Structural Reasoning with Rational Expressions

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Structural Reasoning with Rational Expressions

Dana Steinhorst

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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Many students struggle to make sense of algebraic expressions in math. This lack of understanding results in students making symbolic manipulation errors, hindering their procedural fluency. Researchers believe these errors are linked to students' lack of structural reasoning. While research has shown that students rarely engage in expert structural reasoning, little is known about how students actually reason structurally. In this study, I interviewed six high school calculus students to study the way they identified, matched, and evaluated structures as they solved problems involving rational expressions and equations. I analyzed the participant interviews and outlined the matching process they used and the types of evaluations they made during this matching process. Consequently, I was able to confirm that students were using structural reasoning throughout the tasks and that effective student structural reasoning was characterized by identifying structures using operational hierarchical reasoning and matching them to correct rules. These findings have the potential to help teachers better instruct students on using and identifying structure, leading to less frustration by students and teachers in algebra.

Keywords: structure, algebra, structural reasoning, algebra errors, rational expressions
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TABLE OF CONTENTS

TITLE......................................................................................................................................... i
ABSTRACT .................................................................................................................................... ii
ACKNOWLEDGEMENTS.......................................................................................................... iii
TABLE OF CONTENTS .............................................................................................................. iv
LIST OF TABLES .................................................................................................................... vi
LIST OF FIGURES ................................................................................................................ vii
CHAPTER 1: INTRODUCTION .................................................................................................1
CHAPTER 2: THEORETICAL FRAMEWORK ........................................................................... 4
  Structure Definition ............................................................................................................... 5
  Structural Reasoning Definition ........................................................................................... 7
CHAPTER 3: LITERATURE REVIEW .................................................................................. 11
CHAPTER 4: METHODS ....................................................................................................... 17
  Data Collection .................................................................................................................... 17
    Setting and Participants ................................................................................................. 17
    Data Types ..................................................................................................................... 19
  Data Analysis ................................................................................................................... 22
CHAPTER 5: RESULTS .......................................................................................................... 26
  The Matching Process ....................................................................................................... 26
    Structure Identification and Operational Hierarchical Reasoning (OHR) ..................... 26
    Rule Banks and the Matching Process ........................................................................... 28
  Outcomes of the Matching Process ................................................................................ 30
    Correct Matches ............................................................................................................. 30
    Incorrect Matches ......................................................................................................... 31
    Failure to Construct a Match ......................................................................................... 36
    Useful Matches for Rational Expressions ..................................................................... 45
  The Role of Evaluation ...................................................................................................... 47
LIST OF TABLES

Table 1: Summary of Students’ Correct and Incorrect Matches.................................................36
<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Figure 1</td>
<td>Example of Diagram Used to Code Data</td>
<td>24</td>
</tr>
<tr>
<td>Figure 2</td>
<td>Example of OHR Structure</td>
<td>27</td>
</tr>
<tr>
<td>Figure 3</td>
<td>Example of Non-OHR Structure</td>
<td>27</td>
</tr>
<tr>
<td>Figure 4</td>
<td>Example of the Iterations in the Matching Process</td>
<td>29</td>
</tr>
<tr>
<td>Figure 5</td>
<td>Task 1 First Level of Structure</td>
<td>30</td>
</tr>
<tr>
<td>Figure 6</td>
<td>Task 1 Second Level of Structure</td>
<td>31</td>
</tr>
<tr>
<td>Figure 7</td>
<td>Structure Identified by Parker in Task 2</td>
<td>32</td>
</tr>
<tr>
<td>Figure 8</td>
<td>Structure Identified by Hope in Task 2</td>
<td>34</td>
</tr>
<tr>
<td>Figure 9</td>
<td>Structure Identified by Riley in Task 3</td>
<td>35</td>
</tr>
<tr>
<td>Figure 10</td>
<td>Structure Identified by Hope in Task 3</td>
<td>37</td>
</tr>
<tr>
<td>Figure 11</td>
<td>Initial Structure Identified by Riley in Task 2</td>
<td>40</td>
</tr>
<tr>
<td>Figure 12</td>
<td>Second Structure Identified by Riley in Task 2</td>
<td>41</td>
</tr>
<tr>
<td>Figure 13</td>
<td>Hypothetical Student Solution to Task 3</td>
<td>43</td>
</tr>
<tr>
<td>Figure 14</td>
<td>Hierarchy of Operations Identified by Jake</td>
<td>44</td>
</tr>
<tr>
<td>Figure 15</td>
<td>Structure Identified and Drawn by Jake</td>
<td>44</td>
</tr>
</tbody>
</table>
CHAPTER 1: INTRODUCTION

My student made the following error on a high school math exam: \( \frac{2x^2 - 4}{x+2} = 2x - 2 \). They incorrectly assumed they could divide each pair of terms separately, as they would if there was only one term in the numerator and denominator. When my student made this mistake, I was not surprised. I see a student make this mistake nearly every time we encounter a rational expression like this one. No matter how many times we talk about what a fraction means or look at examples, my students still make this same error. This problem is not unique to my classroom. Matz (1982) recognized that certain types of algebra errors are widespread. She found that not only do students often incorrectly apply a rule they know works to parts of an expression, as in the example above, but there are many other algebra errors that are common among students.

For most universities, students must demonstrate competency with algebra in order to qualify for graduation and to take more advanced science and mathematics courses. Thus, algebra serves as a gatekeeper that holds many students back. An important competency in algebra is knowing how to effectively manipulate algebraic expressions. In higher level math courses, many students are not successful due to algebra errors (Muzangwa & Chifamba, 2012). Thus it is important for students to know how to correctly manipulate algebraic expressions, and prompts the question, how do we help students eliminate these algebra errors?

Kieran (1989) found that students’ difficulties in algebra were usually centered around three topics: the shift of conventions from arithmetic to algebra, the meaning of letters, and the use and identification of structure. A great deal of research surrounds the first two topics. Many researchers have focused on the topic of connecting algebra to arithmetic (Filloy & Rojano, 1989; Linchevski & Livneh, 1999; Yantz, 2013). Researchers have also found that not only do many students have different meanings for letters in algebra, but teachers also use letters in many
different ways (Usiskin, 1988). The difficulties associated with these first two topics are widely accepted and deeply researched.

The last of these topics, using and identifying structure, has received less attention in the mathematics education community. A limited number of researchers have studied the topic of structural reasoning (e.g. Harel & Soto, 2017; Hoch & Dreyfus 2004; Ruede, 2013) and the understanding of it is still very limited. Structure also does not have a widely accepted definition as it pertains to algebraic expressions (Harel & Soto, 2017). Despite not having been well researched, the use of structure is viewed by the mathematics education community as an important skill. With the Common Core Standards, the National Governors Association Center published the Standards for Mathematical Practice (NGA Center, 2010). These standards introduced eight practices that described skills teachers should help students develop across all grade levels and content areas. One of those practices they highlight is to “look for and make use of structure” (para. 8). Structure is an important concept in the mathematics community that is not well explained, but expected to be taught.

In my own classroom, I have seen how my students struggle to identify structure in expressions such as $\frac{1}{x+2} + \frac{2}{x+2}$ or $e^{2x} + 2e^x + 1$. This lack of structural reasoning seems to be a large part of many of the algebraic errors students make. If teachers had a better understanding of how students view and use structure in algebra, they would be more effective in helping students make less errors and develop flexibility in their algebraic problem solving.

The use of structure is even more important when dealing with complex algebraic expressions, as it can simplify solving and re-writing them. One complex algebraic structure often used in higher levels of algebra is rational expressions. Rational expressions foster rich structures as relationships can be seen not only within the numerator and denominator, but also
vertically between the numerator and denominator (Ruede, 2013). This makes rational expressions complex and difficult for students to deal with, while at the same time providing many different structures for students to recognize and reason about. In other words, rational expressions provide opportunities for structural reasoning. Thus we need to determine what algebraic structures students construct when dealing with rational expressions and which are more effective. The purpose of this research is to identify ways students use and see structure in algebraic, specifically rational, expressions.
CHAPTER 2: THEORETICAL FRAMEWORK

Structure is a word used in many areas of research in a variety of ways, but researchers often do not feel the need to define it (Harel & Soto, 2017). For example, in the January 2020 issue of the Journal for Research in Mathematics Education, arguably the premier journal for mathematics education research, all four of the main articles mention structure. Of these four, none of them define what structure means. In these articles, structure is used to refer to a social network (Alcock et al., 2020), a classification of proofs (Czocher & Weber, 2020), the way geometric shapes are related to each other (Groth et al., 2020), and the relationship between participants in a study (Sonnert et al., 2020). In addition to using a variety of meanings for structure, researchers also use different meanings for structural reasoning. Harel and Soto (2017) found that some researchers used structural reasoning synonymously with deductive reasoning, while others viewed it as an activity that is closely related, but not identical, to deductive reasoning. Moreover, some researchers did not even relate structural reasoning to deductive reasoning, but rather viewed it as “re-organizing acquired knowledge into structured schemata” (p. 226). This variety in the uses of structure and structural reasoning in research illustrates not only the need for these words to be explicitly defined, but also the difficulty in doing so.

A lack of clarity and agreement about the meanings for structure and structural reasoning can also be found in documents for teachers and curriculum designers. For example, in the Common Core State Standards (NGA Center, 2010), one of the practice standards is “Look for and make use of structure” (para. 8). In the elaboration of the standard, structure is never explicitly defined, though it seems to be used synonymously with pattern. Rather than giving explicit definitions the authors present examples that demonstrate different ways of thinking about structure. One way is recognizing a discernible mathematical pattern. An example is given
of sorting shapes by the number of sides they have. A second way of using structure is captured in the statement that students should “see complicated things, such as algebraic expressions, as single objects or as being composed of several objects” (para. 8). They cite an example of the way a student could view an expression $5 - 3(x - y)^2$ and know that for any real numbers $x$ and $y$, this expression is less than or equal to five. This is different from the first way they talked about structure, as simply looking for patterns. The *Common Core* is not only unclear about what the word structure means, but they are also inconsistent in the way they refer to reasoning with structure—as looking for regularity in a variety of contexts versus seeing algebraic expressions as being made up of multiple objects. Because of these diverse usages of structure, it is important to be explicit about what is meant when structure is used in the context of making sense of algebraic expressions.

**Structure Definition**

While researchers have varied widely in how they have used the term structure in the literature, a few researchers have defined and used the term in a way that may be helpful in examining how students reason about algebraic expressions and equations. Many of these researchers describe two types of structure: surface structure and systemic structure (to use Kieran's, 1989, terms). Surface structure refers to the arrangement of symbols and operations subject to the order of operations. By this definition, a quadratic expression in standard form versus the same quadratic written as the product of two linear factors would have different surface structure because the symbols and operations have different arrangements. Other researchers have provided definitions of structure similar to Kieran’s surface structure. For example, Harel and Soto (2017) define structure as “‘something made up of a number of parts that are held or put together in a particular way’” (p. 226). Similarly, Hoch and Dreyfus (2004)
define structure in mathematics as “a broad view analysis of the way in which an entity is made up of its parts” (p. 50). These researchers seem to be referring to surface structure in their definitions because they talk about viewing the whole as being made up of parts, and do not view equivalent expressions as having the same structure.

