2022-12-12

# Developing a Quantitative Understanding of U-Substitution in First-Semester Calculus

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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ABSTRACT

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In much of calculus teaching there is an overemphasis on procedures and manipulation of symbols and insufficient emphasis on conceptual understanding of calculus topics. As such students to struggle to understand and use calculus ideas in applied settings. Research shows that learning calculus topics from a quantitative reasoning-perspective results in more powerful and flexible conceptions of calculus topics like integration. However, topics beyond introducing integrals and the Fundamental Theorem of Calculus, like u-substitution, have yet to be explored from a quantity-based perspective.

In this study, I conducted a set of two clinical interviews where we discussed quantitative meanings of integrals, derivatives, and differentials and used those meanings to quantitatively develop u-substitution. This study suggests that given the scaffolding of the quantity-based tasks students can develop the u-substitution structure (substitution of the bounds, the function, and the differential) by applying quantitative reasoning. It also suggests that two-quantity quantitative relationships are critical to students' productive thinking about substitution. Finally, this study offers a theoretical and quantitatively grounded framework for understanding u-substitution.



Keywords: calculus, integration, adding up pieces, u-substitution, quantities

# ACKNOWLEDGEMENTS

First, I wish to thank my family, particularly my incredible parents, for their love, support, and encouragement throughout the process of working through this program and writing this thesis. I thank my committee members Dr. Steve Williams and Dr. Doug Corey for their time and feedback to help me strengthen and improve my study. Finally, I would especially like to thank my advisor Dr. Steven Jones for his patience and support. His expertise, guidance, and encouragement were vital in the creation and completion of this thesis.

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# CHAPTER ONE: INTRODUCTION

Much of calculus teaching at both the secondary and college level is done with an overemphasis on procedures rather than a focus on conceptual development (Tall, 1992). Emphasis on procedure can be problematic and result in students lacking conceptual understanding of calculus topics (Bezuidenhout & Olivier, 2000; Jones, 2015a). In addition to the emphasis on procedures, most approaches (Stewart, 2016; Herman & Strang, 2017) use purely algebraic or geometric ideas as the foundations of integration and differentiation; however, research has shown the importance of developing calculus concepts quantitatively (Dorko & Speer, 2015; Moore et al., 2009, Jones, 2013a, 2013b, 2015a, 2019).

Integration is one of the foundational concepts in calculus; its applications extend beyond pure mathematics into physics, engineering, and science (Jones, 2015a). In fact, Bressoud et al. (2013) found that around 78% of students in calculus classes planned on having careers in engineering, biology, physical and life sciences, computer science, and business. In addition to a procedural understanding of integration being unproductive, Bezuidenhout and Olivier (2000) found that when students' conceptions of the integral were tied primarily to the idea of area they were unable to successfully reason about other integral contexts (see also Jones, 2015a; Sealey, 2006). Students often struggle to reason about integration in real-world problems and make the necessary connections between the mathematics and problem contexts (Hu & Rebello, 2013, Serhan, 2015). Hu and Rebello (2013) found that physics students struggled to productively set up integrals in applied contexts due to overemphasis on computation and symbol manipulation.

Recent literature has shown that reasoning with quantities to develop student understanding of the integral produces a much more powerful and flexible integral conception

(Von Korf & Rebello, 2012; Jones, 2013a;2015; Sealey, 2014; Ely, 2017, Simmons & Othehterman). Jones (2013a; 2015a; Stevens & Jones, in progress) found that while reasoning about the integral as an area under a curve or an antiderivative is more frequently used, conceptualizing the integral as "adding up pieces" (AUP) is more helpful for students in making sense of the integral in applied contexts. The AUP structure represents adding up infinitesimally small bits of a "target quantity," often created from the product of a function and the differential, which correspond to small increments of a partitioned interval of the domain. The structure of AUP provides a more intuitive way to conceptualize the definite integral in context (Jones & Ely, in press).

Because of the importance of quantitative meanings for integrals, researchers have offered learning trajectories for integration. For example, Stevens (2021) utilized Jones' AUP in her integration learning trajectory. She found that introducing the definite integral from the AUP perspective and progressing through integration concepts using various representations resulted in students developing rich understandings of the various layers of integration. However, this learning trajectory began with the introduction to integration and stopped at the Fundamental Theorem of Calculus. There are typically additional ideas in the first-semester chapter on integration, with u-substitution generally being the conclusion of the chapter (e.g., Stewart, 2021). U-substitution is a widely used procedure but is often nothing more than a memorized process for finding an antiderivative. Learning u-substitution in this way presents the same "procedure" problems as mentioned above. Going from a quantitative introduction of integrals to a procedural understanding of u-substitution may cause students to abandon their previously constructed quantitative meaning of the integral for the computational meaning. (Jones et al., 2017).

Typically, the integration unit in first-semester undergraduate calculus includes the following five topics: Riemann sums, the definite integral, the fundamental theorem of calculus, the indefinite integral, and u-substitution. A quantitative reasoning perspective for the first four of these topics has been studied; however, research has not yet examined developing u-substitution quantitatively. In my study, I build on the quantitative work already done in integration and extend the quantitative reasoning approach to encompass u-substitution as the cap to the quantitative treatment of first-semester integral calculus. The purpose of my study is to see what understandings of u-substitution students develop using quantitative reasoning. To achieve this purpose I conducted a set of clinical teaching interviews and analyzed the student thinking that appeared in the interviews.

# CHAPTER TWO: BACKGROUND

In this chapter, I present the existing literature on integrals, derivatives, and differentials. I start with reviewing conceptions of the definite integral, and how they support students in making sense of integrals in applied contexts. I then briefly discuss the literature on derivatives and differentials.

## Productive Conception of the *Definite* Integral

Jones (2013b) identified three conceptualizations of the definite integral: "perimeter and area" based on area under a curve, "function matching" based on the antiderivative, and "adding up pieces" (AUP) based on Riemann sums.

In the perimeter and area conception, the integral is thought of as the area under a curve or the area between the function and the horizontal axis. This conceptualization differs from AUP because rather than viewing the area as a collection of rectangles to be added, here the integral is thought of as one undivided area. While this is a correct graphical meaning, it is not always useful for describing values of the integral in context. Students using this conception with applied integral problems were easily able to draw a picture but struggled to use the picture to make sense of what the area value meant in the problem and felt unsure of their explanations (Jones, 2015a). A similar categorization for three-dimensional shapes was presented by Jones and Dorko (2015) as the boundary of an entire non-segmented shape. Similar to the area conceptions it takes the whole shape as a whole rather than thinking about segments or slices of the shape to find the volume.

The "function matching" conception views the definite integral as an antiderivative. In this conception, for an integral $\int_a^b f(x)dx$ the integrand $f(x)$ is identified as being the derivative of some "original function." Solving the integral is thought of as trying to match the integrand back to the original function. This also means that the derivative of the solution needs to match the integrand (Jones, 2013b). Antidifferentiation is a very important application of the Fundamental Theorem of Calculus and is useful for calculational ease; however, it is not often helpful in making sense of the quantities in contextualized integrals or interpreting their answers. Jones (2015a) found that students who relied on reasoning with antiderivatives displayed a lack of confidence in their answers and in making sense of the problems. They attempted to refer to the antiderivative relationship of velocity and position to reason about their current contexts and units but were not able to identify the multiplicative relationship and struggled to find the applied meaning in the symbol manipulation.

In the AUP conception, similar to the Riemann sum, the integral is viewed as a sum of infinitesimally small quantities. More specifically, in the AUP conception, an interval of the domain quantity is broken up or "partitioned" into infinitesimally small pieces, $d[\,]$. Each piece corresponds to a small bit of a desired or "target quantity," often but not always created by a multiplication of the integrand and differential, and each of the target quantity pieces is added up (Ely, 2017; Jones & Ely, in press). In his study, Jones (2015a) asked students to interpret the meaning of the integral $\int_0^{600} Rdt$ where $R$ is the revolutions per minute of a motor. The AUP perspective proved to be the most productive for students' interpretation of the problem. Using a rectangle to represent a piece of what they were adding, they viewed the differential $dt$ as holding some small quantity, thus having a unit of measure, and were able to make sense of the multiplication of the integrand R and the differential $dt$ to interpret the target quantity of the integral as revolutions.

Related to AUP, Simmons and Oehrtman (2019) further elaborated on the construction of the target quantity describing a "basic" and "local model" for the target quantity. The "basic model" applies to situations with constant quantity values. For example, if an object has uniform density then finding the mass of that object or a part of that object is a simple multiplication. However, if the density is not uniform then we use the "local model" for finding the total mass. After partitioning the object, the density of a partitioned piece is used to find the mass that corresponds with that part of the object. The local model is used for every partitioned piece and the integral sums them to find the total mass.

While all three of the above-listed conceptualizations are useful in understanding decontextualized integrals, the "adding up pieces" symbolic form is shown to be the most productive for students making sense of integral applications and interpreting real-world problems. (Jones, 2013a; 2015a; Sealey, & Oehrtman, 2005).

Although the AUP conception is the most productive for applied definite integrals, students are much more likely to think about integrals as the area under a curve or an antiderivative (Jones, 2015b). This could be because integrals are often introduced in textbooks and classrooms as an area under a curve. (Jones, 2015b; Stewart, 2021). It is true that the AUP concept is closely related to the Reimann sum conception, but being familiar with Reimann sums does not necessarily indicate the students will use AUP. Jones et al. (2017) found that in some cases teachers make instructional moves that actually undermine their previous instruction of Reimann sums. It is possible that rather than students viewing Riemann sums as a way to conceptually understand the integral they view it merely as a computational procedure that is often forgotten after learning the Fundamental Theorem of Calculus.

## Derivative

Stewart's (2021) textbook presents the derivative first as the slope of the tangent line using multiple graphical representations. Once that definition is established this book introduces the derivative as a rate of change connecting it to the average rate of change concept. Bingolbal et al. (2007) investigated the derivative conceptualization of first-year engineering students and first-year mathematics students. They found that the mathematics students preferred to think of the derivative as a tangent line while the engineering students favored thinking about the derivative as a rate of change. This difference could be related to the amount of time the teachers of their respective classes spent on each conceptualization; however, the engineering students expressed that their preference was because the rate conception supported the application of the mathematics in their field (Bingolbali et al., 2007).

Reasoning about rates can be cognitively complex. Milner and Rodríguez (2019) state that the central idea of the derivative is not slope, but quotients of small quantities. Ely (2020) similarly talks about the differentials-based approach like $f(x) = \frac{dx}{dy}$ where the rate is a ratio of the changes of two quantities. The structure of $\frac{dx}{dy}$ is useful in understanding rate and makes the quantities involved more clear than $f'(x)$ notation of the derivative. Jeppson (2019) used the interpretation of the derivative $\frac{dx}{dy}$ as a ratio in her hypothetical learning trajectory of quantitative understanding of the chain rule and implicit differentiation. She found that by thinking about the rates, and how the two quantities changed together, students were able to develop an intuition for the multiplicative nature of the chain rule and implicit differentiation.

### Differential

The differential, symbolized by $d[\,]$, is rarely given significant attention in calculus classrooms. In integration the differential is frequently regarded only as an indicator of the variable of integration or touched on only as a part of the substitution process (Dray & Manogue, 2010, Jones, 2015a). This inattention to the differential may be due to it being difficult to define. Even mathematicians struggle to define the differential or verbally express their own intuition of it (McCarty & Sealey, 2019). When presented with differentials in different contexts and asked to interpret the meaning, they gave a wide variety of interpretations, in some cases even contradicting their own statements from previous contexts. A common interpretation of the differential as reported by McCarty & Sealey (2019), is that it is small but has no specific size, and when presented with the expression $dy = 2xdx$, several mathematicians expressed uneasiness about the differential.

Similar to Leibniz, Ely (2017; 2020) approaches the differential as an infinitesimally small piece of a quantity and treats differential equations like $dy = 2xdx$ as a relationship between two infinitesimal quantities. This conceptualization of the differential gives it more quantitative meaning that supports making sense of definite integrals. It is also a key part of understanding the integral as accumulating or adding up bits of quantity. In Jones' (2015a) findings, students who had productive conceptualizations of various contextualized integrals attended to the quantity of the differential. For example, students with a productive conception of the integral $\int_S \; PdA$, where $S$ is some surface, and $P$ is pressure on a point, recognized $dA$ as a small piece of area that could also be expressed as $dx * dy$. They were able to use these ideas to understand that pressure on a point multiplied by the differential quantity area resulted in a small piece of force, and adding up these pieces would result in finding the total force on the surface.

The benefit of understanding the differential as carrying quantity is further illustrated by Hu and Rebello (2013) who also found that students who were able to reason about the quantity and units of the differential were more successful in interpreting and setting up integral expressions to solve real-world applications of the integral in physics problems (see also Amos & Heckler, 2015; Schermerhorn & Thompson, 2019a; 2019b). Not only does interpreting the differential as having quantity support productive understanding of the integral, but a lack of it can also hinder students' ability to understand applied integrals. Work done by Ngyuen and Rebello (2011) suggested that students' inability to interpret differentials and products using differentials created a significant obstacle to successfully constructing integrals in physics.

## CHAPTER THREE: THEORETICAL PERSPECTIVE

### Images

Tall and Vinner (1981) define a concept image as being all the cognitive structures associated with a concept. This could include mental pictures, associated processes or other structures. Thompson (1994) talks about images as a more strictly cognitive experience or mental operation. In line with both of these ideas I use image to mean the overall understandings, metaphors, experiences and mental operations that students have of a concept or concepts.

### Quantities and Quantitative Reasoning

The framework for my study utilizes Thompson's (1990, 2011) definitions surrounding quantity and quantitative reasoning. It also includes my own quantitative conceptual analysis of u-substitution (Thompson, 2008). The conceptual analysis breaks down the u-substitution of a definite integral from an "Adding Up Pieces" (AUP) perspective, the meaning of the integral and the meaning of quantities of each component part (Jones, 2013).

In this section I first discuss the meanings of quantity, quantitative reasoning, quantitative operation, quantitative relationship, value, and covariational scaling. These ideas of quantitative reasoning formed the basis of my study and served as codes for my data analysis. I selected them because I found these particular aspects of quantitative reasoning to be inherent in the meanings of the components of my conceptual analysis of u-substitution and the types of reasoning I anticipated appearing in student output gathered in teaching interviews.

Thompson (1990) defines a quantity as the "quality of something that one has conceived as admitting some measurement process." He later adds that quantities are "mental constructions," or exist as a result of our conceptualization of a given object or situation (Thompson, 2011). For example, radius is a quantity because it describes the measurement from the center of a circle or sphere to any edge or surface. Thompson (1990) defines the “result” of the measurement process as a value that can be a specific numerical value or some imagined magnitude (Moore et al., 2009).

The definitions of quantitative relationship and quantitative operation are closely related to one another. A quantitative operation is the mental action taken on one quantity with itself or with two or more different quantities to produce a new quantity. For example, multiplicative comparison between energy generated $dE$ and time elapsed $dt$ produces power. A quantitative relationship describes the connections between quantities that exist because of the operation. Using the previous example, time, power, and energy are all related to each other and an operation with any two of the quantities could be used to create the third.

With all the previous definitions in place, we can define quantitative reasoning. Quantitative reasoning includes the mental action of conceiving something and its measurable attributes, or quantities, how multiple quantities are related to each other, and the operations used to form those relationships. A network of these relationships creates a structure for constructing quantitative mathematical understanding of a concept (Thompson 2011, Moore, 2013).

**Covariation**

Closely related to quantitative reasoning is covariation. Two types of covariation have been used in reasoning conceptually about the integral, one involving dynamically changing quantities called dynamic covariation (Carlson et al., 2002; Thompson & Carlson, 2017) and the other as a “zoomable” static continuum called scaling-continuous covariation (Ellis et al., 2020). If we consider the covariation between time and distance from a dynamic perspective, dynamic covariation might involve imagining distance being accumulated as the time passes, both quantities changing in tandem. Conversely, scaling covariational reasoning imagines taking a static picture of a relationship between the quantities of time and distance. Scaling or zooming in on a value of one quantity to any arbitrary size always corresponds to its related value simultaneously being scaled or zoomed into (Ellis et al., 2020). Returning to the example of time and distance, this is like zooming into an increasingly small static interval of time and simultaneously zooming in on the distance associated with that particular amount of time.

Covariational reasoning is a critical aspect of u-substitution for each of the three pieces of substitution: bounds, function, and differential. When considering the change of bounds there is a covariational idea in thinking about how a range of values in one quantity corresponds to a range of values in a new quantity. In the function substitution, given a relationship between two quantities it is important to be able to coordinate values of one quantity to their corresponding value in the second to reason that substituted functions are equivalent. Because my study focuses on infinitesimal pieces of quantities, like adding up infinitesimal bits of the target quantity and the differential being an infinitesimal amount, I utilized the scaling-continuous covariation or zooming covariation conception. This view has been used and promoted as a potentially valuable underlying image of derivatives and integrals based on infinitesimals (Ellis et al., 2020; Ely & Ellis, 2018; Ely & Samuels, 2019). In scaling covariation, one imagines a fixed infinitesimal of one quantity as corresponding to a fixed infinitesimal (of likely a different infinitesimal size) of another quantity. Again, given a relationship between two quantities, a change of one quantity results in a change of the related quantity. From a scaling perspective it is fairly straightforward to think of zooming into smaller and smaller changes to the infinitesimal level maintaining that same relationship between changes.

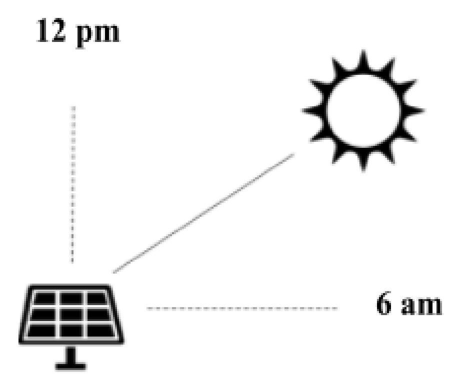
## Conceptual Analysis

To demonstrate the conceptual understandings I want students to develop using quantitative reasoning I discuss the various components of the definite integral and how I quantitatively conceptualized the substitution of each. I use the below solar panel problem to support this discussion. In this problem the quantities are time (hours), energy (watt hours) and power (watts). Watts are often described in the joules of energy (J) per second (s) that is generated, J/s, however, for simplicity I just discuss power in the simple unit of watts. I also note that if power is constant the basic model of a simple product with time will give energy.

