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Vector-Valued Mock Theta Functions

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ABSTRACT

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Ramanujan introduced his now celebrated mock theta functions in 1920, grouping them into families parameterized by an integer called the order. In 2010 Bringmann and Ono discovered generalizations of Ramanujan's mock theta functions for any order relatively prime to 6; this result was later strengthened by Garvan in 2016. It was also shown that by adding suitable nonholomorphic completion terms to the mock theta functions the family of mock theta functions corresponding to a given order constitute a complex vector space which is closed under the action of the modular group. We strengthen the Bringmann, Ono, and Garvan result by constructing a vector-valued modular form of weight $1/2$ transforming according the Weil representation for orders greater than 3 by introducing an algorithm which simultaneously numerically constructs the form and proves its transformation laws. We also explicitly construct the 7th order form and prove analytically that it has the proper modular transformations. It is conjectured the same method will apply for other orders.

Keywords: mock theta functions, mock modular forms, Maass forms, Weil representation

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Contents

Notation

The following notations and conventions are used in this thesis.

$$e(z) := e^{2\pi i z} \tag{1}$$

$$q := e(\tau) \tag{2}$$

$$\mathbb{H} := \{\tau \in \mathbb{C} : \Im\{\tau\} > 0\} \tag{3}$$

$$\zeta_b := e\left(\frac{1}{b}\right) \tag{4}$$

$$(a; z)_0 := 1 \tag{5}$$

$$(a; q)_n := \prod_{m=1}^{n-1} (1 - aq^m) \tag{6}$$

$$(a; q)_\infty := \prod_{m=1}^{\infty} (1 - aq^m) \tag{7}$$

Our convention is that τ is an element of $\mathbb{H}$. Here $(a; q)_n$ is the standard q-Pochhammer symbol. We define $\log(z)$ to be the principal branch of the logarithm function, that is, $\log(z)$ has argument lying in $(-\pi, \pi]$. This resolves the ambiguity in the definition of $\sqrt{z} = z^{\frac{1}{2}}$ by defining $z^c = e^{c \log(z)}$. Hence for $z = |z|\, e^{i\theta}$, we have $\sqrt{z} = e^{\frac{1}{2}\log|z| e^{i\theta}} = |z|^{\frac{1}{2}} \exp\{\frac{i}{2} \arg\left(\log e^{i\theta}\right)\}$, so $\arg(\sqrt{z}) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, identifying $\sqrt{z}$ with the principal branch of the square-root. Other roots and powers of roots are simultaneously resolved with this convention.

Recall that a unitary linear transformation is one which has inverse equal to its Hermitian (or complex-conjugate) transpose. We denote the Hermitian transpose of g by $g^\dagger$.

We also use the Legendre symbols and their generalizations the Kronecker symbols. These are traditionally denoted by $\left(\frac{\cdot}{n}\right)$ and $\left(\frac{n}{\cdot}\right)$. In order to avoid confusion with fractions, care is taken not to write any fractions in parentheses unless they are the argument of a function, as in $e\left(\frac{\cdot}{\cdot}\right)$.

For χ a character we denote by $G(\chi)$ the Gauss sum of χ, that is, $G(\chi) = \sum_{h \pmod{d}} \chi(h) e\left(\frac{h}{d}\right)$, where d is the modulus of the character.

Chapter 1. Introduction

1.1 History of the Mock ϑ-Functions

In January 1920 S. Ramanujan sent his last letter to his mentor G.H. Hardy. In this letter he introduced 17 examples of his now celebrated mock ϑ-functions and gave a loose definition which these functions satisfy. All these functions are Fourier series in the variable $q = e^{2\pi i\tau}$. Ramanujan grouped his functions into 3 families using a parameter he called the order; each of his original 17 functions had order $3, 5$ or 7 respectively. What Ramanujan meant by the order is still mysterious today, though we now have a better understanding of how functions of each order are related. In his 1920 letter Ramanujan also listed some conjectured relations between his mock ϑ-functions. An annotated copy of Ramanujan's letter to Hardy can be found in [BR95, pages 220-224].

Ramanujan died in April 1920 but other mathematicians, such as G.N. Watson and A. Selberg, continued the study of his mock ϑ-functions. In 1976 G. Andrews rediscovered some of Ramanujan's notebooks, one of which contained 10 conjectured identities of the 5th order mock ϑ-functions. These became the mock ϑ-conjectures, and were proved in 1988 by D. Hickerson using q-series methods [Hic88].

A revolution in the study of mock ϑ-functions came in 2002 when S. Zwegers wrote his PhD thesis for Utrecht University. In his thesis Zwegers proved that previous work on the classical Ramanujan mock ϑ-functions follows from the fact that these 17 functions are each simultaneously examples of Lerch sums, quotients of indefinite ϑ functions, and Fourier coefficients of Jacobi forms. This was shown to be equivalent to stating these functions are the holomorphic parts of harmonic Maaß forms of weight $\frac{1}{2}$; the study of analogous functions for other weights is the study of mock modular forms.

In recent years evidence has been accumulating that the most natural perspective from which to study the mock ϑ-functions uses vector-valued weak Maaß forms with Weil representations. In 2010 and 2016 familes of completed mock ϑ-functions were introduced for each order relatively prime to 6 whose span is preserved under modular transformations by $\mathrm{SL}_2(\mathbb{Z})$ [BO10][Gar16]. Garvan's result was later published in 2019 [Gar19]. These functions are completed in the sense that they have modular transformations at the cost of adding a nonholomorphic term to each of the mock ϑ-functions. While Bringmann, Ono, and Garvan did not specify a representation giving the transformation law for each vector-valued form, using this result Andersen was able to reprove the mock ϑ-conjectures by constructing a vector-valued form transforming according to the Weil representation which has as its components the completed 5th order mock ϑ-functions. Andersen proved the mock ϑ-conjectures by realizing them as the difference between two vector-valued forms, which difference corresponds to a Jacobi form lying in a 0 dimensional space [And16]. In their 2017 and 2018 master's theses (later published in *The Ramanujan Journal* [KK21]), D. Klein and J. Kupka constructed weight $\frac{1}{2}$ vector-valued weak Maaß forms with Weil representations, each with components whose holomorphic part is equal to one of 22 mock ϑ-functions of order 2, 3, 6, or 8.

In this thesis we strengthen Bringmann, Ono, and Garvan's result by constructing a weight $\frac{1}{2}$ vector-valued harmonic Maaß form of order 7 whose components are completed mock ϑ-functions, and specify the Weil representation under which the vector-valued form is a weak Maaß form. Our construction generalizes to other prime orders. While our objective overlaps superficially with the Klein and Kupka theses our results do not overlap; our method will produce harmonic weak Maaß forms for orders relatively prime to 6 and so our results are disjoint.

1.2 The Mock ϑ-Functions

Let $\tau \in \mathbb{H}$, where $\mathbb{H}$ is the complex upper half-plane, and define $e(\tau) = e^{2\pi i \tau}$. Each of the mock ϑ-functions is a Fourier series in the variable $q = e(\tau)$. In his last letter to Hardy, Ramanujan proposed a definition for his mock ϑ-functions, specifying that a q-series $f(q)$ which converges for $|q| < 1$ is a mock ϑ-function if it satisfies the following:

(i) infinitely many roots of unity are exponential singularities of f,

(ii) for every root of unity ξ there exists a ϑ-function ϑ_ξ such that $f - \vartheta_\xi = O(1)$ for $q \to \xi$ radially,

(iii) there is no single ϑ-function satisfying condition (ii) for every ξ, implying f is not simply the sum of a ϑ-function with a bounded function.

There is some ambiguity in what Ramanujan meant by a ϑ-function. It wasn't until 2013 that it was proven by M. Griffin, K. Ono, and L. Rolen that Ramanujan's original mock ϑ-functions obey his condition (iii) in which a ϑ-function is essentially a modular form of weight $\frac{1}{2}$ [GOR13]. According to Zwegers, by ϑ-function Ramanujan likely meant a sum, product, or quotient of functions of the form $\sum_{n\in\mathbb{Z}} \epsilon^n q^{an^2+bn}$ for some $a, b \in \mathbb{Q}$ and $\epsilon = 1$ or -1 [Zwe02, page 63]. The main result of Zwegers' thesis is that vector-valued mock ϑ-functions differ from real-analytic modular forms by a bounded function, contradicting a relaxed version of condition (iii).

This thesis is connected to Ramanujan's original mock ϑ-functions by the mock ϑ-conjectures, which relate Ramanujan's 5th order functions to the function

$$M(r; q) := \sum_{n=1}^{\infty} \frac{q^{n(n-1)}}{(q^r; q)_n (q^{1-r}; q)_n}.$$

An example of a mock ϑ-conjecture is the following. Let

$$f_0(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n}. \tag{1.1}$$

This is one of Ramanujan's classical 5th order mock ϑ-functions. Then

$$f_0(q) = -2q^2 M\left(\frac{1}{5}; q^{10}\right) + \theta_4(0, q^5) G(q), \tag{1.2}$$

where θ_4 is a theta function and $G(q)$ is a Rogers-Ramanujan function (definitions can be found in [And16]). The function M is a mock ϑ-function in the modern sense, meaning it is the holomorphic part of a harmonic Maaß form of weight $\frac{1}{2}$. The mock ϑ-functions that Bringmann, Garvan, and Ono found, $N(a, b; \tau)$ and $M(a, b; \tau)$, are generalizations of the M function from the mock ϑ-conjectures.

1.2.1 This Thesis. The results of this thesis can be broadly divided into two categories: the first, numerical, centers on an algorithm constructing a vector-valued form transforming according to the Weil representation, with components which are completed mock ϑ-functions for any prime ≥ 5, using numerical tools for solving systems of linear equations in Mathematica. The program constructing this form produces numerical coefficients for each mock ϑ-function and specifies which components that mock ϑ-function is a part of. It simultaneously constructs the forms and proves their transformation properties, and with enough computing power one could produce numerical forms of any prime order. With our available computing power we were limited to orders 11 and below.

The second category of results, which we may broadly term analytic, concern writing an explicit solution for such vector-valued forms and proving that they transform appropriately without the use of computational aids. We present such a solution for order 7; moreover, we've proven the transformation laws for this solution using identities of character sums. We obtained this solution by first constructing a numerical form using the above mentioned program, then solving for the coefficient of each mock ϑ-function.

It should be emphasized that once one has an analytic solution matching the numerical solution the program has proved the transformation laws for that function. What was desired in proving the transformation laws without the computer was a method which would generalize to other orders. We have a method which we conjecture can be generalized to construct a vector-valued completed mock ϑ-function and prove the transformation law for each order simultaneously.