In contrast, Kieran’s (1989) second type of structure—systemic structure—refers to the nature of mathematical objects or classes of mathematical objects (e.g., function families such as linear functions, quadratic functions, etc.) that can be represented with multiple, equivalent expressions. For example, the standard form and factored form of a particular quadratic have the same systemic structure, because they represent the same mathematical object. Hoch and Dreyfus (2004) also seem to be drawing upon this meaning in their discussion of algebraic structures. They note that the external appearance of an expression can be transformed to reveal the internal order. This idea of internal order aligns with the systemic definition of structure, while the external appearance they refer to is similar to surface structure. They go on to state that equivalent expressions have the same structure, but are different interpretations of that common structure.

In addition to the distinction between surface and systemic structure, a second issue that often arises in definitions of structure is about whether the structure (be it surface or systemic) resides in the expression or equation itself, or if it is constructed by the person who is making sense of the expression or equation. Those who use the idea of systemic structure (Kieran, 1989; Hoch & Dreyfus, 2004) seem to view structure as an inherent property of an expression since it is based on the mathematical system the expression or equation belongs to rather than the way an individual views it. On the other hand, Hoch and Dreyfus (2004) define structure in terms of an “analysis,” which implies that structure is the result of cognitive activity taken by the person
observing the expression. Similarly, Ruede (2013) recognized the central role of human
cognition in the creation of structure. Rather than define structure as a property of an expression,
Ruede (2013) instead talked about “personal structure” (p. 388), which is the product of the
activity through which a person recognizes and forms relationships between the parts of an
algebraic expression. By defining structure as a result of cognitive activity and not a
characteristic of an expression itself, Ruede allows for variations on how people perceive
structure in a single expression.

I am using the following definition for structure in this study: *structure* is the hierarchy of
parts and the relationships between them that are perceived by an individual. With this definition,
I include the term hierarchy to indicate that structure is not limited to a single level of an object;
rather, an object itself can be seen as a whole, as well as its parts or parts of its parts. Note that
my definition is more closely related to the idea of surface structure rather than systemic
structure. This will allow me to analyze the different ways students group symbols and
operations when they view expressions or equations and simplify them. Also, my definition
acknowledges that structure is not a property of an object, but is constructed in the act of
perceiving it. Thus, a structure does not exist separate from the person who is viewing that
structure. Treating structure as cognitively constructed is necessary for my research, because I
hypothesize that the groupings and structure that students perceive in an algebraic expression
will be different depending on the person viewing it. In this research, I use the term *algebraic
structure* to refer to structure in an algebraic expression.

**Structural Reasoning Definition**

In addition to analyzing how students perceive algebraic structure, it is important to think
about how students then reason about that structure. *Structural reasoning* is the action of using
structure to reason about and manipulate mathematical objects and expressions. I will analyze structural reasoning in terms of three actions—identifying, matching, and evaluating. Note that these are not the only actions that are mentioned when scholars talk about structural reasoning, but these are likely the actions that will be helpful in the scope of this study, i.e., in thinking and reasoning about rational expressions.

The first action in structural reasoning is identifying. To engage in structural reasoning, students must first identify algebraic structures in an expression or equation. This may include viewing the expression or part of an expression as a single object, or splitting an expression into parts and seeing factors within those parts. For example, when looking at the expression $2x(x - 1) + 3(x - 1)$ students could think of the expression as the two terms $2x(x - 1)$ and $3(x - 1)$ being added together. Within those terms, they could see factors of those terms and identify that they each have a common factor of $x - 1$. Other students might not think about the expression in this way; rather they might just see the expression as a string of symbols and numbers. The action of identifying has been discussed by other authors, including Hoch and Dreyfus (2004), who recognize two abilities of structure sense as being able to “see an algebraic expression or sentence as an entity” and to “divide an entity into sub-structures” (p. 51). Harel and Soto (2017) also referred to the action of identifying in their description of structural reasoning as the ability to “look for structures” and “recognize structures” (p. 226).

The second action in structural reasoning is matching. Once students have identified structure in an expression or equation, they then have to match that structure with a known rule or relationship. For example, in the expression $2x(x - 1) + 3(x - 1)$, students might see the expression as being two terms with a common factor of $x - 1$, and could match this expression with the distributive property of $ba + ca = (b + c)a$, where $a = x - 1$. This matching would allow
them to later manipulate the expression by factoring. On the other hand, students might see the subexpression $2x(x - 1)$ as a match for the distributive property before the distribution, which would mean they could distribute the $2x$ into the $(x - 1)$ term and the same for the $3(x - 1)$ term. These are both examples of taking the identified structure of the expression and matching it with a known rule. Hoch and Dreyfus (2004) referred to this action as “recognizing an algebraic expression or sentence as a previously-met structure” (p. 51).

The third action in structural reasoning is evaluating. When engaging in structural reasoning, students must evaluate whether the identified structure and match will be productive. It is important that students know the goal of the problem they are trying to solve, because that goal dictates which structures and matches would be more or less productive in that case. This evaluation can be made before or after the manipulations. For example, in the expression $2x(x - 1) + 3(x - 1)$ a student could evaluate the two matches outlined above. Using the first match of seeing the common factor of $x - 1$ in each term, they could rewrite the expression as $(2x + 3)(x - 1)$. Using the second match, they could distribute into both terms to get $2x^2 - 2x + 3x - 3$. Both of these are correct structures and matches, but one might be more effective than the other, depending upon the problem being solved. If the problem required students to write the expression in factored form, then the first match would be more productive; however, if the problem required students to write the expression in standard form, then the second match would be more productive. Hoch and Dreyfus (2004) refer to this third action as “recognizing which manipulations it is useful to perform” (p. 51). Overall, these three actions of identifying, matching, and evaluating will help me classify the types of structural reasoning students are engaging in.
Note that the action of performing the actual computation or manipulation is not present in this framework of structural reasoning. Computation is an action separate from structural reasoning that we will call manipulating. Manipulating is not identified as a structural reasoning action because manipulating involves making changes to the symbols in the expression or equation after the structure has been identified and matched to a rule. This distinction between structural reasoning and manipulating is also made in the literature. For example, Hoch and Dreyfus (2004) made a distinction between students using structure sense and “displaying manipulation skills” (p. 306). Despite this separation of manipulation from structural reasoning, examining the manipulations that students perform can provide insight into the structures they identified and the rules they matched these structures to.

The presence of one or more of the actions identifying, matching, and evaluating signifies that a student is engaged in structural reasoning. However, not all three will necessarily be present in every instance of structural reasoning. For example, a student may identify structures and match them to a particular rule without evaluating whether the match is helpful to solving the problem. In addition, these actions can occur multiple times in any order as a student solves a problem. For example, if a student starts by evaluating and determines the goal of the problem is to factor, then they know what structures to look for and what types of matches to make. When identifying structures, they would be looking at specific terms and the factors in those terms. Students would most likely be matching these factors with the distributive property in order to get a factored form. In this case, the evaluation at the beginning informs the ways identifying and matching occur.
CHAPTER 3: LITERATURE REVIEW

The research on the structural reasoning actions of identifying, matching, and evaluating is limited. One of the few studies on the action of identifying was conducted by Hoch and Dreyfus (2004). In this study, they compared students’ ability to solve equations involving multiple terms, including rational expressions with and without brackets. They wondered if adding brackets that highlighted useful structures (but were not necessary to interpret the meaning of the expression) would cause students to be more likely to identify and use those structures. They found that adding brackets to equations increased students’ identification of useful structures in the expressions by only 11% (17.7% compared to 6.3%). In fact, many students eliminated the brackets almost immediately, making the brackets unhelpful in their identification of structure. This study suggests that many students do not know how to identify useful structures in algebraic expressions, and that the strategy of providing visual cues, such as brackets, in the arrangement of the symbols does little to ameliorate this problem.

The action of matching was addressed by Matz (1982), Vega-Castro et al. (2012), and Kirshner and Awtry (2004). Matz (1982) referred to the matching action as extrapolation techniques. These extrapolation techniques can be very useful, but also can be the cause of many errors when students attempt to extrapolate rules onto expressions when they do not actually apply. Matz (1982) says that to extrapolate correctly, often “top-level matches” (p. 28) must be made instead of “literal-for-literal matches” (p. 28). She describes top-level matches as being able to project a rule onto an expression as a whole, rather than attempting to apply a rule to part of an expression. An example of using literal-for-literal matches instead of top-level matches would be assuming that because \( \frac{ab}{b} = a \), that \( \frac{2x - 6}{x - 3} = 2 + 2 \). In this case, the student applied the rule to \( \frac{2x}{x} \) and \( \frac{-6}{-3} \). If students were to apply top-level matches, they would recognize that they
need the numerator to be a product of two factors, one of which is the same as the denominator, and would factor the numerator, yielding \(\frac{2x - 6}{x - 3} = \frac{2(x-3)}{x-3}\). Factoring the numerator creates a top-level match to the equation \(\frac{ab}{b} = a\), and allows the student to use the rule to conclude that \(\frac{2(x-3)}{x-3} = 2\). Note that students who make top-level matches not only make different matches than those who make literal-for-literal matches, but also identify different structures.

Vega-Castro et al. (2012) gave students a complex rational expression, had them simplify it, and then asked them to create a rational expression with similar structure. The researchers found that over four different tasks, many students struggled to create a new expression that kept the same structure, especially when given more complicated expressions. Students who used literal-for-literal matches to create similar rational expressions were successful in recreating rational expressions with simple structure. But once compound terms were introduced, these students using literal-for-literal matches were unable to create rational expressions with similar structure because they made incorrect matches. Overall, many students were able to successfully recreate simple rational expressions where a literal-for-literal match could be made, while the expressions that contained compound terms and required students to make top-level matches were rarely recreated correctly by students. Therefore, when matching it is important that students are able to make top-level matches.

In addition to extrapolating incorrectly by using literal-for-literal matches instead of top level matches, it is common for students to match structures to incorrect rules that appear to be true but actually are not. Often these untrue rules are caused by incorrect extensions of linear patterns. For example, after learning that \(2(a + b)\) is equivalent to \(2a + 2b\), many students will make the mistake of thinking \((a + b)^2\) and \(a^2 + b^2\) are equivalent. This error was discussed by
Matz (1982) and also by Kirshner and Awtry (2004), who explained this error in terms of visual salience. Visually salient rules are those that have a visual coherence that makes the left and right sides look naturally equal. They identify rules such as \( x(y + z) = xy + zy \) as visually salient and rules like \((x + y)^2 = x^2 + 2xy + y^2\) as non-visually salient. Students sometimes use visual salience to create their own rules for matching, such as equating \((a + b)^2 = a^2 + b^2\), and then match that rule to the expression that they are working with. The match they make may be either a literal-for-literal or a top-level match. Thus, errors in matching may be due to using incorrect rules, and not just to the incorrect matching of parts in the rule to parts in the expression.