**Figure 1**

*Solar Panel Context*

A solar panel collects power in watts, and the amount of power the panel generates is dependent on where the sun is in relation to the panel. It will reach its maximum output when the sun is directly above it (at noon). The generated wattage of the solar panel can be thought of as a function of time as the sun moves across the sky. It can be modeled well by a sine function (Solar Panels, 2021), and for simplicity we'll use the basic function $P(t) = 250sin\left(\frac{\pi}{12}t\right)$with *t* time in hours since sunrise. On the day we're measuring, the sun rises at 6 am and sets at 6 pm.
How much energy in watt hours has the panel generated in the first 6 hours of the day (from 6am to 12pm?



In this problem I utilize the “adding up pieces" (AUP) conception to discuss the total energy generated by the solar panel (Jones, 2015). As mentioned previously “AUP is a *structure* comprised of three elements: a partition, a target quantity and a sum.” (Jones & Ely, in press). I will now discuss each of the elements of the AUP structure in relation to this solar panel task.

In our problem context we are making a *partition* of our time interval of the first six hours of the day into infinitesimally small but substantive quantities of time represented by $dt$. The *target quantity* is constructed by a multiplicative relationship between power generated in watts $P(t) = 250sin\left(\frac{\pi}{12}t\right)$ over small amount of time $dt$. The product of these two components gives us a tiny amount of energy in watt hours, our *target quantity.* Finally, we sum up each of these infinitesimal target quantities for each of our partitioned $dt$ pieces in our interval $\int_0^6$ . The summation of all of the small pieces of energy gives us the total energy in watt hours generated by our solar panel in the first six hours of the day, $\int_0^6 250sin\left(\frac{\pi}{12}t\right) dt$.

Using this same AUP conception I treat the substitution of each individual component. As we consider the context, the passage of time is an abstract way to measure the position of the sun in relation to the solar panel. A more direct way to describe the situation is to measure the angle of the sun. Reasoning with the quantities of the problem context we know that the sun rises on the horizon at an angle of 0 radians and since the sun sets at 6pm after six hours at noon the sun will be directly above the panel at an angle of $\frac{\pi}{2}$ radians. The range of the angles the sun traces out in six hours is $[0,\frac{\pi}{2}]$, making the substituted bounds of integration $t = 0 \rightarrow \theta = 0$ and $t = 6 \rightarrow \theta = \frac{\pi}{2}$. The integral with this first substitution is $\int_{\theta = \mathbf{0}}^{\boldsymbol{\theta} = \frac{\boldsymbol{\pi}}{2}} 250sin\left(\frac{\pi}{12}t\right) dt$. The integral in this form has tiny bits of energy that are a result of the multiplicative relationship to form the target quantity; however, in the form we have our target quantity formed in relation to power for every tiny bit of time being added up over a span of angles. This combination of units is difficult to make sense of.

To clarify reasoning about the integral expression we next consider a substitution in target quantity. In the original quantitative relationship of the target quantity, we get a bit of energy from the multiplicative relationship of a little bit of power $P(t)$ and a little bit of time $dt$. We want to take this quantitative relationship and describe it in terms of angle rather than time, multiplying a little bit of power by a little bit of angle to get the desired quantity of energy. We will now look at each piece of the quantitative relationship separately.

Starting with the power function, we want to represent $P(t) = 250sin\left(\frac{\pi}{12}t\right)$ in terms of angle in radians. Reasoning again with the problem context and bounds of integration, $t = 0 \rightarrow \theta = 0$ and $t = 6 \rightarrow \theta = \frac{\pi}{2}$, we can derive the covariational relationship between the time in hours and the angle of the sun as $\theta = \frac{\pi}{12}t$. This substitution makes the new substituted power function $P(\theta) = 250sin(\theta)$ and the integral $\int_0^{\frac{\pi}{2}} 250sin(\theta)\frac{\pi}{12}dt$. In the original partition we have small amounts of time scaled down to the infinitesimal quantity $dt$. In this new integral we are partitioning the range of angles into infinitesimal amounts of angle $d\theta$. Using the relationship $\theta = \frac{\pi}{12}t$ from above, we can use that relationship and think about scaling it down to be infinitesimally small amounts of $\theta$ and $t$. The result is the differential covariation $d\theta = \frac{\pi}{12}dt$, which we can also be conceptualized as for each little bit of time, the related amount of angle is $\frac{\pi}{12}$ times that amount of time. Another way to conceptualize the differential relationship is with the derivative. If we think about the derivative as a ratio of small amounts of covarying quantities, taking the derivative of the relationship $\theta = \frac{\pi}{12}t$ gives $\frac{d\theta}{dt} = \frac{\pi}{12}$ and we can similarly arrive at the differential relationship $d\theta = \frac{\pi}{12}dt$. We now have a substitution for both pieces in the quantitative relationship of our substituted target quantity expressed in terms of angle $\theta$: $250sin(\theta) * d\theta$. As with our original problem each infinitesimal pieces of target quantity will be summed over the interval of angles $[0, \frac{\pi}{2}]$ and our final substituted integral is $\int_0^{\frac{\pi}{2}} 250sin(\theta)\frac{12}{\pi}d\theta$.

In the process of this analysis, I found that successfully reasoning quantitatively about u-substitution requires background understanding of the integral as adding up infinitesimally small pieces of some quantity constructed by a multiplicative quantitative relationship between a rate function and a differential. Additionally, a knowledge of covariational scaling and the derivative as a ratio of differentials is important to support a quantitative conception of u-substitution. I have taken this background knowledge into consideration and designed the first teaching interview to develop this understanding to prepare participants for a quantitative introduction to u-substitution.

## Research Questions

As discussed previously, there is nothing in the literature that describes how students reason about u-substitution specifically from a quantitative approach. In order to build on the research previously done about quantitative reasoning in first-semester calculus topics in this study I purposefully scaffold a teaching process to direct students to use quantities in their reasoning to help them develop the idea of substitution. Within the structure of this context I propose the following research questions:

1) What quantities and quantitative reasoning did students exhibit given they were expected and directed to engage with quantities?

2) How did the quantities and quantitative reasoning connect specifically to the three parts of u-substitution (bounds, function, and differential), given they were expected to learn these three parts?

In answering these questions I hope to gain insight into the understandings of u-substitution that students can develop using quantitative reasoning.

# CHAPTER FOUR: METHODS

## Teaching Interviews

Principles of quantitative reasoning adhere to the idea that students are constantly in the process of constructing their mathematical ideas and the teaching experiment aligns with this perspective (Engelhardt et al., 2004; Moore, 2013). Steffe and Thompson (2000) describe the purpose of using a teaching experiment, saying it is, "for researchers to experience, firsthand, students' mathematical learning and reasoning." They go on to explain that being immersed in the teaching experience provides a basis for understanding students' construction of mathematical operations and concepts (Steffe & Thompson, 2000). The aims of the researcher in a teaching experiment are to build a feasible model of student understanding, explore ways of influencing students' mathematical knowledge and understanding, and record where and how shifts in understanding occur (Moore, 2013; Steffe & Thompson 2000). Due to the limited time of the master's program, I was not able to engage with a full teaching experiment; however, I based my clinical task-based interviews on the idea of a teaching experiment because my research questions align with the purposes and aims of the teaching experiment. In this study I investigate what understandings of u-substitution students develop and explore how using quantities and quantitative reasoning impacts the knowledge about u-substitution students emerge with.

## Participants and Data Collection

The participants for my study were students recruited from first-semester undergraduate calculus classes. The students were recruited from classes whose instructors had attended to quantity in their teaching throughout the semester. I planned to use students who had not taken a calculus class prior to the one they were currently enrolled in to get a better idea of how students developed the mathematical ideas of u-substitution without influence from previous exposure to the procedure; however, due to the student population that volunteered to participate and students' availability three of my participants had previously taken a calculus class and three had not. Students were interviewed in pairs, in one pair both students had taken calculus before (Jackson and Bently), in another pair one student had and one had not taken calculus (Nate and Liam), and in the third pair, both were first-time calculus students (Andres and Ellie). A summary of the student participants is given in Table 1. Students were in pairs with the intent that they would help each other bring out the key concepts of the lessons and allow for more student thinking to be visible as they worked together and communicated ideas with one another as well as with me. The students were interviewed after they had an introduction to Riemann sums, integration, and the Fundamental Theorem of Calculus, but before they had been introduced to u-substitution as a technique of integration.

**Table 1**

*Student Participant Information*

| Pseudonym | Gender | Group | Taken calculus before? |
| --- | --- | --- | --- |
| Bently | M | 1 | Yes |
| Jackson | M | 1 | Yes |
| Nate | M | 2 | No |
| Liam | M | 2 | Yes |
| Ellie | F | 3 | No |
| Andres | M | 3 | No |

I conducted a set of two teaching interviews, each around 60 minutes, over 2-3 days. The first teaching interview introduced covariational change between two quantities from a scaling perspective, the quantitative AUP conception of the integral, and the derivative as a ratio of differentials. In this interview, students were given problem contexts that support reasoning about the integral as a summation of small pieces of the target quantity rather than as an antiderivative or area under a curve. The problems given were also designed to promote reasoning about the differential as having some quantity rather than strictly as an infinitesimal. I chose these topics based on what I found in my conceptual analysis to be the foundational prior understandings needed to build a quantitative understanding of u-substitution.

The second interview began by having the students recap/review what they learned from the first interview. We then developed the concept of u-substitution using meaningful contexts and quantitative reasoning. The contexts and questions drew on the quantitative understanding of the integral, derivative, and differential developed in the first interview to support the students' learning. These ideas of substitution were further explored in a second contextual situation with a pure mathematics problem to explore time permitting. Both interviews were videotaped to capture students' work and gestures. The video footage, audio transcription, and students' written work were used in the analysis.

## Teaching Interview Lesson Plans

This section details the lesson plans and tasks given to the students in both of the teaching interviews. The tasks and questions are based on the above conceptual analysis, and definitions relating to quantitative reasoning.

## Lesson One

### Scaling Covariation

I began with having a brief conversation about the relationship between height and volume of a box. I presented students with a table labeled with a $\Delta ft$ column and a $\Delta V$ column and asked the students to list out a few values value pairs (ie. 1 $ft$, 4 $ft^3$).

Key Questions:

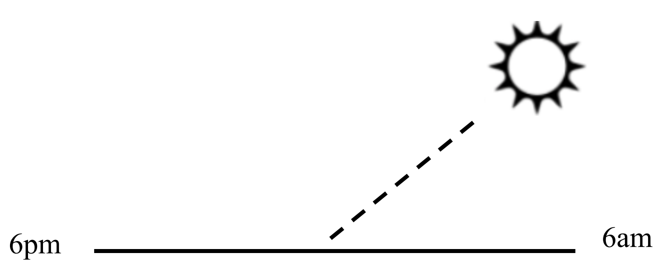
- If I have a height that changes by a half a foot what is the equivalent change in volume? What about a change of ¼ of a foot? 1/100th of a foot?
- If we have an infinitesimally small change in length in feet what is the equivalent change in inches?
- Can you describe what the change in inches will be in relation to any change of length in feet?
- What symbols would you use to represent that relationship?

I begin with this simplistic example and questions to help students develop the idea of covariational scaling and focus on the quantities involved. Scaffolding the relationship in this way is meant to help students avoid the common misconception of expressing the relationship as $a\ ft = 12b\ in$ rather than $12a\ ft = b\ in$. I do not anticipate students having difficulty with this problem; however, it will help them to have dealt with the quantities in this familiar context as we move to the next more complex situation below.

**Figure 2**

*Solar Differential Relationship Context*

On the equator the sun rises around 6 am, is directly overhead at noon, and sets on the horizon at 6 pm. If you are standing on the equator how much has the angle of the sun in relation to you changed from 6 am to 6 pm? From 6 am to noon?




| $\Delta t$ hours | $\Delta\theta$ angle |
|---|---|
| 12 | |
| 6 | |
| $\frac{1}{2}$ | |
| $\frac{1}{100}$ | |
| dt | |

Key Questions:

- If you are standing on the equator how much has the angle of the sun in relation to you changed from 6 am to 6 pm? From 6 am to noon?
- How big of an angle change corresponds to one hour of time elapsed?
- What if we only let a half hour elapse? One-tenth of an hour? A thousandth of an hour?
- If I continued to scale this down to the change of a fraction of a second, what would the change in angle be?
- Can you describe what the angle change will be in relation to any elapsed amount of time?
    - What symbols can we use to represent that relationship?
    - Does this make sense for this relationship to hold for change in time at any part of the day?
- Is this relationship still valid as the change in time gets infinitesimally small?
- What do $dt$ and $d\theta$ represent?

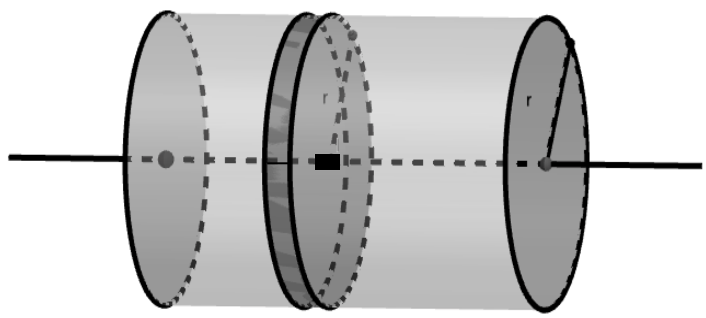
The goal of my questions is to guide students to think about the differential as representing a small piece of a quantity that can be expressed in terms of the numerical value of another differential quantity. As with the first context, I anticipated that students would answer the questions without too much difficulty.

**AUP Integrals**

**Figure 3**

*Volume of a Cylinder Context*

The volume of a cylinder is $V = \pi r^2 l$ where $r$ is the radius and $l$ is the length of the cylinder.




Key Questions:

- Let's use a symbol to represent the volume of the pictured slice of the sphere. What symbol should we use? (if no "d", ask: "how could we suggest it's a very thin slice?)
- What does the $dV$ mean? (or the equivalent symbol they use to represent the volume)
- How do you interpret the integral $\int_a^b dV$?
- What are the quantities that make up this slice of volume? What symbols can we use to represent these quantities?
    - Is this integral $\int_a^b dV$ the same as this one $\int_a^b \pi r^2 dl$?
- Given the integral $\int_a^b \pi r^2 dl$ what does each piece mean? What does the integral mean all together?

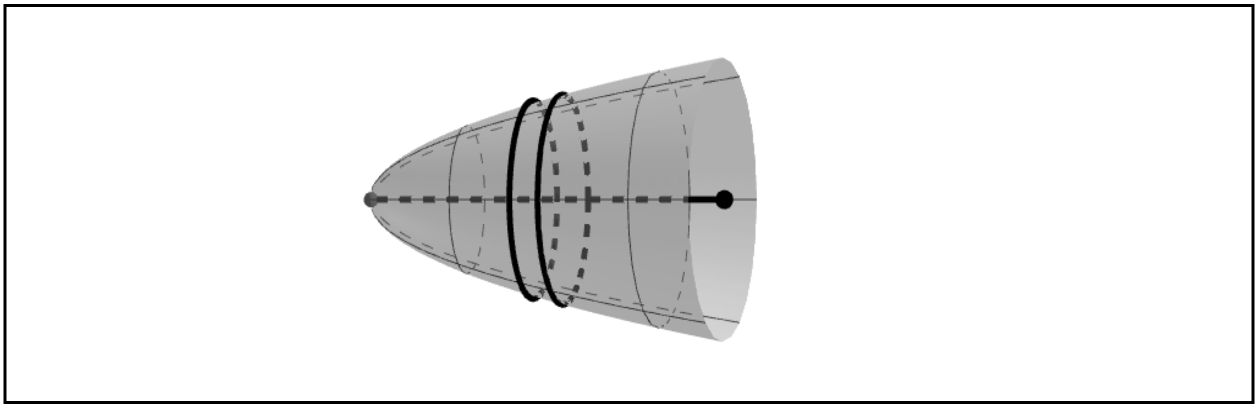
The goal of this problem is to introduce the AUP structure of the integral. The first three questions are meant to guide students to think about the target quantity being summed up.

The purpose of the last two questions is to introduce the multiplicative structure of the target quantity.

**Parabolic shape**

**Figure 4**

*Volume of a Solid Context*



Key Questions:

- We can take volume slices of this shape similar to the previous cylinder shape, how would you describe the slices of volume ($dV$) for this new shape?
- Like the last shape, let's think of the axis being made up of lots of little $dl$'s does each $dl$ have an associated $dV$?
- Are all the $dV$s the same?
- Can we use an integral like we did with the cylinder to find the total volume of the shape? Write the integral expression.
    - How are you thinking about this integral?
- What do $dl$ and $dV$ mean? How are they related?
- Using the ideas that we talked about from these two problems, can you describe how you're thinking about the integral in general?

The main focus of these questions is for the students to identify the quantitative structure as they add up small pieces of volume. Asking the students to write the integral helps to solidify their understanding of the components and to make connections to their symbolic understanding.

**Derivative as a Ratio of Differentials**

For the last portion of this interview, I use some of the pieces previously developed to have a conversation about the derivative as a ratio of differentials of two quantities. In the first question, we establish the differential relationship between time elapsed and the angle of the sun, $d\theta = \frac{\pi}{12} dt$. I use this relationship to ask the following questions.

- If we take the $dt$ and divide it over to the other side to make the ratio $\frac{d\theta}{dt}$, what does this ratio mean?
- What does it mean that that ratio is $\frac{\pi}{12}$?

I then present the students with the following context: A spherical balloon with a small heater inside has been filled with gas. As the gas is heated it expands, increasing the volume of the balloon. At any given radius $r$ the relationship between the radius of the sphere and the temperature $T$ of the gas in degrees Kelvin is given by $r = \sqrt{T} + 5)$

Key Questions:

- Compute the derivative of this equation with respect to temperature.
- What does this derivative mean in this context?
- What do $dr$ and $dT$ mean individually?
- What does it mean that the derivative is a function rather than a number?
- For different temperatures what does that tell you about how fast the sphere is growing?
- If we multiply the $dT$ over to the other side we get $dr = \frac{1}{2\sqrt{T}} dT$. Is this valid?

- What does this new expression mean?