Chapter 2 introduces the completed mock ϑ-functions and their transformation laws while chapter 3 introduces the Weil representation and vector-valued forms. Many of the results depend on evaluations of exponential sums (such as Gauss sums); a small selection of elementary lemmata related to exponential sums is found in appendix A.

In chapter 4 we prove a series of identities which, for a vector-valued mock ϑ-function, are equivalent to proving that it transforms according to the Weil representation. Chapter 5 is devoted to proving that a 7th order vector-valued mock ϑ-function transforms according to the Weil representation, while in section 5.1 we provide a conjectured formula for certain coefficients for forms of general order.

Chapter 2. Mock ϑ-Functions and their Non-holomorphic Completions

2.1 Nonholomorphic Completions of Mock ϑ-Functions

This chapter is included to record the Bringmann, Ono, and Garvan results generalizing mock ϑ-functions to any order and giving their modular transformations [BO10][Gar19]. Most of the material can be found in [Gar19, sections 2 and 3]. Our perspective is less general than Garvan's, and so we adapt his notation to streamline it for our application. Notably we suppress the order c and make an identification between his $M\left(\frac{a}{c}; z\right)$ and $M(a, 0, c; z)$, and similarly for N. We use $\tilde{N}, \tilde{M}$ for the completed functions instead of $\mathcal{G}_1, \mathcal{G}_2$. We also use an identity of Garvan's (relating his $\Theta_1\left(\frac{a}{c}; z\right)$ to $\Theta_2(0, -a, c; z)$) to reduce the number of ϑ-functions introduced [Gar19, page 13].

Let $c > 0$, $(c, 6) = 1$ and let $(a, b) \in (\mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/c\mathbb{Z}) \setminus \{(0, 0)\}$, with a, b having least non-negative residue. Let $\tau \in \mathbb{H}$. We first define the mock ϑ-functions of order c. Define

$$M(a, b; \tau) := \frac{1}{(q; q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n+a/c}}{1 - \zeta_c^b q^{n+a/c}} q^{\frac{3}{2}n(n+1)} \tag{2.1}$$

$$\tag{2.2}$$

and

$$k(b, c) := \begin{cases} 0 & \text{if } 0 < \frac{b}{c} < \frac{1}{6}, \\ 1 & \text{if } \frac{1}{6} < \frac{b}{c} < \frac{1}{2}, \\ 2 & \text{if } \frac{1}{2} < \frac{b}{c} < \frac{5}{6}, \\ 3 & \text{if } \frac{5}{6} < \frac{b}{c} < 1. \end{cases}$$

Note this is well defined because $(c, 6) = 1$. Then for $a \neq 0$ let

$$N(a, 0; \tau) := \frac{1}{(q; q)_\infty} \left(1 + \sum_{n=1}^{\infty} \frac{(-1)^n (1 + q^n)(2 - 2\cos\frac{2\pi a}{c})}{1 - 2\cos\frac{2\pi a}{c} q^n + q^{2n}} q^{\frac{1}{2}n(3n+1)}\right) \tag{2.3}$$

and for $b \neq 0$ define

$$N(a, b; \tau) := \frac{1}{(q; q)_\infty} \left(\frac{i\zeta_{2c}^{-a} q^{b/2c}}{2(1 - \zeta_c^{-a} q^{b/c})} + \sum_{n=1}^{\infty} K(a, b, n; \tau) q^{\frac{n(3n+1)}{2}}\right), \tag{2.4}$$

where

$$K(a,b,n;\tau) := \frac{(-1)^n \left(q^n \sin\left(\frac{\pi a}{c} - \pi\tau\left(\frac{b}{c} - 2nk(b,c)\right)\right) + \sin\left(\frac{\pi a}{c} - \pi\tau\left(2nk(b,c) + \frac{b}{c}\right)\right)\right)}{1 - 2q^n \cos\left(\frac{2\pi a}{c} - \frac{2\pi b\tau}{c}\right) + q^{2n}}.$$

These are the generalized mock ϑ-functions of order c. M and N must be completed to have proper modular transformations; defining their completions requires the introduction of period integrals of certain ϑ-functions. Define for $0 \leq f < d$, both integers, the ϑ-function

$$\theta(f,d;\tau) := \sum_{m=-\infty}^{\infty} (dm+f) e\left(\frac{\tau(dm+f)^2}{2d}\right). \tag{2.5}$$

Define the ϑ-functions

$$\Theta_1(a,b;\tau) := \zeta_{c^2}^{3ab} \zeta_{2c}^{-a} \sum_{m=0}^{6c-1} (-1)^m \sin\left(\frac{\pi}{3}(2m+1)\right) e\left(\frac{-ma}{c}\right) \theta(2mc - 6b + c, 12c^2; \tau) \tag{2.6}$$

and

$$\begin{aligned}\Theta_2(a,b;\tau) := \sum_{m=0}^{2c-1} \Big((-1)^m e\left(\frac{-b(6m+1)}{2c}\right) \theta(6cm + 6a + c, 12c^2; \tau) \\ +(-1)^m e\left(\frac{-b(6m-1)}{2c}\right) \theta(6cm + 6a - c, 12c^2; \tau)\Big).\end{aligned} \tag{2.7}$$

Further, define

$$\varepsilon(a,b;\tau) := \begin{cases} 2\zeta_c^{-2b} \exp\left\{-3\pi i\tau\left(\frac{a}{c} - \frac{1}{6}\right)^2\right\} & \text{if } 0 \leq \frac{a}{c} < \frac{1}{6}, \\ 0 & \text{if } \frac{1}{6} < \frac{a}{c} < \frac{5}{6}, \\ 2\exp\left\{-3\pi i\tau\left(\frac{a}{c} - \frac{5}{6}\right)^2\right\} & \text{if } \frac{5}{6} < \frac{a}{c} < 1. \end{cases} \tag{2.8}$$

The M, N functions are completed by the addition of nonholomorphic period integrals of ϑ-functions. The integrals are

$$T_1(a,0;\tau) := -\frac{1}{2c\sqrt{3}} \int_{-\bar{\tau}}^{i\infty} \frac{\Theta_2(0,-a;z)}{\sqrt{-i(z+\tau)}} dz, \tag{2.9}$$

$$T_2(a,0;\tau) := \frac{i}{3c} \int_{\bar{\tau}}^{i\infty} \frac{\Theta_1(0,-a;z)}{\sqrt{-i(z+\tau)}} dz, \tag{2.10}$$

and for $b \neq 0$

$$T_1(a,b;\tau) := \frac{\zeta_{2c}^{-5b}}{3c} \int_{-\bar{\tau}}^{i\infty} \frac{\Theta_1(a,b;z)}{\sqrt{-i(z+\tau)}} dz, \qquad \text{and} \tag{2.11}$$

$$T_2(a,b;\tau) := -\frac{\zeta_{2c}^{-5b}}{2i\sqrt{3}c} \int_{\bar{\tau}}^{i\infty} \frac{\Theta_2(a,b;z)}{\sqrt{-i(z+\tau)}} dz. \tag{2.12}$$

We can now define the completed versions of the mock ϑ-functions N and M. Define

$$\tilde{N}(a,0;\tau) := \frac{1}{\sin\frac{a\pi}{c}} q^{-1/24} N(a,0;\tau) - T_1(a,0;\tau) \qquad \text{and} \tag{2.13}$$

$$\tilde{M}(a,b;\tau) := 2q^{\frac{3a}{2c}\left(1-\frac{a}{c}\right)-\frac{1}{24}} M(a,b;\tau) + \varepsilon(a,b;\tau) - T_2(a,b;\tau). \tag{2.14}$$

Finally, for $b \neq 0$, we have

$$\tilde{N}(a,b;\tau) := 4e\left(\frac{-a}{c}k(b,c)\right) e\left(\frac{3b}{2c}\left(\frac{2a}{c}-1\right)\right) \zeta_c^{-b} q^{\frac{b}{c}k(b,c)-\frac{3b^2}{2c^2}-\frac{1}{24}} N(a,b;\tau) \tag{2.15}$$

$$- T_1(a,b;\tau).$$

2.2 Modular Transformations of Completed Mock ϑ-Functions

We can now state the modular transformations of the completed mock ϑ-functions [Gar19, theorem 3.1]. Recall $T : \tau \mapsto \tau + 1$ and $S : \tau \mapsto -\frac{1}{\tau}$. We have for the T transformation:

$$\tilde{N}(a,b;\tau+1) = \begin{cases} \zeta_{2c^2}^{3b^2} \zeta_{24}^{-1} \tilde{N}(a-b,b;\tau) & \text{if } a \geq b, \\ -\zeta_{2c^2}^{3b^2} \zeta_c^{-3b} \zeta_{24}^{-1} \tilde{N}(a-b+c,b;\tau) & \text{otherwise} \end{cases} \tag{2.16}$$

$$\tilde{M}\,(a,b;\tau+1) = \zeta_{2c}^{5a} \zeta_{2c^2}^{-3a^2} \zeta_{24}^{-1} \tilde{M}(a, a+b \bmod c;\tau). \tag{2.17}$$

Now S switches $\tilde{M}$ and $\tilde{N}$ and preserves their a,b indices.

$$\frac{1}{\sqrt{-i\tau}} \tilde{M}\left(a,b;\frac{-1}{\tau}\right) = \tilde{N}(a,b;\tau) \tag{2.18}$$

$$\frac{1}{\sqrt{-i\tau}} \tilde{N}\left(a,b;\frac{-1}{\tau}\right) = \tilde{M}(a,b;\tau) \tag{2.19}$$

Chapter 3. Vector-Valued Weak Maass Forms and the Weil Representation

3.1 The Metaplectic Group

Let $\gamma \in \Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Define the Möbius fractional linear transformation for $\tau \in \mathbb{H}$ and $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ as the transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \tau : \tau \mapsto \frac{a\tau + b}{c\tau + d}.$$

A function transforms with automorphy factor $J(\gamma, \tau)$ with respect to Γ if for all $\gamma \in \Gamma$ we have $f(\gamma\tau) = J(\gamma, \tau)f(\tau)$. For integers k, a weight k modular form on $\mathrm{SL}_2(\mathbb{Z})$ transforms with automorphy $J\left(\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \tau\right) = (c\tau + d)^k$. There is a difficulty in extending this definition to half-integer weight $k + \frac{1}{2}$; this arises from ambiguity in the choice of the square-root function. Because the group action resulting from the Möbius transformation is associative it is possible, for any congruence subgroup Γ of $\mathrm{SL}_2(\mathbb{Z})$, to act on τ by matrices γ_1 and $\gamma_2 \in \Gamma$ so that $J(\gamma_1\gamma_2, \tau) = -J(\gamma_1, \tau)J(\gamma_2, \tau)$. Hence there are no non-zero half-integer weight forms whose factor of automorphy is $(c\tau + d)^k$ [Kob93, page 178].