The action of evaluating was explored in a study by Ruede (2013). In this study, he examined the personal structures formed in rational expressions by both experts and novices. The novices in his study were 9th and 10th year students in Switzerland who had at least 3 years of algebra instruction, and the experts were mathematics teachers with degrees in mathematics and many years of teaching experience. He found that experts formed personal structures in many different ways. Thus, there was not one correct interpretation of an expression, but there appeared to be some that were more effective than others. He found that the less effective structures were based purely on the syntactical attributes of the expression or when the subjects changed the expression just to change it with no goal in mind. The more sophisticated structures were those that helped reinterpret and classify the expressions to reach a specific end goal. Ruede found that experts usually used these latter methods of structuring, while novices rarely did and usually used the less sophisticated methods. He argued that teachers should talk about personal structures in mathematics classrooms more explicitly and often so students develop the ability to solve and simplify these expressions in more sophisticated, creative ways instead of following rote procedures. Thus, there is a need to identify ways of reasoning structurally that are more
accessible to students and lead them to evaluate whether structures they are identifying are useful.

Overall, research suggests that while structural reasoning is uncommon among students, those who engage in structural reasoning get answers more quickly and accurately (Hoch & Dreyfus 2004; Hoch & Dreyfus 2006). This was seen in Hoch and Dreyfus’ (2004) study on the use of brackets in algebra problems discussed previously, as they found that those students who used structural reasoning were better able to manipulate rational expressions correctly. They also did a study examining the relationship between structural reasoning, which they referred to as structure sense in their study, and manipulation skills when dealing with polynomial equations and expressions (Hoch & Dreyfus, 2006). In this study, they found that about half of the students used structure sense, and those who did made less errors. When students used structure sense, they ended up solving simpler problems that required less steps compared to those who did not use structure sense. In other words students who used structure sense were able to convert the expressions they were working on into ones that required fewer and less complex computations. This could explain why the students who used structure sense made less errors, and why those students who were classified as using low structure sense all displayed low manipulation skills.

Note that the researchers classified students as having low manipulation skills if they made a significant number of errors in their computations. But the students who did not use structure sense often ended up working with expressions that required them to do different, more complicated manipulations than the manipulations used to simplify expressions that had been modified using structure sense. These findings help illustrate that students need structure sense because it allows them to solve problems in simpler, more efficient ways that require less complicated manipulations, often leading to less errors.
In a separate line of research, scholars have examined the connection between how students think about complex algebraic expressions and similar arithmetic expressions, with the hope that teachers might be able to leverage students’ arithmetic understanding to help them reason about algebraic expressions (Booth, 1989; Kieran, 1989; Linchevski & Linvneh, 1999). This research has shown little correlation between students’ success in solving arithmetic expressions and their success in solving structurally similar algebraic problems. Specifically, Yantz (2013) studied how students relate solving rational expressions to rational numbers. She gave students two problems to simplify that had the same structure, but one was an algebraic expression and the other was a numeric expression with no variables. She found that students’ success in simplifying the algebraic expressions was not related to their success in simplifying the numeric expression. This line of research suggests that the body of literature on rational numbers is insufficient to explain students’ reasoning with algebraic rational expressions, and that more research is needed to find out how students perceive and reason about rational expressions and equations.

Overall, the research on structural reasoning shows that students lack the ability to consistently use structural reasoning by identifying, matching, and evaluating structures effectively. It also shows that though it is rare, structural reasoning leads to more efficient and correct solutions. However, the current research on structural reasoning mostly focuses on what students are unable to do; it consists of multiple deficit studies that show students are not using sophisticated or expert ways of reasoning structurally. This research does not tell us how high school students are actually thinking about structure and using structural reasoning to make sense of and manipulate rational expressions, but rather that high school students are not thinking about structure the same ways as experts or researchers. Research is needed to identify what
understandings of structure and ways of structurally reasoning high school students have, so that teachers might know what student conceptions they might build on to help students develop the more advanced structural reasoning sought after in the research literature. This type of research requires a close examination of the initial or non-normative ways high school students approach rational expressions, including what structures they identify, what matches they make, and how they evaluate these matches. Thus, my research question is the following: How do high school students reason structurally as they simplify rational expressions or solve equations involving rational expressions?
CHAPTER 4: METHODS

In this study, to investigate the way students were reasoning about rational expressions, I conducted semi-structured interviews with participants where I presented them with tasks in order to get a view of their use of structural reasoning. During these interviews, I had them explain their thinking as they were solving problems involving rational expressions and evaluating hypothetical student solutions. This enabled me to analyze instances when they were using structural reasoning to identify, match, and evaluate as they were simplifying and solving rational expressions and equations. In this chapter, I describe my data collection process by discussing the setting, participants, and data types used, and then outline how I conducted my data analysis.

Data Collection

Setting and Participants

In the *Common Core State Standards for Mathematics, Integrated Track* (NGA Center, 2010), students are expected to interpret the structure of rational expressions in the Secondary Mathematics III course. Thus I chose participants in high school who have already completed this course and were enrolled in an AP Calculus course. This allowed me to interview students who have already learned to simplify rational expressions, solve equations involving rational expressions, and graph rational functions. I chose to interview students who were taking the subsequent course to the one where they learn to interpret structure in rational expressions to give the students time and experience using rational expressions beyond the rational expression unit. I anticipated that this time would allow students to get a better idea of what structures were useful in rational expressions as they saw them in a variety of contexts. Also, I was concerned
that if I had interviewed students right after they learned about rational expressions, they might use more rote procedures rather than engaging in structural reasoning.

Of those students taking the AP Calculus course, I chose participants who showed varying success and methods when solving algebra problems. Since the goal of this study is to get an idea of how students view and use structure, I wanted to make sure I choose a variety of participants who represent all levels of students at the AP calculus level, from those who are very successful to those who struggle with algebra. By interviewing students with varying levels of success, I hoped to get a broader view of how students who have taken advanced high school mathematics courses use structural reasoning. To choose my participants, I recruited volunteers from a calculus class in a public Utah high school to do an initial task in their math class that was similar to the interview Task 1 (see Appendix A). Based on their work on this initial task, I selected six participants to interview who displayed a variety of success and methods used. I also chose students who had communicated their thinking clearly in this initial task. It was important that the participants I selected had good communication skills so I could better identify the structural reasoning they were engaging in.

In these interviews, I wanted to make sure I interviewed enough participants to get an answer to my research question of how students reason structurally about rational expressions. I was able to conduct six interviews in which students cooperated and showed a variety of reasoning. Six participants were few enough to reasonably analyze their thinking effectively, but enough to get an idea of how students might vary in using structural reasoning. Though I anticipate I would have seen a wider variety of structural reasoning from interviewing more than six participants, interviewing more than six participants would be beyond the scope of a master’s thesis and six was sufficient to answer my research question.
**Data Types**

To collect data for this study, I conducted a single interview with each of the six participants. Each interview was 45 to 60 minutes long. The goal of these interviews was to gather samples of students’ structural reasoning with rational expressions and solicit a wide range of structural reasoning so I could compare these instances of structural reasoning. I found that one, hour long interview was enough time to gather a variety of instances of structural reasoning from each student. I conducted one interview with a single participant first and analyzed the data so I could make sure that my methods work as I anticipated. I found that my methods worked well with my first participant, so I did not need to adjust my methods based on that first interview.

Each interview consisted of two main parts: solving tasks and analyzing hypothetical student solutions (see Appendix A for the tasks). In the solving tasks part of the interview, I presented participants with three different rational expressions to simplify or rational equations to solve, one at a time. During the first two interviews, before having them actually simplify or solve, I asked them what their initial general solution strategy is. I thought that getting an idea of how they were approaching the task would help me make sense of what structures they were looking for so I could better anticipate how they were viewing the rational expressions and ask useful follow up questions. After hearing their general strategy, I had them actually solve the task and asked questions to hear their reasoning as they engaged with the task. During the first two interviews, participants struggled to explain their initial general solution strategy, and asked if they could just solve the task. Because of this difficulty, I determined that the initial question about their general solution strategy was unhelpful. Consequently, I had the other participants just solve the tasks, asking them clarifying questions as they went. I repeated this procedure for
all three tasks. I chose three tasks because they each required different solution paths, and thus might reveal a different type of structural reasoning. Two of the tasks asked the participants to simplify an expression, while the other task asked them to solve an equation.

The first task had the participant solve an equation with two rational terms in it. Solving the equation was different from just simplifying an expression because it required participants to see the structure of not only a single expression, but two expressions with an equality relationship. The second task asked participants to simplify an expression that involved the addition of two rational expressions, where one of the rational expressions had a rational term within it. Due to the complexity of this expression, there were multiple structures that could be identified. The third task asked them to simplify a rational expression that had factors in the numerator and denominator that could be canceled and other terms that could not be canceled until after being factored. This task allowed for a variety of potential top level and literal-for-literal matches, which could lead to variety in the ways participants engaged in structural reasoning. It also allowed me to see if they could differentiate between what matches were useful when terms were being divided versus when factors were being divided.

Throughout these tasks, I examined the participants’ use of structural reasoning by looking at how they were identifying, matching, and evaluating. I determined the structures they were identifying and matching based on the manipulations they made and by asking follow up questions when their manipulations were not clear to me. To assess their use of evaluation, I asked follow-up questions of why they chose the manipulation and structures they did and how that action helped them solve the task.

After solving the three tasks, participants were given hypothetical student solutions for two of the tasks they solved in the first part of the interview. I created two hypothetical student
solutions for each of the two tasks, one correct and one problematic. The solutions were based on
many of the solutions I have seen students construct as they reason about rational expressions. I
only presented hypothetical student solutions for two of the tasks so the interview was not too
long. By only asking about two, I had time to go into detail and ask probing questions about the
hypothetical student solutions rather than feeling rushed. I chose two instead of one because it
allowed me to see if students were consistent in their structural reasoning or not. I presented a
method that was different from the one they originally used to see if they could use structural
reasoning to make sense of the structures and matches used in the hypothetical solution.
Presenting a different solution helped me understand their structural reasoning abilities and if
they were able to make sense of another way of viewing and reasoning about the structure of the
expressions in the tasks.

The hypothetical student solution I chose to present depended on which method the
participant originally used to solve the task. If they initially solved the task correctly, I presented
them with an incorrect method that used literal-for-literal matches. This way, I could see if they
were tempted to say that a different, incorrect method worked. If they made a structural
reasoning error in their own solving method, I presented a correct hypothetical solution for them
to analyze. This way, I could see if they were able to make sense of the correct structures that
could be used to solve the task. This second part of the interview helped answer my research
question because as I presented hypothetical student solutions, I was able to assess if participants
realize there were multiple ways of interpreting structure and see if they could make sense of or
catch the errors in these other solutions.

During the interviews, I collected data by video recording the interview so I could later
analyze both the written work the participant wrote down and the verbal explanations they give.
Recording the interviews was important because I used the gestures, written work, and verbal reasoning used by the participants in my analysis. I recorded over the participants’ shoulders so their faces were not shown, but I could still get a good view of their written work.

**Data Analysis**

I analyzed this data by conducting a video analysis on the interviews based on the structural reasoning actions of identifying, matching, and evaluating as well as the manipulations they made. The unit of analysis I used was a participant’s attempted match. Attempted matches refers to the instances when a student identified a structure and a rule then tried, successfully or unsuccessfully, to match them to each other. I chose this unit of analysis because these segments of data often included the related identifying, matching, and evaluating actions that comprised a single decision made through structural reasoning. By analyzing solutions one match at a time, I could still analyze the correct structural reasoning that took place after an error occurred. In my analysis, I also took into consideration the matches made before and after each instance to determine the productivity of their matches. Using this unit of analysis, I was able to create a coherent description of all of the structural reasoning that occurred throughout the task, particularly if the participant moved back and forth between different ways of structural reasoning in their solution.