This is meant to help the students understand when they take a derivative like $\frac{dr}{dT}$ and multiply the differential in the denominator over to the other side of the equation($dr = \frac{1}{2\sqrt{T}}dT$) that quantitatively the equation will still have meaning. I anticipate that students may describe things non-quantitatively, using just numerical values or symbols without reference to quantities. In these cases, I directed them to think about the quantities involved and what they mean. I do not mean to indicate that non-quantitative reasoning is bad or unproductive; however, because I am interested in how quantities relate to the understandings they develop I follow non-quantitative descriptions with questions guiding the students to the quantities.

**Lesson Two**

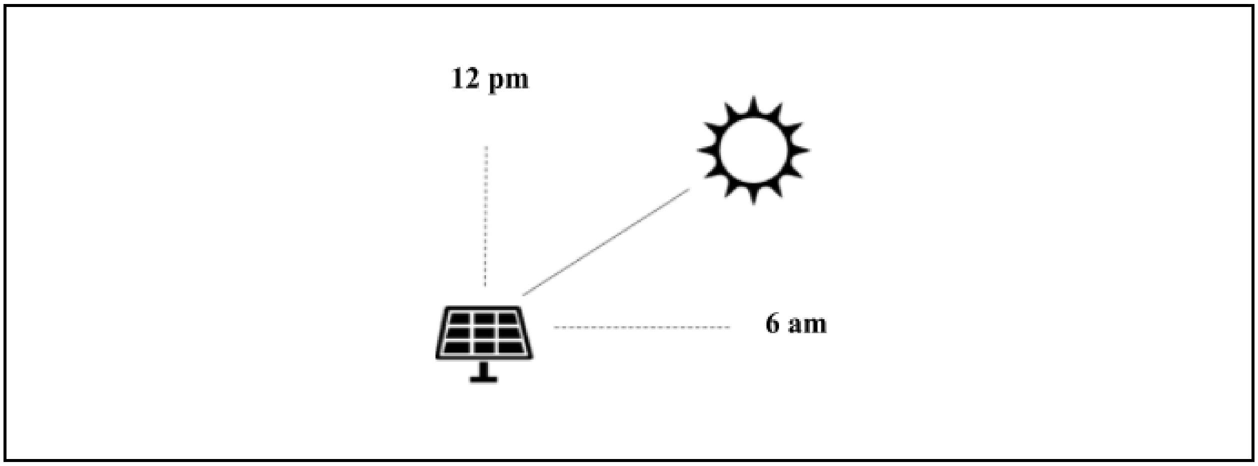
I began this lesson by asking students to interpret the meaning of the following integral to review the ideas developed in the first interview $\int_a^b v(t)dt$.

**Solar Panel task**

**Figure 5**

*Solar Panel Context*

| |
|---|
| A solar panel collects power in watts, and the amount of power the panel generates is dependent on where the sun is in relation to the panel. It will reach its maximum output when the sun is directly above it (at noon). The generated wattage of the solar panel can be thought of as a function of time as the sun moves across the sky. It can be modeled well by a sine function (Solar Panels, 2021), and for simplicity, we'll use the basic function $P(t) = 250sin\left(\frac{\pi}{12}t\right)$with *t* time in hours since sunrise. On the day we're measuring, the sun rises at 6 am and sets at 6 pm.<br>How much energy in watt hours has the panel generated in the first 6 hours of the day (from 6am to 12pm? |




Key Questions:

- What are the quantities present in this problem?
- Can you create an integral that would give us the amount of energy accumulated during the first six hours of the day?
- Using the AUP idea from our first interview, describe what this integrals means
    - What is the partition?
    - What does the integrand mean? What is the quantity of the little pieces that we're adding up?

These questions are meant to help the students refresh the AUP meaning of the integral developed in the first interview. Additionally, unpacking this integral and its pieces may help the students more easily recognize the various pieces that will be substituted.

To motivate the substitution we consider picturing the context, where time is difficult to visualize while viewing the angle of the sun in relation to the solar panel is more simple to conceptualize. To begin the discussion about the substitution I ask the students the following question

- Could we change the integral to be in terms of the angle of the sun in relation to the solar panel rather than the time of day?

As needed I prompt the students to think about the bounds substitution with the following questions

- What is the angle range that corresponds to our time interval?
    - What are the units that we have in the integrand?
- What is the quantity that we are now breaking into small pieces?

Next, we move on to substituting the target quantity beginning with the integrand, and I ask the following questions to help the students with the substitution of the target quantity.

- What might the power function look like in terms of angles
- What is the relationship between $t$ and $\theta$ in the function?

We now focus on the substitution of the differential. Now that we have a function in terms of $\theta$ we need the differential to be in terms of $\theta$ for the quantities/units to make sense.

- How does a little bit of time relate to a little bit of angle?
- What if we use our previous partition and scale it down to infinitesimals, what is a tiny bit of $dt$ equal to in angles?

The next questions are meant to check for the students' understanding of what this new integral represents. They will help me know how the students are thinking about the quantities involved and if they are making the connection that we still end up with the desired quantities after the substitution. Asking the students to review the work they have done will also start to build the foundation for them to abstract and generalize the ideas of substitution.

- What is the quantity that we are adding up here?

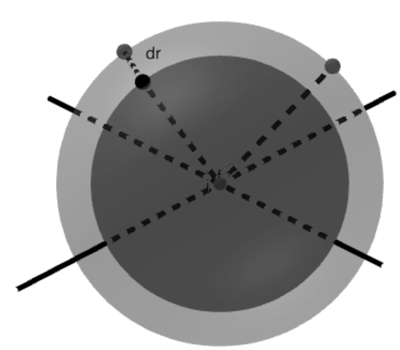
- Let's compare the original integral in terms of time and the new integral in terms of angle. How are things being added up here now that we've done these substitutions?
    - Are these two integrals adding up the same quantity?
    - Describe how this new expression is adding up energy like we initially intended it to,
- Let's review the work that we've just done. Can you list the different substitutions that we made?

**Sphere task**

Following the solar panel context, I introduce the equation for the volume of a sphere, $V = \frac{4}{3}\pi r^3$ and use Geogebra to help students visualize how the volume of a sphere increases with an increase in the radius. Specifically, to demonstrate that for a small increase in radius $r$ another spherical shell or layer of volume is added and as $r$ is scaled to be infinitesimally small, the precise volume $dV$ added is given by surface area $S$ multiplied by the change in radius $dr$, $dV = S * dr$.

**Figure 6**

*Volume of a Sphere Context*



After this introduction I remind the students that from our first problem we had an integral that was a bit more complex and we found a way to make it more simple. I tell them up front that in this problem we will work out the simple volume of a sphere integral and a more complex integral in parallel. The purpose of doing this is to help them see the connections between each form with the goal that given a complex integral to begin with, as they will in the last questions, they will be able to change it into a simpler version.

Key Questions:

- We established that $dV = S * dr$, what does $dV$ mean? What does $dr$ mean?
- Given $\int_{r=10}^{r=15} dV = \int_{r=10}^{r=15} S * dr$, what does each integral represent?
- Can you solve the integral? What does the answer mean within the context?

The purpose of the first two questions is to prompt the students to think about the actual quantities involved in this integral and what they mean. It will also assess how much they are still thinking of the integral as AUP. I ask the third question to emphasize that we want to solve this integral in its simple form. Working through the simple integral first will give students a reference to help them construct the more complicated integral later.

Next, I present the students with the balloon context used in the first lesson: A spherical balloon with a small heater inside has been filled with gas. As the gas is heated it expands, increasing the volume of the balloon. At any given radius the relationship between the radius of the sphere and the temperature of the gas in degrees Celsius is given by $r = \sqrt{T} + 5$ (As the gas heats the up the radius is a function of the temperature and expands)

Key Questions:

- Could we rewrite this integral where everything is in terms of what's happening with the temperature rather than what's happening with the radius?

- What temperature value corresponds to the start radius (r=10)? The end radius (r=15)
- What is the surface area $S$ in terms of temperature $T$?
- $dr$ represents a small change in $r$ in our original integral, what is the equivalent quantity needed in the integral with respect to temperature?
    - What is $dr$ equal to in this context?
    - What is $dT$ equal to in this context?
- How are you thinking about what each piece of the complicated integral $\int_{T=25}^{T=100} 4\pi(\sqrt{T}+5)^2 \frac{1}{2\sqrt{T}} dT$ means?

These questions are meant to guide students to use the relationship $r = \sqrt{T} + 5$ in all their substitutions. The purpose of the last question is to see if the students are still thinking about the quantities involved in the substitution.

Key Questions:

- Compare these two integrals to each other $\int_{T=25}^{T=100} 4\pi(\sqrt{T}+5)^2 \frac{1}{2\sqrt{T}} dT$ and $\int_{r=10}^{r=15} \pi r^2 dr$ What do you notice about them? What similarities do you see?
- Can you explain to me why these two integrals are equal to each other? How are they the same thing? $\int_{T=25}^{T=100} 4\pi(\sqrt{T}+5)^2 \frac{1}{2\sqrt{T}} dT = \int_{r=10}^{r=15} \pi r^2 dr$
- Can you list the substitutions that we made?
- What were the key pieces of information you needed to make those substitutions?
- What similarities do you see between the substitutions in the solar panel problem and this sphere problem?
- In general, how are you thinking about the process of going from a complex to a simple integral?

This part of the problem is meant to help the students start to generalize the key components of the substitution. As part of the conversation, I emphasize to the students that the key substitution relationship was nested inside another function for both problems.

In the event of additional time remaining in the interview after the first two tasks I prepared a third pure math substitution task for students to explore.

**Pure Math Substitution Task**

**Figure 7**

*Pure Math Substitution Context*

| |
|---|
| Do a substitution to write the below integral in a simpler form. $$\int_2^4 5x^2\sqrt{1+x^3}dx$$ |

Key Questions:

- In our previous problems, we identified an "inside" piece that described the relationship between two different quantities. What is an inside piece here that we can use to do a substitution?
- What are the key components that need to be switched from one variable to another?
  - What is the differential in terms of u
  - What are the bounds in terms of u
- What does this substituted integral mean?
- Having done this problem is there anything you would like to add to your previous summary/comparison between the sphere and solar questions?

I ask the students to describe the meaning of the substituted integral to see if they are still attending to quantities even without a context. These questions are meant to solidify the conception of u-substitution developed in the first two problems and support generalizing that the substitution structure

## Data Analysis

The data analyzed for this study are the videotapes of students' gestures and work, as well as transcriptions of the audio recordings, and artifacts of student work. In this section, I describe how I coded and analyzed data—video, audio transcription, and student work—from the first interview to form an idea of students' understanding of quantity and the discrete components of the integral. I then describe the analysis and codes that I used on data from the second interview to answer my research questions. The initial unit of analysis for both interviews was student speaking turns, which were further broken down according to content as described in the next subsection.

### Lesson One

For the analysis of the first interview, I began by identifying and coding all instances of student speaking turns that had some connection to an image of differentials, derivatives, or integrals. The codes in this case were simply to label the speaking turn as "differential," "derivative," and/or "integral." Along with the code, I made a note of the contents of the image that the specific speaking turn showed evidence of. If students were working with multiple intertwined concepts in one speaking turn (e.g. discussing the target quantity of the integral with scaling covariation), those units of data were coded for both ideas (See Appendix B). Next, I coded instances of student speaking turns that contained some aspect of quantities, covariation of quantities, quantitative relationships, quantitative operations, and units. For example, if a student described the derivative as “a ratio between a small amount of two different quantities producing a third rate quantity,” I coded it as a quantitative relationship. As another example, if a student explained that as one quantity changed (e.g., time) another quantity also changed (e.g., power), I coded that as a covariation of quantities.

In the process of coding for quantitative relationships, I found multiple speaking turns that had the feel of quantitative relationships, but were between two quantities rather than Thompson’s (1990; 2011) three-quantity quantitative relationships. Because of that, I added a new code for two-quantity quantitative relationships. In the process of creating this new code I found that there were a few distinct types of two-quantity relationships, a basic relationship, an equivalence, and function or input/output relationship. As with the image codes, I made note of the contents of the two quantity relationship the speaking turn showed evidence of along with the code.

Following the initial coding, I went back to identify which portion of the speaking turn was specifically relevant to each code. For instance, the following speaking turn was initially coded under two-quantity relationships, image of differential, and image of derivative.

> Bently: So, because your function is your radius equaling the square of your temperature plus five, right? When you take the derivative, which is showing the rate between the two, right, $r$ becomes $dr$ and then $dT$ doing derivative rules that you know. But it's multiplied by the change, the infinitesimal smallness of t. And the way you write the rate if you show that $d$ the change of $r$ over the change of $t$. So that's technically what happens and you just move it over.

To refine this code for the image of the derivative I identified this portion of the speaking turn which was categorized as a rate image of the derivative: “When you take the derivative, which is showing the rate between the two, right, $r$ becomes $dr$ and then $dT$ doing derivative rules that you know."

These codes were used to formulate an idea of how students were thinking about the three foundational calculus concepts and what quantitative reasoning they engaged with. Following the second pass of coding, I looked across the codes for each concept to determine which conceptualizations were the most prominent among the students.

**Lesson Two**

I now describe the codes and analysis I used for the second interview data to answer each of my research questions.

To answer my first research question, there were two acts of analysis I completed. First, since the research question dealt with how the students used quantitative reasoning I coded instances of student speaking turns that contained some aspect of quantities, covariation of quantities, quantitative relationships, quantitative operations, and units in the same way as the first interview described above. For example, if a student talks about $S * dr$ as taking a surface area times a small bit of radius to create a small bit of volume I coded it as a quantitative operation. As part of classifying these turns as either quantitative or non-quantitative, I looked for both indications and contraindications of each (Moore, 2019). For instance, an indication of quantitative reasoning would be if a student is discussing the derivative $\frac{d\theta}{dt} = \frac{\pi}{12}$ and describes this as a ratio of how big the angle is to time and that the ratio will always be $\frac{\pi}{12}$. Conversely, a contraindication of quantitative reasoning would be if the students describe going from $d\theta = \frac{\pi}{12}dt$ to $\frac{d\theta}{dt} = \frac{\pi}{12}$ as merely division or symbolic manipulation of the equation without reference to the quantities.

The second analysis activity for the first research question involved identifying the connections students made between the quantities and the symbolic integral expression. I made note of all instances of students working with or discussing symbolic representations of the integral simultaneously denoting each instance as students either being attentive or not attentive to quantities using the same analysis methods mentioned earlier.

Lastly, to answer my second research question and identify the understandings of u-substitution that the students develop, I conducted an open coding of the second interview data. Based on my conceptual analysis and the process of creating the questions in the lesson plans, I anticipated that the codes would fall roughly into the following categories: the relationship between variables in the substitution ($\theta = \frac{\pi}{12}t$ ), the relationship applied to differential substitution ($d\theta = \frac{\pi}{12}dt$), the relationships in the bounds substitution ($t = 6 \rightarrow \theta = \frac{\pi}{2}$), and overarching ideas about what u-substitution is (any statements about the process of changing integral from one quantity to another). Because students made some general comments about substitution I added a fourth "general substitution" code to capture those ideas. For example, students recapping the process saying, "I have like my normal integral, over here, where we have $4\pi r^2 dr$ and taking all the points where our input was, *r* in this case, those are each of the points that have to change."

After the interview was coded, I went back and identified the specific portion of the "turn" that was relevant to each code. To further refine the substitution codes, I conducted a secondary analysis for each of the three parts substitution and looked for similar types of thinking or student moves then grouped common ideas together to see what ideas emerged and were most common (See Appendix B). Due to the greater variety and complexity of ideas within the differential substitution codes I also compared the students' thinking within each pair from one task to the next for the differential substitution. This comparison allowed me to see the ideas each pair of students developed at each stage of the interview, and what reasoning students maintained or did not maintain from the first task to the second. Analyzing the coded data this way provided a larger picture of how the participants' quantitative understanding of u-substitution developed.

## CHAPTER FIVE: RESULTS

In this chapter I summarize the reasoning displayed by the students in the interviews. I start with describing the general results across both interviews in terms of students' usage of quantities and quantitative reasoning. Following these general results, I then focus on interview #1 to describe the students' concept images for the derivative, differential and integral. I then move to focus on interview #2 by describing the images that students developed of u-substitution--specifically substitution of bounds, substitution of the integrand, and substitution of the differential.

### Use of Quantity

I start my results section by speaking generally across both interviews in terms of how the students used and attended to quantity in the tasks from both interviews. I first talk about the students' use of symbols as representing quantities, then about the quantitative relationships that emerged between two quantities and finally students' quantitative relationships and quantitative operations as defined by Thompson.

#### Symbol as Quantity

Redish (2005) talks about loading meaning onto mathematical symbols and how those meanings can help students reason about mathematics problems (see also Dray & Manogue, 2005). Throughout the interviews students repeatedly referred to symbolic expressions as a quantity demonstrating the ways they were loading meaning onto the symbols.

Ellie: This right here, the π/12, it's giving us the like, it gives us the radians at that time.

"that equation [$\frac{1}{2\sqrt{T}}dT$] is essentially equal to bit of radius"

Bently: So this is your function [points to $sin(\frac{\pi}{12}t)$], this is showing how much jewels you're getting per second right. And this [points to $dt$] is showing small time in seconds.

So it's [gestures to the whole integral expression] just getting like all of the power that you're getting, all of the jewels that you're getting.

Viewing symbolic expressions as quantities helped the students make sense of and keep track of the quantities as they completed the substitutions.

Jackson: Basically, cuz this is the angle right here, right? Like that's a sin of, or the sin of theta is our power output and this [circles $\frac{\pi}{12}t$ in the sin function] would be what theta would be equal to

**Two Quantity Relationship**

As mentioned earlier, in Thompson's (1990) definitions surrounding quantity he describes a quantitative relationship as existing between three quantities, any two of which could be used to find the third under quantitative operation. This can be thought of as having a triangular structure with each vertex being a quantity connected by quantitative operations. Carlson et. al. (2002) describe mental actions of covariational reasoning, the first mental action being "coordinating the value of one variable with changes in the other." In my interviews I found that students frequently reasoned with and about relationships between two quantities in ways that did not quite fit in either Thompson's definition of quantitative relationships or Carlson et. al.'s covariational ideas. The student reasoning had the feel of Thompson's quantitative relationship but between two quantities rather than three, while having some covariational aspects. That is, while they certainly used some amount of covariational reasoning mental actions, because they were describing a relationship between two variables the reasoning was more about the quantitative relationship rather than the covarying relationship. Because of that I created a code for their two-quantity relationships and categorized each instance in one of three ways: basic relating of two quantities, equivalence, and a function or input/output relationship. I will unpack each of these in turn. I note for all six of the students evidence of these three types of two-quantity relationship appeared in their thinking.