One way to resolve this difficulty is to pass from $\mathrm{SL}_2(\mathbb{Z})$ to a double cover which is agnostic with respect to the choice of square-root. This group is the metaplectic group $\mathrm{Mp}_2(\mathbb{Z})$.

Definition 3.1. The metaplectic group on $\mathbb{R}$ is defined by the set

$$\mathrm{Mp}_2(\mathbb{R}) := \left\{(M, \phi(\tau)) : M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{R}),\ \phi(\tau)^2 = c\tau + d \text{ and } \phi \text{ is a holomorphic function}\right\} \tag{3.1}$$

along with the group operation

$$(M_1, \phi_1(\tau))(M_2, \phi_2(\tau)) = (M_1M_2, \phi_1(M_2\tau)\phi_2(\tau)). \tag{3.2}$$

Here $M\tau$ is the Möbius transformation.

We see $\mathrm{Mp}_2(\mathbb{R})$ is a double cover of $\mathrm{SL}_2(\mathbb{R})$. Let $\mathrm{Mp}_2(\mathbb{Z}) = \tilde{\Gamma}$ be the inverse image of $\mathrm{SL}_2(\mathbb{Z})$ under the covering map. We have the following theorem for $\tilde{\Gamma}$.

Theorem 3.2. $\tilde{\Gamma}$ *is finitely generated by* $(T, 1)$ *and* $(S, \sqrt{\tau})$*, where* $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$.

This theorem follows from the fact that $\mathrm{SL}_2(\mathbb{Z}) = \langle T, S\rangle$ and $(S, \sqrt{\tau})^4 = (I, -1)$, where I is the 2×2 identity matrix. The significance of theorem 3.2 is that computing the modular transformations of a function amounts to computing the results of the transformations

$$T\tau \mapsto \tau + 1 \text{ and} \tag{3.3}$$

$$S\tau \mapsto \frac{-1}{\tau}. \tag{3.4}$$

3.2 The Weil Representation

We introduced $\tilde{\Gamma} = \mathrm{Mp}_2(\mathbb{Z})$ to resolve ambiguity arising from the choice of the square-root, however, the modular group $\mathrm{SL}_2(\mathbb{Z})$ is much more tractable because it is a group of linear transformations. In this section we introduce a unitary representation of $\tilde{\Gamma}$ as a subgroup of $\mathrm{GL}(V)$ for a vector space V. A representation of a group G is a group homomorphism $\rho : G \to \mathrm{GL}(V)$ and is a *unitary* representation if it maps elements of G to unitary matrices (so $\rho(g)^\dagger = \rho(g)$ for all $g \in G$). The construction of the Weil representation is given in [Bru02, pages 15-16].

Let L be an even lattice, that is, a finitely generated $\mathbb{Z}$-module together with a symmetric bilinear form $(\cdot, \cdot) : L \times L \to \mathbb{Z}$ whose associated quadratic form $q(x) = \frac{1}{2}(x, x)$ is such that $q(x) \in \mathbb{Z}$ for all $x \in L$. We can extend L to a vector space over $\mathbb{Q}$ by taking its tensor product $L \otimes_{\mathbb{Z}} \mathbb{Q}$ and extending $(., .) : L \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$. Note that $L \subset L \otimes_{\mathbb{Z}} \mathbb{Q}$ by inclusion $\iota : l \mapsto (l, 1)$. Using this we can construct the dual lattice L'.

Definition 3.3. If L is an even lattice then $L' = \{x \in L \otimes_{\mathbb{Z}} \mathbb{Q} : (x, y) \in \mathbb{Z} \text{ for all } y \in L\}$ is the *dual lattice* of L.

Since by definition $(., .) : L \times L \to \mathbb{Z}$ under restriction we have $L \subset L'$.

Definition 3.4. The quotient group L'/L is the *discriminant group.*

The discriminant group is a finite Abelian group. The group ring $\mathbb{C}[L'/L]$ has a standard basis conventionally denoted by $\{\mathfrak{e}_\gamma : \gamma \in L'/L\}$. Define the inner product $\langle \mathfrak{e}_\gamma, \mathfrak{e}_{\gamma'} \rangle = \delta_{\gamma,\gamma'}$, δ the Dirac delta function and extend by linearity.

Because $(.,.)$ is a symmetric bilinear form it can be represented by a symmetric matrix, say B, so $(x, y) = y^t Bx$. Because a real symmetric matrix has real eigenvalues we can define n_- to be the number of negative eigenvalues, n_+ to be the number of positive eigenvalues, and n_0 to be the dimension of the null space of B. The eigenvalues do not depend on the choice of basis, so the triple (n_0, n_-, n_+) — called the *signature* of L — is invariant of the matrix representation B. If $(.,.)$ is nondegenerate then $n_0 = 0$ and we write the signature as (n_-, n_+). If $n_- = 0$ then $(.,.)$ is *positive definite*, similarly for negative definite. We assume $(.,.)$ is nondegenerate.

Definition 3.5. The *Weil Representation* of $\tilde{\Gamma}$ on L is defined on the generators $(T, 1)$, $(S, \sqrt{\tau})$ by the linear transformations with action on $\{\mathfrak{e}_\gamma\}$ given by

$$\rho_L((T,1))\mathfrak{e}_\gamma = e(q(\gamma))\mathfrak{e}_\gamma, \tag{3.5}$$

$$\rho_L((S,\sqrt{\tau}))\mathfrak{e}_\gamma = \frac{e\left(\frac{n_- - n_+}{8}\right)}{\sqrt{|L'/L|}} \sum_{\gamma \in L'/L} e(-(\gamma,\gamma'))\mathfrak{e}_{\gamma'}, \tag{3.6}$$

where (n_-, n_+) is the signature of the lattice L.

Let $f : \mathbb{H} \to \mathbb{C}[L'/L]$ and fix $k \in \frac{1}{2}\mathbb{Z}$. We define the action of $\tilde{\Gamma}$ on f.

Definition 3.6. Let $(M, \phi) \in \tilde{\Gamma}$. The Petersson slash operator on $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is then [Bru02, equation 1.7]

$$(f \mid_k (M,\phi))(\tau) = \phi(\tau)^{-2k}\rho_L((M,\phi))^{-1} f(M\tau).$$

Vector-valued modular forms are defined with respect to the dual lattice arising from taking the Hermitian transpose. Our representation, however, is unitary, so the theory is invariant under the operator †. We may now define a vector-valued modular form transforming according to the Weil representation [Bru02, definition 1.2].

Definition 3.7. Let $f : \mathbb{H} \to \mathbb{C}[L'/L]$ and fix $k \in \frac{1}{2}\mathbb{Z}$. A function $f : \mathbb{H} \to \mathbb{C}[L'/L]$ is a vector-valued modular form of weight k with respect to the representation ρ_L if

(i) $f |_k (M, \phi) = f$ for all $(M, \phi) \in \tilde{\Gamma}$,

(ii) f is holomorphic on $\mathbb{H}$, and

(iii) f is holomorphic at the cusp ∞.

We will often use the following equivalent formulation of condition (i).

(i)′ $f(M\tau) = \phi(\tau)^{2k}\rho_L((M, \phi))f(\tau)$.

3.2.1 An Example.

Let $d \in \mathbb{Z}_{>0}$ and $L = \mathbb{Z}$ with bilinear form $(x, y) = 2dxy$. The associated quadratic form $q(x) = dx^2$ is even and L has dual lattice $L' = \{x \in \mathbb{Q} : 2dxy \in \mathbb{Z} \text{ for all } y \in L = \mathbb{Z}\}$. The $2dxy \in \mathbb{Z}$ for all $y \in \mathbb{Z}$ means $2dx \in \mathbb{Z}$ so the denominator of x divides $2d$. Hence $L' = \frac{1}{2d}\mathbb{Z}$. Then $L'/L \cong \mathbb{Z}/2d\mathbb{Z}$.

While the elements of L'/L are of the form $\frac{h}{2d} + \mathbb{Z}$ for $0 \le h < 2d$, we will write $\mathfrak{e}_h$ instead of $\mathfrak{e}_{\frac{h}{2d}} + \mathbb{Z}$. Note $\dim(\mathbb{C}[L'/L]) = 2d$. The matrix representing $(.,.)$ is $(2d)$ in the basis $\{1\}$, so $n_- = 0$, $n_+ = 1$. The Weil representation on $\mathbb{C}[L'/L]$ is given by

$$\rho_L((T, 1))\mathfrak{e}_h = e\left(\frac{h^2}{4d}\right)\mathfrak{e}_h$$
$$\rho_L((S, \sqrt{\tau}))\mathfrak{e}_h = \frac{1}{\sqrt{2id}} \sum_{h' \pmod{2d}} e\left(\frac{-hh'}{2d}\right)\mathfrak{e}_{h'}.$$

Now let $\eta(\tau)$ be the Dedekind eta function

$$\eta(\tau) := q^{1/24} \prod_{n=1}^{\infty}(1 - q^n).$$

Theorem 3.8. *Let $d = 6$ in the construction above. Then*

$$f(\tau) := \sum_{h \pmod{12}} \left(\frac{12}{h}\right) \eta(\tau)\mathfrak{e}_h$$

is a vector-valued modular form of weight $\frac{1}{2}$ and representation ρ_L, where $\left(\frac{12}{\cdot}\right)$ is the Dirichlet character given by the Kronecker symbol.

Before proving this theorem, we record the following identity for the character $\left(\frac{12}{\cdot}\right)$. It is used frequently elsewhere in this thesis. We have

$$G\left(\left(\frac{12}{\cdot}\right)\right) = 2\sqrt{3}, \tag{3.7}$$

where $G(\chi)$ is the Gauss sum of the character χ. See appendix A. We can now prove the modular transformations for f above.

