.3}$, f_1, and f_2



Algebraically, this can be seen from the fact that these epicycloids are parameterized as

$$z(\theta) = \left(\frac{kc}{n}\right)^{-\frac{k}{n+k}} \left(\frac{kc}{n}e^{in\theta} + ce^{ik\theta}\right) - 1 = c^{\frac{n}{n+k}} \left(\frac{k}{n}\right)^{-\frac{k}{n+k}} \left(\frac{k}{n}e^{in\theta} + e^{ik\theta}\right) - 1.$$

Because c and its exponent $\frac{n}{n+k}$ are both positive, we see that the size of the epicycloid gets larger as c increases. Notably, as c gets arbitrarily large, so will the size of the epicycloid.

We prove some geometric facts necessary to our later arguments.

Lemma 3.8. *Consider the epicycloid parametrized as in Equation 3.1 with* $gcd(n, k) = 1$.

- *This epicycloid has* $n - k$ *cusps.*
- *Arcs of the epicycloid connect cusps that are* k *apart; we call this value* k *the* cuspal distance.
- *This epicycloid is symmetric across the real axis.*

- *The value $\theta = 0$ corresponds to the farthest-right intersection of the epicycloid with the real ray $\mathbb{R}_{>-1}$, and it also corresponds to the point halfway along the arc connecting two cusps.*

Proof. For the first two claims, we consider the undilated, untranslated epicycloid parametrized by

$$z(\theta) = \frac{kc}{n}e^{in\theta} + ce^{ik\theta}.$$

To determine the number of cusps, we consider the ratio of the inner radius to the outer radius:

$$\frac{c(\frac{n-k}{k})}{c} = \frac{n-k}{k}.$$

Because $\gcd(n,k) = 1$, this fraction is in reduced terms, so we see that the epicycloid has $n-k$ cusps.

Now, these cusps are determined by the point on the circle of radius $R = c$ rolling around the inner circle with radius $r = \left(\frac{n-k}{k}\right)c$. Because the outer circle does not change size, the cusps are evenly spaced apart on the inner circle. Comparing the two circumferences, we see that

$$\frac{2\pi R}{2\pi r} = \frac{1}{\frac{n-k}{k}} = \frac{k}{n-k}.$$

Thus arcs connect cusps that are $\frac{2\pi k}{n-k}$ radians apart. We know that there are $n-k$ cusps, evenly spaced, so consecutive cusps (not necessarily connected by a single arc) are $\frac{2\pi}{n-k}$ apart. Then arcs connect cusps that are k times this distance apart, so cusps connect arcs that are k apart.

We show that the conjugate of any point on the epicycloid is also on the epicycloid. Observing that

$$\overline{\frac{kc}{n}e^{in\theta} + ce^{ik\theta}} = \frac{kc}{n}e^{in(-\theta)} + ce^{ik(-\theta)},$$

we see that taking the conjugate parametrizes the same epicycloid (in the opposite direction). Then it is symmetric across the real axis.

For the final claim, consider the dilated, translated epicycloid

$$\left(\frac{kc}{n}\right)^{-\frac{k}{n+k}}\left(\frac{kc}{n}e^{in\theta}+ce^{ik\theta}\right)-1.$$

Note that $\theta = 0$ gives the value

$$\left(\frac{kc}{n}\right)^{-\frac{k}{n+k}}\left(\frac{kc}{n}+c\right)-1.$$

which is real-valued and greater than -1. Noticing that

$$\left|\frac{kc}{n}e^{in\theta}+ce^{ik\theta}\right| \leq \frac{kc}{n}+c,$$

and noticing that $\theta = 0$ gives this value, we see that $\theta = 0$ corresponds to the farthest-right intersection between the epicycloid and the ray $\mathbb{R}_{>-1}$.

Finally, we claim that $\theta = 0$ corresponds to the point halfway between two cusps along an arc. To see this, note that, since it is the maximal distance away from the center of the epicycloid, it must be distance $2R = 2c$ away from the intersection between the inner circle and $\mathbb{R}_{>-1}$. Geometrically, the outer circle has completed half its rotation between two cusps when the fixed point is a full diameter away from the inner circle; but $\theta = 0$ is such a point, and so it must land exactly halfway along the arc between two cusps. □

In order to count the winding number, we also need the following:

Lemma 3.9. *When $gcd(n, k) = 1$, the epicycloid parameterized as in Equation 3.1 is traversed exactly once as θ ranges between 0 and 2π. It is traversed in the counterclockwise direction.*

Proof. We consider the epicycloid with the dilation factor or the translation; so, consider the function

$$f(\theta) = \frac{kc}{n}e^{in\theta} + ce^{ik\theta}.$$

We claim that f has period 2π; if this is the case, then the epicycloid is traced exactly once as θ ranges from 0 to 2π.