***Basic Relationship***

In the instances where students talked about relationships between quantities in fairly non-specific ways, I coded as a basic relationship. Within this category students exhibited different levels of specificity about the relationship as shown by the following student excerpts.

> Andres: Well, it's just a much more direct relationship between radius and volume whereas the relationship between temperature and volume isn't as direct, and so that's why it's a much more complex relationship.

Here Andres is acknowledging there exists some relationship between each pair quantities but is not describing any type of covariation or specifics of the nature of that relationship. As mentioned earlier and as seen in the following quotes, sometimes the students would incorporate some covariational reasoning while discussing how the two quantities change in relation to each other.

> Bently: your $dr$ is how much it's growing which means if your $dr$… if your radius changes, your volume's gonna change. But this is showing that as your radius changes very slightly, shows how your volume changes very slightly.

> Liam: We also want, you know, the relationship between $dt$ and $d\theta$

Again students would talk about the changes in one quantity affecting the other, but not initially describe any specifics of what those changes were. Bently later went on to say that the change in volume was “dependent on your radius times the surface area” engaging in a different type of two quantity relationship that I will describe later.

### *Equivalence*

The two quantity equivalence codes were used when students directly stated either two symbolic expressions, quantities or some combination of the two were the same. I note that these instances appeared most frequently when students were describing the change of quantity relationship, or when comparing the target quantities of the original and substituted integral.

I will first talk about some different ways students equated the quantities themselves. All three pairs at some point equated *values* of quantities and said things like “one hour is equal to π over 12.” Some students simply equated the quantities themselves without reference to the specific relationship values. For example, in the below excerpt Andres equated the quantities without attention to the symbology or measure of the equivalence.

> Andres: Yeah, it's just, I mean, it's the same because they're both measuring, well [pause] they don't measure the same thing, but I would say they both measure the same thing without measuring the same thing. I think they're both, they both represent the same thing, which is the position of the sun.

Similarly, equating quantities also appeared when comparing an integrand to its substituted version as seen during this sphere task substitution.

> Bently: ‘Cause this function is the same thing as the other one. Like we're both changing at the same time, it's just you’re calling this one in terms of temperature instead of radius even though they both happen at the same time.

In both of these instances the students are referring to the quantities of the position of the sun and the size of the balloon as being represented by two different measures. In these instances we can see the students are also attending to some type of covariation, recognizing that if the quantities are equivalent they also must change together. I also note in these instances students seem to be thinking about the covariation as being dynamic: changing time and angle as the sun sweeps across the sky, and radius and temperature changing at the same time.

Interestingly, while talking about changing the quantity from time to radians, Ellie did not initially see the relevance of changing the variable because she had already mentally equated the symbols $\frac{\pi}{12}t$ with being the quantity of radians not considering the .

> Ellie: I feel like that would make it a lot more difficult. I don't know why we would switch it into radians. Um, because this right here, the π/12 is giving us the like, it gives us the radians at that time.

It was not till after she and her partner had completed the full u-substitution that she was able to see the utility of changing the variable. This highlights that understanding how students load meaning onto the symbols is a non-trivial aspect to consider when using quantity-based tasks (Dray & Manogue, 2005; Redish, 2005).

The other instances of equivalence I coded were when students used some symbolic expression as part of their equivalence statement. Similar to Ellie's thinking the students were interpreting the symbols as representing the quantities, although unlike Ellie still seemed to keep them distinct.

> Bently: Your temperature is being changed with respect to, all of this stuff [$\frac{1}{2\sqrt{T}}dT$], which like, as he said, it's kind of the same thing as radius.
>
> Liam: I was just thinking just like the same things as before with the difference in volume is equal to $\pi r^2$ and then the difference in length [indicating $dV = \pi r^2 * dl$].

***Function***

For the student thinking that I coded as a functional relationship, the students were describing some type of input/output or operational thinking when talking about how two quantities were related to each other. I will illustrate this type of relationship in the following student excerpts. First, consider Nate who explained that the original integral and substituted integral for the sphere problem were the same by noting that putting in temperature will result in the desired radius for the integrals to be equivalent.

> Nate: We found the relationship between r and T, so uh, we know that like for whatever T we put in there, it's gonna come up to like the right r to get the same result as this one [points to the original integral]

In a second example, Liam is describing the quantities being used in the solar panel task and says, "When we plug in time, it gives us the angle." In this excerpt Liam seems to also be indicating that there is some underlying structure or some operation being done, but the operation is done on a single quantity once it has been "plugged in" rather than an operation between two quantities. In this example there also is some covariational reasoning in Liam recognizing that there is an angle corresponding to a given time.

> Bently: I think you could lowkey just write $d\theta$ or changes in $\theta$ as a function of $t$. Right? You already know that your theta is gonna be π whenever $t$ is 12. Right? So you can write that as what, at one hour you have π over 12, so $f(t) = \frac{\pi}{12} * t$

> Jackson: Instead of our tiny, tiny changes in, um, radius we're substituting that with, you know, what our tiny change in temperature times-ing one over the square root of that temperature.

I note that the function two-quantity relationship is related to quantitative operation and can have some overlap with covariational reasoning. However, it is somewhat in between the two in that some operation is typically done on a single quantity to produce the second, and the students seem to be thinking of it more as the result of "plugging in" a quantity than of the two quantities varying together.

## Usage of 3-Quantity Operations and Relationships

Thompson (2011) defines quantitative operation and relationships as typically involving three quantities. As mentioned earlier quantitative operation is the mental action taken on two or more quantities to produce a third quantity and quantitative relationship describes the connections between quantities that exist because of the operation. Although much of the students' quantitative reasoning happened with the two quantity relationships, they did still reason about and use quantitative operations and relationships in the way Thompson defined them. I also note that students frequently utilized the units to make sense of the integrals and will point out those instances in conjunction with their reasoning about the quantitative operation and relationships.

### *Operation*

Students' use of quantitative operation was primarily in reasoning about the target quantity in conjunction with units to check that the basic model was correct. At one point in the interviews all three pairs used the idea of units canceling to think about or justify their reasoning for the target quantity of the integral as in the following two excerpts.

> Bently: So the way I see is like the velocity in terms of time, right, that's giving you a rate, how much something is changing in terms of time. $dt$ shows a very small portion of time… so if you multiply those together you're just going to get a distance, 'cause the times will cancel out.
>
> Andres: 'Cause $v(t)$ is meters per second and it'd be times $t$ … so it would just be $t$ in seconds… over seconds. So just, you are just getting meters.

It also seems that the quantitative operations and units may have supported students' use of scaling covariational reasoning to move from a basic to a local model. Because the units worked out for the basic model that same line of reasoning was extended to the local model.

> Nate: Well it's [meaning a small slice of volume of a cylinder] the area times like the length. So it's like the area of a circle at 'a' (meaning point a) multiplied by a really small length to give you a really small volume.
>
> Jackson: This dark circle is your sphere, and your $dr$ is how much it's growing, which means if your $dr$…if your radius changes, your volume's gonna change. But this is showing that as your radius changes very slightly shows how your volume changes very slightly, which is dependent on your radius times the surface area

Jackson has a solid grasp on the basic model of getting volume with radius and surface area$r * S = V$. In talking about the local model zoomed in to small changes in radius and volume he referred back to the operation and local model that radius and surface area produce volume.

***Relationship***

Reasoning about the quantitative relationships and operations along with the units that form the target quantity helped students to reason about why the original and the substituted integral both add up to the desired quantity.

> Liam: I think you get at the same thing, but this would be different units. As you do this [pointing to $\int_0^6 sin(\frac{\pi}{12}t)dt$] because this will give joules, and then this $dt$ kind of helps with like the joules per second. This [pointing to $sin(\theta)d\theta$] would be like joules per degree, and then degrees. So you'd get joules out of both of them and you get the same thing, but this $P$ [meaning the power function $P$] significantly would be in different units.

Here he unpacks the two different quantitative relationships and the units for the target quantities of the two equivalent integrals. For the first, he is thinking about the function quantity as a rate in units of joules per second multiplied by the time $dt$ to get the desired energy in joules and for the second that the rate function is in joules per degree multiplied by degree $d\theta$. I note that although he uses the word degree here he previously had correctly referred to the unit of the angle as radians.

***Units of Energy***

There is a peculiarity within the solar panel context with regard to the unit of the target quantity which is watt hours. In practice watt hours is a fairly common unit; however, the unit of watts is joules per second meaning the unit of watt hours is also $\frac{Joule}{second} hours$, thus we have two distinct time measures happening simultaneously in the single target quantity. For some of the students this was not an issue and as seen in Liam's reasoning in the previous section where he took $dt$ to be in units of seconds. For the pair who had not taken calculus previously it mattered that the time measures present were different. However, they did not have a good way to grapple with the discrepancy they found. In the process of making sense of the basic model Ellie hit on this complexity.

Ellie: Um, I guess it's still just not making a whole lot of sense to me. Because this function that we have here… which we know is watts, which we know is joules per second, so we know that this in here [indicating the function $250sin(\frac{\pi}{12}t)$]is going to be joules per second. And then we're multiplying it by dt, which we're saying is hours.

She was wrestling to think about how the function could input hours and output joules per second as well as how the units of joules per second multiplied by a small amount of hours could result in energy. The following excerpts demonstrate how her partner Andres responded to these ideas.

> Andres: I think because somewhere else in this function, it compensates for that probably with this constant in the front. That with that constant front, it'll compensate for the fact that it's [the input] in hours and then with that, it'll give us joules per second.

Andres initially reasoned that the constant in the function would have some units that would convert the input unit of hours to the desired power units of joules per second. This is similar to what Redish (2005) says about constants in physics rarely being just numbers, but that they indicate a connection with something physical. Andres was reasoning that the constant's connection to the physical would take care of any unit discrepancy. Another way Andres tried to grapple with this was by reasoning with a combination of quantities and units.

> Andres: So how you can see it too, is it's just joules per a certain amount of time, times time.

We see he is combining units (joules) and quantities (time) and canceling the quantity of time to perform a "unit check" to make sense of the multiplication resulting in the target quantity of energy in joules.

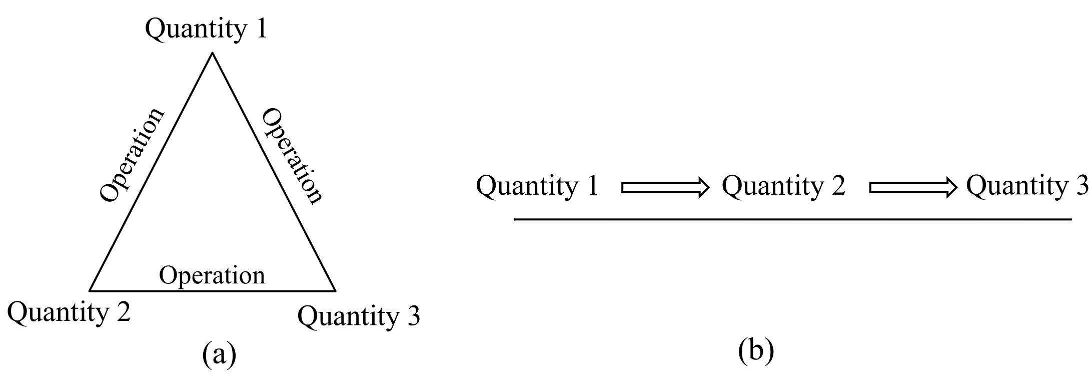
This exchange draws attention to an important aspect of using quantitative based tasks in mathematics teaching. As seen from the students' work, while this may seem like a simple unit difference, the context imported a complexity that was nontrivial for students to make sense of. As part of creating and using quantity-based tasks it can be easy to inadvertently import some scientific conventions that are non-trivial and not realize it until after the fact. It is important to be aware of the potential complexities of the context and notice when those complexities interfere with the mathematics. I did not anticipate this complexity in creating the task and although there is nothing incorrect about having the two distinct time measures, I would define the power function to be in joules per hour for simplicity in future usage of the task.

**A 3-Quantity Structure Different from Thompson's "Triangle"**

I end this section by noting that sometimes the three-quantity relationship did not have the same structure as defined by Thompson. As noted earlier Thompson's three quantity relationship has a triangular structure whereas students in the interview talked about the relationships between three quantities more as a linear structure, or nested function composition structure, like Jones' nested multivariation (Jones, 2022) (Figure 8).

**Figure 8**

*Comparison of 3-Quantity Structures (a) Thompson's Triangular Structure and (b) Jones' Nested Multivariation Structure)*




Andres: So you go from relationship with temperature to volume, to temperature, to radius. And so it's just substituting to find a relationship of one thing in relation to another. Just finding the relationships between two different, um, things or rates of change and relating them to another relationship of two rates of change.

Interviewer: What are the little pieces that I'm adding up?

Nate: The energy at that time at that angle.

Nate went on to elaborate that he was thinking the sun at a certain time gives an angle and the little pieces are some amount of energy for that given angle value. In both these instances students are recognizing a string of relationships $temperature \rightarrow radius \rightarrow volume$ and $time \rightarrow angle \rightarrow energy$. This quantitative nested mulivariational structure similarly appears in the chain rule. It makes sense that this structure would appear here since u-substitution "undoes" the chain rule. I will speak about this more in the discussion section.

## Interview 1: Images of Differential, Derivative, and Integral

In this section I now move to describe the images for the differential, derivative, and integral that students had or developed in the first interview and which of these did or did not carry on to the second interview.

### Derivative

The image for derivative that was the most prominent was the derivative as a rate of change with all six students calling on this image multiple times. When asked what various derivative expressions meant (i.e., $\frac{dV}{dh} = 4$, $\frac{d\theta}{dt} = \frac{\pi}{12}$, $\frac{dr}{dT} = \frac{1}{2\sqrt{T}}$) students responded primarily with reasoning based on rate.

> Ellie: Um, it's going to mean…Four is the rate at which the volume is changing in relationship to the height
>
> Liam: The instantaneous rate of change of the angle relative to the change in time
>
> Nate: The way I interpret it is like the change of the radius with respect to the temperature is this derived function where the temperature can be anything. And you get the rate from that temperature.

Since we were building these ideas from differentials, and as is shown in the following section on differentials students' image of differentials was very much small change or small amount, they did begin to use this differentials-based idea of derivatives as a ratio of small amounts or changes as we went.

Interviewer: So then this $dr/dT$ what does that mean?

Andres: Um, the rate at which radius gets, uh -- the way which radius gets infinitely smaller as the temperature also does.

Andres was seeing the derivative as consisting of two infinitely small quantities that form a rate. One student specifically used the differential relationship to describe what derivative meant.

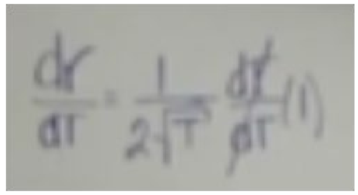
Bently: When you take the derivative, which is showing the rate between the two [meaning between the two quantities radius and temperature] *r* becomes *dr*, and then $dT$, doing derivative rules that you know becomes that [points to $\frac{1}{2\sqrt{T}}$], but it's multiplied by the change-- the infinitesimal smallness of t [$dr = \frac{1}{2\sqrt{T}}dT$]. And the way you write the rate is you show that d the change r over change of T. So that's technically what happens and you just move it over …I typically write this [$\frac{dr}{dT}$] first cause I know that's what it's going to be, but this is to me technically that is what it is.

This comment prompted further conversation where Bently's partner Jackson about the notation for the derivative with respect to the differentials.

Jackson: For me, I sometimes when I do this, um, like I'll add this in here to show that, um, you know, I'm still, it's looking like that on top, but I'm still taking all my variables in respect to $dT$ even, or as well as my $dT$ over $dT$. So, this just goes to one.

**Figure 9**

*Jackson's Symbolic Work for Describing the Derivative*



In the student population I recruited from, I wanted students with some quantitative experience from their class background, which meant that I did not see a lot of slope of tangent images and in the two places where the slope of tangent line conception appeared it was never in isolation, but was connected to a rate description. I also wish to note here that the lack of slope images could be due to the nature of the tasks.

> Ellie: Um, it's finding, well on like a graph it's finding the slope or the rate of change of a function.
>
> Bently: the rate at which it's changing is your slope right here. And *dT* is just kind of the *x* of your tan line. It's just *x* is moving so small, which means *y* also is moving so small, but it is increasing at that rate [pointing to $\frac{1}{2\sqrt{T}}$].

Although his explanation does not use the quantities of the problem specifically, Bently seems to be using reasoning similar to Weber et al.'s (2012) calculus triangle where he is visualizing the slope as being created by horizontal changes in the quantity of temperature and its corresponding change in radius.

**Differential**

The following concept images that appeared in students' work and reasoning relating to differentials: changes, small amounts, almost zero, collapsed to nothing, and incorrectly stated as derivative. I wish to note that I use the differential here and in the interviews as an informal infinitesimal. This is frequently done in practice and researchers have argued that the informal approach is preferable and more conceptually useful for first-semester calculus students (Milner and Rodriguez, 2020).

For most of the students, they were unfamiliar with the term “differential,” although they were quite familiar with the notation $d[\,]$ and the differential being small. Students most frequently talked about the differential as a small change or a “difference.” This is likely impacted by their background with thinking about the derivative as a rate of change.

> Liam: I'm thinking because you're saying it's like a very, very small slice so that's like an infinitely small thing which is like $dl$ which is an infinitely small change and that leads to the difference in volume.

> Bently: so $dr$ is the infinitesimal smallness change of the radius

One student repeatedly referred to the differential in both interviews as a derivative, but still maintained the concept of it being a small change.

I note that throughout the interviews the concept image that I as the interviewer drew on was that of differential as a tiny amount; however, in the process of analyzing my data I found that the language that I used throughout the interviews to talk about the differential was “change.” It appears that the students' images for the differential may have followed the language I used, and the “change” concept image of the differential appeared the most frequently throughout the interviews.

In conjunction with talking about the differential as a change, students also frequently referred to the differential as being close to zero.

> Andres: I think it becomes small. It would mean it's just approaching zero to be practically zero and just, yeah as close as possible to a change of zero.

Jackson: It might be a limit idea, right? Uh, for taking it approaching a very, very, very small amount getting close to zero.

Bently: It approaches zero, but it's always four times as much as the change of height.

In the first interview none of the students talked about the differential collapsing to nothing, rather just becoming small. Interestingly, for the most part this did not hold true in the second interview when students were doing the differential substitution. This indicates that although thinking about the differential as an amount is part of their mental image, it may not be the most prominent and the tasks could have directed them away from a collapse metaphor.