Proof. Note that because $\tilde{\Gamma} = \langle (T,1), (S,\sqrt{\tau}) \rangle$ it is sufficient to determine that $f \mid_{\frac{1}{2}} (T,1) = f$ and similarly $f \mid_{\frac{1}{2}} (S,\sqrt{\tau}) = f$. Now $\eta(\tau+1) = e\left(\frac{1}{24}\right)\eta(\tau)$. Then

$$f(\tau+1) = \sum_{h \pmod{12}} \left(\frac{12}{h}\right) e\left(\frac{1}{24}\right) \eta(\tau)\mathfrak{e}_h.$$

It is easily verified that $\left(\frac{12}{h}\right) e\left(\frac{h^2}{24}\right) = \left(\frac{12}{h}\right) e\left(\frac{1}{24}\right)$. Hence

$$\begin{aligned} f(\tau+1) &= \sum_{h \pmod{12}} e\left(\frac{1}{24}\right)\left(\frac{12}{h}\right)\eta(\tau) \\ &= \sum_{h \pmod{12}} e\left(\frac{h^2}{24}\right)\left(\frac{12}{h}\right)\eta(\tau) \\ &= \rho_L((T,1))f(\tau) \end{aligned}$$

as required. The modular transformation for η under S is given by $\eta(S\tau) = \eta\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau}\eta(\tau)$. We therefore have

$$f\left(\frac{-1}{\tau}\right) = \sqrt{-i\tau} f(\tau).$$

Now the Weil representation yields

$$\begin{aligned} \sqrt{\tau}\rho_L((S,\sqrt{\tau})) \sum_{h \pmod{12}} \left(\frac{12}{h}\right)\eta(\tau)\mathfrak{e}_h &= \sqrt{\frac{\tau}{12i}} \sum_{h \pmod{12}} \left(\frac{12}{h}\right)\eta(\tau) \sum_{h' \pmod{12}} e\left(\frac{hh'}{12}\right)\mathfrak{e}_{h'} \\ &= \sqrt{\frac{\tau}{12i}} \sum_{h' \pmod{12}} \left[\sum_{h \pmod{12}} \left(\frac{12}{h}\right) e\left(\frac{hh'}{12}\right)\right]\eta(\tau)\mathfrak{e}_{h'}. \end{aligned}$$

We can apply lemma A.4 because $\left(\frac{12}{h+6}\right) = \left(\frac{12}{h+4}\right) = -\left(\frac{12}{h}\right)$ for all $h \in \mathbb{Z}/12\mathbb{Z}$. Then $\sum_{h \pmod{12}} \left(\frac{12}{h}\right) e\left(\frac{hh'}{12}\right) = \left(\frac{12}{h'}\right) G\left(\left(\frac{12}{\cdot}\right)\right)$, because $\left(\frac{12}{\cdot}\right)$ is a real character. Hence on substitution we obtain

$$\begin{aligned}\sqrt{\tau}\rho_L((S,\sqrt{\tau}))\sum_{h \pmod{12}}\left(\frac{12}{h}\right)\eta(\tau)\mathfrak{e}_h &= \sqrt{\frac{\tau}{12i}}G\left(\left(\frac{12}{\cdot}\right)\right)\sum_{h \pmod{12}}\left(\frac{12}{h}\right)\eta(\tau)\mathfrak{e}_h\\ &= \sqrt{-i\tau}f(\tau)\\ &= f\left(\frac{-1}{\tau}\right).\end{aligned}$$

Thus $f(\tau)$ transforms as a vector-valued modular form of weight $\frac{1}{2}$ and representation ρ_L. □

3.2.2 Vector-Valued Weak Maaß Forms.

The forms we construct are not holomorphic, however, they are eigenfunctions of the differential operator Δ_k when considered component-wise. This leads us to introduce harmonic Maaß forms. See, for example, [BFOR17, definition 18.9].

Definition 3.9. Let $\tau = x+iy = \Re(\tau)+i\Im(\tau)$. The weight k hyperbolic Laplacian is defined by

$$\Delta_k := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right).$$

The advantage of Δ_k is its invariance under the Möbius fractional linear transformation. If $\gamma \in \mathrm{SL}_2(\mathbb{R})$, then

$$\Delta_k(f\mid_k \gamma) = \Delta_k(f)\mid_k \gamma.$$

A vector-valued weak Maaß form f of weight k and representation ρ_L satisfies definition 3.7 slightly relaxed.

Definition 3.10. A vector-valued weak Maaß form $f : \mathbb{H} \to \mathbb{C}[L'/L]$ of weight k and representation ρ_L on $\tilde{\Gamma}$ satisfies:

(i) $f\mid_k (M,\phi) = f$ for all $(M,\phi) \in \tilde{\Gamma}$,

(ii) There exists a constant λ such that, for all τ, $\Delta_k f(\tau) = \lambda f(\tau)$ (Δ_k is applied componentwise), and

(iii) f has at most linear exponential growth at the cusp at ∞.

A harmonic weak Maaß form is one with $\lambda = 0$. This is a generalization of definition 3.7 because holomorphic functions are harmonic functions.

In the following sections we will construct vector-valued modular forms transforming according to a Weil representation. In [Gar19, corollary 3.2], stated below as in the original, Garvan proves the $\tilde{N}$ and $\tilde{M}$ functions satisfy (ii) and (iii); he also determines the modular transformations of these functions. A similar theorem appears in [BO10, theorem 3.4].

Theorem 3.11. *Suppose c is a fixed positive integer with $(c, 6) = 1$. Then*

$$\mathfrak{D}_c := \left\{ \tilde{N}(a, b; \tau), \tilde{M}(a, b; \tau) : (a, b) \in (\mathbb{Z}/c\mathbb{Z} \times \mathbb{Z}/c\mathbb{Z}) \setminus \{(0, 0)\} \right\}$$

is a vector-valued Maaß form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$.

Chapter 4. Mock ϑ-Functions and their Vector-Valued Forms

The functions we construct will be vector-valued forms transforming according to a Weil representation. By equating the transformation laws for the Weil representation in definition 3.5 with those of Bringmann-Ono and Garvan in section 2.2 we obtain a system of linear equations that we can solve for the coefficient of each mock ϑ-function. In what follows let $(c, 6) = 1$ and c be prime; we do this so T has full span as a linear transformation on the complex vector spaces $\text{Span}\{\tilde{N}(j,k) : 0 \leq j, k \leq c \text{ and } (j,k) \neq 0\}$, similarly for $\tilde{M}$.

First we need a representation of $\tilde{\Gamma}$. Let $(c, 6) = 1$ and $L = \mathbb{Z}$ have bilinear form $(x, y) = -12c^2xy$. Then L has signature $(1, 0)$ and dual lattice $L' = \frac{1}{12c^2}\mathbb{Z}$. The Weil representation on $\mathbb{C}[L'/L]$ is given by transformations on the basis vectors $\mathfrak{e}_h$ by

$$\rho_L((T,1))\mathfrak{e}_h = e\left(\frac{-h^2}{24c^2}\right)\mathfrak{e}_h, \tag{4.1}$$

$$\rho_L((S,\sqrt{\tau}))\mathfrak{e}_h = \frac{1}{\sqrt{-12ic^2}} \sum_{h' \pmod{12c^2}} e\left(\frac{hh'}{12c^2}\right)\mathfrak{e}_{h'}. \tag{4.2}$$

Let $\sum'$ be the restricted sum

$$\sum_{j,k}{}' = \sum_{\substack{0 \leq j,k \leq c-1 \\ (j,k) \neq (0,0)}}.$$

Represent a vector-valued mock ϑ-function $\vec{H}(\tau)$ in the standard basis $\mathfrak{e}_h$ by

$$\vec{H}(\tau) = \sum_{h \pmod{6c^2}} \sum_{j,k}{}' \left[a_h(j,k)\tilde{N}(j,k;\tau) + b_h(j,k)\tilde{M}(j,k;\tau)\right](\mathfrak{e}_h - \mathfrak{e}_{-h}). \tag{4.3}$$

Here $\vec{H}(\tau)$ has the property $\vec{H}(\tau)\mathfrak{e}_h = -\vec{H}(\tau)\mathfrak{e}_{-h}$ because $\rho_L((S^2, i))\mathfrak{e}_h = i^{n_- - n_+}\mathfrak{e}_{-h}$ [Bru02, equation 1.4]. Then $\vec{H}(\tau)$ transforms according to the Weil representation on L if

$$\vec{H}(T\tau) = \rho_L((T,1))\vec{H}(\tau) \tag{4.4}$$

and

$$\vec{H}(S\tau) = \sqrt{\tau}\rho_L((S,\sqrt{\tau}))\vec{H}(\tau). \tag{4.5}$$

In the next sections we substitute the Bringmann-Ono and Garvan transformations from section 2.2 into $\vec{H}(\tau)$.

4.1 The Fundamental Relations

4.1.1 The Fundamental Relations for $(T,1)$. Let

$$\zeta_M(a) := \zeta_{2c}^{5a}\zeta_{2c^2}^{-3a^2}\zeta_{24}^{-1}.$$

Also, let

$$\zeta_N(a,b) := \begin{cases} \zeta_{2c^2}^{3b^2}\zeta_{24}^{-1} & \text{if } a \geq b \\ -\zeta_c^{-3b}\zeta_{2c^2}^{3b^2}\zeta_{24}^{-1} & \text{otherwise.} \end{cases}$$

Writing $\vec{H}$ in the standard basis, note that $\vec{H}(\tau)$ transforms according to the Weil representation for $(T,1)$ if

$$\vec{H}(T\tau) = \rho_L((T,1))\vec{H}(\tau);$$

substituting equation (4.1) and the identities in section 2.2 into this relation yields

$$\sum_{h \pmod{6c^2}} \sideset{}{'}\sum_{j,k} \left[\zeta_N(j,k)a_h(j,k)\tilde{N}(j-k \pmod{c}, k;\tau) + \zeta_M(j)b_h(j,k)\tilde{M}(j, j+k \pmod{c};\tau)\right] \times (\mathfrak{e}_h - \mathfrak{e}_{-h})$$

$$= \sum_{h \pmod{6c^2}} \sideset{}{'}\sum_{j,k} e\left(\frac{-h^2}{24\cdot 49}\right)\left[a_h(j,k)\tilde{N}(j,k;\tau) + b_h(j,k)\tilde{M}(j,k;\tau)\right](\mathfrak{e}_h - \mathfrak{e}_{-h}).$$

Note that $j+k \pmod 7 \geq k$, taking least nonnegative residues, is true if and only if $j+k<7$. Similarly $j+k \pmod 7 < j$ is true if and only if $0 \leq j+k < j$. Knowing this allows us to rewrite the definition of ζ_N below. Equating coefficients of the $\tilde{M}(j,k;\tau)(\mathfrak{e}_h - \mathfrak{e}_{-h})$ and $\tilde{N}(j,k;\tau)(\mathfrak{e}_h - \mathfrak{e}_{-h})$ summands yields the fundamental relations for $(T,1)$ given in the following theorem.