Let m be the period of f, and observe that

$$\begin{aligned} f(\theta+2\pi) &= \frac{kc}{n}e^{in(\theta+2\pi)} + ce^{ik(\theta+2\pi)} \\ &= \frac{kc}{n}e^{in\theta} + ce^{ik(\theta)} = f(\theta). \end{aligned}$$

Then $m \leq 2\pi$.

To show that $m \geq 2\pi$, suppose that there is some $0 < m < 2\pi$ for which

$$f(\theta+m) = f(\theta).$$

Then

$$\frac{kc}{n}e^{in\theta}e^{inm} + ce^{ik\theta}e^{ikm} = \frac{kc}{n}e^{in\theta} + ce^{ik\theta},$$

or

$$\frac{k}{n}e^{in\theta}(e^{inm}-1) = e^{ik\theta}(1-e^{ikm}).$$

Now, either $e^{inm} = 1$ or not. If not, then we can rewrite the equation

$$e^{i(n-k)\theta} = \left(\frac{k}{n}\right)\left(\frac{1-e^{ikm}}{e^{inm}-1}\right).$$

Then the right-hand side is constant, while the left-hand side is not. Indeed, consider $\theta = 0$ and $\theta = \frac{\pi}{n-k}$, which give the respective values

$$\left(\frac{k}{n}\right)\left(\frac{1-e^{ikm}}{e^{inm}-1}\right) = 1 \quad \text{and} \quad \left(\frac{k}{n}\right)\left(\frac{1-e^{ikm}}{e^{inm}-1}\right) = -1.$$

This is a contradiction, so it must be that $e^{inm} = 1$. Then

$$\frac{k}{n}e^{in\theta}(e^{inm}-1) = 0 = e^{ik\theta}(1-e^{ikm}),$$

which means that $e^{ikm} = 1$. Then e^{im} is both an nth root of unity and a kth root of unity.

Now, consider the multiplicative group generated by e^{im}, and let d be the order of this group. This group is a subgroup of the nth roots of unity and the kth roots of unity, so by Lagrange's Theorem, $d \mid n$ and $d \mid k$. Since n and k are relatively prime, it must be that $d = 1$, which means that $e^{im} = 1$. This is possible exactly when m is an integer multiple of 2π; but $0 < m < 2\pi$, so m cannot be an integer multiple of 2π, a contradiction.

Thus, $m \geq 2\pi$, and so $m = 2\pi$, as desired.

To see that these epicycloids are traversed in the counterclockwise direction, consider $0 < \theta < \frac{\pi}{n}$. For these values of θ,

$$\frac{kc}{n}\sin(n\theta) + c\sin(k\theta) > 0,$$

so the imaginary part of

$$\frac{kc}{n}e^{in\theta} + ce^{ik\theta}$$

is positive. The epicycloid does not reverse direction as it is traced out, and since it starts on the real axis (with $\theta = 0$) and then has positive imaginary part for small positive values of θ, this means that the epicycloid is traversed in the counterclockwise, or positive, direction. □

Because the epicycloid is traversed exactly once for $0 \leq \theta \leq 2\pi$, we will not have to worry about any multiplicity issues when counting the winding number. Since the epicycloid is traversed in the positive direction, we know that the winding number will be positive.

Using these preliminary results, we determine characteristics of the cusps that will allow us to compute the winding number of the critical curve. We will need two more results:

Lemma 3.10. *Consider the epicycloid parameterized as in Equation 3.1 with gcd*$(n, k) = 1$.

(i) If n and k are both odd, the epicycloid has no cusps on the real axis.

(ii) If n is odd and k is even, there is a cusp on the real axis, and it is to the right of the center of the epicycloid.

(iii) If n is even and k is odd, there is a cusp on the real axis, and it is to the left of the center of the epicycloid.

Proof. Because $\theta = 0$ marks the halfway point between two cusps, and two cusps connected by an arc are $\frac{2\pi k}{n-k}$ radians apart, this means that there is a cusp at

$$\frac{1}{2}\left(\frac{2k\pi}{n-k}\right) = \frac{k\pi}{n-k}.$$

Consecutive cusps (according to the labelling, not necessarily connected by a single arc) are $\frac{2\pi}{n-k}$ radians apart; it follows, then, that all cusps correspond to

$$\frac{k\pi}{n-k} + \frac{2m\pi}{n-k} = \left(\frac{k+2m}{n-k}\right)\pi, \quad m \in \mathbb{Z}.$$

Then there is a cusp on the real axis if and only if $\frac{k+2m}{n-k}$ is an integer; further, a cusp on the ray $\mathbb{R}_{>-1}$ corresponds to this fraction being an *even* integer, and a cusp on the ray $\mathbb{R}_{<-1}$ corresponds to this fraction being an *odd* integer.