I took each of the participant’s attempted matches and analyzed them by creating a diagram that outlined the structures they identified, matches they considered with those structures, and evaluations they made. In these diagrams, I illustrated the different structures the participants identified by boxing the different parts of the expression or equation that they grouped together. Since students often identified multiple levels of structures, there were often nested boxes to show all the levels of the substructures students identified. I diagramed matches
by writing out the rules students attempted to match these identified structures to. When writing out the rules, I used letters to represent each of the structures in the original expression that the student was connecting to the rule. I coded each of these attempted matches by whether the student was actually able to make the match and whether the rule and structure they used was mathematically correct or not. I used codes SC, UC, SI, and UI to signify successful correct, unsuccessful correct, successful incorrect, and unsuccessful incorrect matches, respectively. The distinction between correct and incorrect matches was helpful during my analysis because it allowed me to easily recognize errors students made. The distinction between a successful and unsuccessful match was helpful because I found that students often considered correct rules, but were unable to match them to the structures they saw in the rational expression. Those considered matches helped me understand their structural reasoning. I diagramed the participants’ evaluation of their matches and rules by writing quotes from the participants next to the outlined match that indicated the participants were engaged in evaluating. Due to the structure, these diagrams were a clear way to illustrate the hierarchical nature of structural reasoning that was taking place, enabling me to quickly analyze their reasoning without needing to go back to the original videos. An example of such a diagram is shown in Figure 1.
After I diagrammed each participant’s solutions, I was able to easily look at each diagram to see all the structures students identified, matches they attempted, and evaluations they made. Next, I coded the data by relating the attempted matches to characteristics of those matches, such as correctness of the matches, correctness of the matched rules, level of matching (i.e., high-level matches vs literal-for-literal), and blocks and supports of structural reasoning. By coding the data based on these characteristics, I was able to start seeing common themes emerge in the participants’ structural reasoning.

Next, I went back to the data and classified which instances related to these themes. Two such themes that emerged were using structures that identified division as the highest level in a rational function and errors caused by literal-for-literal matches across the fraction with terms. As I went back to the data, I identified all the attempted matches related to each of these two
themes. For example, the first instance in Figure 1 aligns with the theme of identifying the division of the rational expression as the highest level operation.

I also organized these themes across my three structural reasoning actions of identifying, matching, and evaluating structures. For example, the first theme of identifying structure with division as the highest level operation was related to the structural reasoning action of identifying. The second theme of errors caused by literal-for-literal matches across the fraction with terms was organized under matching. Another theme that emerged was students comparing complicated expressions to simpler ones with the same structure to test whether rules worked or not. This theme was organized under the structural reasoning action of evaluating, because it was a way that students were evaluating their rules.

After analyzing these themes, some ended up collapsing into one new theme. For example, the two themes identified above—structures with division as the highest level operation in a rational expression and errors caused by literal-for-literal matches across the fraction with terms—turned into the collapsed theme of operational hierarchical reasoning. I collapsed these two themes because I realized that they were both caused by students’ ability to identify structures and create matches based on the highest level operation present in the expression. After collapsing my themes, I once again went back to the data to make sure these themes aligned with the structural reasoning instances from the tasks.

Overall, my analysis consisted of coding the data based on characteristics, then finding themes across these characteristics. Then I organized these themes by structural reasoning actions and collapsed them into bigger, overarching themes. At this point, I was able to write about these themes and use them to identify and describe the important characteristics of students’ structural reasoning that were elicited by the tasks.
CHAPTER 5: RESULTS

In this section I discuss three main results that were found from classifying the themes that emerged from analyzing the samples of structural reasoning. The first result is about the matching process, which outlines a theory about the way students identify structures and make matches to those identified structures. The second result is the outcomes of the matching process, where I discuss patterns in the correct and incorrect matches students found. The third result is the role evaluation played in this matching process, where I discuss the main types of evaluation that occurred in the interviews. After presenting these three results, I discuss these findings and how they answer my research question.

The Matching Process

Structure Identification and Operational Hierarchical Reasoning (OHR)

When solving the tasks in the interviews, participants started by identifying structures in the expressions. The most useful structures participants identified were structures that seemed to take into account the hierarchy of operations present in the expressions. This included identifying which operations were the highest level operations (HLO) within each structure and breaking up structures into substructures that were related by the HLO in the original structure. I define operational hierarchical reasoning (OHR) as the recognition of the structure of the expression that correctly takes into account the HLO of each level of structure and breaks up the expression into substructures using these HLOs. I use the term OHR structures to refer to structures participants identify using OHR. To illustrate OHR, consider the rational expression with multiple terms in the numerator and denominator in Figure 2. The operation of division is the HLO in the expression because the entire numerator is being divided by the entire denominator, leading to the first level structure. Within the numerator and denominator groups, the second
level of substructures would be the terms. This substructure is made by recognizing that the operations of addition and subtraction between terms are the HLOs within the numerator and denominator. A third level of substructures could be seen within the term $2x$, breaking down this term into its factors. I demonstrate identified structures using diagrams like in Figure 2.

**Figure 2**
*Example of OHR Structure*

This structure is an example of an OHR structure. In contrast, the term *non-OHR structures* refers to structures students make without using OHR. For example, a student could use a structure that grouped the terms across the fraction (Figure 3) which would be a non-OHR structure because they are not making substructures that correctly take into account the HLO of division.

**Figure 3**
*Example of Non-OHR Structure*
Rule Banks and the Matching Process

After identifying structures, participants tried to match these structures to a known rule. Participants seemed to have rules that they considered matching with certain structures, and I will refer to the set of rules that participants considered for a particular structure as a *rule bank*. Note that a rule bank corresponds to a particular structure, so rule banks vary from structure to structure. Within a rule bank, the student might have multiple, one, or possibly no rules to match to the structure they identified. I used participants’ responses to infer which rules were in their rule banks. However, I also realized that participants may have considered and rejected rules without giving me any indication of having done so. Consequently, it is possible that participants occasionally had additional rules in their rule banks that I was unaware of.

Participants constructed matches between a structure and a rule by identifying a one-to-one, onto mapping between parts of the structure and parts of the rule. I use *match* to refer to a structure that has been paired with a rule, and *matching process* to refer to the set of actions a student engaged in to construct a correspondence between the rule and the structure. When participants were unable to construct a correspondence to create a match, they often started the matching process over again with another rule, structure, or both.

Participants often used the matching process in an iterative manner. This occurred when participants identified a structure and a rule, but could only create a partial correspondence between them. Participants would then try to manipulate the non-matching part of the structure so that it would fit the rule. They did this by trying to construct a second match, this time between the non-matching substructure and another rule that resulted in the desired form. They could only make the first match with the original structure if they were able to make the second match with the substructure. The matching process often went through one or two such
iterations, but could go through more. For example, in a rational expression like \( \frac{x+2}{x^2-5x-14} \), the matching process might involve identifying the original structure of the numerator divided by the denominator and looking for a correspondence to the rule \( \frac{a}{ab} = \frac{1}{b} \). In order to make a match between this structure and rule, the denominator would have to be in factored form. Then, a student might identify the substructure in the denominator of three terms being added together and consider the rule \((x \pm a)(x \pm c) = x^2 \pm (a + b)x \pm ab\). If they were able to match the trinomial substructure to the factored form, then they could also match the original structure to \( \frac{a}{ab} = \frac{1}{b} \). If they were unable to make this match with the substructure, then they also would be unable to match the original structure. This example is illustrated in Figure 4.

Figure 4

Example of the Iterations in the Matching Process
Outcomes of the Matching Process

When participants were able to construct a match, there were two cases that occurred. They either made a correct match or an incorrect match. I first discuss correct matches, and then incorrect matches.

Correct Matches

Correct matches required participants to use OHR structures and find a correct rule to match these OHR structures to. All participants showed evidence that they were capable of constructing correct matches. One example of this was in Task 1 when participants were solving $\frac{7}{x+2} = 2 - \frac{3}{x-2}$. Every participant was able to identify subtraction as the highest operation on the right side of the equation, and use this HLO to view the structure of the equation as one term equal to two terms being subtracted (Figure 5).

Figure 5

Task 1 First Level of Structure

\[
\begin{align*}
\frac{7}{x+2} & = 2 - \frac{3}{x-2} \\
\end{align*}
\]

This identified structure led to different manipulations depending on the match the participants made. Four of the participants rearranged these terms by adding $\frac{3}{x-2}$ to both sides of the equation. In this case, they matched the equation to the rule, if $a = b - c$, then $a + c = b - c + c$. The other two participants multiplied both sides of the equation by $x + 2$, distributing this multiplication to each of the three terms they identified. These two participants actually saw a second level of structure, identifying the division within the term on the left side of the equation (Figure 6).
Figure 6

Task 1 Second Level of Structure

Then they matched this second level with the rule of $\frac{a}{b} \cdot b = a$, using inverse operations to cancel out the division by multiplying by $x + 2$. They also matched the multiplication to the first level structure, realizing they needed to multiply each term on the right side of the equation by $x + 2$ using the distributive property, $a(c + d) = ac + ad$. This example demonstrates that all participants were capable at times of using OHR and making correct matches, as they were able to group symbols together in a useful way using the correct hierarchy of the operations in this equation, then correctly match these structures to rules from their rule banks.

Incorrect Matches

Incorrect matches occurred when participants did not correctly match a structure and a rule. This happened in two different ways. First, sometimes participants identified OHR structures but matched the structures to a rule in their rule bank that was incorrect. In this case their structure was correct, but their rule was wrong. Incorrect rules are rules that are not mathematically correct, such as $(a + b)^2 = a^2 + b^2$. Second, at times participants identified non-OHR structures and matched them (usually with a literal-for-literal match) to a correct rule. In this case, the match was incorrect because the rule did not actually match the OHR structure of the expression. I give examples of both types of errors below.

OHR Structures and Incorrect Rules

Some participants had rules in their rule bank that were mathematically incorrect. One example of this was seen when Parker was solving Task 2, where he was trying to simplify the
expression \( \frac{1}{x-1} - \frac{1}{x^2-1} \). He seemed to use OHR to identify the OHR structure of seeing the subtraction between \( \frac{1}{x-1} \) and \( \frac{1}{x^2-1} \) as the highest level operation, which led him to view the expression as two terms being subtracted. Within those terms, he saw the division as the next highest level operation. The third level he identified was the terms within the denominator (Figure 7).

**Figure 7**

*Structure Identified by Parker in Task 2*

He matched these second and third level structures to the incorrect rule \( \frac{a}{a} + \frac{c}{c} = \frac{a}{a} + \frac{a}{a} \), which led him to manipulate the expression to get \( \frac{1}{x-1} - \frac{1}{x^2-1} = \frac{1}{x} - \frac{1}{x} - \frac{1}{x^2} - 1 \). To justify his manipulation, he said that he knows \( \frac{3x+x}{x} = 3 + 1 = 4 \), which he inferred to mean that his manipulation worked. He seemed to conflate the two rules \( \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \) and \( \frac{a}{b+c} = \frac{a}{b} + \frac{a}{c} \), the second of which is incorrect. In this case, Parker had a rule in his rule bank that was incorrect, so even though he used OHR, he still made an error because of this incorrect rule he had in his rule bank. Other participants were able to match this same structure with correct rules, such as \( \frac{a}{b} = \frac{ac}{bc} \) to make common denominators between the two terms.