Theorem 4.1. *Take all residues modulo 7 to be the least nonnegative residue. Let*

$$\vec{H}(\tau) = \sum_{h \pmod{6c^2}} \sideset{}{'}\sum_{j,k} \left[a_h(j,k)\tilde{N}(j,k;\tau) + b_h(j,k)\tilde{M}(j,k;\tau)\right](\mathfrak{e}_h - \mathfrak{e}_{-h}).$$

Then

$$\vec{H}(T\tau) = \rho_L((T,1))\vec{H}(\tau).$$

if and only if

$$a_h(j,k) = \begin{cases} \zeta_{2c^2}^{-3k^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot c^2}\right) a_h(j+k \pmod 7, k) & \textit{if } j+k<7 \\ -\zeta_c^{-3k}\zeta_{2c^2}^{3k^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot c^2}\right) a_h(j+k \pmod 7, k) & \textit{otherwise} \end{cases} \tag{4.6}$$

and

$$b_h(j,k) = \zeta_{2c}^{5j}\zeta_{2c^2}^{-3j^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right) b_h(j, k-j \pmod 7) \tag{4.7}$$

for all $h, j,$ and k.

4.1.2 The Fundamental Relations for $(S, \sqrt{\tau})$. Writing $\vec{H}$ in the standard basis and applying the Weil transformation for $(S, \sqrt{\tau})$ to it yields

$$\sqrt{\tau}\rho_L((S,\sqrt{\tau}))\vec{H}(\tau)$$
$$= \frac{\sqrt{\tau}}{\sqrt{-12ic^2}} \sideset{}{'}\sum_{j,k} \sum_{\substack{h \pmod{6c^2} \\ h' \pmod{12c^2}}} e\left(\frac{hh'}{12c^2}\right) \left[a_{h'}(j,k)N(j,k;\tau) + b_{h'}(j,k)\tilde{M}(j,k;\tau)\right](\mathfrak{e}_h - \mathfrak{e}_{-h}).$$

From section 2.2 we see that

$$\vec{H}(S\tau) = \sqrt{-i\tau} \sum_{h \pmod{6c^2}} \sideset{}{'}\sum_{j,k} \left[a_h(j,k)\tilde{M}(j,k;\tau) + b_h\tilde{N}(j,k;\tau)\right](\mathfrak{e}_h - \mathfrak{e}_{-h}).$$

Equating the coefficients of $M(j,k;\tau)(\mathfrak{e}_h - \mathfrak{e}_{-h})$ and $N(j,k;\tau)(\mathfrak{e}_h - \mathfrak{e}_{-h})$ yields the fundamental relations for $(S, \sqrt{\tau})$ given in the following theorem.

Theorem 4.2. *If*

$$\vec{H}(\tau) = \sum_{h \pmod{6c^2}} \sideset{}{'}\sum_{j,k} \left[a_h(j,k)\tilde{N}(j,k;\tau) + b_h(j,k)\tilde{M}(j,k;\tau)\right](\mathfrak{e}_h - \mathfrak{e}_{-h})$$

then

$$\vec{H}(S\tau) = \sqrt{\tau}\rho_L((S,\sqrt{\tau}))\vec{H}(\tau).$$

if and only if

$$\frac{i}{\sqrt{12c^2}} \sum_{h' \pmod{12c^2}} e\left(\frac{hh'}{12c^2}\right) a_{h'}(j,k) = b_h(j,k) \tag{4.8}$$

and

$$\frac{i}{\sqrt{12c^2}} \sum_{h' \pmod{12c^2}} e\left(\frac{hh'}{12c^2}\right) b_{h'}(j,k) = a_h(j,k) \tag{4.9}$$

for all $h, j,$ and k.

The significance of the fundamental relations lies in the observation that $\tilde{\Gamma} = \langle (T,1), (S, \sqrt{\tau}) \rangle$; using this fact we have the following theorem.

Theorem 4.3. *The function*

$$\vec{H}(\tau) = \sum_{h \pmod{6c^2}} \sum_{j,k}{}' \left[a_h(j,k) \tilde{N}(j,k;\tau) + b_h(j,k) \tilde{M}(j,k;\tau) \right] (\mathfrak{e}_h - \mathfrak{e}_{-h})$$

is a vector-valued modular form transforming according to the Weil representation with lattice $L = \mathbb{Z}$ and bilinear form $(x,y) = -12c^2xy$ if and only if the coefficients $a_h(j,k)$ and $b_h(j,k)$ are related by equations (4.7) - (4.9).

We have numerically solved the system of equations (4.7) - (4.9) for orders $c = 5, 7,$ and 11.

Chapter 5. Analytic Proof of Transformation Laws for the Seventh Order Vector-Valued Form

The goal of this chapter is to prove, using the theory of exponential sums, the following theorem.

Theorem 5.1. *Let*

$$\varepsilon_\alpha(h) = \begin{cases} 1 & \text{if } h \equiv \alpha \pmod{7}, \\ -1 & \text{if } h \equiv -\alpha \pmod{7}, \\ 0 & \text{otherwise.} \end{cases}$$

Define the vector-valued form $\vec{H}_7(\tau)$, $\tau \in \mathbb{H}$ by

$$\begin{aligned}
\vec{H}_7(\tau) =& \frac{1}{2\sin\frac{\pi}{7}} \sum_{\substack{h \pmod{6\cdot 49} \\ 0\le\alpha\le 6}} \left(\frac{12}{h}\right)(\mathfrak{e}_h - \mathfrak{e}_{-h}) \\
&\times\Bigg\{\varepsilon_1(h)\left[ie\left(\frac{57\alpha - \alpha h^2}{4\cdot 49}\right)\tilde{M}(1,\alpha;\tau) + e\left(\frac{5}{14}\right)e\left(\frac{\alpha+\alpha h^2}{4\cdot 49}\right)\tilde{N}(\alpha,1;\tau)\right] \\
&-\varepsilon_2(h)\left[ie\left(\frac{-40\alpha - 4\alpha h^2}{4\cdot 49}\right)\tilde{M}(2,\alpha;\tau) + e\left(\frac{10}{14}\right)e\left(\frac{86\alpha+4\alpha h^2}{4\cdot 49}\right)\tilde{N}(\alpha,2;\tau)\right] \\
&+\varepsilon_3(h)\left[ie\left(\frac{-60\alpha - 12\alpha h^2}{4\cdot 49}\right)\tilde{M}(3,\alpha;\tau) + e\left(\frac{15}{14}\right)e\left(\frac{-102\alpha+12\alpha h^2}{4\cdot 49}\right)\tilde{N}(\alpha,3;\tau)\right] \\
&-\varepsilon_4(h)\left[ie\left(\frac{-24\alpha - 16\alpha h^2}{4\cdot 49}\right)\tilde{M}(4,\alpha;\tau) + e\left(\frac{20}{14}\right)e\left(\frac{46\alpha+16\alpha h^2}{4\cdot 49}\right)\tilde{N}(\alpha,4;\tau)\right] \\
&+\varepsilon_5(h)\left[ie\left(\frac{-142\alpha - 10\alpha h^2}{4\cdot 49}\right)\tilde{M}(5,\alpha;\tau) + e\left(\frac{25}{14}\right)e\left(\frac{-44\alpha+10\alpha h^2}{4\cdot 49}\right)\tilde{N}(\alpha,5;\tau)\right] \\
&-\varepsilon_6(h)\left[ie\left(\frac{6\alpha - 6\alpha h^2}{4\cdot 49}\right)\tilde{M}(6,\alpha;\tau) + e\left(\frac{30}{14}\right)e\left(\frac{-8\alpha+6\alpha h^2}{4\cdot 49}\right)\tilde{N}(\alpha,6;\tau)\right]\Bigg\} \\
&+\frac{1}{\sin\frac{\pi}{7}} \sum_{\substack{h \pmod{6\cdot 7} \\ 0\le\beta\le 6}} \left(\frac{12}{h}\right)\sin\frac{h\beta\pi}{7}\left\{e\left(\frac{5\beta}{14}\right)\tilde{M}(0,\beta;\tau) + i\tilde{N}(\beta,0;\tau)\right\}(\mathfrak{e}_{7h} - \mathfrak{e}_{-7h}).
\end{aligned}$$

Then $\vec{H}_7(\tau)$ is a vector-valued weak Maaß form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ transforming according to the Weil representation with lattice $L = \mathbb{Z}$ and bilinear form $(x, y) = -(12\cdot 49)xy$.

5.1 Boundary Terms

The $\tilde{N}(j,k;\tau)$ and $\tilde{M}(j,k;\tau)$ functions each have an especially simple transformation under T when one of their indices is 0, but the dependence on index is different for $\tilde{N}$ and $\tilde{M}$. This property makes $\tilde{N}(\beta,0;\tau)$ and $\tilde{M}(0,\beta;\tau)$ eigenfunctions of T. The same cannot be said for $\tilde{N}(0,\beta;\tau)$ and $\tilde{M}(\beta,0;\tau)$. To keep track of the eigenfunction property we say $\tilde{N}(j,k;\tau)$ is a boundary term of $\vec{H}_c(\tau)$ if $j=0$ and $\tilde{M}(j,k;\tau)$ is a boundary term if $k=0$.