So, we break into three cases, depending on the parity of n and k. (Note that n and k cannot both be even, because they are relatively prime.) First, suppose that n and k are both odd. Then $n-k$ is even and $k+2m$ is odd, so $\frac{k+2m}{n-k} \notin \mathbb{Z}$. In this case, there are no cusps on the real axis.

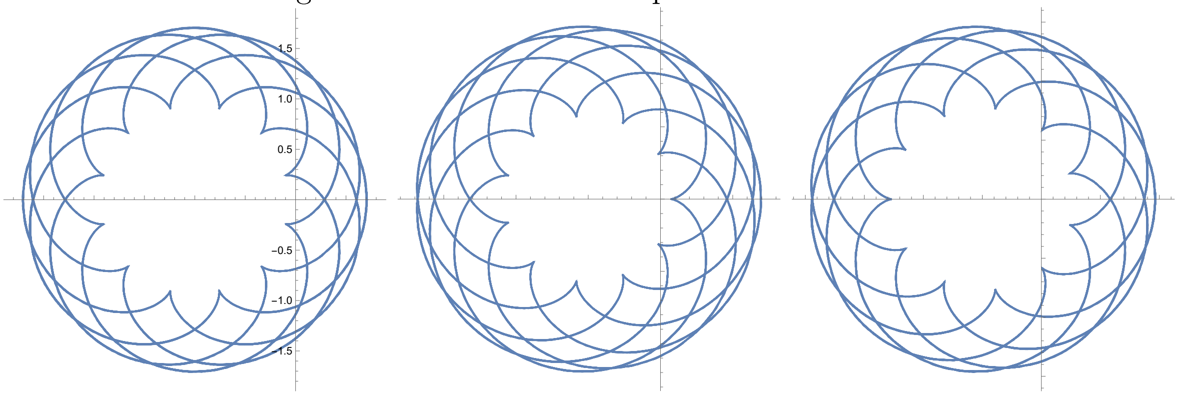
Now, suppose that n is odd and k is even. Then $n-k$ is odd and $k+2m$ is even. We can find an m for which $(n-k) \mid (k+2m)$ (for instance, take $m = n - \frac{3}{2}k$, which is an integer since k is even). Then there is a cusp on the real axis; but since $n-k$ is odd, it cannot divide 2 (which divides $k+2m$), so $\frac{k+2m}{n-k}$ gives an even integer (when it is an integer). Then, in this case, there is a cusp on the ray $\mathbb{R}_{>-1}$.

Finally, suppose that n is even and k is odd. Then both $n-k$ and $k+2m$ are odd. We can find an m for which $(n-k) \mid (k+2m)$ (for instance, take $m = \frac{n}{2} - k$, which is an integer since n is even). In such a case, $\frac{k+2m}{n-k}$ must be give an odd integer, and so there is a cusp on the ray $\mathbb{R}_{<-1}$. □

Figure 3.3 shows instances of the first, second, and third cases. From left to right, we have the respective functions

$$f_1(z) = z^{17} + \frac{1}{\bar{z}^5} - 1$$
$$f_{1.5}(z) = z^{17} + \frac{1.5}{\bar{z}^6} - 1$$
$$f_2(z) = z^{16} + \frac{2}{\bar{z}^5} - 1.$$

Figure 3.3: Instances of Cusps on the Real Axis



In order to compute the winding number of the image of the critical curve, we determine the number of distinct intersections of the epicycloid with $\mathbb{R}_{>-1}$. We first determine the number of non-distinct intersections:

Lemma 3.11. *Consider the epicycloid parameterized as in Equation 3.1, with gcd*$(n, k) = 1$*. This epicycloid has a total of* k *intersections with the ray* $\mathbb{R}_{>-1}$*.*

Proof. By Lemma 3.8, we know that the epicycloid has $n-k$ cusps; further, the cuspal distance is k, and consecutive cusps are $\frac{2\pi}{n-k}$ radians from each other.

Suppose that k is odd; then by Lemma 3.10, there are no cusps of the epicycloid on $\mathbb{R}_{>-1}$. Label the cusps j in the positive direction, starting with 1 and ending with k, giving a cusp multiple labels if necessary, and label arcs according to the cusp they *end* at. We consider the intersections of the cusps labelled $1 \leq j \leq k$ with $\mathbb{R}_{>-1}$; because the cuspal distance is k, any arcs which intersect $\mathbb{R}_{>-1}$ must fall in this labelling.