**No OHR**

Though participants often used OHR to identify OHR structures, there were other times that participants failed to use OHR at all. In these cases, participants failed to correctly take into
account the hierarchy of the operations in the equation and consequently identified non-OHR structures. These non-OHR structures seemed to be caused by one or more of three different factors. The first factor was an operation mismatch where the participant treated addition/subtraction like multiplication/division. This usually looked like using a rule that would work for factors in a fraction that were being multiplied or divided, but applying it to terms that were actually added or subtracted. The second factor was the presence of “seductive” features—visual similarities or instances of sameness participants noticed between elements of an expression—that seemed to override participants’ OHR. The third factor was the belief that it is acceptable to use literal-for-literal matches rather than top-level matches. Recall that Matz (1984) describes top-level matches as being able to project a rule onto an expression as a whole, rather than attempting to apply a rule to part of an expression (a literal-for-literal match). In the interviews, it was impossible to identify which of these three factors caused particular instances of matching with non-OHR structures, because all three factors seemed to play some part in the matches participants made with non-OHR structures. Below, I give examples of participants forming matches with non-OHR structures and explain how each of these three factors could have played a part in the participants’ use of non-OHR structures.

Hope identified a non-OHR structure in Task 2 when she was presented with the problem of simplifying $\frac{1}{x-1} - \frac{1}{x^2-1}$. She saw the first level structure of two terms separated by subtraction, then within the first term she grouped the 1 in the numerator with the $x$ in the denominator, and the $\frac{1}{x}$ in the denominator as a separate group (Figure 8).
She then matched these two groups of $\frac{1}{x}$ to the rule $\frac{1}{x} = x^{-1}$, leading her to manipulate the first term in the expression from $\frac{1}{x-x^{-1}}$ to $\frac{x^{-1}}{x-x^{-1}}$. In this case, Hope was not using OHR because she was not taking into account the fact that the division was the second highest operation (the whole numerator divided by the whole denominator), and subtraction between the terms in the denominator was the HLO of a third level. This non-OHR structure seems to have all three factors present—operation mismatch, seductive structure, and literal-for-literal matches. If the terms in the denominator had been multiplied, the structures she had identified would have been OHR structures. Therefore, her error could have been caused by simply conflating the rules for subtraction with those for multiplication, an operational mismatch. There was also a seductive feature in this expression—the two $\frac{1}{x}$’s. It was tempting to just match each $\frac{1}{x}$ to $\frac{1}{x} = x^{-1}$ rather than trying to take into account the operations present. It also could have been the result of thinking that she was allowed to apply the rule $\frac{1}{x} = x^{-1}$ to parts of the expression, making a literal-for-literal match.

Another example of a non-OHR structure was seen in Task 3 when multiple participants identified structures by matching common terms in the numerator and denominator. For example, Riley constructed structures by grouping the like terms across the fraction (Figure 9).
She proceeded to match these structures to the rule $\frac{ab}{ac} = \frac{b}{c}$ to cancel out the variables, simplifying the expression to $\frac{1}{1} - \frac{4}{3} - \frac{2}{5}$. Riley was not using OHR in this case because she failed to use the fact that the division was the HLO to determine the structure she was identifying. The non-OHR structure could have been a result of an operation mismatch, where she just thought of the addition and subtraction signs as having some of the same properties as multiplication. In this case, if the terms had been multiplied this would have been an OHR structure and correct match. Riley’s error could have also been influenced by seductive features—the presence of the same like terms, written in the same order, across the numerator and denominator. The seductive, non-OHR structure Riley identified led to a literal-for-literal match because while it is true that $\frac{x^2}{x^2} = 1$, she matched this rule to part of the expression rather than making a high-level match that matched the structure based on the HLO of the entire expression. Her identified structure could be based on the belief that she could correctly apply this rule to parts of the expression.

Overall, four of the six participants made incorrect matches at least once in their interview. These incorrect matches were a mix of incorrect rules and non-OHR structures, but the majority (12/17) of these incorrect matches were caused by non-OHR structures that seemed
to be influenced by the above three factors: operational mismatches, seductive structures, or the belief that literal-for-literal matches worked.

Table 1

*Summary of Students’ Correct and Incorrect Matches*

<table>
<thead>
<tr>
<th>Participant</th>
<th>Task 1 Correct Matches</th>
<th>Task 1 Incorrect Matches</th>
<th>Task 2 Correct Matches</th>
<th>Task 2 Incorrect Matches</th>
<th>Task 3 Correct Matches</th>
<th>Task 3 Incorrect Matches</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hope</td>
<td>15</td>
<td>1</td>
<td>7</td>
<td>2</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>Parker</td>
<td>13</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>7</td>
<td>2</td>
</tr>
<tr>
<td>Dustin</td>
<td>12</td>
<td>1</td>
<td>8</td>
<td>3</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>Jake</td>
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<td>0</td>
<td>4</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>Riley</td>
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<td>4</td>
<td>1</td>
<td>4</td>
<td>2</td>
</tr>
<tr>
<td>Curtis</td>
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<td>0</td>
<td>10</td>
<td>0</td>
<td>5</td>
<td>0</td>
</tr>
</tbody>
</table>

**Failure to Construct a Match**

Participants were not always successful in constructing a correspondence between structures and rules. This often happened when participants seemed to be “missing” a rule from their rule bank that would have fit their structure and made their solution simpler. In this section, I discuss missing rules and what happened when participants were unable to make a match due to missing these rules. I will also discuss the characteristics of the participants who refused to make non-OHR structures when they were unable to make matches.

**Missing Rules**

Many times in the interviews, participants attempted to match their OHR structures to rules, but seemed to be missing a rule in their rule bank that would allow them to make progress
on the problem. One example of this was in Hope’s interview when she was trying to simplify

\[
\frac{x^2-4xy+4y^2}{2x(x^2+3xy-10y^2)}
\]

She used OHR to identify three levels of structure in the expression (Figure 10).

The first level she saw was the division of the numerator and denominator. The second level she saw was the factors in the denominator. Lastly, the third level was the terms within the numerator and within the second factor in the denominator.

**Figure 10**

*Structure Identified by Hope in Task 3*

Hope seemed to look for a correspondence between this first structure and the rule \( \frac{ab}{ac} = \frac{b}{c} \) where she needed to have common factors in the numerator and denominator to be able to cancel factors out. To create this match, she knew she needed to find a way to factor the trinomials in the numerator and denominator. The first rule she attempted to match to the trinomial structure was \( ab + ac + ad = a(b + c + d) \), but she realized that there were not common factors in all three terms. She decided this rule would not match and searched her rule bank for another rule.

At this point, she said “You can’t factor because like there are two variables, so it wouldn’t make sense.” This quote suggests that she was trying to match the structure to a factoring rule like \((ax \pm b)(cx \pm d)\), but she realized this match did not work because of the \( y \) variable. At this point, she gave up on factoring, thus giving up her original structure and rule of canceling out a common factor.
In solving this problem, Hope seemed to be missing a rule from her rule bank that would have been very useful—factoring trinomials to binomials of the form \((ax \pm by)(cx \pm dy)\). This missing rule from her rule bank halted her progress on simplifying the expression, since she was unable to create a correspondence between her first level structure and the rule \(\frac{ab}{ac} = \frac{b}{c}\). Many other participants ran into this same missing rule, as only two of the participants were able to match this expression to the factored form \((ax \pm by)(cx \pm dy)\). In this case, Hope was using OHR to identify OHR structures (both the quotient and the trinomials) and even correctly identified a potential match for the first level structure. However, to make this match, she needed to be able to factor the numerator and denominator. Unfortunately, she was missing the rule that would help her factor the trinomials in the numerator and denominator.

**Three Results of Missing Rules**

When participants used OHR to identify OHR structures but were missing a rule from their rule bank, they typically experienced conflict, because by this time they had exhausted the possible OHR structures that could be considered and the rules that might be correctly paired with those structures. Participants responded to this conflict by taking one of three possible paths. The first was to incorrectly change the expression so it would fit one of the rules they had in mind. Second, some participants abandoned OHR and identified a non-OHR structure. Third, other participants refused to change the expression or abandon OHR, and gave up on solving the problem. I give examples and discuss each of these three different paths below.

**Changing the expression.** When some participants were stuck because they were missing rules, they changed the expression to fit a rule they did have in one of their rule banks. One example of this was when Dustin was solving Task 2 and he was asked to simplify the
expression $\frac{1}{x-\frac{1}{x}} - \frac{1}{x^2-1}$. He started by using OHR to identify structures in the expression, even making a correct match to factor the denominator of the second term. He was able to change the expression to $\frac{1}{x} - \frac{1}{x} - \frac{1}{(x-1)(x+1)}$, but then he got stuck and did not know what match to make next.

I believe Dustin got stuck because he was missing rules that would help him proceed, like using the rule $a - \frac{b}{c} = \frac{ac - b}{c}$ to make a common denominator between $x$ and $\frac{1}{x}$, or using the rule $\frac{a}{b} = \frac{ca}{c}$ to change the first fraction from a complex fraction into a regular fraction.

Since he was unsure what to do next, he fell back on the correct match that he used previously in Task 1 to simplify the fractions in an equation by multiplying both sides of the equation by the denominator of one of the fractions. He seemed to realize the structures in these two tasks were different, as Task 2 involved an expression rather than an equation. To resolve this difference, he simply set the expression in Task 2 equal to zero, changing the expression to an equation. He did not seem to realize that creating an equation fundamentally changed the problem. He proceeded to use the rule, if $a = b$ then $ac = bc$, to multiply both sides of the equation by $x^2 - 1$ and $x - \frac{1}{x}$ to simplify the equation down to $x^2 - x - \frac{2}{x} = 0$. He then claimed that $\frac{1}{x^2-1} - \frac{1}{x} = x^2 - x - \frac{2}{x}$ (in the process of simplifying, he also made the error of equating $1$ with $\frac{1}{x}$ when trying to make a common denominator). Though he initially identified OHR structures, as soon as he was missing a rule and unsure how to proceed, he changed the expression so the structure would match a rule he had used previously. During the six interviews, two participants who were interviewed changed an expression to fit a match they had in mind.
Abandoning OHR. Other participants abandoned OHR structures for non-OHR structures when their progress was halted because they were missing a rule. An example of this occurred when Riley was attempting to simplify the expression $\frac{x-1}{x^2-1}$ from Task 2. At first, she identified the OHR structure of a binomial divided by a binomial, seeing the first level separated by division, and the second level as terms separated by subtraction (Figure 11).

**Figure 11**

*Initial Structure Identified by Riley in Task 2*

![Initial Structure](image)

She then tried to find a correspondence between this structure and the rule $\frac{a}{ab} = \frac{1}{b}$, but realized in order to make this match she would need to factor the denominator so that it contained a factor of $x - 1$. To do this, she multiplied $x$ by $x - 1$, realized it did not equal $x^2 - 1$, and then concluded that $x^2 - 1$ could not be factored. In this case, she seemed to have very few rules in her rule bank to match the $x^2 - 1$ to. Many participants recognized that this structure could be matched to the difference of squares factoring rule or tried to use polynomial long division to divide $x^2 - 1$ by $x - 1$ to factor the denominator. Unfortunately, Riley’s rule bank seemed to be limited to this single rule of the factored form $x(x - 1)$.