**Integral**

The images for the integral that the students had were grouped according to Jones' (2015a) concept images: area under a curve, antiderivative, and adding up pieces (AUP). I included any instance of students talking about partition, target quantity or sum with AUP.

In the first interview students were asked to talk about what the basic integral form $\int_a^b f(x)dx$ meant. From this initial question there were three students who used the idea of area under a curve to describe what was happening in the integral. However, their explanations were not purely as area under a curve as is shown in the following student excerpts.

Liam: you're basically finding the total change... Kind of like an area, but it's not always the area, but like an area underneath the curve.

Nate: You're taking a function and finding like a spot and b spot and you're adding up all the really small rectangles underneath it to get like, that area… this is the base of the rectangle, and the function is the output, which would give you --- in cartesian at least -- would give you a height. And so, you're saying your base is like infinitely small, and then your height is whatever the function outputs.

Liam acknowledged that while area under a curve was one representation of the meaning of the integral that not every integral had to have that meaning. Within his area ideas Nate was more focused on the Riemann sum of the little areas. He also seemed to be thinking about the are rectangles as having finite withe then zooming in to an infinitesimally small base for each. After this initial question both the tasks and the questions I asked were designed to lead students towards using the AUP conception. This was unproblematic and students were able to easily talk about the integrals as adding up little pieces of quantity.

Andres: It's adding up each infinitesimally small little slice, the volume, small little slice of volume in that whole shape

Nate: So it's like the area of circle at 'a' multiplied by a really small length to give you really small volume, and you get all the small points between ‘a’ and ‘b’.

Within the adding up pieces thinking, the partition and the sum seemed straightforward for the student and the larger part of their reasoning was with the target quantity, spending a lot of time looking at local models to conceptualize the target quantity.

Andres: Yeah, I think it would be $\pi r^2$ times that distance l. [pause] It would be the volume within this range. [pause] I think it would just $\pi r^2 dl$…If the volume is infinitesimally getting smaller and that means something about the volume also has to be getting infinitesimally smaller and π can't get smaller, it's a constant and so the only other thing I could get smaller is the length.

Andres started out with the local model to conceptualize volume of the cylinder and then was able to think about the length becoming smaller and smaller to the infinitesimal $dl$ size. Ellie similarly thought about zooming in to smaller and smaller lengths giving smaller volumes

Ellie: Then the $\pi r^2 dl$ would be equal to the change in volume, because as the length of the missing piece of cylinder (talking about the slice of volume) decreases, the volume will as well.

Moving from the basic cylinder to the second shape, the students were also able to effectively reason about how infinitesimal changes of radius as well as length would impact the integral as they thought about each target quantity slice.

Ellie: So $r(l)$ is going to be the radius... When I know what point of $l$ I'm looking for then that's going to tell me the point that my radius is at…And so if I plug in my different lengths, then it's going to give me out what the radius is and it's therefore going to give me, um, the volume.

Liam: But if we know those two points, $dl$ would be relative to like $dr$. Because as you increase at any point (indicating moving along the length of the shape), the $dr$ is like, kind of relative to that.

The students' conceptual reasoning about the integral at the end of the first interview was largely where I wanted it to be with AUP as the most prominent conception. Because there are many ways that students can conceptualize the integral it was a possibility that students would revert to an anti-derivative of area under a curve conception in the second interview; however, I found that students maintained AUP as their primary conception in the second interview as they reasoned about the integrals both before and after the substitution.

Andres: Basically, what the integral is summing up is all of the instantaneous amount of power it collects throughout those six hours… So the $dt$ is just infinitely small amounts of time to give you, um, all of the wattage at every moment in time in those six hours to add 'em all together.

Jackson: The output of that multiplication would be your volume at that temperature. So we're adding up different volumes all the way up from, or from our beginning temperature to our ending temperature

The antiderivative conception only appeared a couple of times in one student's reasoning in the second interview, which could be due to the fact that the focus of the tasks was on the set up of the integrals rather than their evaluation. Notably, even in the instances where the student did mention antiderivatives, they still maintained the connection between the quantities and the antiderivative.

Andres: It (the power function) gives you energy per second so the amount of energy in that total time frame would be the antiderivative of that [points to the integral]. So when you solve for this you get the amount of energy in that specific range you accumulated.

Two of the students had difficulty with the interpreting integral $\int_a^b dV$, wanting there to be some type of multiplication visible in the integral.

Andres: Integral is a multiplication of… its two things being multiplied and it's the summation of something and there's no multiplication there. So I'm just trying to figure out what's being multiplied, what's being added.

Liam: I'm just a little confused 'cause you just wrote $dV$ and there's no nothing else, like a $dl$ or something like that, that would, that would represent the multiplying.

Notice that these students are focused on the multiplication which is in line with the multiplicatively based summation (MBS) thinking described by Jones (2015a), whereas in Jones and Ely (in press) we see that AUP is more general and does not always have to have a product.

It is true that many integrals do have a multiplicative element and because it is a common structure there can be a tendency for students to overgeneralize and assume that is the only integral structure. However, not all integrals have that structure so it may be important for students to be able to recognize that though there often is some multiplication there does not always have to be.

## Interview 2: Images of U-Substitution

I now move to describe the results pertinent to the second interview, focused on u-substitution. To orient the reader for the results of each of the three pieces of substitution (bounds, function, and differential), I first provide an overview of how the students progressed through each substitution in the two tasks.

### Overall Flow of Student Work

In both the solar panel task and the sphere task, the students were asked if they could change the integral to be in terms of a new quantity and the order that the students completed the substitutions of bounds, function, and differential varied from pair to pair and in some cases from one task to the next. After completing each of the tasks, students were asked to recap again what substitutions they made and I additionally made note of the order they described these substitutions. The below tables (Tables 2 and 3) outline the progression of each pair of students through the two substitution tasks.

I note that in the solar panel task two of the pairs started with the function substitution, and in the sphere task all three pairs began there. In the case of the solar panel task the two groups who began with the function substitution had a written-out change of quantity expression $\theta = \frac{\pi}{12}t$ while the group who started with bounds substitution did not. Similarly, the relationship $r = \sqrt{T} + 5$ was given to all the students and written out on the board. It seems that having the change of variable relationship made the function substitution easiest and most familiar, so students began there. For the Nate and Liam who did not start with the function substitution on the first task I believe they started with the bounds because they were using the bounds as part of their reasoning about the relationship between $t$ and $\theta$.

**Table 2**

*Solar Panel Task Component of Substitution Flow*

| Pair | Solar Panel Task Flow | Order of recap |
|---|---|---|
| J & B | Function → differential → bounds | function → bounds → differential |
| N & L | bounds → function → differential | function → differential → bounds |
| E | function → bounds → differential | bounds→ function → differential |
| A | Function → differential → bounds | |

**Table 3**

*Sphere Task Component of Substitution Flow*

| Pair | Sphere Task Flow | Order of recap |
|---|---|---|
| J & B | B Function → J differential → bounds | function → bounds → differential |
| N & L | Function → bounds → differential | bounds → function → differential |
| E & A | A function → E Differential → both bounds | bounds→ function → differential |

**Bounds Substitution**

None of the students had trouble identifying the need to change the bounds and five of the six students were able to successfully draw on the change of quantity relationship in both tasks to perform that substitution. For the solar panel task, some students used the visual of the sun moving across the sky to reason about the bounds changing as shown in the following excerpts.

> Jackson: I mean, we're taking angle measurements here. We're starting at zero, uh ending up with a straight vertical line, which is π halves. So that's how I'd explain this, uh, bounds right there from zero to π halves
>
> Liam: Well ... You would need different bounds 'cause these bounds are in terms of t. You have this value t. So you'd have to say when the sun is at -- or like at like zero degrees relative or to 90 degrees so you'd have to change that.

Conversely some students reasoned more with the symbolic relationship between time and angle. They recognized that plugging in a time to the function gave an angle, and so to check what the angles related to $t = 0$ and $t = 6$.

> Bently: So what's happening here is you're lowkey, trying to find your theta, which is going to be your time starting at zero, right? Which gives you sin of zero. And you put in six, it's going to be sin of $\frac{\pi}{2}$, which is what you want your theta to be $\frac{\pi}{2}$. Right. You'll calculate all of the angles from that time period.

In the sphere task five of the six of the students again had no trouble thinking about the bounds substitution and tracking the quantities through their computations.

> Liam: We would need the equivalent temperatures as the bounds because we can't do relative to $r$ 'cause we're not going to have $r$ in this equation. So we'd have to do it from when $r = 10$. So you'd have to solve, so $10 = \sqrt{T} + 5$ which would be $5 = \sqrt{T}$ would be $25 = T$. So you go from T = 25. And then you're just gonna get $10 = \sqrt{T}$ if you go through that again with 15 [meaning r = 15] you're just gonna subtract the five over. So then you're gonna get T = 100. So this is gonna be $T = 100$

Bently: And then all you gotta do for the bound is, you know that $r$ equals that [points to $\sqrt{T}+5$]. So then you can plug in what your bounds are. So 10 equals all that stuff [ solves for $T$] So that means $r=10$ means $T=25$ and when $r=15, T=100$. And that would be your bounds instead, 25 to 100.

One of the students however, did initially have a difficulty with reasoning through what the change of bounds would be for the sphere task. While he recognized that they needed to be changed, stated that he didn't know what to do with the bounds. His partner was able to help him reason through how to find what the change of bounds should be.

Ellie: So what are our bounds? They need to now be in terms of temperature so we already have this equation [pointing $r=\sqrt{T}+5$. And so, I mean, I think that we can just like, plug in the numbers.

Following her suggestion Andres figured out the new temperature bounds, and was able to explain that the new bounds were the temperature equivalent of the radius bounds.

**Function Substitution**

For all of the students, the substitution of the function seemed to be a fairly easy and almost intuitive substitution. There was less cognitive work that students did surrounding the function substitution and having the change of variable relationship (i.e.. $\theta=\frac{\pi}{12}t$) was an important precursor for them to make the function substitution.

Jackson: Basically, 'cause this *is* the angle right here [points to $\frac{\pi}{12}t$], right? Like that's the sin of theta is our power output and this [again gestures towards $\frac{\pi}{12}t$] would be what theta would be equal to.

Nate: We found the relationship between r and T, so we know that like for whatever T we put in there, it's gonna come up to like the right r to get the same result as this one [$4\pi r^2$]

It is interesting to note that of all six students only one talked about the function substitution as specifically as substituting what was "inside the function" which is one of the ways u-substitution is typically introduced.

As mentioned earlier in the flow of the three pieces of u-substitution, the pairs who started with the function substitution in the solar panel task had the relationship $\theta = \frac{\pi}{12}t$ explicitly written out on the white board where the group that began with the bounds did not. While it may seem like the students had merely completed a successful symbolic substitution when asked about why they made that substitution the students were able to provide quantitative justification for the substitution or how the two functions were equivalent.

> Bently: It means that your sin of theta is going to follow the same pattern as shown here [Gestures to the function $sin(\frac{\pi}{12}t)$] As it moves along, it's moving along the same way as the time is.
>
> Liam: So it's basically just skipping the step where you multiply this by this [$\frac{\pi}{12} * t$] and just directly plugging in the angles.

The students reasoned about the inputs being equivalent to justify the function equivalence, again highlighting the importance of students having a quantitative understanding of the change of quantity relationship.

After completing the full substitution for the solar panel task, one pair talked about the different units of the substituted function.

> Liam: I think, I think you get at the same thing, but this would be different units. As you do this [pointing to $\int_0^6 sin(\frac{\pi}{12}t)dt$] because this will give joules, and then this $dt$ kind of helps with like the joules per second. This [pointing to $\int_0^{\pi/2} sin(\theta)d\theta$] would be like

joules per degree, and then degree. So you'd get joules out of both of them and you get the same thing, but this $P$ [meaning the function $P$] significantly would be in different units.

**Differential Substitution**

During the solar panel task, when asked if they could write the integral $\int_0^6 250sin(\frac{\pi}{12}t)dt$ in terms of the angle rather than time five of the six students (Jackson, Bently, Nate, Liam, and Andres) directly substituted $d\theta$ for $dt$. To help the students reason through the differential substitution by comparing their integrals and the pieces they had substituted I pointed out to the students that based on what they had written that $d\theta = dt$. Their responses to this observation were varied and yielded some interesting results. I will discuss the process of each pair reasoning about and resolving this issue.

***Jackson and Bently***

As soon as I had pointed out that as their integrals stood it meant that $d\theta = dt$, Jackson and Bently immediately recognized this was incorrect. J wrote the correct differential relationship on the board and gave the following reasoning.

Jackson: So, I mean, we just take the derivative of both sides, right. Um, so I mean, this is the derivative of theta is $d\theta$, the little change of theta, and then we apply, um, we take the derivative of the right-hand side, which is just a constant times a multiple. So we can do the constant out here times the little change in time out here.

Bently: And then you can solve for $dt$ yeah. Plug that in too

Both of these students had taken calculus before and seemed to be drawing on previous calculus experience. Saying “take the derivative of both sides” then incorrectly stating $d\theta$ as the derivative of $\theta$ is indicative of a previously learned procedure. However, when prompted the students were able to use quantities to describe why their calculations made sense.

Bently: For every one portion of time, I'm gonna say, like one hour is going to be $\pi/12$ radians. Which means, like, your small change in theta shows that's your small change in time, except it's going to be multiplied by $\pi/12$. So every like small thing of theta moved your time is moving by that amount multiplied by pi/12

Here I note that although the language he used was $d\theta * \frac{\pi}{12} = dt$ rather than the other way around, this was not his meaning based on his work and gestures as he was speaking. However, this highlights again how difficult articulating that multiplicative relationship is and shows the prevalence of the well known x times as many students as teachers problem (Clement, 1982). In a different setting I might have addressed this issue, however it would have required more focused intervention to straighten out the language and it was clear he had the correct meaning of the relationship if not the correct language.

Jackson made sense of the differential relationship in terms of larger changes rather than infinitesimal ones but demonstrated that he believed this relationship still held on the infinitesimal level and is evidence of scaling covariational reasoning.

Jackson: As the sun rotates it's gonna be rotating every hour at a constant rate of, $\pi/12$. So yeah, just describing that changing rotation with respect to time.

In the next task this pair again used similar language and reasoning to complete the differential substitution but seemed to keep the ideas from the first integral substitution and did not have the same problem with directly equating the differentials.

Jackson: If we take the derivative of both sides, we know that we can solve for $dr$. So our $dr$. would be one over two root the temperature ($\frac{1}{2\sqrt{T}}$).

### *Nate and Liam*

Both Nate and Liam recognized that the differential needed to be substituted for the integral to be in terms of angle rather than time.

> Liam: "because…we don't have $t$ anymore because we've replaced $t$, so we can't use $dt$ which is like a very small change in t so the way you would have to measure it is you'd have to multiply that by a very small change in the angle."

Although they both agreed on why $dt$ needed to be substituted Nate and Liam had two different responses to the question about $d\theta$ and $dt$'s equivalence.

> Liam: Equal as in they're going to be infinitesimally small. They're gonna be, like $d\theta$ goes from zero to π over two. So that's a shorter range, but because it's broken up, infinitesimally you can't compare infinitesimally small pieces… I think they, they would both be the same as you're like evaluating integral but they would change differently.

There seem to be two reasons for Liam's incorrect equating of $d\theta$ and $dt$. The first is he appears to be doing something similar to what Ohertman (2009) describes as the collapse metaphor when thinking about the differential (see also Hu & Rebello, 2013). He sees $d\theta$ and $dt$ as being incomparable since they are both infinitesimally small. The other reason is that he is numerically equating the intervals of integration of the two integrals rather than thinking about the quantities and how they were related saying that 6 is bigger than π/2 so that is how they were potentially different. Nate, however, did not agree with Liam's argument for $d\theta$ and $dt$ being equal.

> Nate: Yeah. I mean, I think they'd be proportional, but I don't know if they'd be exactly the same… Like one hour is equal to π over 12, but like, they don't, in my mind, they

don't like mean the same thing. ‘Cause $\pi$ over 12 is just like a ratio and one hour is like a unit.

He went on to explain that he viewed $\frac{\pi}{12}$ as $\frac{\pi\, rad}{12\, hours}$ and in multiplying by $dt$ the units of hours cancel so the value of radians would be left. Later, in the sphere problem Liam maintained the concept that differentials are not all equivalently infinitesimally small and used Nate’s idea of proportionality in his reasoning.

> Liam: You get the really small things the way I was thinking about it before about it kind of like derivatives, you wouldn't be as simple as just making them really small pieces ‘cause they're proportional, but they're not directly linear, like in a linear fashion proportional.

He seemed to be using scaling covariation maintaining the relationship between the quantities while thinking about the pieces of the quantities becoming smaller and smaller.

***Andres and Ellie***

Andres and Ellie also initially thought of $dt$ and $d\theta$ as being the same. Ellies’s response was similar to Liam that it was okay to directly substitute $d\theta$ in for $dt$ as long as everything else in the integral had switched to be in terms of theta. Andres was less sure about the equivalence of the two differentials and was using quantities to try to make sense of the relationship.

> Andres: Yeah, it's just, I mean, it's the same because they're both measuring, well... [brief pause] they don't measure the same thing, but I would say they both measure the same thing without measuring the same thing. I think they both represent the same thing, which is the position of the sun.

He correctly reasoned that the different quantities had different measures but seemed to justify the equivalence by reasoning that the quantities both measured the position of the sun.

This highlights an important distinction between comparing quantities and the values of the quantities.

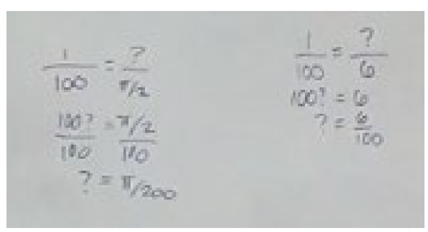
To push them further I asked if $\frac{1}{100}th$ of an hour was equal to $\frac{1}{100}th$ of a radian. This launched a discussion about the relationship between small amounts of each quantity. I note that this conversation lasted for about 12 minutes. It is not reasonable to go through all of the details of that conversation here so for brevity I highlight here the parts of their reasoning that lead them to the correct substitution of the differential.

Ellie interpreted my question to mean is one one hundredth of the interval of 6 hours equal to one one hundredth of the interval of $\frac{\pi}{2}$ radians, and she was able to accurately compute and interpret the value of $\frac{1}{100}th$ of the interval of each quantity.