Because $\theta = 0$ marks a halfway point between two cusps and the difference in radians between two consecutive cusps is $\frac{2\pi}{n-k}$, we know that the first cusp corresponds to $\frac{\pi}{n-k}$, the second cusp corresponds to $\frac{\pi}{n-k} + \frac{2\pi}{n-k} = \frac{3\pi}{n-k}$, and in general, the jth cusp corresponds to $\frac{(2j-1)\pi}{n-k}$. Then the jth arc corresponds to the interval

$$\left[\frac{(2j-1)\pi}{n-k} - \frac{2\pi k}{n-k}, \frac{(2j-1)\pi}{n-k}\right] = \left[\frac{(2j-1-2k)\pi}{n-k}, \frac{(2j-1)\pi}{n-k}\right].$$

We count the number of even multiples of π in each of these intervals, which will count the number of intersections of the jth arc with $\mathbb{R}_{>-1}$.

When $1 \leq j \leq k$, we see that $\frac{2j-1-2k}{n-k} \leq -\frac{1}{n-k} < 0$ and $\frac{2j-1}{n-k} > \frac{1}{n-k} > 0$, so each arc $1 \leq j \leq k$ intersects $\mathbb{R}_{>-1}$ at least once. If $k \leq \frac{n}{2}$, then

$$\frac{2j-1}{n-k} \leq \frac{2k-1}{n-k} < \frac{2(n-k)}{n-k} = 2,$$

and

$$\frac{2j-1-2k}{n-k} \geq \frac{1-2k}{n-k} > \frac{2(k-n)}{n-k} = -2,$$

so

$$\Big[\frac{(2j-1-2k)\pi}{n-k}, \frac{(2j-1)\pi}{n-k}\Big] \subseteq (-2\pi, 2\pi).$$

This means that, when $k \leq \frac{n}{2}$, each arc $1 \leq j \leq k$ has *exactly* one intersection with $\mathbb{R}_{>-1}$. None of the cusps were labelled more than once, since $k \leq \frac{n}{2}$ is equivalent to $k \leq n-k$, so there are k intersections of the epicycloid with the ray.

Now suppose that $\frac{n}{2} < k < n$. In this case, at least the first cusp $1 \leq j \leq k$ has been labelled twice, since $n-k < k$. We claim that the number of intersections of the jth arc modulo $n-k$ corresponds to the number of labels between 1 and k it has. To see this, observe that giving an additional label corresponds to adding $n-k$ to j modulo $n-k$. Doing this adds 2π to the right endpoint of the interval of the arc:

$$\frac{(2(j+n-k)-1)\pi}{n-k} = \frac{(2j-1)\pi}{n-k} + 2\pi,$$

meaning that we can traversed the epicycloid once by relabeling. This gives an extra intersection with $\mathbb{R}_{>-1}$. Conversely, if we traverse the epicycloid once, or add 2π to the interval of the arc, and continue to label arcs as we do so, then we have given each arc a single extra label. This shows that the total number of labels of arcs j modulo $n-k$ counts the total number of intersections with the ray; but we have labelled $1 \leq j \leq k$, giving us k total intersections.

Now, suppose that k is even and n is odd. By Lemma 3.10, there is a cusp of the epicycloid on $\mathbb{R}_{>-1}$. Label this cusp 1, and the label the cusps j in the positive direction, up to k; label each arc according the cusp it *ends* at.

The first cusp corresponds to $\theta = 0$, and so we know that the second cusp corresponds to

$\theta = \frac{2\pi}{n-k}$, and in general, the jth cusp corresponds to $\frac{2(j-1)\pi}{n-k}$. Then the jth arc corresponds to the interval

$$\left[\frac{2(j-1)\pi}{n-k} - \frac{2\pi k}{n-k}, \frac{2(j-1)\pi}{n-k}\right] = \left[\frac{2(j-1-k)\pi}{n-k}, \frac{2(j-1)\pi}{n-k}\right].$$

Since $1 \leq j \leq k$, we see that $\frac{2(j-1-k)}{n-k} \leq -\frac{2}{n-k} < 0$ and $\frac{2(j-1)}{n-k} \geq 0$, so each arc intersects $\mathbb{R}_{>-1}$ at least once. If $k \leq \frac{n}{2}$, then

$$\frac{2(j-1)}{n-k} \leq \frac{2k-2}{n-k} \leq \frac{2(n-k)}{n-k} = 2,$$

and

$$\frac{2(j-1-k)}{n-k} \geq -\frac{2k}{n-k} \geq -\frac{2(n-k)}{n-k} = -2,$$

so

$$\left[\frac{2(j-1-k)\pi}{n-k}, \frac{2(j-1)\pi}{n-k}\right] \subseteq (-2\pi, 2\pi).$$

As in the previous case, then, when $k \leq \frac{n}{2}$, each arc $1 \leq j \leq k$ has *exactly* one intersection with $\mathbb{R}_{>-1}$, and since there are no cusps labelled more than once, we have k intersections of the epicycloid with the ray.