Recall that we have numerical forms of order $5,7$, and 11. We can write the boundary terms of $\vec{H}_5, \vec{H}_7$, and $\vec{H}_{11}$ in a more unified way. Define the boundary of a linear combination of $\tilde{N}, \tilde{M}$ terms to be

$$\begin{aligned}&\mathrm{Bd}\left(\sum a_h(j,k)\tilde{N}(j,k;\tau)+b_h(j,k)\tilde{M}(j,k;\tau)\right)(\mathfrak{e}_h-\mathfrak{e}_{-h})\\&=\sum_{\beta} a_h(\beta,0)\tilde{N}(\beta,0;\tau)+b_h(0,\beta)\tilde{M}(0,\beta;\tau)(\mathfrak{e}_h-\mathfrak{e}_{-h}).\end{aligned}\tag{5.1}$$

The boundary term of $\vec{H}_5$ is

$$\mathrm{Bd}(\vec{H}_5)(\tau)=\frac{1}{\sin\frac{\pi}{5}}\sum_{\substack{h \pmod{6\cdot 5}\\0\le\beta\le 4}}\left(\frac{12}{h}\right)\sin\frac{h\beta\pi}{5}\left\{e\left(\frac{5\beta}{10}\right)\tilde{M}(0,\beta;\tau)+i\tilde{N}(\beta,0;\tau)\right\}(\mathfrak{e}_{5h}-\mathfrak{e}_{-5h}).\tag{5.2}$$

Similarly we have

$$\mathrm{Bd}(\vec{H}_7)(\tau)=\frac{1}{\sin\frac{\pi}{7}}\sum_{\substack{h \pmod{6\cdot 7}\\0\le\beta\le 6}}\left(\frac{12}{h}\right)\sin\frac{h\beta\pi}{7}\left\{e\left(\frac{5\beta}{14}\right)\tilde{M}(0,\beta;\tau)+i\tilde{N}(\beta,0;\tau)\right\}(\mathfrak{e}_{7h}-\mathfrak{e}_{-7h})\tag{5.3}$$

and

$$\mathrm{Bd}(\vec{H}_{11})(\tau)=\frac{1}{\sin\frac{\pi}{11}}\sum_{\substack{h \pmod{6\cdot 11}\\0\le\beta\le 10}}\left(\frac{12}{h}\right)\sin\frac{h\beta\pi}{10}\left\{e\left(\frac{5\beta}{22}\right)\tilde{M}(0,\beta;\tau)+i\tilde{N}(\beta,0;\tau)\right\}(\mathfrak{e}_{11h}-\mathfrak{e}_{-11h}).\tag{5.4}$$

From the order $5,7$, and 11 cases we may conjecture $\mathrm{Bd}(\vec{H}_c)(\tau)$ is the following for

$(c, 6) = 1$.

$$\mathrm{Bd}(\vec{H}_c)(\tau) = \frac{1}{\sin\frac{\pi}{c}} \sum_{\substack{h \pmod{6\cdot c} \\ 0\le\beta\le c}} \left(\frac{12}{h}\right) \sin\frac{h\beta\pi}{c} \left\{ e\left(\frac{5\beta}{2c}\right) \tilde{M}(0,\beta;\tau) + i\tilde{N}(\beta,0;\tau) \right\} (\mathfrak{e}_{ch} - \mathfrak{e}_{-ch}). \tag{5.5}$$

The importance of the boundary terms is in the following lemma.

Lemma 5.2. *Let c be a prime, $(c, 6) = 1$. Every $\tilde{N}(j,k;\tau)$ and $\tilde{M}(j,k;\tau)$ is of the form $r\sqrt{\tau}^u \tilde{M}(0,\beta;(T^v S^u)\tau)$ or $r\sqrt{\tau}^u \tilde{N}(\beta,0;(T^v S^u)\tau)$ for $0 \le u \le 1$, $0 \le v \le c$, and $r \in \mathbb{C}$.*

Proof. First consider if $k = 0$. Then $\tilde{N}(j,k;\tau)$ is a boundary term and we are done. Otherwise $k > 0$. If $j = 0$ then $\tilde{M}(0,k;S\tau) = \sqrt{-i\tau}\tilde{N}(0,k;\tau)$ and we are done. Suppose $j \ne 0$. Then $j \in (\mathbb{Z}/c\mathbb{Z})^\times$ and k generates the units modulo c. Define v by $vk \equiv j \pmod{c}$. Using section 2.2 we see that $\tilde{N}(j,k;\tau)$ is a constant multiple of $\tilde{N}(0,k;T^v\tau)$. Hence $\tilde{N}(j,k;\tau) = r\sqrt{\tau}^u \tilde{M}(0,k;(T^v S^u)\tau)$ for $0 \le u \le 1$, $0 \le v \le c$, and $r \in \mathbb{C}$.

The proof for $\tilde{M}(j,k;\tau)$ is identical. □

The fact that c is prime is essential here. Suppose for example that $c = 25$, the smallest composite relatively prime to 6. Then $\tilde{N}(2,5;\tau)$ is not of the form $r\sqrt{\tau}\tilde{M}(0,\beta;(T^v S)\tau)$ for any β because $2 \notin \langle 5 \rangle \subset \mathbb{Z}/25\mathbb{Z}$.

Here we present a sample calculation using lemma 5.2 together with the relations imposed by equations (4.6) - (4.9) to write a general coefficient of $\vec{H}_7(\tau)$ in terms of a boundary coefficient, supposing Theorem 5.4. Consider $a_h(1,4)$. Because $5-4 \equiv 1 \pmod{7}$ and $5 \ge 4$ we have

$$a_h(1,4) = \zeta_{2\cdot 49}^{3\cdot 4^2} \zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right) a_h(5,4).$$

Similarly we have

$$a_h(5,4) = -\zeta_7^{-3\cdot 4} \zeta_{2\cdot 49^2}^{3\cdot 4^2} \zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right) a_h(2,4).$$

Continuing in like manner the chain[1] of coefficients obtained is $a_h(1,4) \mapsto a_h(5,4) \mapsto a_h(2,4) \mapsto a_h(6,4) \mapsto a_h(3,4) \mapsto a_h(0,4)$. In this chain there are 3 terms $a_h(j,4)$ — other

[1]Here chain is meant in the usual sense; that is, as an ordered subset of a partially ordered set. The order on the set is the number of applications of the T transformation until a boundary term is reached.

than $a_h(1,4)$ — with $j < 4$. Hence

$$a_h(1,4) = \left(-\zeta_7^{-3\cdot 4}\right)^3 \left(\zeta_{2\cdot 49^2}^{-3\cdot 4^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\right)^5 a_h(0,4).$$

Now substituting the fundamental relations imposed by $(S, \sqrt{\tau})$ on $a_h(0,4)$ yields the following identity for $a_h(1,4)$ if $\vec{H}_7(\tau)$ is a vector-valued modular form with appropriate representation.

$$a_h(1,4) = \left(-\zeta_7^{-3\cdot 4}\right)^3 \left(\zeta_{2\cdot 49^2}^{-3\cdot 4^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\right)^5 \frac{i}{\sqrt{12\cdot 49}} \sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) b_{h'}(0,4).$$

Hence if $\vec{H}(\tau)$ transforms according to the Weil representation we must be able to write $a_h(1,4)$ using a boundary term. Note the chain from $a_h(0,4)$ to $a_h(1,4)$ consists of 5 applications of identity (4.6). In each application of the identity the first index changes by 4; the reason there are 5 applications of the identity is then seen to be because 5 is the solution to $4n+1 \equiv 0 \pmod 7$ with least nonnegative residue. The number 3 above is also an important factor, and determining it is an important step in the calculation of $\vec{H}_c(\tau)$. We present the problem explicitly below.

Question 5.3. *Fix c a prime and let $a, b \in \mathbb{Z}$ with $0 \le a < c$, $0 < b < c$. Define $n \equiv -a\bar{b} \pmod c$ with $0 \le n < c$, $\bar{b}b \equiv 1 \pmod c$. How many elements x of the set*

$$\{a + lb \pmod c : 0 < l \le n\}$$

satisfy $x < b$?

Label the number of such elements $v(a,b)$. Note that n is the number of applications of the identity in 4.6; in each application the first index changes by a, so n is the least nonnegative residue such that $an + b \equiv 0 \pmod 7$. Using $v(a,b)$, lemma 5.2, and the relations imposed by equations (4.6) - (4.9) we have the following theorem allowing us to write a general coefficient in terms of boundary coefficients.

Theorem 5.4. *Let $\alpha \neq 0$. Let $m(\alpha,\beta) \equiv \bar{\alpha}\beta \pmod 7$ with $0 \le m(\alpha,\beta) < 7$. Similarly define $n(\beta,\alpha) \equiv -\beta\bar{\alpha} \pmod 7$ with $0 \le n(\beta,\alpha) < 7$. Let $v(\beta,\alpha)$ be the number of elements x in the set $\{\beta + l\alpha \pmod 7 : 0 < l \le n(\beta,\alpha)\}$ with $x < \alpha$. Then $\vec{H}_7(\tau)$ is a vector-valued* *form if and only if*

$$a_h(\beta,\alpha) = \frac{i\left(-\zeta_7^{-3\alpha}\right)^{v(\beta,\alpha)}\left(\zeta_{2\cdot 49}^{3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\beta,\alpha)}}{\sqrt{12\cdot 49}} \sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) b_{h'}(0,\alpha) \quad (5.6)$$

and

$$b_h(\alpha,\beta) = \frac{i\left(\zeta_{14}^{5\alpha}\zeta_{2\cdot 49}^{-3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)\right)^{m(\alpha,\beta)}}{\sqrt{12\cdot 49}} \sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) a_{h'}(\alpha,0). \quad (5.7)$$

Using the notation just introduced we have the following computationally verified identity for $\vec{H}_7(\tau)$ which is more amenable to proving Theorem 5.4.

$$\begin{aligned} &\vec{H}_7(\tau) \qquad (5.8)\\ &= \frac{1}{2\sin\frac{\pi}{7}} \sum_{\substack{h \pmod{6\cdot 49}\\ 1\le\alpha\le 6\\ 0\le\beta\le 6}} (-1)^{\alpha+1}\varepsilon_\alpha(h)\left(\frac{12}{h}\right)\Bigg\{ i\left(\zeta_{14}^{5\alpha}\zeta_{2\cdot 49}^{-3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)\right)^{m(\alpha,\beta)}\tilde{M}(\alpha,\beta;\tau)\\ &+ e\left(\frac{5\alpha}{14}\right)\left(-\zeta_7^{-3\alpha}\right)^{v(\beta,\alpha)}\left(\zeta_{2\cdot 49}^{3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\beta,\alpha)}\tilde{N}(\beta,\alpha;\tau)\Bigg\}(\mathfrak{e}_h - \mathfrak{e}_{-h})\\ &+ \frac{1}{\sin\frac{\pi}{7}} \sum_{\substack{h \pmod{6\cdot 7}\\ 0\le\beta\le 6}} \left(\frac{12}{h}\right)\sin\frac{h\beta\pi}{7}\Bigg\{e\left(\frac{5\beta}{14}\right)\tilde{M}(0,\beta;\tau) + i\tilde{N}(\beta,0;\tau)\Bigg\}(\mathfrak{e}_{7h} - \mathfrak{e}_{-7h}). \end{aligned}$$

Equivalently, if $a_h(\beta,\alpha)$ is the coefficient of $\tilde{N}(\beta,\alpha;\tau)$ in $\langle \vec{H}_7(\tau), \mathfrak{e}_h\rangle$ and similarly for $b_h(\alpha,\beta)$ and $\tilde{M}(\alpha,\beta;\tau)$, we have the identities

$$a_h(\beta,\alpha) = \begin{cases} \frac{(-1)^{\alpha+1}\varepsilon_\alpha(h)}{2\sin\frac{\pi}{7}}\left(\frac{12}{h}\right)e\left(\frac{5\alpha}{14}\right)\left(-\zeta_7^{-3\alpha}\right)^{v(\beta,\alpha)}\left(\zeta_{2\cdot 49}^{3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\beta,\alpha)} & \text{if } \alpha\neq 0,\\ \frac{-i\sin\frac{h\beta\pi}{49}}{\sin\frac{\pi}{7}}\left(\frac{12}{h}\right) & \text{if } \alpha = 0 \text{ and } 7\mid h,\\ 0 & \text{otherwise} \end{cases} \quad (5.9)$$

and

$$b_h(\alpha,\beta) = \begin{cases} \frac{i(-1)^{\alpha+1}\varepsilon_\alpha(h)}{2\sin\frac{\pi}{7}}\left(\frac{12}{h}\right)\left(\zeta_{14}^{5\alpha}\zeta_{2\cdot 49}^{-3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)\right)^{m(\alpha,\beta)} & \text{if } \alpha\neq 0,\\ \frac{\sin\frac{h\beta\pi}{7}}{\sin\frac{\pi}{7}}e\left(\frac{5\beta}{14}\right)\left(\frac{12}{h}\right) & \text{if } \alpha = 0 \text{ and } 7\mid h,\\ 0 & \text{otherwise.} \end{cases} \quad (5.10)$$

5.2 Proving Identities (4.6) and (4.7) for $\vec{H}_7$

Recall that in proving Theorem 5.4 it is sufficient to verify identities (4.6) - (4.9). In this section we verify (4.6) and (4.7). There are 2 cases to consider: the boundary terms and the nonboundary terms.