Ellie: When $dt$ is equal to $\frac{6}{100}$, $d\theta$ is equal to $\frac{\pi}{200}$

**Figure 10**

*Ellie's Board Work for Formulating the Differential Relationship*




However, when she tried to generalize she became confused with the symbols and equated dt with $\frac{1}{100}$ rather than her original reasoning with one one hundredth of the interval of each quantity. Andres was able to build off her original reasoning and ratios to set up an equivalence with differentials $\frac{dt}{6} = \frac{d\theta}{\pi/2}$ and describe its quantitative meaning.

Andres: $dt$ is infinitesimally getting smaller over a period of six hours, whereas that's where $d\theta$ is getting infinitely smaller over the range of π/2 So that's how they're equal to each other.

From there the students were able to use that expression to find the differential relationship $d\theta = \frac{\pi}{12}dt$ and make the substitution. In line with Andres's earlier statement about both time and angle measuring the position of the sun they were most successful when their reasoning was based on an equivalence of quantities rather than reasoning based on the relationship of the values of the quantities.

An interesting note about the solar panel task, students reasoned with the units of $dt$ for the solar panel task and treated the differential $dt$ in a couple of different ways. As mentioned previously one of the pairs wrestled with the units of dt being hours and the power function units being joules per second. However, two of the pairs thought about the $dt$ as being a small amount of time in seconds even after having talked about the relationship between time and angle in terms of hours.

Bently: So this is your function, this is showing how much joules you're getting per second right. And this [pointing to the differential] is showing small time in seconds. So it's just getting like… all of the joules that you're getting

This is likely because the units of the power function are watts which can also be expressed as joules per second and students recognize joules as a unit of their target quantity of energy. Since they talked about dt being a small amount of time they had no problem with calling it seconds to make the units work for their desired target quantity.

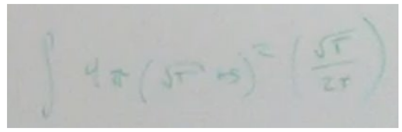
### *Sphere Task Differential Substitution*

In the second task all three pairs eventually used the idea of taking the derivative of the change of quantity relationship in their process of making the differential substitution. Because of the nature of the second task, many of these same problems did not arise; however, there are a couple areas of note. First the problem of equating differentials $d\theta = dt$ that appeared in the first task did not appear in this second. This likely happened for a couple of reasons. Firstly, some of the students referred back to their thinking from the first problem saying that even though both were infinitesimally small there would still be some explicit relationship between them, maintaining the differential as an infinitesimal quantity. And secondly, in the set up of the task the students were asked to take the derivative of the change of variable relationship, where doing so in the first interview could have influenced the students' approach to this part of the substitution. Jackson, Bently, and Liam, the students who had all taken calculus before, again said "take the derivative of both sides" as they were doing their computations and similarly wrote $dr = \frac{1}{2\sqrt{T}}dT$ not actually taking the derivative of both sides indicating that they were drawing on their previous calculus experience. Although the three students who had not taken calculus before drew on differential substitution ideas from the first task, this was still the most difficult part of the substitution to conceptualize.

For example, Ellie tried to follow some of the ideas from the solar task and the idea of taking the derivative to find the differential equation. However, in trying to replicate the process she ended up with $dr = \frac{1}{2\sqrt{T}}$ and made the substitution with that relationship.

**Figure 11**

*Ellie and Alex's Work for Substituting the Differential in the Sphere Task*



$$\int 4\pi(\sqrt{T}+5)^2(\frac{\sqrt{T}}{2T})$$

*Note: Typed below for clarity*

When prompted to consider if they were still getting the desired target quantity of volume Andres noticed that her work was not quite right.

Andres: 'Cause this isn't, this is just for one amount of temperature. It's not as we're getting smaller, it's just for this just temperature.

While somewhat mistaken in what $\frac{\sqrt{T}}{2T}$ means in the context he was able to recognize that something was missing. He seems to have been thinking about the process of going from a basic to local model, and that when zooming in to a local model there needed to be an infinitesimal amount of temperature $T$, demonstrating that scaling covariation was a productive part of his reasoning about differential substitution. The students needed some prompting to resolve this but in the end were able to correct the differential term.

Nate also had difficulty thinking about the differential substitution in the sphere task. Liam was able to help Nate think about it with the following explanation.

Liam: My, my main explanation would be something along the lines of like, let's say we're just like, pretending, like it's a graph and we're using like, I don't know, [draws an arbitrary curve] I'm not an artist, there's a reason I like math, but let's say we're using like five rectangles. If it's five rectangles, the change dr would just be like, would just be

> one. But because we solved for the change in like temperature, if we did it into five rectangles, that would be like 75 divided by five. And this is kind of, this will be equivalent to like this piece, but then this piece [pointing to the $\frac{1}{2\sqrt{T}}dT$ written on the board] would be much bigger than that one.

Zooming out from the infinitesimal level of the relationship to compare finite values of the quantities and their individual partitions helped Nate to make sense of the differential relationship for this problem.

**General Ideas About Substitution**

I, as the interviewer, played the role of instructor presenting the tasks and asking questions to direct the students' attention; I wish to emphasize that all six students by the end of the second interview had developed an understanding of the u-substitution structure (bounds, function, and differential). They were each able to successfully develop this by applying quantitative reasoning in the context of the systematic way the tasks were designed. Early on in the interviews, students were able to identify the three components that needed to be substituted. This was accomplished by prompting the students to think about how they would reframe the integral in terms of a new quantity (angle rather than time and temperature rather than radius). As shown in the above sections as they worked through each substitution students used the quantities to successfully reason about the substitutions for each of the three parts and to resolve difficulties they encountered.

At the end of each task the students were asked to summarize the substitutions they had done throughout the task. All three pairs mentioned the need for the two integrals (the original and the substituted integral) to be equivalent, saying things similar to the following statement from Jackson.

Jackson: if we're going to go from one relationship to the other like radius to temperature or from, you know, time to, um, degrees or I guess radians, we had to change our function bounds and our differential. We had to make sure that they were still equivalent, um, statements.

Notice that Jackson specifically developed a personal "function-bounds-differential" schema for u-substitution. Jackson, Bently, and Nate all specifically mentioned that the change of quantity relationship led to figuring out the values for each of the substitutions.

Jackson: We need a Relationship of how radius relates to temperature,

Bently: We Needed this [Points to $r = \sqrt{T} + 5$]. And Everything else came from That.

Nate: Um, so we take the initial thing, we find the variable that changes it. ($r = \sqrt{T} + 5$) We find, like the link between the two, change the bounds, and then find the link between the small stuff. And then you get this [points to the substituted integral].

As shown by the above statements the students remained focused on the quantities and quantitative relationships in their descriptions of the meaning of substitution.

# CHAPTER SIX: DISCUSSION

In this chapter, I first summarize my findings in answer to my research questions. I then discuss how this study connects to and builds on the existing literature in this area, and finally will examine the limitations of my study and ideas for future research.

## Answering the Research Questions

As a reminder to the reader my two research questions are (1) How do students use quantities and quantitative reasoning in building an understanding of the three parts of u-substitution? and (2) What resulting understanding of u-substitution do students develop and are those understandings connected to quantitative reasoning?

### Answering Research Question #1

In answer to my first research questions, the data showed that students engaged in quantitative reasoning throughout their work on the tasks about the integrals, derivatives, and differentials in the first interview. This quantitative reasoning carried through to their sense-making of each piece of the substitution in the second interview which I will discuss in more detail in the following section.

A few different types of quantitative relationships helped students as they reasoned through the tasks. First, students exhibited a two-quantity relationship that was different from Thompson's triangle (1990, 2011) and Thompson and Carlson's covariation (2017). The relationship was key for students reasoning about each component of the integral and their substitution and manifested as a "basic" relationship, as an "equivalence", and as a "function" function relationship, which I will discuss in more detail later in the contributions section.

I also saw three-quantity relationships displayed (Thompson, 1990, 2011). This appeared most often relating to the target quantity in conjunction with quantitative operation and reasoning with units to note that the multiplication of a function quantity with a small amount of the differential quantity produced the desired target quantity. Yet, the three-quantity relationship was often nested multivariation, like students describing power being for a given angle at a given time, rather than Thompson's triangle relationship structure (Thompson, 1990, 2011; Jones 2022).

Quantitative reasoning was used to think of derivatives as rates and ratios of small changes, rather than the slope of a tangent line (Ely, 2020). But it did not show up in u-substitution as strongly as anticipated, it only appeared in terms of calculating d[]/d[] for the differential substitution and did not appear to play much of a role in the cognitive load. Rather, having a quantitative conception of the differential was more important (Jones, 2015; Ely, 2017; Simmons & Oehrtman, 2019). The idea of the differential as an infinitesimal amount of quantity proved key to successful differential substitution and the data showed other conceptions being problematic as will be described in the next section (Ely, 2017, 2020).

It was crucial for students to have the quantitative meaning of AUP for integrals to engage with these tasks (Jones, 2015b; Jones & Ely, in press). An area-only (or antiderivative-only) meaning would not have provided the resources for understanding the conversions between all three integral components: bounds, function, and differential. Thus, such activities need to be based on AUP understandings.

**Answering Research Question #2**

In answer to my second research question, I discuss the meanings for each of the three parts of substitution that students developed. For the majority of the students, the substitution of both the bounds and the function was fairly straightforward. In substituting the bounds students had no difficulty keeping track of the quantities involved particularly for the solar panel task.

Some of the students reasoned about the quantities of the solar panel directly to make the bounds substitution and some relied more on the change of quantity relationship, and all three were able to use the change of quantity relationship in the sphere task to algebraically find the change of bounds relationship.

The function substitution did seem to be primarily driven by symbolic manipulation; however, students were able to track the quantities when asked without too much problem. When asked how he thought about the function substitution after it was Nate said, “We found the relationship between *r* and *T*, so we know that like for whatever *T* we put in there, it's gonna come up to like the right r to get the same result as this one [$4\pi r^2$].” In other words, quantitative reasoning was more backgrounded here, and the students used other types of reasoning (symbolic and algebraic) as their primary means of doing the substitution. Yet, it seemed important for the students to be asked to track the quantities for the entire shift from the initial integral to the substituted integral to be wholly sensible within a quantitative paradigm. Otherwise, the quantitative relationships between the two integrals might have been less obvious.

During discussions about the differential substitution, it became clear that this was where the majority of the cognitive load of substitution resides. This is in contrast to the way that differentials are often portrayed in common curricula – as nothing more than a notational device (Thomas et al., 2020; Stewart, 2021). Additionally, it was evident that being able to use the differential as an amount conception (Hu & Rebello, 2013) was critical in finding the differential substitution relationship and by extension making sense of that substitution. Research indicates that thinking about the differential as collapsing to nothing or having no size can lead to problems (McCarty & Sealey, 2019; Oehrtman, 2009). My data shows that this is also true for making sense of the differential in substitution. Liam stated that the differentials were the same because they are so small, they can't be measured, and based on work from other students it's likely they had similar thinking. Reasoning this way led students to leave out important components in the substitution to make the integrals equivalent. Students need some direction to steer them away from this misconception. By asking about equating some small value of each quantity students were able to use the quantities and quantitative reasoning to fix this mistake in the substitution.

Because the students had justified all of the substitutions along the way when comparing the original and newly-substituted integral they were confident that the two were the same. For example, Ellie reasoned about it saying, “I think that is true as long as you're switching all of the parts of the equation so that they all are consistent.” By the end of the second task when recapping the substitutions they made the students easily identified the need for some relationship between the new and old quantities as well as the three pieces that needed to be substituted. The two students who had taken calculus before and who did the pure math problem generalized that each substitution was related to the change of variable relationship.

## Contributions of the Study

### Expanding the Notion of Quantitative Relationships

The first contribution of this thesis to the literature is the inclusion of different types of quantitative relationships that are related to but distinct from Thompson's (1990; 2011) quantitative relationships and Carlson et al.'s (2002) levels of covariation, namely two quantity quantitative relationships and nested function composition relationships (Jones, 2022).

There are three types of two-quantity relationships that I identified from student work. The first type of two-quantity relationship exhibited was a basic relating of two quantities without specific definition of the relationship or detailed reference to how they covary. Students would talk about changes in one quantity meaning changes in another, like acknowledging that there is, "a much more direct relationship between radius and volume whereas the relationship between temperature and volume isn't as direct."

The second type was equating two quantities. This thinking was sometimes exhibited as students indicating that some expression was "kind of the same thing" as the quantity being substituted. At other times the equivalence was described only with quantities like Andres saying "they both measure the same thing without measuring the same thing…they both represent the same thing, which is the position of the sun."

The last type of two-quantity relationship that appeared in my results was the function relationship. In this relationship, the students described some type of input/output or operation on one quantity to produce the second. When talking about the differential relationship Bently said, "you could lowkey just write $d\theta$ or changes in $\theta$ as a function of $t$," and went on to create a function to describe the relationship saying, "*f* of *t* is π 12ths times *t*".

These two quantity relationships were very common in student reasoning across the different tasks in both interviews. They were particularly important in their substitution reasoning since students had to either formulate or unpack a relationship between two quantities for each part of the substitution to ensure the substituted integral remained the same as the original. In the bounds substitution, students equated the quantities of the time and angle to identify the new bound in terms of angle in the solar panel task and used the functional relationship $r = \sqrt{T} + 5$ to compute the substituted bounds in the sphere task. Students used the equivalence of the change of quantity or change of variable relationship to reason that substituting one side of the change of variable expression ($\sqrt{T} + 5$) in for the other ($r$) would not change the integral function. The differential substitution was sometimes talked about as specifically as one differential being a function of another and at other times spoken of more generally as a small change in one differential quantity corresponding to a small change in the other differential quantity.

In his research where he defines and uses the three-quantity relationship, Thompson was focused on creating a quantitative reasoning-based algebra class (1990, 2011). Thus the three-quantity structure makes sense as a fairly algebra-oriented structure where equations often have two (or more) quantities producing a third. However, it may be that for a quantitative-based study of calculus these other types of relationship structures, two quantity and nested function composition relationships are more common and important in student reasoning than the three quantity "triangle" structure.

If we consider that u-substitution is in fact an "undoing" of the chain rule it should not be surprising that nested multivariational relationships would appear. In fact, it is important that the students think about the relationship between quantities in this way. Returning to the solar panel example, the chain rule would take power as a function of angle as a function of time $P(\theta(t))$. Thus, taking the derivative of power with respect to time involves imagining how fast the power changes as the angle changes, but also how fast the angle changes as time changes. This reasoning leads to the chain rule $\frac{dP}{dt} = \frac{dP}{d\theta} * \frac{d\theta}{dt}$ (Jeppson, 2019). Thus, for u-substitution, it makes sense that we would be traversing this relationship in the reverse nested multivariation order $time \rightarrow angle \rightarrow Power$. This quantitative relationship between u-substitution and the chain rule adds power connecting the quantitative and conceptual understandings with the procedural aspects of both.

## The Central Role of Differentials in Substituting

Research has indicated that it is difficult to articulate intuition about and define differentials (McCarty & Sealey, 2019). Consistent with what the research suggests, I found that differential substitution was the most cognitively demanding aspect of u-substitution in both tasks. While students might think of the differential as a marker of the variable of integration or as a part of the perimeter of a shape (Dray & Manogue, 2010, Jones 2015a), research has suggested that it is much more powerful to conceptualize the differential as a tiny or infinitesimal amount of a quantity (Ely 2017; 2020; Hu and Rebello, 2013 Amos & Heckler, 2015; Schermerhorn & Thompson, 2019a; 2019b). My data likewise demonstrates the significance of conceptualizing the differential as an amount when making sense of the differential substitution. The students encountered difficulties when thinking about the differential as being collapsed or having no specific size (McCarty & Sealey, 2019) and were unable to find a correct substitution relationship using that conception. Once prompted to reason with the differential as being an amount students were able to formulate and make sense of the differential equations needed for the substitution. In fact, I believe such differential-focused thinking may also be an important factor in understanding other types of substitutions, such as in integration by parts, trigonometric substitutions, and change of variables with Jacobians.

Interestingly, and unexpectedly, the derivative did not play a very prominent role in students' substitution reasoning. Students did not appear to think about the integrand function as a derivative function or as a rate. This could be due to the set-up of the tasks themselves and the focus on the differentials. In the instances where derivatives did appear students used derivative rules and the conceptualization of the derivative as a ratio of small changes or differentials (Ely, 2020) needed for students to be able to form the differential equation or differential relationship.

**Strengthening the Case for Scaling-Continuous Covariation**

Two different types of covariational reasoning that can be used in thinking about the integral are scaling covariational reasoning, zooming in on a static relationship, and dynamic covariational reasoning, where the changes are being traced out simultaneously (Ely & Ellis, 2018). Evidence of both types of reasoning was present in student thinking in the interviews. Students used scaling reasoning in their process of thinking about the target quantity of the integral. They seemed to naturally start by thinking about the basic model of the target quantity and had no trouble thinking about shrinking a quantity down to be small. For example, for the basic integral structure, Liam thought about shrinking down rectangles, "$dx$ is like, you're acting as if that's the width, but because it's $dx$ it's getting infinitely smaller. So you're making smaller and smaller rectangles."

Students used dynamic reasoning most often in conversations about rates or the change of quantity relationship. For example, thinking about time and angle dynamically moving together as the sun moves across the sky. This reasoning was useful for students reasoning about those relationships but did not appear to be used as much when students were making sense of the target quantity and the integral itself.

**Providing a Theoretical Framework for Quantities-Based U-Substitution**

The last, and perhaps most important, contribution of this thesis is a theoretical unpacking of u-substitution into its component parts of substitution of bounds, substitution of the function, and substitution of the differential. The following table summarizes this quantitative theoretical unpacking. For clarity, I wish to add a few notes about the table. The first row describes each of the three pieces of initial integral in terms of quantities and provides some symbolic interpretation that will be useful to describe the substituted integrals. The second row attempts to capture the cognitive work it takes to think of translating the initial integral into the new quantity. The third row aims to summarize the substitution in a more computational and formally mathematical way. To help illustrate the quantitative substitutions the final row uses the example described in the conceptual analysis to show the quantitative substitution of each piece in context.