Now suppose that $\frac{n}{2} < k < n$. As before, we claim that the number of intersections of the jth arc modulo $n-k$ corresponds to the number of labels between 1 and k it has. Adding a label to j adds 2π to the right endpoint of the interval of the arc:

$$\frac{2(j+n-k-1)\pi}{n-k} = \frac{2(j-1)\pi}{n-k} + 2\pi.$$

Conversely, by adding 2π to the right endpoint of this interval, or traversing the epicycloid once, and continuing to add labels as we do so, adds an extra label to the given arc. This shows that the total number of labels of arcs j modulo $n-k$ counts the total number of intersections with the ray; but we have labelled $1 \leq j \leq k$, giving us k total intersections.

In all cases, then, the epicycloid has k total intersections with the ray $\mathbb{R}_{>-1}$. □

We now come to the culminating result in determining the winding number of the epicycloids for given values of c.

Lemma 3.12. *Consider the epicycloid parameterized as in Equation 3.1, with gcd$(n,k) = 1$.*

(a) *If n is odd and k is even, there are $\frac{k+2}{2}$ distinct intersections of the epicycloid with the ray $\mathbb{R}_{>-1}$, the farthest right and farthest left (on $\mathbb{R}_{>-1}$) having multiplicity one and all intermediate intersections having multiplicity two.*

(b) *If k is odd (and regardless of the parity of n), there are $\frac{k+1}{2}$ distinct intersections of the epicycloid with the ray $\mathbb{R}_{>-1}$, the farthest right having multiplicity one and all others having multiplicity two.*

In either case, there are $\lceil \frac{k+1}{2} \rceil$ distinct intersections of the epicycloid with the ray $\mathbb{R}_{>-1}$.

Proof. By Lemma 3.11, there are k non-distinct intersections of the epicycloid with the ray $\mathbb{R}_{>-1}$.

(a) Suppose n is odd and k is even. By Lemma 3.10, there is a cusp on $\mathbb{R}_{>-1}$. Label all $n-k$ cusps in the positive direction, starting by labelling the cusp on $\mathbb{R}_{>-1}$ as 1. Label each arc according to the cusp it ends at. Throughout the following arguments, we treat the labels modulo $n-k$.

Consider now the mapping that describes reflection across the real axis, $(x, y) \to (x, -y)$. Because the epicycloid is symmetric across the real axis, cusps are sent to cusps. Specifically, the cusp labelled 1 gets sent to itself, the cusp labelled 2 gets sent to the $(n-k)$th cusp, and in general, the jth cusp gets sent to the $(n-k-j+2)$th cusp. However, this mapping sends the arc *ending* at the jth cusp to the arc *starting* at the $(n-k-j+2)$th cusp; recalling that the cuspal distance is k we see that adding k to the arc starting at the $(n-k-j+2)$th cusp gives us that $(n-j+2)$th cusp. The jth arc then gets mapped to the $(n-j+2)$th arc. This process is illustrated in Figure 3.4.

We can then describe this mapping of arcs as the function $\varphi : \mathbb{Z}/(n-k)\mathbb{Z} \to \mathbb{Z}/(n-k)\mathbb{Z}$ defined by $\varphi(j) = n-j+2$. First, observe that $\varphi(\varphi(j)) = j$, so the mapping has order 2 in the group of mappings $\mathbb{Z}/(n-k)\mathbb{Z} \to \mathbb{Z}/(n-k)\mathbb{Z}$. This means that, if we perform the reflection across the real axis twice, we end up at the arc we started on. The intersections of these arcs with $\mathbb{R}_{>-1}$ are fixed under $(x, y) \mapsto (x, -y)$, and so arcs that are paired under this reflection must intersect $\mathbb{R}_{>-1}$ at the same point. This implies that all intersections of the arcs with $\mathbb{R}_{>-1}$ are *at most* double intersections.