After identifying an OHR structure that she was missing rules for, she abandoned the OHR structure she had constructed and identified a new, non-OHR structure. She saw the structure of grouping the terms across the fraction (Figure 12).
She then matched each part of this structure to division rules $\frac{a}{ab} = \frac{1}{b}$ and $\frac{a}{a} = 1$ respectively, saying that $\frac{x-1}{x^2-1} = \frac{1}{x} \cdot 1$. In this case, she abandoned her OHR structure when she was missing rules to help her make progress in the problem. If she had the difference of squares rule in her rule bank, it would have been a very simple problem for her to solve. But since she was missing this rule, she identified a different, non-OHR structure that resulted in an error. Four of the six participants interviewed made this error of abandoning their OHR to identify non-OHR structures when they were missing a rule and unsure how to proceed.

**Stopping when missing rules.** Two of the participants interviewed refused to alter the problem or identify structures that did not follow OHR, even when they were missing rules and did not know what manipulation to do next. One example of this was Jake when he was solving Task 1. He correctly simplified the equation down to $0 = \frac{2x^2-10x-16}{x-2}$, and identified the structure of an equation where the right side had the division as the HLO. He attempted to match this structure to the rule $\frac{ab}{a} = b$ and tried to divide the entire numerator by the entire denominator using polynomial long division. When he got a remainder, he realized he could not match the fraction to the rule $\frac{ab}{a} = b$ using long division. Next, he attempted to match the substructure of the numerator trinomial to the rule $ab + ac + ad = a(b + c + d)$, and factored out the common factor of 2. He seemed to be trying to form a match between the fraction and the rule.
He realized factoring a 2 out of the denominator \( x - 2 \) would lead to fractional coefficients, so he abandoned this rule as well. Next, Jake seemed to consider the rule, \( \frac{a}{b} = 0 \) then \( b \cdot \frac{a}{b} = 0 \cdot b \), because he claimed that multiplying both sides of the equation by \( x - 2 \) was not a correct rule. He explained that it sometimes led to an incorrect answer. He was also missing the rule, \( \frac{a}{b} = 0 \) iff \( a = 0 \), and did not realize that only the numerator had to equal zero for the whole fraction to equal zero. After being unable to find a correct rule to successfully match to his OHR structure, he proceeded to say that he could not do anything else with the problem. He was not tempted to make a non-OHR match to make progress. Through all the interviews, there were three instances where students did not completely solve the problem, but were not tempted to make a non-OHR structure to reach their goal.

**Students Who Only Use OHR**

Jake and Curtis were the only participants who did not make non-OHR matches, and I believe that these two students did not make these errors because they believed that structure must always take into account OHR and that literal-for-literal matches were not allowed. They were not tempted by seductive structures, and avoided operation mismatches by having a clear understanding of the different rules for multiplication/division versus addition/subtraction. Below, I describe evidence of these beliefs and describe the OHR structures that students with these beliefs used the most often.

In the second part of the interview, I presented participants with an hypothetical student solution to Task 3 where terms were divided across the fraction (Figure 13).
Three of the six participants immediately said that this manipulation was not allowed. These participants went on to discuss that dividing across the fraction only worked for objects being multiplied, not those being added. These participants were able to recognize that the given solution was not correct, since the manipulation did not match up with the hierarchy of the operations in the expression. They were also able to realize the operation mismatch that could be made here was not valid and were not tempted by the seductive structures, the common terms across the fraction.

Jake was able to articulate further why this manipulation was not correct, and he showed evidence that he believed that you must use OHR to identify structures and that correct matches are always high-level matches. He described and drew tiers of operations (Figure 14), where the lowest tier was addition and subtraction, the next was multiplication and division, followed by exponents as the top tier. He said that the rules are different for each tier. “Canceling, which you can do like that (pointed to the second line in Figure 13) if it was multiplication or division
between all these, does not work on this (+,-) step, and you can do it with exponents but it’s different.” This explicit breakdown of the hierarchy of operations was something I did not see from any other participant, and it demonstrates that Jake was using OHR and recognizing the difference between the rules for the different operations. He was able to identify when the numerator had multiplication as the HLO (Figure 13, line 1) versus when it had addition as its HLO (Figure 13, line 2), and use these operations to determine which matches were allowed. He was not only considering the operations in the expression, but also the hierarchy of them. His clear understanding of the order of operations and the fact that rules that apply to each operation were different was sufficient for helping him avoid operation mismatches and ignore seductive features as well.

**Figure 14**

*Hierarchy of Operations Identified by Jake*

\[
\begin{array}{c}
2, -3 \\
\times, \div \\
+,-
\end{array}
\]

**Figure 15**

*Structure Identified and Drawn by Jake*

\[
\begin{array}{c}
+,-
\end{array}
\]

\[
\begin{array}{c}
+,-
\end{array}
\]

\[
\begin{array}{c}
+,-
\end{array}
\]
Jake also showed this belief about OHR and its necessity when solving Task 3. He saw the expression \( \frac{x^2-4xy+4y^2}{2x^3+6x^2y-20xy^2} \), and he described the structure as subsections of addition and subtraction (the numerator and denominator) that are being separated by division. He also drew a diagram showing the structure he saw in the expression, as seen in Figure 14. From this diagram, it was evident that he was identifying division as the HLO.

He also knew that this structure must be matched to rules that fit this HLO. He talked about how the rules that work for addition and subtraction were being “blocked” by the division in this structure (Figure 15). He said, “The rules are now different with the division, so for these [rules for addition/subtraction] to transfer, they have to be in a form that is cool with the division rules.” From this statement, it was evident that he understands that it was not accurate to make literal-for-literal matches in this case because he talked about the rules changing due to the division, the HLO of the equation. He had to find a match that fit the HLO, rather than just the smaller parts of the equation.

**Useful Matches for Rational Expressions**

Using Jake’s idea of structures that are “cool with division rules,” it seemed like there were a limited number of these useful matches for structures such as those shown in Figure 13, where a fraction has multiple terms in the numerator and denominator. The first was matching the structure to the rule \( \frac{ab}{ac} = \frac{b}{c} \), which required students to match the substructures of the numerator and denominator to a factored form. Every participant used this match at some point in their interview, though there were varying possible rules to match to factoring trinomials in each participant's rule bank. Those with a greater number of possible rules for factoring seemed to be more successful and made less errors because they were able to make progress with this first form. For example, Curtis was the only student who factored by grouping, and was also the
only student who successfully factored Task 3. Having this additional factoring rule in his rule bank made this task more accessible to him than the other students.

Another match that was used by many participants was what I will call unadding the fraction, where they matched the structure to the rule \( \frac{a+b}{c} = \frac{a}{c} + \frac{b}{c} \), split the numerator up by terms, but kept the denominator the same. Jake showed this match in task 3 when he said

\[
\frac{x^2-4xy+4y^2}{2x^3+6x^2y-20xy^2} = \frac{x^2}{2x^3+6x^2y-20xy^2} - \frac{4xy}{2x^3+6x^2y-20xy^2} + \frac{4y^2}{2x^3+6x^2y-20xy^2}.
\]

This match was correct, but did not help participants to simplify the expression. They always said this resulted in a more complicated form. For Jake, this seemed to be the only other match he could think of for a rational expression like this when he was unsuccessful at factoring.

The most common incorrect match that participants used was when they matched the fraction structure to the incorrect rule \( \frac{a+b}{c+d} = \frac{a}{c} + \frac{b}{d} \), like

\[
\frac{3x^2-12xy+12y^2}{6x^4+18x^3y-60x^2y^2} = \frac{3x^2}{6x^4} - \frac{12xy}{18x^3y} - \frac{12y^2}{60x^2y^2}.
\]

I call this dividing terms across the fraction. Three of the six participants used such a match. One student was confident that this was valid, while the two other students were hesitant. In Task 3, Riley and Parker both ended up starting the problem over three different times because they would divide the terms across the fraction, then evaluate, which led them to doubt the validity of this match. Interestingly, they both went from this incorrect match with a non-OHR structure, to identifying an OHR structure where they were missing a rule, then back to this incorrect match and non-OHR structure. They clearly had some hesitation about the match’s correctness, but ended up dividing terms across the fraction anyway when they got stuck. This was a contrast to Jake, who would stop and say he could not solve the problem instead of using a non-OHR structure.
One characteristic the two correct matches above have in common that differs from the incorrect match was the acknowledgment that the denominator has to stay together and cannot be split across separate fractions. In both factoring and un-adding fractions, students see the structure of the denominator as a single group that they might be able to factor, but that they cannot split up when it is made of terms. The same restriction does not apply to the structure in the numerator, where each term can be split into its own fraction (un-adding the fraction). Participants who were more successful at simplifying rational expressions were confident that the denominator of a fraction had to be treated differently from the numerator. Jake explained this difference as the numerator being more “mutable” than the denominator, because more can be done with it. This belief helps explain why he never made the error of dividing terms across fractions. I believe that the awareness that the numerator can be split across fractions, but the denominator cannot, is essential for students to be successful at simplifying fractions.

Overall, participants who were successful at finding matches for rational expressions with polynomials in the numerator and denominator seemed to realize that matching the expression with the rule $\frac{ab}{ac} = \frac{b}{c}$, then matching the numerator and denominator substructures to a factoring rule was typically the most useful match. Though un-adding fractions was also a correct match, it often did not lead to progress in solving the problem. Participants also benefited from identifying structures that kept the denominator together, thinking of it as one group. The most common non-OHR structure and match participants used with fractions was when they divided terms across the fraction.

**The Role of Evaluation**

Throughout the interviews, participants showed evidence that they were evaluating whether the structures, rules, and matches they were using were helpful and correct. They
seemed to evaluate at two different points in the matching process. Participants would evaluate matches, trying to determine if their correspondence between the structure and the rule was one-to-one and onto, and trying to determine if their match would help them make progress. Participants would also evaluate rules when they were unsure if they were correct or not. Below, I discuss these two types of evaluation—evaluating matches and evaluating rules—and give examples of each.

**Evaluating Matches**

Once students have identified a rule and a structure, they need to make a match by finding a one-to-one and onto correspondence between the rule and the structure. In this process, students had to evaluate whether they could make this correspondence or not. Participants often showed evidence that they had identified a rule, then through evaluation determined that the rule did not match the structure. For example, when Hope was solving Task 3, she had simplified the expression to $\frac{3x^3 - 12x^2y + 12xy^2}{6x^4 + 18x^3y - 60x^2y^2}$ and she identified the common factor of $y^2$ in the last term in the numerator and denominator. Then, she tried to create a correspondence between the rational expression and the rule $\frac{a(b+c+d)}{a(e+f+g)} = \frac{b+c+d}{e+f+g}$. To do this, she had to factor a $y^2$ out of each term in the numerator and denominator. After evaluating this rule and structure, she determined she could not make a match between them, saying “I can’t because there’s not a $y^2$ in anything else.” Through evaluation, she determined that there was not a correspondence between each term in the numerator and denominator and a factored form because each term did not contain a factor of $y^2$. This evaluation led her to abandon this rule and look for another one to match the structure to.
Participants also evaluated matches based on the match’s usefulness in making progress in the problem, which I will refer to as progress evaluations. At times in the interviews, participants would identify a structure and a rule, but determine that the match would not help them make progress on the task. One example of this was when Hope was solving Task 1. She had simplified the equation to $5x = x^2$ and was unsure how to solve for $x$. She first thought about taking the square root of both sides of the equation, and recognized that this was not a helpful match because it did not help her get the equation to $x$ equals a number. Instead, it would leave her with $x = \sqrt{5x}$. Hope was able to evaluate the match, and look ahead to realize it did not help her towards an answer in the form $x = a$, where $a$ is a number.