**Table 4**

*Theoretical Quantity-based Unpacking of U-Substitution*

| | **Bounds** | **Function** | **Differential** |
|---|---|---|---|
| Quantitative meaning of initial integral $\int_{a_0}^{a_1} Q(a)da$ (*quantity Q is a function of quantity a*) | Range of values of quantity "$a$" to be partitioned; notated by the range's endpoints: $[a_0\,,a_1]$ | One quantity input (the independent variable $a$) maps to another quantity output (the function value $Q$): $a \rightarrow Q$ | Tiny amounts of or little pieces of the partitioned quantity: $dq$ |
| Quantitative meaning of substitution $Q(a) \rightarrow Q(b)$ *(from quantity a to quantity b)* | Find an equivalent range of values in a new quantity of measure using the endpoints.<br><br>Quantity 1 range: $[a_0\,,a_1]$<br>Substitution: $[a_0\,,a_1] \rightarrow [b_0\,,b_1]$ | These two quantities ($a$and $Q$) exist in a relationship with a third quantity $b$ such that $a \leftrightarrow b$ allowing substitutions from $a \rightarrow b \rightarrow Q$ (units are helpful in doing this substitution)<br><br>Function output quantity : $Q$<br>Substitution: $Q(a) \rightarrow Q(b(a) \rightarrow Q(b)$ | A differential amount of quantity $a$ corresponds to another differential amount of quantity $b$ through some covariational relationship. $da \leftrightarrow db$<br><br>Substitution: $da \rightarrow$ [factor] $* \, db$ |

| | | | |
|---|---|---|---|
| Formal mathematical notation | Original bounds $x_0$ to$x_1$ with substitutions<br>$u_0 = f(x_0)$<br>$u_1 = f(x_1)$ | $x \rightarrow u \rightarrow f$<br>$f(x) \rightarrow f(u(x)) \rightarrow f(u)$ | $\frac{du}{dx} = g'(x)$<br>$du = g'(x)dx$ |
| Solar Example<br>$\int_{t=0}^{t=6} 250sin(\frac{\pi}{12}t)dt$<br>$P(t) = 250sin(\frac{\pi}{12}t)$ | $t_0 = 0 \rightarrow \theta_0 = 0$<br>$t_1 = 6 \rightarrow \theta_1 = \frac{\pi}{12}$<br>$\theta_0 = \frac{\pi}{12} * t_0$<br>$\theta_1 = \frac{\pi}{12} * t_1$ | $t \leftrightarrow \theta: \frac{\pi}{12}t = \theta$<br>$P(t) \rightarrow P(\theta(t)) \rightarrow P(\theta)$<br>$250sin(\frac{\pi}{12}t) \rightarrow$<br>$250sin(\theta)$ | $\frac{d\theta}{dt} = \frac{\pi}{12}$<br>$d\theta = \frac{\pi}{12}dt$ |

I previously discussed different types of quantitative relationships that appeared throughout the interviews. Integrals seem to always consist of Thompson's (1990; 2011) three-quantity relationship between the function quantity, the differential quantity, and the resulting target quantity (see triangle in Figure 12a). Combining this structure with the nested multivariation three-quantity relationship (Jones, 2022; Figure 12b) we observed in the student data offers a quantitative structure of u-substitution. If we place the nested relationship along that triangle edge, u-substitution can be seen as shifting one vertex of the original integral triangle to form a new triangular relationship structure of the substituted integral (Figure 13a). In Figure 13b, I also use the power example to show how the integrand and target quantity vertices (power and energy) remain the same, while the independent variable vertex (time) shifts to a new vertex for the new independent variable (angle). This combination of a triangle and the nested relationship along one edge helps show why the two integrals are equivalent.

**Figure 12**

*(a) Thompson's Original Triangular Quantitative Relationship and (b) Jones' Nested Multivariation*

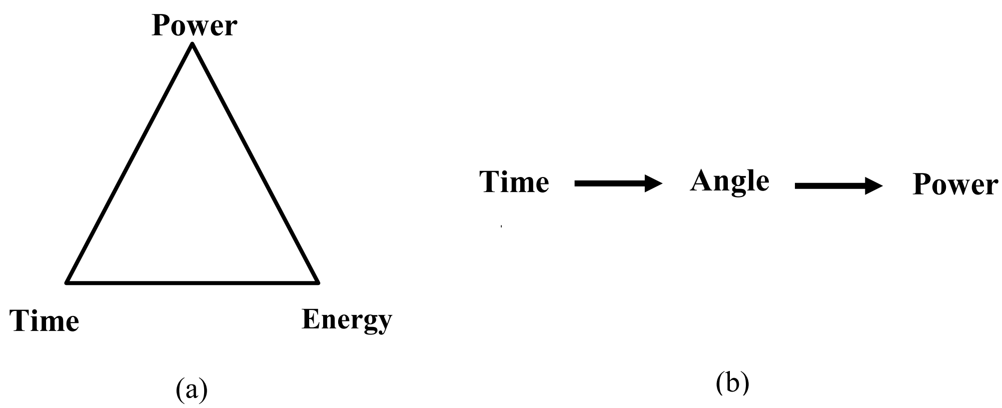
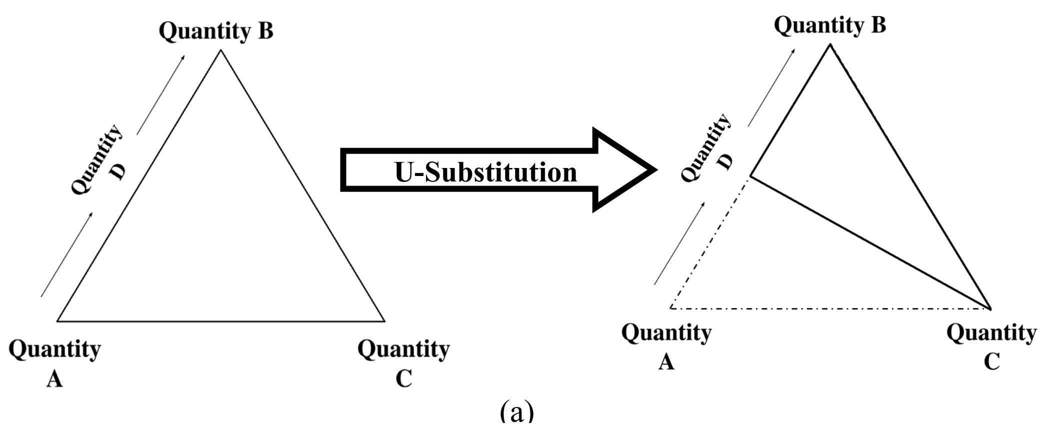
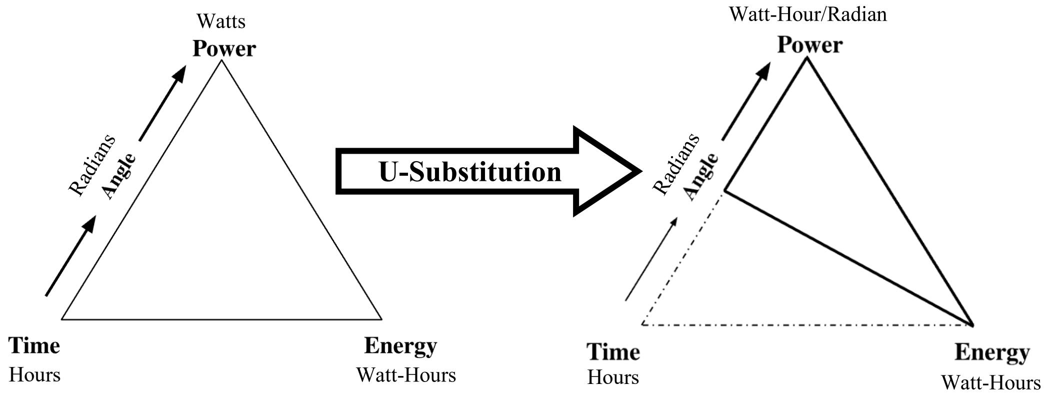



**Figure 13**

*(a) Quantitative Structure of U-substitution and (b) Example of Quantitative Structure with Time, Power, and Energy*






(b)

Throughout the u-substitution tasks, students primarily used the AUP conception of the integral rather than an antiderivative or area under a curve conception. U-substitution can be thought of as an antiderivative technique (Stewart, 2021); however, the main goal of this study was to use quantities to develop strong meanings for what actually happens during the u-substitution process. In fact, Ely (2017) differentiated between two "registers" of working with integrals, the setting up or modeling process and the working-it-out or evaluating process. The modeling process involves making sense of the integral and its component parts, and AUP is the most useful for making sense of the integral (Jones, 2013a; 2015a; Sealey, & Oehrtman, 2005). which was my focus. Because of that my tasks and questions were centered on the AUP conception of the integral rather than the antiderivative conception that is used in the working-it-out process.

Lastly, as stated earlier, previous work on integration within a quantitative reasoning perspective has been focused on introducing or developing the integral, but later topics of techniques of integration like u-substitution are less developed. By applying quantities and quantitative reasoning to the final section of the integral unit, my study completes the first-semester integration chapter within a quantitative reasoning perspective.

## Implications for Teaching

U-substitution is often thought of and used as nothing more than an antiderivative procedure, or technique for making integrals more simple to compute, but students can actually develop and make sense of u-substitution. Typically, the focus of the u-substitution instruction is on finding the "inside" function followed by performing a series of symbol manipulations, which causes confusion for students trying to identify the inside function and handle any constants or extra variables that appear in the integral. The approach described here helps frame what is actually happening with the change of variable and can help students be more sophisticated in their ability to work with substitution. This approach gives them a three-part structure to keep track of: bounds, function, and differential. Breaking it down in this way helps students track each piece and understand how it transforms to the new variable. Making sense of each of these three substitution relationships makes it easier to see the equivalence and comparison between the original integral and the substituted integral. Thus, even if u-substitution is viewed as an antiderivative procedure, students will have a better idea of how to enact the procedure.

Approaching the teaching of u-substitution with this framing of the three pieces can also help an instructor structure the lesson on how students might learn this process. For example, similar to what was done in the actual interviews, an instructor can lead students to identify the original quantitative structure and the nested relationship that leads to the substituted quantitative structure. Once this relationship is identified, the instructor can help students focus on each piece of the substitution by itself, which can make them aware of that specific aspect of the relationship between the original and substituted variables. This framing can also help instructors assess whether students are fully comprehending the entire u-substitution process. Rather than trying to identify if students are simply doing the entire process "correctly or not," the instructor can look for which of the three pieces of the u-substitution process the students are doing and perhaps which they are overlooking. This allows the instructor to do more targeted scaffolding and directing, rather than simply seeing that students “did not do it correctly.” For example, students may have readily substituted the function and bounds but may be missing the differential substitution. In this way, their work can be viewed for the productive elements it contains, and the instructor can then focus on the remaining aspect of u-substitution. This framing also helps an instructor assess whether students can describe *why* the two integrals are equivalent to each other.

While this study only addressed definite integrals, this approach can also potentially be helpful for understanding indefinite integrals. This approach can develop strong understandings for the change of variable relationships for each of the three parts of substitution, and that understanding would make switching back into the original variable of integration for indefinite integrals make more sense and have meaning.

Lastly, one crucial implication is that the differential is typically ignored in most mathematical texts and by most mathematicians. It is primarily just used as a symbol or the “period at the end of the integral” or the marker of the variable of integration. If one wants to make sense of integrals and substitution specifically, more emphasis needs to be placed on developing the concept of the differential as an infinitesimal amount and subsequently the role of the differential in forming the target quantity. It is not simply a “bookend” to the integral expression, but represents one of the main quantities in the quantitative relationship. This is supported by the fact that a significant aspect of the students’ work was in grappling with the differential and the appropriate substitution to the new differential.

## Limitations of the Study and Future Research Directions

There are several limitations of the study to be addressed. First, the sample size of student selection was small. The results described represent the thoughts and reasoning of six students all of whom came from classes whose instructors had used some quantitative reasoning in developing derivative and integral concepts. Thus students coming from classes with different instructors or backgrounds of derivatives and integrals may not have been as prepared to reason with quantities or as readily used the ideas of derivative as a rate, or the adding up piece conception of the integral. Additionally, the small sample size, and the resulting lack of variation of characteristics like gender, university major, and background, limits generalizability to a larger group of students. However, the work of these students still provides useful insight into the reasoning and difficulties involved in learning u-substitution.

Another limitation is that my study only focused on developing a conceptual quantitative understanding of the change of variable relationship and setting up a substituted integral. This leaves other aspects of substitution unexplored at the moment. Future research could examine how students would go on to handle pure math substitutions after a quantitative introduction, how students identify the change of variable relationship ("inside" piece) for a given function, the u-substitution of indefinite integrals, and additional substitution techniques like trig substitution.

## Conclusion

In calculus teaching, there is an overemphasis on procedures and manipulation of symbols and not enough emphasis on conceptual understanding of calculus topics (Tall, 1992). Because of this students struggle to understand and use ideas like integration ideas in applied settings. Research has shown that learning calculus topics from a quantitative reasoning perspective results in more powerful and flexible conceptions of topics like integration. While this has been for introducing or developing the integral, there is a lack of using quantitative reasoning-based approach for other integration topics like techniques of integration. I specifically focused on u-substitution and explored a quantitative-based approach to introducing u-substitution. Based on the clinical interviews, given quantitative relationships, the substitution of the bounds and function was straightforward for students to develop, but developing an intuition for and understanding of the differential relationship was a critical and cognitively demanding aspect of substitution. Overall through the interviews students gained a conceptual understanding for how to reason through substituted integrals being equivalent to the original using quantitative relationships for the bounds, function, and differential of the integral.

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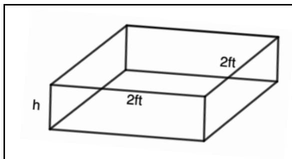
APPENDIX A: INTERVIEW PROTOCOL

**Interview One:**

<u>Scaling covariation</u>

**Given to students:**




| | | | | | | |
|---|---|---|---|---|---|---|
| $\Delta h$ $feet$ | | | | | | |
| $\Delta V$ $cubic$ $ft$ | | | | | | |

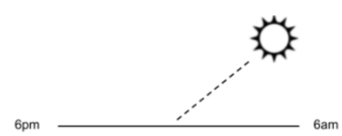
**Interview Questions:**

- If I have a height that changes by a half a foot what is the equivalent change in volume?
    - What about a change of ¼ of a foot? 1/100th of a foot?

  *Have students use the table to keep track of these relationships*
- Can you describe how you're thinking about the change in the amount of volume as we think about smaller and smaller changes in height?
- Can you describe what the change in volume will be in relation to any change of length in height?
- What symbols would you use to represent that relationship?
- If we zoom this in further to have an infinitesimally small change in height can we find the infinitesimally small change in volume of our box?

**Given to students:**

1. On the equator the sun rises around 6 am, is directly overhead at 12 pm and sets on the horizon at 6 pm.




| | | | | | | |
|---|---|---|---|---|---|---|
| $\Delta t$ $hours$ | | | | | | |
| $\Delta \theta$ $angle$ | | | | | | |

**Interview Questions:**

- If you are standing on the equator how much has the angle of the sun in relation to you changed from 6 am to 6 pm? From 6 am to noon?
- How big of an angle change corresponds to one hour elapsing?

- What if we only let a half hour elapse? One tenth of an hour? A thousandth of an hour?

- If I continued to scale this down to the change of a fraction of a second, what would the change in angle be?

  *Have students use the table to keep track of these relationships*

- Can you describe what the angle change will be in relation to any elapsed amount of time?
  - What symbols can we use to represent that relationship?
  - Does this make sense for this relationship to hold for change in time at any part of the day?
  - Does the time of day effect how the angle changes? (Ex: if I look at the change in angle from 9-10 am is that different than the angle from 3-4pm?)

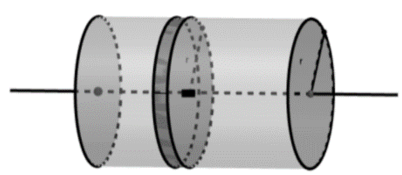
    *Diagram out the idea of a time number line and a range of angles*

- Is this relationship still valid as change in time gets infinitesimally small?
  - How would you represent the infinitesimally small relationship?
- What do $dt$ and $d\theta$ represent?

*Instruction on development of infinitesimals as having an amount*

<u>AUP Integrals</u>

**Given to students:**



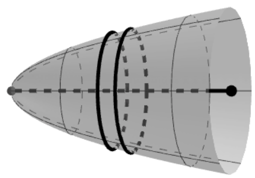
1. The volume of a cylinder is $V = \pi r^2 l$ where $r$ is the radius and $l$ is the length of the cylinder.

**Interview Questions:**

- Let's use a symbol to represent the volume of the pictured slice of the sphere. What symbol should we use? (if no "d", ask: "how could we suggest it's a very thin slice?)
- What does the $dV$ mean? (or the equivalent symbol they use to represent the volume)
- How do you interpret the integral $\int_a^b dV$?
- What are the quantities that make up this slice of volume? What symbols can we use to represent these quantities?
  - Is this integral $\int_a^b dV$ the same as this one $\int_a^b \pi r^2 dl$?

- Given the integral $\int_a^b \pi r^2 dl$ what does each part of the integral mean? What does the integral mean all together?

**Given to students:**



**Interview Questions:**

- We can take volume slices of this shape similar to the previous cylinder shape, how would you describe the slices of volume ($dV$) for this new shape?
- Similar to the last shape, let's think of the axis being made up of lots of little $dl$'s does each $dl$ have an associated $dV$?
- Are all of the $dV$s the same?
    - How can we differentiate between the different $dV$'s or represent the $dV's$?
- Can we use an integral like we did with the cylinder to find the total volume of the shape?
- Write the integral expression.
    - How does this integral represent the total volume of the shape?
- What do $dl$ and $dV$ mean for this shape? How are they related?
- Using the ideas that we talked about from these two problems, can you describe how you're thinking about the integral in general?

*Instruction on development of adding up pieces*

*Recap the adding up pieces structure of partition target quantity and sum. Highlight that an infinitesimal amount of the target quantity corresponds to a specific tiny piece of length*

**Derivative as a ratio of differentials**

**Given to students:**

1. Refer back to the table in the first question and the found relationship $d\theta = \frac{\pi}{12}dt$

**Interview Questions:**

- Can we take the $dt$ and divide it to the other side to make the ratio $\frac{d\theta}{dt}$? Is that valid?
- What does the ratio $\frac{\pi}{12}$ it mean?