We claim, however, that there are instances with single intersections. To determine these instances, we consider the fixed points under φ. Suppose that $\varphi(j) = n-j+2$, or $j \equiv n-j+2 \pmod{n-k}$. Rewriting, this is equivalent to solving the equation

$$2j \equiv n + 2 \pmod{n-k}.$$

By Theorem 5.1 in [1], this has d solutions, where $d = \gcd(2, n-k)$. Since n is odd and k is even, $n-k$ is odd, implying that 2 and $n-k$ are relatively prime. Then there is exactly one solution (modulo $n-k$) to $2j \equiv n-j+2 \pmod{n-k}$. We claim that this solution is $j = \frac{k+2}{2}$, which is seen as follows (noting that $n \equiv k \pmod{n-k}$:

$$2\Big(\frac{k+2}{2}\Big) = k + 2 \equiv n + 2 \pmod{n-k}.$$

Thus, the arc corresponding to $\frac{k+2}{2}$ is fixed under φ, so its intersection with $\mathbb{R}_{>-1}$ is a single intersection. (We are justified in claiming that the intersection is with $\mathbb{R}_{>-1}$ rather than $\mathbb{R}_{<-1}$ because $\frac{k+2}{2} \leq k$ for all $k \geq 2$, and any such arcs under are labelling must intersect $\mathbb{R}_{>-1}$.) Further, the $\left(\frac{k+2}{2}\right)$th arc in this case corresponds to the interval

$$\Big[\frac{2(j-1-k)\pi}{n-k}, \frac{2(j-1)\pi}{n-k}\Big] = \Big[-\frac{k\pi}{n-k}, \frac{k\pi}{n-k}\Big],$$

the midpoint of which is $\theta = 0$. Then this is the arc mentioned in Lemma 3.8, and so is the arc with the farthest-right intersection with $\mathbb{R}_{>-1}$.

Altogether, then, the farthest right intersection of the epicycloid with $\mathbb{R}_{>-1}$ is a single intersection. The farthest left intersection is the cusp, which is unaffected by the mapping $(x, y) \mapsto (x, -y)$, and so is a single intersection. All other intersections correspond to arcs that are not fixed points of φ, and so are double intersections. This gives 2 intersections with multiplicity one and $\frac{k-2}{2}$ distinct intersections of multiplicity two; there are then $\frac{k-2}{2}+2 = \frac{k+2}{2}$ distinct intersections, with the desired properties.

(b) Now, suppose that k is odd. By Lemma 3.10, there is no cusp on $\mathbb{R}_{>-1}$. Label all $n-k$ cusps in the positive direction, starting by labeling the cusp corresponding to $\frac{\pi}{n-k}$

radians as 1. Label each arc according to the cusp it ends at.

As before, consider the mapping $(x, y) \mapsto (x, -y)$. Under this mapping, cusps are sent to cusps. Specifically, the cusp labelled 1 gets sent to the $(n-k)$th cusp, the cusp labelled 2 gets sent to the $(n-k-1)$st cusp, and in general, the jth cusp gets sent to the $(n-k-j+1)$st cusp. Also as in the previous case, we add k to ensure that arcs under this mapping are sent to arcs with the correct labels.

Then consider the mapping $\varphi : \mathbb{Z}/(n-k)\mathbb{Z} \to \mathbb{Z}/(n-k)\mathbb{Z}$ defined by $\varphi(j) = n-j+1$. We observe that $\varphi(\varphi(j)) = j$, so this mapping also has order 2, implying that all intersections are *at most* double intersections.

We again determine the arcs fixed under this mapping or the j for which $j \equiv n - j + 1 \pmod{n-k}$. This is equivalent to solving the equation

$$2j \equiv n + 1 \pmod{n-k}.$$

By Theorem 5.1 in [1], this has $d = \gcd(2, n-k)$ mutually incongruent solutions. If n is even, then $n-k$ is odd, and so 2 and $n-k$ are relatively prime. Then there is only a single solution, and it must be $j = \frac{k+1}{2}$:

$$2\Big(\frac{k+1}{2}\Big) = k + 1 \equiv n + 1 \pmod{n-k}.$$

We note that $1 \leq \frac{k+1}{2} \leq k$, so this intersection with the real axis will be on the ray $\mathbb{R}_{>-1}$.