5.2.1 Boundary Terms: $a_h(\beta, 0)$ and $b_h(0, \beta)$.

Consider $a_h(\beta, 0)$. Substituting into equation (4.6) we must prove

$$a_h(\beta, 0) = \zeta_{24}^{-1} e\left(\frac{h^2}{24 \cdot 49}\right) a_h(\beta, 0).$$

It is sufficient to prove that $h^2 \equiv 49 \pmod{24 \cdot 49}$ whenever $a_h(\beta, 0) \neq 0$. Note $a_h(\beta, 0) \neq 0$ only if $(h, 6) = 1$ and $7 \mid h$. It so happens that $a_h(\beta, 0) \neq 0$ implies $7 \mid\mid h$, but that is a stronger condition than we will need. Write $h = 7x$ with $(x, 6) = 1$. $(x, 6) = 1$ is true if and only if $x^2 \equiv 1 \pmod{24}$, so we may write $x^2 = 1 + 24l$ for some $l \in \mathbb{Z}$. Then $h^2 = 49(1 + 24l)$ and $h^2 \equiv 49 \pmod{24 \cdot 49}$. We have therefore proven identity (4.6) for $a_h(\beta, 0)$; the proof for $b_h(0, \beta)$ is identical. Then we have the following theorem.

Theorem 5.5. *We have*

$$a_h(\beta, 0) = \zeta_{24}^{-1} e\left(\frac{h^2}{24 \cdot 49}\right) a_h(\beta, 0)$$

and

$$b_h(0, \beta) = \zeta_{24}^{-1} e\left(\frac{h^2}{24 \cdot 49}\right) b_h(0, \beta).$$

5.2.2 Nonboundary Terms.

We record an identity of $v(j, k)$, easily verified in Mathematica, which is useful in determining the desired transformation law for $a_h(j, k)$.

Theorem 5.6. *Take all residues modulo 7 to be the least nonnegative residue. Then*

$$v(j + k \pmod{7}, k) = \begin{cases} v(j, k) & \text{if } j + k < 7 \\ v(j, k) - 1 & \text{if } j + k > 7 \\ 0 & \text{otherwise} \end{cases}$$

and

$$v(j, -j \pmod 7) = 1.$$

Now consider $a_h(\beta, \alpha)$ with $\alpha \neq 0$. Our aim is to verify

$$a_h(\beta, \alpha) = \zeta_{2\cdot 49}^{3\alpha^2} \zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right) a_h(\beta+\alpha, \alpha). \tag{5.11}$$

Suppose that $\beta + \alpha < 7$. On substitution we see equation (5.11) is true if and only if

$$\frac{(-1)^{\alpha+1}\varepsilon_\alpha(h)\left(\frac{12}{h}\right) e\left(\frac{5\alpha}{14}\right)}{2\sin\frac{\pi}{7}}(-\zeta_7^{3\alpha^2})^{v(\beta,\alpha)}\left(\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\beta,\alpha)}$$
$$= \zeta_{2\cdot 49}^{3\alpha^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\frac{(-1)^{\alpha+1}\varepsilon_\alpha(h)\left(\frac{12}{h}\right) e\left(\frac{5\alpha}{14}\right)}{2\sin\frac{\pi}{7}}(-\zeta_7^{3\alpha^2})^{v(\beta+\alpha,\alpha)}\left(\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\beta+\alpha,\alpha)}.$$

This is true if and only if

$$v(\beta+\alpha, \alpha) = v(\beta, \alpha)$$
$$n(\beta+\alpha, \alpha) + 1 = n(\beta, \alpha).$$

The v identity is true by Theorem 5.6. The identity for n follows on substitution: $n(\beta + \alpha, \alpha) = -\overline{\alpha}(\beta+\alpha) = -\overline{\alpha}\beta - 1$. Hence equation (5.11) is an identity, as desired.

Now consider the case $\beta + \alpha > 7$. In this case we must verify

$$a_h(\beta, \alpha) = (-\zeta_7^{-3\alpha})\zeta_{2\cdot 49}^{3\alpha^2} \zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right) a_h(\beta+\alpha-7, \alpha). \tag{5.12}$$

On substitution equation (5.12) is equivalent to

$$\frac{(-1)^{\alpha+1}\varepsilon_\alpha(h)\left(\frac{12}{h}\right) e\left(\frac{5\alpha}{14}\right)}{2\sin\frac{\pi}{7}}(-\zeta_7^{3\alpha^2})^{v(\beta,\alpha)}\left(\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\beta,\alpha)}$$
$$= (-\zeta_7^{-3\alpha})\zeta_{2\cdot 49}^{3\alpha^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\frac{(-1)^{\alpha+1}\varepsilon_\alpha(h)\left(\frac{12}{h}\right) e\left(\frac{5\alpha}{14}\right)}{2\sin\frac{\pi}{7}}(-\zeta_7^{3\alpha^2})^{v(\beta+\alpha-7,\alpha)}\left(\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\beta+\alpha-7,\alpha)},$$

which is true if and only if

$$v(\beta+\alpha-7, \alpha) = v(\beta, \alpha)$$
$$n(\beta+\alpha-7, \alpha) + 1 = n(\beta, \alpha).$$

Our previous work establishes these identities. Hence equation (5.12) is an identity.

Our final case for a_h is when $\alpha+\beta=7$. The desired identity is

$$a_h(7-\alpha,\alpha)=(-\zeta_7^{-3\alpha})\zeta_{2\cdot 49}^{3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)a_h(0,\alpha).$$

On substitution this identity is true if and only if $v(7-\alpha,\alpha)=1$ and $n(7-\alpha,\alpha)=1$. The first is true by Theorem 5.6. The second is true because $n(-\alpha \pmod{7},\alpha)\equiv 1 \pmod{7}$.

The case for $b_h(\alpha,\beta)$ with $\alpha\neq 0$ follows in a near-identical manner, save there is no need to apply Theorem 5.6. Hence we have the following theorem.

Theorem 5.7. *Suppose $\alpha\neq 0$. Then*

$$a_h(\beta,\alpha)=\begin{cases}\zeta_{2\cdot 49}^{-3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)a_h(\beta+\alpha \pmod{7},k) & \text{if } \beta+\alpha<7,\\ -\zeta_7^{-3\alpha}\zeta_{2\cdot 49}^{3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)a_h(\beta+\alpha \pmod{7},k) & \text{otherwise},\end{cases} \tag{5.13}$$

and

$$b_h(\alpha,\beta)=\zeta_{14}^{5\alpha}\zeta_{2\cdot 49}^{-3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)b_h(\alpha,\beta-\alpha \pmod{7}) \tag{5.14}$$

for all h.

5.3 Proving Identities (4.8) and (4.9) for $\vec{H}_7$

In this section we verify (4.8) and (4.9). There are 3 cases to consider:

(i) $a_h(\beta,0)$ or $b_h(0,\beta)$, which are boundary terms;

(ii) $a_h(0,\alpha)$ and $b_h(\alpha,0)$, are terms which are related to boundary terms by equations (4.8) and (4.9); and

(iii) $a_h(\beta,\alpha)$ and $b_h(\alpha,\beta)$ with neither α nor β equal to 0.

5.3.1 Boundary Terms: $a_h(\beta,0)$ and $b_h(0,\beta)$.

Our aim is to prove

$$\frac{i}{\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right)b_{h'}(0,\beta)=a_h(0,\beta). \tag{5.15}$$

Then on substitution the left hand side of equation (5.15) becomes

$$\frac{i}{\sin\frac{\pi}{7}\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 7}} e\left(\frac{hh'}{12\cdot 7}\right)\left(\frac{12}{h'}\right)e\left(\frac{5\beta}{14}\right)\sin\frac{h'\pi\beta}{7}$$
$$= \frac{e\left(\frac{5\beta}{14}\right)}{2\sin\frac{\pi}{7}\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 7}}\left(\frac{12}{h'}\right)\left(e\left(\frac{h'(h+6\beta)}{12\cdot 7}\right)-e\left(\frac{h'(h-6\beta)}{12\cdot 7}\right)\right).$$

Define $J(x) := \sum_{h' \pmod{12\cdot 7}} e\left(\frac{h'x}{12\cdot 7}\right)\left(\frac{12\cdot 49}{h'}\right)$. Then equation (5.15) becomes

$$\frac{e\left(\frac{5\beta}{14}\right)}{2\sin\frac{\pi}{7}\sqrt{12\cdot 49}}(J(h+6\beta)-J(h-6\beta)) = a_h(0,\beta). \tag{5.16}$$

Note we can use the Kronecker symbol $\left(\frac{12\cdot 49}{h'}\right)$ instead of $\left(\frac{12}{h'}\right)$ because when $7 \mid h'$ we have both $\left(\frac{49}{h'}\right) = 0$ and $\sin\frac{h'\pi\beta}{7} = 0$. To determine $J(x)$ there are 3 cases for x, not all disjoint but which together cover every possibility.

(i) $(x, 6) > 1$,

(ii) $(x, 6\cdot 7) = 1$, and

(iii) $(x, 7) > 1$.