Next, she considered matching the equation $5x = x^2$ to the power rule by taking the derivative of both sides of the equation. She realized this would not help her get to the goal of solving the equation for $x$, so she quickly abandoned the power rule as a possible rule to match her structure to. Hope considering the power rule was very interesting because derivatives were not even closely related to the goal of the problem. I anticipate that as a current calculus student, Hope used the power rule often, which is why it came to mind. Progress evaluations were used by participants throughout the interviews, as they considered whether matches would help them towards the goal of the problem.

Evaluating Rules

As participants constructed matches, they often were unsure of the validity of their rules, so they evaluated whether the rules were correct rules. For example, in Parker’s interview, he was struggling to know if a rule worked or not. He said, “I need to consult the rule book. If I knew what the rule was I could just divide these and simplify it out.” This is evidence that participants sometimes had rules in their rule bank that they were unsure of. Two types of rule
evaluation that often occurred were creating a test structure, a new expression with a similar structure to test their rule; and engaging in post-manipulation evaluation, simply doing the manipulation to see if the result made sense. I give examples of these two types of evaluation below.

**Test Structures**

Often, when participants were trying to figure out if a rule was valid, they would create their own expression that had a similar, but simpler, structure to test a rule. Test structures were numeric in some cases, where participants tested the rules using expressions that contained only integers. Other times participants used test structures that had variables in them, but were simpler than the structures in the problem. Test structures did not always match the structure of the expression, even though the participants thought they did. The use of test structures seemed to yield two different outcomes: the participant would either gain confidence in the validity of the rule being tested or the participant would not trust the result of their test and become confused. I provide examples of each of these results.

Test structures helped participants gain confidence in their match when participants were able to create a numeric expression that matched the structure of the algebraic expression. For example, Hope was trying to simplify $\frac{-x^{-1}}{x^{-1}}$ and thought she could just divide the variables, but was unsure. She was able to confirm that $\frac{-5}{5} = -1$, which let her to believe that her original idea that $\frac{-x^{-1}}{x^{-1}} = -1$ was correct. In this case, the simpler expression she created had the same structure as the original expression, so the evaluation helped her gain confidence that the rule $\frac{-a}{a} = -1$ was correct.
There were other times where the participants seemed to doubt the results they got from a test structure, resulting in them being more confused after the test. In Task 3, Parker was trying to determine if \( \frac{a+b+c}{e+f+g} = \frac{a}{e} + \frac{b}{f} + \frac{c}{g} \) was a correct rule to make the manipulation:

\[
\frac{3x^2-12xy+12y^2}{6x^4+18x^3y-60x^2y^2} = \frac{3x^2}{6x^4} - \frac{12xy}{18x^3y} - \frac{12y^2}{60x^2y^2}.
\]

He created a simpler expression with the same structure but with integers so he could test this rule. He was able to verify that \( \frac{1+4+3}{4+5-7} \neq \frac{1}{4} + \frac{4}{5} - \frac{3}{7} \), but this did not convince him that his manipulation in the original expression was incorrect. He said “I don’t think it’s the same concept at this point because we are working with variables instead of numbers.” Even though he was able to correctly verify that this rule did not work with an integer expression of the same structure, he was unconvinced that the rule was incorrect for an expression that included variables. He ended up starting the problem over because of his confusion about this manipulation, but later came back to this same incorrect rule. Overall, when participants created expressions of a similar structure to evaluate their rules, it usually helped them gain confidence in their rule, but not always.

**Post-Manipulation Evaluation**

When participants were evaluating, sometimes they voiced hesitation in their rule, but decided to do the manipulation anyway to see if their result made sense. In this case, they decided to perform the manipulation then evaluate their rule based on the validity of the expression after the manipulation. I will call this type of evaluation *post-manipulation evaluation*. For example, in Task 2, Parker matched each term in the expression \( \frac{1}{x-2} - \frac{1}{x^2-1} \) to the rule \( \frac{a}{b+c} = \frac{a}{b} + \frac{a}{c} \). He seemed unsure if this rule was correct, and said, “It feels wrong to do it here but I guess if we get a coherent answer…” He proceeded to use the incorrect rule to perform the
incorrect manipulation $\frac{1}{x-1} - \frac{1}{x^2 - 1} = \frac{1}{x} - \frac{1}{x} - \frac{1}{x^2} - 1$. Next, he simplified this expression to get $x^{-1} - x - x^{-2} - 1$, and then made another error. He used the incorrect rule $x^n = x(1^n)$ to factor the $x$ out of the expression $x^{-1} - x - x^{-2} - 1$ to get $x(1^{-1} - 1 - 1^{-2} - 1)$. At this point, he concluded that he had performed an invalid manipulation because every term left in the parentheses was just a one or negative one. In this case, his evaluation after the manipulation helped him catch his second mistake and the incorrect rule he used. This example illustrates that post-manipulation evaluation was not always effective, as Parker did not catch his first mistake.

Throughout all the interviews, three of the six participants used post-manipulation evaluations to successfully catch mistakes in their interviews.

Overall, participants evaluated matches by looking for correspondence between a rule and a structure or making progress evaluations by determining if the match helps them progress towards their goal. Participants evaluated rules by making test structures or doing post-manipulation evaluations. Evidence of some type of evaluation was present in each participants’ interview. Some participants vocalized their evaluations throughout the process, while others seemed to evaluate in their heads and did not explicitly vocalize their evaluation until they were asked about it.

**Discussion**

Through this research, I have been able to gain insight into how high school calculus students reason structurally and have findings that support and extend our knowledge of previous research in this area. In this section, I discuss the evidence of structural reasoning by the students interviewed and the connections between this evidence and the previous research in algebraic structural reasoning.

**How Students Reason Structurally**
Many researchers have found that structural reasoning is rare among students (Hoch & Dreyfus 2004), but my research suggests otherwise. Each student interviewed showed evidence of structural reasoning in their interview. As shown in Table 1, each student made at least three matches in each task they solved, which included identifying a structure and matching it to a rule. Even when students made incorrect matches, they were still engaging in structural reasoning. Thus these results support the idea that students are reasoning structurally when they are solving problems involving rational expressions. I identified the ways students reason structurally and organized my results by the matching process, outcomes of the matching process, and the role of evaluation. Here, I summarize my findings from each of these categories of how students displayed structural reasoning.

In the matching process, participants identified structure in the expression then tried to find a correspondence between that structure and a rule from their rule banks. They tried to make this correspondence by mapping the parts of the identified structure to the parts of the rule. This matching process often happened iteratively as students would identify structures, then identify structures within that structure to match rules to. The most useful structures students identified were OHR structures, which were structures that took into account the highest level operation (HLO) of the expression and the subsequence HLO’s in each level of the equation.

There seemed to be three outcomes of the matching process: correct matches, incorrect matches, or failure to construct a match. Each participant showed they were capable of making correct matches, and they each made more correct matches than incorrect matches (see Table 1). Four of the six students made incorrect matches at least once in their interview. These incorrect matches seemed to be caused by incorrect rules or students using non-OHR structures. When students failed to make a match between a structure and a rule, they were more likely to turn to
an incorrect match. The failure to make a match was usually caused by missing rules that would have matched the structures students identified. Two students refused to use non-OHR structures and showed evidence of the belief that matches had to be made with OHR structures.

Previous research defines evaluation in structural reasoning as students evaluating the usefulness of matches and manipulations (Hoch & Dreyfus, 2004; Ruede, 2013). In this research, I saw students doing this type of evaluation. When evaluating matches, they tried to determine the usefulness of a match through a progress evaluation, where they evaluated whether a match would help them progress towards their goal of the problem. This type of progress evaluation has been talked about multiple times in the previous research.

Students also used evaluation in a different way throughout their interviews—to evaluate their matches and rules—which is not discussed in the literature. Participants evaluated matches when trying to create a correspondence between a rule and a structure, trying to determine if such a match could be made. Students evaluated rules by using test structures, where they created a new expression of a similar structure that usually consisted of integers that was easier to test rules with. They also evaluated whether a rule was correct by manipulating the expression, then determining if the result made sense or not. Evaluation was an important part of the structural reasoning students engaged in during the interviews, and these results show that evaluating was not limited to the progress evaluations discussed in the literature.

The Need for OHR Structures

While the literature has described students’ struggles to make correct matches, it has not captured the complexity of the decision process, nor offered a clear guideline for how students can construct correct matches. For example, Matz (1982) discussed a process similar to the matching process participants used in this study, where students try to construct a one-to-one
correspondence between a known rule and an algebraic expression. Matz suggested that students can construct correct matches by making top-level matches instead of literal-for-literal matches. However, not all matches fit neatly into these two categories.

For example, consider a student who is trying to factor an expression like \((x - 3)^2 + 2(x - 3) + 1\). They could match this expression to the rule \((a + b)^2 = a^2 + 2ab + b^2\), where they have to match the \(a\) to the \(x - 3\) (a top-level, multiple-symbols-for-one symbol match) and match the \(b\) to 1 (a literal-for-literal, one-symbol-for-one-symbol match). In this case, they would have to make a literal-for-literal and top-level match at the same time. Thus Matz’s (1982) distinction between top-level matches and literal-for-literal matches is not enough to describe the complexity of the matching process nor does it suggest how students decide when a top-level match is needed versus when a literal-for-literal match is sufficient.

In this study, I noticed that useful, correct matches always involved matching OHR structures to correct rules. When students identified structure based on the HLO’s in the expression, they were able to determine whether it was appropriate to map multiple symbols in the expression to one symbol in the rule, versus when to map one symbol in the expression to one symbol in the rule (a multiple-to-one versus a one-to-one symbol correspondence). In the example above, a student using OHR would recognize the first level structure of three terms in the expression, making it appropriate to make a multiple-to-one symbol correspondence with \(x - 3\) and a one-to-one symbol correspondence between 1 and \(b\). Thus, the need to use OHR structures when constructing a correct match provides a clearer criterion for when to make a multiple-for-one versus a one-for-one symbolic match between structure and rule than Matz’s notions of top-level and literal-for-literal matches.
Confirmation of Prior Studies

Multiple findings in these results confirm prior research findings. Some of those findings include Kirshner and Awtry’s (2004) idea of visually salient rules, Ruede’s (2013) concept that experts evaluate with a goal in mind, and the idea that students do not connect algebraic and arithmetic expressions. I will discuss these findings from prior research and how results from this study support them.