On the other hand, if n is odd, then $n-k$ is even, and so $\gcd(2, n-k) = 2$. We then have 2 solutions; as above, $j = \frac{k+1}{2}$ is a solution. And, clearly $\frac{n+1}{2}$ is a solution:

$$2\Big(\frac{n+1}{2}\Big) \equiv n + 1 \pmod{n-k}.$$

Note that the solutions $\frac{n+1}{2}$ and $\frac{k+1}{2}$ are incongruent to each other in this case; otherwise,

$$\frac{n+1}{2} - \frac{k+1}{2} = \frac{n-k}{2} \equiv 0 \pmod{n-k},$$

so $n-k$ divides $\frac{n-k}{2}$. This is possible only if $n-k = 1$, or $n = k+1$; but n and k are both odd, a contradiction. Then we have found our two solutions, or the two fixed points of φ. When $n \geq 2k$, we see that

$$\frac{n+1}{2} \geq \frac{2k+1}{2} > \frac{2k}{2} = k,$$

so this second fixed arc does not intersect $\mathbb{R}_{>-1}$. On the other hand, if $\frac{n}{2} < k < n$,

$$\frac{n+1}{2} = \frac{n}{2} + \frac{1}{2} < k + \frac{1}{2},$$

so $\frac{n+1}{2} \leq k$. Notice, however, that the interval corresponding to the arc $j = \frac{n+1}{2}$ is

$$\left[\frac{(2j-1-2k)\pi}{n-k}, \frac{(2j-1)\pi}{n-k}\right] = \left[\frac{(n-2k)\pi}{n-k}, \frac{n\pi}{n-k}\right],$$

which has midpoint π; then the only place the $\left(\frac{n+1}{2}\right)$th arc can have a single intersection is on $\mathbb{R}_{<-1}$. All other intersections of this arc must be double intersections (and the arc will intersect itself in such cases).

Then the only fixed arc in question is the $\left(\frac{k+1}{2}\right)$th arc, which has interval

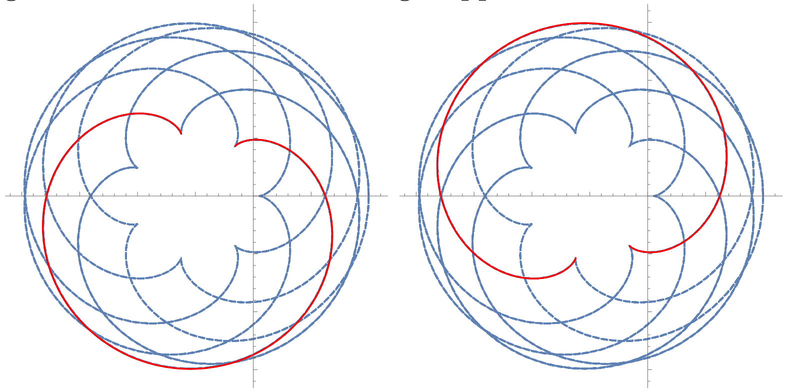
$$\left[\frac{(2j-1-2k)\pi}{n-k}, \frac{(2j-1)\pi}{n-k}\right] = \left[-\frac{k\pi}{n-k}, \frac{k\pi}{n-k}\right],$$

which has midpoint $\theta = 0$. As in the previous case, this corresponds to the farthest-right intersection of the epicycloid with $\mathbb{R}_{>-1}$.

Since there is no cusp on $\mathbb{R}_{>-1}$, there is only one single intersection, namely the farthest-right intersection. All other $\frac{k-1}{2}$ distinct intersections are double intersections, and so we have $\frac{k-1}{2} + 1 = \frac{k+1}{2}$ distinct intersections, with the desired properties. □

Figure 3.4 illustrates the effect of φ on the arc labelled 2 for the epicycloid with $n = 13$ and $k = 6$.

Figure 3.4: The First Arc Being Flipped for $n = 13$ and $k = 6$



The second arc is shown in red on the left; the image of the second arc, or the sixth arc, is shown in red on the right. Notice that the second *cusp* is not sent to the sixth cusp, but rather to the seventh cusp, and the image of the second arc *starts* at the seventh cusp but ends at the sixth. This motivates our definition of φ in the proof of Lemma 3.12.

Finally, with this in hand, we prove the main theorem:

Theorem 3.13. *Consider the family of functions*

$$f_c(z) = z^n + \frac{c}{\bar{z}^k} - 1,$$

where $gcd(n,k) = 1$ *and* $n > k$*. Label the critical values* $0 < c_1 < c_2 < \cdots < c_N$*, where* $N = \lceil \frac{k+1}{2} \rceil$*. Then*

(a) if $0 < c < c_1$*,* f_c *has* $n + k$ *zeros.*

(b) if $c_{j-1} < c < c_j$*, where* $2 \leq j \leq N$*,* f_c *has* $n + k + 6 - 4j$ *zeros.*

(c) if $c > c_N$*,* f_c *has* $n - k$ *zeros.*

Moreover, $c_1 = \left(\frac{k}{n}\right)^{\frac{k}{n}} \left(\frac{n}{n+k}\right)^{\frac{n+k}{n}}$*, and* $c_N \leq \left(\frac{k}{n}\right)^{\frac{k}{n}} \left(\frac{n}{n-k}\right)^{\frac{n+k}{n}}$*. This upper bound is sharp when* n *is odd and* k *is even.*

Proof. By Lemma 3.12, there are $N = \lceil \frac{k+1}{2} \rceil$ distinct intersections of the epicycloid with the ray $\mathbb{R}_{>-1}$. Critical values correspond exactly to these intersections.