**Given to students:**

2. A spherical balloon with a small heater inside has been filled with a gas. As the gas is heated it expands, increasing the volume of the balloon. At any given radius $r$ the relationship between the radius of the sphere and the temperature $T$ of the gas in degrees Celsius is given by $r = \sqrt{T} + 5$

**Interview Questions:**

- Compute the derivative of this equation with respect to temperature.
- What does this derivative mean in this context?
- What do $dr$ and $dT$ mean individually?
- What does it mean that the derivative is a function rather than a number?
- For different temperatures what does that tell you about how fast the sphere is growing?
- If we multiply the $dT$ over to the other side we get $dr = \frac{1}{2\sqrt{T}}dT$**.** Is this valid?
- What does it mean that there is a variable in this relationship (that its not constant like the previous problem $d\theta = \frac{\pi}{12}dt$?)
  - What does this new expression mean?

*Instruction on derivative as a ratio of differential.*

*Since each differential represents an amount we can multiply or divide the differential. The derivative is a ratio of those differentials or small changes. The ratio $\frac{dr}{dT}$ describes how radius changes as temperature changes. The relationship $dr = \frac{1}{2\sqrt{T}}dT$ means that the numerical value of a small change in radius is going to be $\frac{1}{2\sqrt{T}}$ times the size of the numerical change in temperature T, but those changes are dependent on the value of the temperature. (This is like we said with the second shape, the dV depended on where the slice was, and for this the dr depends on what the temperature is for the change we're looking at.*

**Interview Two**

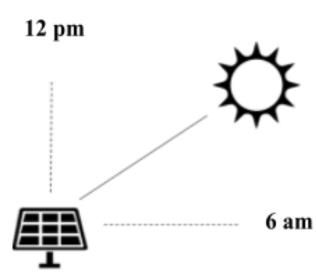
*Follow up on the AUP conception of the integral. Ask students to describe the meaning of the components of the velocity integral $\int v(t)dt$ to refresh AUP conceptions of the integral and the partition, target quantity, and sum.*

Solar Panel

**Given to students:**

1. A solar panel collects power in watts, which is a unit that describes the joules of energy ($J$) per second (s) that is generated, $J/s$. The amount of power the panel generates is dependent on where the sun is in relation to the panel. It will reach its maximum output when the sun is directly above it (at noon). The generated wattage of the solar panel can be thought of as a function of time as the sun moves across the sky. It can be modeled well by a sine function (Solar Panels, 2021), and for simplicity we'll use the basic function $P(t) = 250\sin\left(\frac{\pi}{12}t\right)$ with *t* time in hours since sunrise. On the day we're measuring, the sun rises at 6 am and sets at 6 pm.

*(Note that the time in the unit of power, $J/s$, is separate from the hours in the day)*




**Interview Questions:**

- What are the quantities involved in this context?
- Using the AUP idea from our first interview, can you create an integral that would answer this question?
- Again using the AUP ideas describe what this integrals means as a whole
    - o What is the quantity that we are dividing up (partitioning)?
    - o What does the integrand mean? What is the quantity of the little pieces that we're adding up?
    - o How do we get each of those little pieces of quantity?

*To motivate the substitution note that it's not very intuitive to visualize time passing, however we can see clearly from our picture the angles of the sun in as it moves across the sky. What if we were to rethink our problem in terms of the angle of the sun in relation to the solar panel instead of in terms of time?*

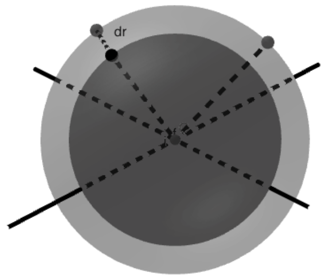
- What is the angle range that corresponds to our time interval?
  - What are the units that we have in the integrand?
- What is the quantity that we are now breaking into small pieces?
- What are the bounds of the integral in terms of the range of angles?
- Now we have the bounds in terms of the angle in radians, what else do we need to change for this to be all in terms of angle rather than time?
- What might the power function look like in terms of angles
- What is the relationship between $t$ and $\theta$ in the function?
- What does the resulting integral look like after this substitution?
- How does a little bit of time relate to a little bit of angle?
- If we use our previous partition and scale it down to infinitesimals, what is a tiny bit of $dt$ equal to in angles?

*Prompt students to keep the quantities in mind throughout.*

- What is the integral resulting from this substitution?
- What is the quantity that we are adding up in the substituted integral?
- Let's compare the original integral in terms of time and the new integral in terms of angle. How are things being added up here now that we've done these substitutions?
  - Are these two integrals adding up the same quantity?
  - Describe how this new expression is adding up energy like we initially intended it to?
- Let's review the work that we've just done. Can you list the different substitutions that we made?

**Given to students:**

1. $dV = S * dr$,




*Instruct students about the addition of a bit of volume being a shell of volume and an infinitesimal amount volume being equivalent to the surface area* radius*

**Interview Questions:**

- We established that $dV = S * dr$, what does $dV$ mean? What does $dr$ mean?
- How could I represent the total change in volume from one radius value to another?
- We're adding up all of our little bits of volume $\int_{r=10}^{r=15} dV$, and we've established that that $dV$ is found from the multiplication $dV = S * dr$ so adding up the volume is $\int_{r=10}^{r=15} S * dr$. Could we write everything in terms of the radius instead of having surface area?
    - We've written this integral in a few different ways now, so to remind ourselves what does this integral represent?
- Can you solve the integral? What does the answer of $\frac{9500\pi}{3}$ mean?

**Given to students**:

2. A spherical balloon with a small heater inside has been filled with a gas. As the gas is heated it expands, increasing the volume of the balloon. At any given radius $r$ the relationship between the radius of the sphere and the temperature $T$ of the gas in degrees Celsius is given by $r = \sqrt{T} + 5$ (As its heating the up the radius is a function of the temperature an expands

**Interview Questions:**

- Could I rewrite this integral where everything is in terms of what's happening with the temperature rather than what's happening with the radius
- What temperature value corresponds to the start radius (r=10)? The end radius (r=15)

- Using the structure of the integral $\int_{r=10}^{r=15} 4\pi r^2 dr$ construct an integral to find how much the volume of the balloon increases as the temperature increases from 25ºC to 100ºC.

*As needed prompt students to consider each piece (bounds, function, differential) that needs to be substituted*

  - What is the temperature when r = 10
  - What is the surface area $S$ in terms of temperature $T$?
  - $dr$ represents a small change in $r$ in our simple version, what is the equivalent quantity needed in the more complex integral with temperature?
    - What is $dr$ equal to in this context?
    - What is $dT$ equal to in this context?
- How are you thinking about what each piece of the complicated integral $\int_{T=25}^{T=100} 4\pi(\sqrt{T}+5)^2 \frac{1}{2\sqrt{T}}\, dT$means?

*As needed guide students to use the relationship $r = \sqrt{T} + 5$ in all their construction of the integral.*

- So now we have these two integrals up here side by side can you compare them*?*
  - Can you explain why these two integrals are equal to each other? How are they the same thing? $\int_{T=25}^{T=100} 4\pi(\sqrt{T}+5)^2 \frac{1}{2\sqrt{T}}\, dT = \int_{r=10}^{r=15} 4\pi r^2 dr$
    - What do you notice about them? What similarities do you see?
    - (As they point out things write down what they're saying
- Can you list the substitutions that we made?
- What were the key pieces of information you needed to make those substitutions?
- What similarities do you see between the substitutions in the solar panel problem and this sphere problem?
- In general, how are you thinking about the process of going from a complex to a simple integral?

*As part of their generalization, emphasize to the students that the key substitution relationship was nested inside another function for both problems. In both of these problems we had some type of relationship between quantities, and we used that relationship to make a substitution in the bounds of the integral in the function and in the differential. (Highlight each of these three pieces from their recap of what they did.) What you've constructed what we call a substitution* *technique in calculus, that tells me in order to go from one variable to the other we need to transform the bounds, function, and differential*

<u>Pure math substitution task</u>

**Given to students:**

1. Using some of the ideas we've developed today tell me how you would approach doing a substitution to write the below integral in a simpler form.

   $$\int_2^4 3x^2 e^{1+x^3}\, dx$$

**Interview Questions:**

- In our previous problems we identified an "inside" piece that described the relationship between two different quantities. What is an inside piece here that we can use to do a substitution?

*Inform students of conventional notation of "u" as the substitution variable.*

- What are the key components that need to be switched from one variable to another?
  - What is the differential in terms of $u$
  - What are the bounds in terms of $u$
- What does this substituted integral mean?
- Having done this problem is there anything you would like to add to your previous summary/comparison between the sphere and solar questions?

## APPENDIX B: EXAMPLES OF ANALYSIS

### Image Codes

Each speaking turn was coded by marking the related code column with an x as shown in the "Image" columns in the table below. The "Contents" column gives a brief description of the evidence of the code. If the code was only related to a specific part of the speaking turn, I highlighted the relevant sections in the "Speaking Turn" column.

| | Speaking Turns | Image Deriv | (Contents) | Image Diff | (Contents) | Image Int | (Contents) |
|---|---|---|---|---|---|---|---|
| A: | Integral is a multiplication of Two things being multiplied and it's the summation of something and there's no multiplication there. So I'm just trying to figure out what the, what's being multiplied, what's being added. | | | | | x | Integrals involving multiplication (AUP) |
| N: | Immediately this seems like a related rates problem which I'm not Good at, but somehow we have to do the change in volume, as it relates to the change in length. You have to relate that to the change in like length as relates to the change in $r$ It's like this somehow equals this | x | Related changes in two quantities (Rate) | | | | |

| | | | | | |
|---|---|---|---|---|---|
| L: | Velocity is kind of like a measure of how fast you're going at a very specific time. but if you find kind of the area of that, so if you're going 25 miles an hour for a second, and you're just for that distance, if you like, just like assume that like just for a second, it's like linear can find the distance that you travel and then you'd go like a second over. But this is kind of like the thing that we're making very, very small and we're just trying to find the total change and that would represent, um, just position for change in position. | | | x | Integrals involving (Area) |
| J: | It's the total change in volume between or, between radius 15 and 10 --- radius. 10 | x | (Change) | | |
| A: | Yeah. I think, you know, dt is just as time gets infinitely smaller and infinitesimally small amounts of time and this is infinite, infinitetes... Infinitely small amounts of theta in relation to like the same amount of time so it's not just infinitely small amounts of theta, however much theta you want, but it's infinitely small amounts of data within the timeframe of 12 π radians | x | Small amount of time and theta (Amount) | | |
| B: | The pieces, what they all mean is showing that your degree Celsius is gonna start 25 degree Celsius and going to a hundred degree Celsius. And it's adding up all of this in terms of your temperature that is being changed, right? Your temperature is being changed with respect to, to All of this stuff, which like, As he said, it's kind of the | | | x | Bounds give starting stopping point of summation (AUP) |

| | |
|---|---|
| | same thing as radius. 'Cause this function is the same thing as the other one like, we're both changing at the same time. It's just you calling this one in terms of temperature instead of radius, even though they both happen at the same time. Right. And so you're pretty much just adding up All of them. Okay. From 25º Celsius to 100º Celsius, |

## Quantity Codes

Each speaking turn was coded by marking the related code column with an x as shown in the "Quant" columns in the table below, again with the "Contents" column describing the evidence for that code. If the code was only related to a specific part of the speaking turn, I highlighted the relevant sections in the "Speaking Turn" column.

| | **Speaking Turns** | **2 Quant Rel** | **(Content)** | **Quant Rel** | **(Content)** | **Quant Op** | **(Content)** | **Covar** | **(Content)** |
|---|---|---|---|---|---|---|---|---|---|
| E: | d of t is equal to six over 100, d of ø that is π over 200. | x | (Equivalence) | | | | | | |
| N: | Well, if this gives us… well, that equation is essentially equal to bit of radius, they're shoved in there and that's equal to radius shoved in there. so together you get volume, I guess. | x | Equating temp equation with radius (Equivalence) | x | Radius with the surface area give volume (Thompson) | | | | |

B: So that means, Well, this is how you can rewrite surface area in terms of temperature, instead of in terms of radius

x

One quantity “in terms of” another (Function)

B: So this $[4\pi(\sqrt{T}+5)^2]$is your surface area multiplied by your change in temperature[dT] plus this. So like this is still your surface area just written in terms of temperature.

x

Surface area times dT to get volume (Thompson)

A: Yeah. I think, you know, dt is just as time gets infinitely smaller and infinitesimally small amounts of time and this is infinitely small amounts of theta in relation to like the same amount of time. So it's not just infinitely small amounts of theta, however much theta you want, but it's infinitely small amounts of data within the timeframe of 12 π radians.

x

Infinitesimal amount of time in relation to amount of theta

I: So once we’ve done this multiplication here, what are the little pieces that I’m adding up?

N: The energy at that time at that angle.

x

Energy at angle of a given time (Nested Multivariation)

| | | | |
|---|---|---|---|
| B: | It shows you how much your temperature is changing when your radius would've been changing. | x | Change in temp corresponds to change in radius |

**Substitution codes**

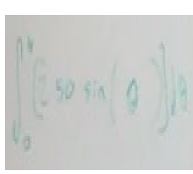
Each speaking turn was coded by marking the related code column with an x as shown in each of the "Substitution" columns in the table below. As with the previous tables, if the code was only related to a specific part of the speaking turn, I highlighted the relevant sections in the "Speaking Turn" column.

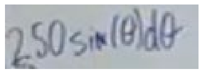
| | Speaking Turns | Bounds Substitution | Function Substitution | Differential Substitution | General Substitution |
|---|---|---|---|---|---|
| E: | Okay. Um, so our, our 10 to 15 turned into 25 to 100. Okay. And our input went from $r$ to $\sqrt{T}+5$. | x | x | | |
| A: | I think both just changed the time to temperature, you put just T | x | x | | |
| A: | 'Cause we're not…yeah, 'cause we just input the temperature | | x | | |
| E: | Right, right, to temperature. And then our, the derivative we changed from the derivative of the radius to the derivative of temperature. | | | x | |
| J: | Cool. If we're going to change anything, if we're going to go from one relationship, one relationship to the other like radius to temperature or from, you know, time to, um, degrees or I guess radians we had to change our function bounds and our differential. We had to make sure that they were | | | | x |

| | | | |
|---|---|---|---|
| | still equivalent, um, statements. | | |
| L: | We are using time using the time where like the sun is and given we also need, we need, uh, Actually, no, it gives us the angle. When we plug in time, it gives us the angle and then it gives us the whole function outputs jewels per second. | x | |
| N: | Yeah. I mean, I think they'd be proportional, but I don't know if they'd be exactly the same you would get. Yeah. That's all I'm gonna say. | | x |

**Secondary Analysis of Differentials**

The following table demonstrates how I compiled "differential" related quotes together so I could see the different types of thinking related to the differential substitution and any commonalities across students.

| | Differential relationship is proportional | | Directly sub in dø for dt | | General how refer back to differential substitution | | Differential Relationship |
|---|---|---|---|---|---|---|---|
| L: | Okay. Yes, I'll do that. So from Zero to a π/2, 250 And then, sin($\vartheta$) $d\vartheta$ | L: | Because, well, we don't have *t* anymore because we've replaced *t* so we can't use it *dt,* which is like very small change *t* so the way you would have to measure it is you'd have to multiply that by very small change the angle | B: | Means that your sign of theta is going to follow the same pattern as shown here, right. As it moves along, it's moving along the same way as the π is. But your change in theta is actually changing by 12 over π instead of normally as the time, because the differences in time of theta are with that rate π over 12. | A: | I think that's what the relationship between the two are, is that D of T is, getting infinitesimally smaller. over a period of six hours, which is whereas that's where $d\vartheta$ is getting infinitely smaller over the range of π/2. So that's how they're equal each other. |
| B: | And then you can solve for $dt$ yeah. Plug that in too . | A: |  | J: | So I mean, what we did is just do a major substitution. I mean, we substituted a $dt$ with what we came up with for a $d\vartheta$,, which is 12 over π times, ,$d\vartheta$, we substitute, | E: | Wait a minute. When you say that, that way it gets $\pi/12$ which between what? One Hour of, So yeah. Oh yeah. No, that makes sense. So one, one of the six hours equal to π/12, which we |

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| | | | | | uh, what's inside the sin function with $\vartheta$ and we set it equal to that. And then we changed our bounds from being respected time to being in respect to the, or to angle measurements. | | already knew |
| J: | So, I mean, I mean, we just take the derivative of both sides, right. Um, so this is… the derivative of theta is $d\vartheta$, the little change of theta. And then we apply, um, we take the derivative of the right hand side, um, which is, um, if we're.., or it is just a constant times a multiple. So we can do the constant out here times the little change in time out here. | L: |  | A: | Yeah. I think, you know, dt is just as time gets infinitely smaller and infinitesimally small amounts of time and this is infinite, infinite to infinitetes... Infinitely small amounts of theta in relation to like the same amount of time, so it's not just infinitely small amounts of theta, however much theta you want, but it's infinitely small amounts of data within the timeframe of 12 $\pi$ radians. | B: | So basically this one is going in terms of time, right. And your time is moving from zero hours, to, six hours, right? Yeah. Your time is .. Like is being shown. Your small changes in time is being shown within the sin graph because it follows the same sin pattern as we, we heard in here. Right. And that changes as time moves along. It it's being rotated kind of. I don't really know if $\pi$ always means rotation, but kind of means rotation in mathematics half the time. And so it is changing within this at $\pi/12$ right. Which is the same thing as if your degree, as we have shown here changes, right. Is being changed by the same thing, which means because your change in time is growing at a faster rate, Is that faster rate that's actually smaller rate, but is changing with $\pi$ 12$^{th}$s for every small changein theta. |
| L: | Because, well, we don't have $t$ anymore because we've replaced $t$ so we can't use it $dt$, which is like very small change $t$ so the way you would have to measure it is you'd have to multiply that by very small change the angle . | | | L: | So we're kind of, we're kind of with this logic, with this piece, we're just changing this, so that it's equivalent l here, but we also want, You know, the relationship between $dt$ and $d\vartheta$,. So we went over here. We multiplied both sides by, | N: | We found the relationship between $r$ and $T$ Um, so, uh, we know that like for whatever $T$ we put in there, it's gonna come up to like the right $r$ to get the same result as this one |
| J: | So I mean, what we did is just do a major substitution. I mean, we substituted a $dt$ with what we came up with for a $d\vartheta$, which is 12 over $\pi$ times, dø, | | | E: | So We need to Plug this in so that it's 12 $d\vartheta$ over $\pi$ | | |

we substitute, uh, what's inside the sin function with ø and we set it equal to that. And then we changed our bounds from being respected time to being in respect to the, or to angle measurements.