Kirshner and Awtry’s (2004) idea of visually salient rules is similar to my result of seductive structures. The previous research defined visually salient equations as those that have a visual coherence that makes the left and right sides look naturally equal. Researchers found that students often used visually salient rules that were not correct, but students thought they were correct because they look like they should be. Similarly, in my research I found that many students used incorrect structures that were based on seductive structures—visual similarities or instances of sameness identified by students between elements of an expression. In both visually salient rules and seductive structures, students are deciding the mathematical correctness of a rule or match based on the way it looks, rather than how it makes sense mathematically. This research supports the idea that students make such decisions.

Ruede (2013) compared expert and novice structural reasoning and found that experts evaluated with a goal in mind more than the novices. This goal impacted the structures they saw and rules they matched to. In my research, I also found that students often reasoned structurally with a goal in mind. In the interviews, students used progress evaluations to determine whether a match would help them progress towards their goal or not. Contrary to Ruede’s findings, I found that all the students interviewed used these types of evaluations, and seemed to have a goal in
mind as they made matches. Even though the subjects I interviewed were not experts, they still seemed to be impacted by what they perceived the goal of the question to be.

Many researchers discuss a lack of connection between the way students think about algebraic and arithmetic expression (Booth, 1989; Kieran, 1989; Linchevski & Linvneh, 1999). In my research interviews, many students used test structures, where they took an algebraic expression and created an arithmetic expression of a similar structure to test if rules worked or not. Students making such test structures showed that they were seeing some connection between the algebraic and arithmetic expression, but not completely. The example of Parker discussed on page 51 is evidence that some students do not think about algebraic and arithmetic expressions the same way. In this example, Parker was able to create an arithmetic expression as a test structure for a complex, algebraic rational expression. These two expressions had the same structure, but when Parker proved that the rule did not work with the arithmetic expression, he was not convinced that the rule would not work for the algebraic expression. He expressed doubt that the same rules applied to the arithmetic expression and algebraic expression, which supports the previous research on the lack of connection students make between such expressions.

Overall, many aspects of previous research were verified in my interviews including students’ use of visually salient rules, evaluations made with a goal in mind, and connections between algebraic and arithmetic expressions.
CHAPTER 6: CONCLUSION

Algebraic manipulation is an important skill in mathematics that many students struggle with. Researchers have suggested that a lack of structural reasoning is one of the reasons for this difficulty with algebra (Kieran, 1989). The current research shows that students who use structural reasoning are less likely to make errors while working with algebraic expressions and equations; however, students who use structural reasoning are rare (Hoch & Dreyfus 2004, 2006). In this study, I interviewed six students to study structural reasoning in the context of rational expressions and equations. A review of the literature revealed many structural reasoning actions that students engage in, but for this research I focused on identifying, matching, and evaluating structures. I analyzed the participant interviews and outlined the matching process I saw in the interviews and the types of evaluations they made during this matching process. Consequently, I was able to confirm that students were using structural reasoning throughout the tasks and identify effective structural reasoning as students identifying OHR structures and matching them to correct rules.

Contributions

The results of this study add three main contributions to the existing research and understanding of how high school students use structural reasoning. The first contribution from these results is a new method researchers can use to understand students’ structural reasoning. The previous research has examined students’ reasoning for instances of correct or expert structural reasoning (Hoch & Dreyfus, 2004; Ruede, 2013). In this research, I analyzed students’ reasoning to understand how they saw and used structure. By assuming that structures were not contained in the expressions themselves, rather created by the individual viewing them, I was open and able to recognize the structures and matches the students were constructing even when
their reasoning did not match that of experts. By acknowledging and studying the structural reasoning the students were engaging in, I was able to describe their matching process, which outlined the way students identify structures and match these structures to rules.

The second contribution these results make is an understanding of how advanced high school students reason structurally. Though previous research found that students rarely use structural reasoning, I found that the participants were using structural reasoning very often. In each task, students had two to sixteen matches that they made (Table 1). I found that students were making these matches by identifying structures as they grouped parts of expression, then matching these structures to rules in their rule banks. Through this matching process, students were evaluating at multiple points. Like participants in past studies, students used progress evaluations to determine whether their matches were helping them progress towards their goal in the problem (Ruede, 2013). In this study, participants also used evaluations to determine whether their rules were correct and if their matches were valid, which was not discussed in previous literature. These new types of evaluation found in this research suggest that students are constantly judging the correctness of their structural reasoning in a variety of ways.

The third contribution these results make is a better description of how good matches are made. I found that correct matches consisted of three key components—an OHR structure, a correct rule, and an evaluation of the rules and matches used. The construct of OHR structures suggests perhaps a simpler, more concise criterion for choosing structures that lead to correct matches than has been suggested in past research (e.g., Harel & Soto, 2017; Matz, 1989). The second, necessary component for a good match is having multiple correct rules to match to these OHR structures (which I call their rule banks). I found that often students were able to identify OHR structures, but did not have the correct rule in their rule bank to match it to. Thus to make a
good match, it was important that students had a variety of rules available to them. The third component is a way to evaluate matches and rules. I found that throughout the matching process, students were evaluating whether their rules were correct and whether their matches helped them make progress in the problem. Overall, these three components—OHR structures, correct rules, and evaluations—contribute to a better description of how good matches are made.

**Implications**

The results and contributions of this study provide implications for both researchers and teachers. I first describe the implications for researchers and how this study should affect the way we research structural reasoning. Then, I describe what the implications of this research are for teachers and the way they teach about structure in the classroom.

The main implication of this study for researchers is that more research is needed that focuses on how students are actually reasoning about structure rather than just examining whether students can engage in expert structural reasoning. This research showed that while students may not always engage in expert structural reasoning, they were nevertheless always engaged in some form of structural reasoning as they worked with rational expressions and equations. More research is needed to understand how students see structure and how they construct matches as they progress through secondary mathematics. By studying the structures and matches students create, rather than those they fail to create, the research community can gain a deeper understanding of the variety of structural reasoning that students engage in. This understanding is essential to designing instruction that supports the development of structural reasoning.

The main implication of this study for teachers is that teachers need to be talking more about structures and HLO in the classroom. This implication supports the findings of Ruede
(2013), who concluded that teachers need to be talking explicitly about structure in their classroom. I echo and amplify Ruede’s suggestion to say that it is imperative that teachers start explicitly discussing and teaching structure in the classroom. Currently, little time is spent in mathematics curriculums and classrooms teaching students to identify and use structures, despite it being a main goal given in the Common Core Standards. This research gave us a glimpse at how students are using structural reasoning, which can be used by teachers to discuss and teach students to identify OHR structures and match them to correct rules so students can become more proficient in algebra.

In addition to this finding that teachers need to teach about structure, my research suggests a more specific way to talk about structure with students. In particular, teachers should be discussing HLOs present in expressions and teaching students to break down expressions using OHR structures rather than assuming they will see OHR structures naturally. Diagrams similar to the ones used in this research where different symbols are boxed to show the structure identified could be used in the classroom as a way for teachers to explicitly illustrate and discuss structure with students. This practice could even be implemented into mathematics classrooms as a norm for students to use when performing algebraic manipulations, where students are required to identify and explicitly outline structures before performing any manipulations. I anticipate that this practice would help students start thinking more about structure and would teach them to more easily view OHR structures.

Teachers should also discuss rule banks by talking about the variety of rules that match a particular structure and how these different matches could be useful. This could be done by having students list multiple rules that match a given identified structure as a class, and having students evaluate to discover which matches are correct and which matches help make progress
in the problem. This modeling of the matching process has the potential to teach students to be better able to see a variety of structures and matches available, leading to more correct and concise algebraic manipulations.

An additional implication for teachers is the way they address algebra errors made by students. Many teachers become frustrated with algebra errors made by students and think these errors are made because of mindless manipulations. This research suggests otherwise, that students may be thoughtfully evaluating the correctness and usefulness of their matches as they solve problems. For teachers, this means that instead of viewing errors as a complete lack of structural reasoning, they should view errors as indicating either an incorrect rule that the student has or a lack of using OHR structures, each of which can be addressed by the teacher. As a result of this research, hopefully teachers can pinpoint exactly where the error is coming from to give helpful feedback on the root of the error instead of just saying it is an error.

**Limitations and Directions for Future Research**

This research was limited in the scope of the participants studied and types of tasks used. I only interviewed six students who were all studying calculus. The types of tasks I used all included rational expressions and only included three different problems. This lack of variety in level of student and question type was appropriate for this study, but limited the results to this very specific group of students reasoning about this specific type of problem.

Though this provides some insight into advanced mathematics students’ structural reasoning relative to rational expressions and equations, additional research is needed to obtain a more complete picture of the structural reasoning of students in secondary schools. Research should be done with students at different levels of mathematics with different types of algebraic expressions. For example, a similar study with beginning algebra students, where they are
interviewed solving problems such as one-step algebra problems or basic distributive property problems, would give us more information about how students understand structures and matches when they first learn algebra. Similar studies at different levels would help researchers and teachers to understand how structural reasoning is developed throughout all stages of students learning mathematics. By taking the perspective that students are using structural reasoning at each stage, though it might not be the same as expert structural reasoning, researchers can come to a greater understanding of how structural reasoning changes and grows as students learn.

**Conclusion**

The purpose of this study was to better understand the way students solve algebra problems, specifically the way they use structural reasoning to solve problems involving rational expressions. The results of this research showed that students are engaging in structural reasoning, which led to the concept of the matching process to help make sense of the way high school calculus students think about and use structure in rational expressions and equations. As we come to better understand structural reasoning with a variety of students and types of problems, we will be able to build on this understanding to obtain a wider view of how students develop and use structural reasoning at all levels. This understanding has the potential to help teachers better instruct students on using and identifying structure, leading to less frustration by students and teachers in algebra.
References


http://www.corestandards.org/Math/Practice/


Appendix A

Initial task given in class to choose participants.

Solve for \( x \) in the following equation. Even if you don’t know how to solve it, try your best! Make sure to show all your work.

\[
\frac{2}{x} - \frac{6}{x-1} = 1
\]

Tasks given in Part 1 of the interviews.

Task 1:

Solve the equation for \( x \).

\[
\frac{7}{x+2} = 2 - \frac{3}{x-2}
\]

Task 2:

Simplify the following expression.

\[
\frac{1}{x} - \frac{1}{x^2 - 1}
\]

Task 3:

Simplify the following expression.

\[
\frac{3x(x^2 - 4xy + 4y^2)}{6x^2(x^2 + 3xy - 10y^2)}
\]
Hypothetical student solutions presented in Part 2 of the interview.

a. Hypothetical student solution for Task 1

\[
\frac{3x(x^2 - 4xy + 4y^2)}{6x^2(x^2 + 3xy - 10y^2)}
= \frac{x^2 - 4xy + 4y^2}{2x(x^2 + 3xy - 10y^2)}
= \frac{1}{2x} - \frac{4}{3} - \frac{4}{10}
= \frac{1}{2x} - \frac{40}{30} - \frac{12}{30}
= \frac{1}{2x} - \frac{52}{30}
= \frac{1}{2x} - \frac{26}{15}
\]

b. Hypothetical student solution for Task 2

\[
\frac{x}{x-\frac{1}{x}} - \frac{1}{x^2-1} = \frac{x}{x^2-1} - \frac{1}{x^2-1} = \frac{x-1}{x^2-1} = \frac{x-1}{(x-1)(x+1)} = \frac{1}{x+1}
\]