(a) When $0 < c < c_1$, the epicycloid does not contain the origin at all. Thus, the winding number of the critical curve is 0, so

$$Z_{f,C} = P_{f,C} = -k,$$

telling us that we have $-k$ zeros in the sense-reversing region. Counting order without sign, we have k zeros in the sense-reversing region. Since the sum of the orders of the zeros is $n - k$, there must be n zeros in the sense-preserving region, giving us a total of $n + k$ zeros.

(b) For $c_{j-1} < c < c_j$, we have increased the winding number by 1 (if $j = 2$), and then by an additional 2 for each $2 < k \leq j$. Thus, the winding number is $1 + 2(j - 2) = 2j - 3$, and so

$$Z_{f,C} = P_{f,C} + 2j - 3 = -k + 2j - 3.$$

This gives $-k+2j-3$ zeros in the sense-reversing region, or $k-2j+3$ zeros, if we count without sign. Then, to ensure that the sum of the orders of zeros is still $n-k$, it must be that there are $n-2j+3$ zeros in the sense-preserving region, giving us a total of

$$n-2j+3+k-2j+3 = n+k+6-4j \text{ zeros.}$$

(c) If $c > c_N$, then the winding number is k. Hence,

$$Z_{f,C} = P_{f,C} + k = -k + k = 0,$$

so the sense-reversing region contains no zero. Hence, the sense-preserving region has all $n-k$ zeros, for a total of $n-k$ zeros.

To see why $c_1 = \left(\frac{k}{n}\right)^{\frac{k}{n}}\left(\frac{n}{n+k}\right)^{\frac{n+k}{n}}$, remember that the first critical value corresponds to the c-value at which the farthest right intersection between the epicycloid and $\mathbb{R}_{>-1}$ crosses the origin. This in turn corresponds to when

$$\left(\frac{kc}{n}\right)^{-\frac{k}{n+k}}\left(\frac{kc}{n}e^{in\theta} + ce^{ik\theta}\right) - 1 = 0,$$

when $\theta = 0$. Then we want the value c for which

$$\left(\frac{kc}{n}\right)^{-\frac{k}{n+k}}\left(\frac{kc}{n} + c\right) = 1,$$

or, equivalently,

$$c^{\frac{n}{n+k}} = \left(\frac{k}{n}\right)^{\frac{k}{n+k}}\left(\frac{n}{n+k}\right).$$

Taking appropriate roots of both sides, we see that

$$c_1 = \left(\frac{k}{n}\right)^{\frac{k}{n}}\left(\frac{n}{n+k}\right)^{\frac{n+k}{n}}.$$

Finally, notice that the last critical value cannot occur for any c-values past which the inner circle contains the origin. The inner circle has radius $r = \left(\frac{kc}{n}\right)^{-\frac{k}{n+k}}\left(\frac{n-k}{n}\right)c$; then, if this value is greater than 1, we have passed the last critical value, or

$$c_N^{\frac{n}{n+k}} \leq \left(\frac{k}{n}\right)^{\frac{k}{n+k}}\left(\frac{n}{n-k}\right).$$

Taking the appropriate root yields

$$c_N \leq \left(\frac{k}{n}\right)^{\frac{k}{n}}\left(\frac{n}{n-k}\right)^{\frac{n+k}{n}},$$

as desired. There is a cusp on the inner circle exactly when n is odd and k is even, which gives us the case for which this upper bound is sharp. □

Chapter 4. Areas of Further Research

In this thesis, we restricted values of c in f_c to $c > 0$. However, one might wish to know the number of zeros of f_c in the more general case $c \in \mathbb{C} \setminus \{0\}$. Many of the proofs given remain fundamentally unchanged in the more general case; one can simply replace c with $|c|$. However, certain aspects of the geometry of the epicycloids *will* change: namely, the epicycloids will be rotated by a factor dependent on the argument of c. The full statement and proof of the number of zeros of f_c in this more general case is ongoing research.

More generally, we were focused on a specific family of complex-valued harmonic rationals. This family had geometric properties which were essential to applying the Harmonic Argument Principle. It is fair to wonder if there are other families with nice geometric properties, both of the critical curve and its image, beyond those that have already been studied.

Most generally, one might wish to determine the number of zeros of complex-valued harmonic functions that do *not* have such nice geometric properties. This, of course, makes the problem considerably more complicated.

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