The First and Second Cases. In these first two cases our approach is similar to the proof of lemma A.4. Suppose $(x, 6) > 1$. Then $p \mid x$ for some $p \in \{2, 3\}$. Note that $\left(\frac{12}{h+\frac{12\cdot 7}{p}}\right) = -\left(\frac{12}{h}\right)$ for all h. Now $x = py$ for some $y \in \mathbb{Z}$, so

$$J(x) = J(py) = \sum_{h' \pmod{12\cdot 7}} e\left(\frac{ph'y}{12\cdot 7}\right)\left(\frac{12\cdot 49}{h'}\right)$$
$$= \sum_{h' \pmod{12\cdot 7}} e\left(\frac{p(h'+\frac{12\cdot 7}{p})y}{12\cdot 7}\right)\left(\frac{12\cdot 49}{h'+\frac{12\cdot 7}{p}}\right)$$

under the translation $h' \mapsto h' + \frac{12\cdot 7}{p}$. Then

$$J(py) = -\sum_{h' \pmod{12\cdot 7}} e\left(\frac{ph'y}{12\cdot 7}\right)\left(\frac{12\cdot 49}{h'}\right) = -J(py)$$

so $J(x) = -J(x) = 0$.

Now suppose $(x, 6\cdot 7) = 1$. Then x is invertible modulo $12\cdot 7$; let $\overline{x}x \equiv 1 \pmod{12\cdot 7}$.

Then

$$
\begin{aligned}
J(x) &= \sum_{h' \pmod{12\cdot 7}} e\left(\frac{h'x}{12\cdot 7}\right)\left(\frac{12\cdot 49}{h'}\right) \\
&= \left(\frac{12\cdot 49}{x}\right) \sum_{h' \pmod{12\cdot 7}} e\left(\frac{h'}{12\cdot 7}\right)\left(\frac{12\cdot 49}{h'}\right) \\
&= \left(\frac{12\cdot 49}{x}\right) G\left(\left(\frac{12\cdot 49}{\cdot}\right)\right),
\end{aligned}
$$

under the translation $h' \mapsto h'\overline{x}$, which is a bijection of $\mathbb{Z}/12\mathbb{Z}$, noting that $\left(\frac{12\cdot 49}{x}\right) = \left(\frac{12\cdot 49}{\overline{x}}\right)$. But $G\left(\left(\frac{12\cdot 49}{\cdot}\right)\right) = \mu(7)\left(\frac{12}{7}\right) G\left(\left(\frac{12}{\cdot}\right)\right)$, from theorem A.3. Then for $(x, 6\cdot 7) = 1$ we have

$$
J(x) = \left(\frac{12}{x}\right) G\left(\left(\frac{12}{\cdot}\right)\right) = \left(\frac{12}{x}\right) 2\sqrt{3}. \tag{5.17}
$$

The Third Case. We now want to compute $J(7x)$. There are two possibilites for x : $(x,6) = 1$ and $(x,6) > 1$. If $(x,6) > 1$ then $J(7x) = 0$ by case (i). So suppose $(x,6) = 1$. Note that

$$
\sum_{h' \pmod{12\cdot 7}} e\left(\frac{7h'x}{12\cdot 7}\right)\left(\frac{12\cdot 49}{h'}\right) = \sum_{\substack{h' \pmod{12\cdot 7} \\ (h',7)=1}} e\left(\frac{h'x}{12}\right)\left(\frac{12}{h'}\right)
$$

because for every h' divisible by 7 we have $\left(\frac{49}{h'}\right) = 0$.

Using the Chinese Remainder Theorem we can, for each h', find a $y \pmod 7$ and a $z \pmod{12}$ such that $h' = 12y + 7z$; conversely, every $y \pmod 7$ and $z \pmod{12}$ yields a unique $h' \pmod{12\cdot 7}$. Because $e\left(\frac{12yx}{12}\right)\left(\frac{12}{7z+12y}\right) = \left(\frac{12}{7z}\right)$ we have

$$
\begin{aligned}
J(7x) &= \sum_{\substack{h \pmod{12\cdot 7} \\ (h,7)=1}} e\left(\frac{h'x}{12}\right)\left(\frac{12}{h'}\right) = \sum_{\substack{y \pmod 7 \\ y\neq 0}} \sum_{z \pmod{12}} e\left(\frac{7zx}{12}\right)\left(\frac{12}{7z}\right) \\
&= \left(\frac{12}{x}\right) \sum_{\substack{y \pmod 7 \\ y\neq 0}} \sum_{z \pmod{12}} e\left(\frac{z}{12}\right)\left(\frac{12}{z}\right) = 6\left(\frac{12}{x}\right) G\left(\left(\frac{12}{\cdot}\right)\right),
\end{aligned}
$$

using the transformation $z \mapsto \overline{7x}z$ with $(7x)\overline{(7x)} \equiv 1 \pmod{12}$.

Hence $J(7x) = 6\left(\frac{12}{x}\right) G\left(\left(\frac{12}{\cdot}\right)\right)$ is true for both $(x,6) = 1$ and $(x,6) > 1$.

Condensing our results for J we obtain

$$J(x) = \begin{cases} \left(\frac{12}{x}\right) 2\sqrt{3} & \text{if } (x, 6\cdot 7) = 1, \\ -\left(\frac{12}{x}\right) 12\sqrt{3} & (\text{if } x, 7) > 1, \\ 0 & \text{otherwise.} \end{cases} \tag{5.18}$$

It can be verified that

$$\left(\frac{12}{h \pm 6\beta}\right) = (-1)^{\beta} \left(\frac{12}{h}\right). \tag{5.19}$$

Recall our objective is to prove equation (5.15), which requires computing $J(h+6\beta) - J(h-6\beta)$. There are again 3 cases:

(i) $(h, 6) > 1$,

(ii) $(h, 6) = 1$ and $h \equiv \pm\beta \pmod{7}$,

(iii) $(h, 6) = 1$ and $h \not\equiv \pm\beta \pmod{7}$.

The First Case. When $(h, 6) > 1$ we have $J(h \pm 6\beta) = 0$, so

$$\frac{i}{\sqrt{12\cdot 49}} \sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) b_{h'}(0,\beta) = \frac{e\left(\frac{5\beta}{14}\right)}{2\sin\frac{\pi}{7}\sqrt{12\cdot 49}}(J(h+6\beta) - J(h-6\beta)) = a_h(0,\beta)$$

because $\left(\frac{12}{h}\right) = 0$.

The Second Case. Suppose $(h, 6) = 1$ and $h \equiv \beta \pmod{7}$. Then $h + 6\beta \equiv 0 \pmod{7}$ and $h - 6\beta \not\equiv 0 \pmod{7}$, so

$$\begin{aligned} J(h+6\beta) - J(h-6\beta) &= -\left(\frac{12}{h+6\beta}\right) 12\sqrt{3} - \left(\frac{12}{h-6\beta}\right) 2\sqrt{3} \\ &= (-1)^{\beta+1} \left(\frac{12}{h}\right) 14\sqrt{3} \end{aligned}$$

using equation (5.19). Hence

$$\begin{aligned} \frac{i}{\sqrt{12\cdot 49}} \sum_{h' \pmod{12\cdot 7}} e\left(\frac{hh'}{12\cdot 7}\right) b_{h'}(0,\beta) &= \frac{e\left(\frac{5\beta}{14}\right)}{2\sin\frac{\pi}{7}\sqrt{12\cdot 49}}(J(h+6\beta) - J(h-6\beta)) \\ &= (-1)^{\beta+1} \frac{e\left(\frac{5\beta}{14}\right)}{2\sin\frac{\pi}{7}\sqrt{12\cdot 49}} \left(\frac{12}{h}\right) 14\sqrt{3} \\ &= (-1)^{\beta+1} \frac{e\left(\frac{5\beta}{14}\right)}{2\sin\frac{\pi}{7}} \left(\frac{12}{h}\right). \end{aligned}$$

Similarly if $(h,6)=1$ and $h\equiv -\beta \pmod 7$ then

$$J(h+6\beta)-J(h-6\beta) = -(-1)^{\beta+1}\left(\frac{12}{h}\right)14\sqrt{3}$$

and

$$\frac{i}{\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 7}} e\left(\frac{hh'}{12\cdot 7}\right) b_{h'}(0,\beta) = -(-1)^{\beta+1}\frac{e\left(\frac{5\beta}{14}\right)}{2\sin\frac{\pi}{7}}\left(\frac{12}{h}\right).$$

Hence if $(h,6)=1$ and $h\equiv \pm\beta \pmod 7$ then

$$\begin{aligned}\frac{i}{\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 7}} e\left(\frac{hh'}{12\cdot 7}\right) b_{h'}(0,\beta) &= \varepsilon_\beta(h)(-1)^{\beta+1}\frac{e\left(\frac{5\beta}{14}\right)}{2\sin\frac{\pi}{7}}\left(\frac{12}{h}\right)\\ &= a_h(0,\beta).\end{aligned}$$

The Third Case. Now suppose $(h,6)=1$ and $h\not\equiv \pm\beta \pmod 7$. Then $7\nmid(h\pm 6\beta)$ and

$$\begin{aligned}J(h+6\beta)-J(h-6\beta) &= 2\sqrt{3}\left(\frac{12}{h+6\beta}\right)-2\sqrt{3}\left(\frac{12}{h-6\beta}\right)\\ &= 0 = a_h(0,\beta).\end{aligned}$$

Note that $a_h(\beta,0)$ and $b_h(0,\beta)$ differ by a constant. We have therefore proven the identities in (4.8) and (4.9) for the boundary terms.

Theorem 5.8. *We have*

$$\frac{i}{\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) b_{h'}(0,\beta) = a_h(0,\beta)$$

and

$$\frac{i}{\sqrt{-12\cdot 49}}\sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) a_{h'}(\beta,0) = b_h(\beta,0)$$

for all h and all β, $0<\beta\le 6$.

5.3.2 Nonboundary Terms with One 0 Index: $a_h(0,\alpha)$ and $b_h(\alpha,0)$.

Consider $b_h(\alpha,0)$, $0<\beta\le 6$. Our aim is to prove equation (4.9) for this coefficient. On substitution equation (4.9) becomes

$$\frac{i(-1)^\alpha}{2\sin\frac{\pi}{7}\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right)\varepsilon_\alpha(h')\left(\frac{12}{h'}\right) = \begin{cases} -i\left(\frac{12}{h}\right)\frac{\sin\frac{\pi h\alpha}{49}}{\sin\frac{\pi}{7}} & \text{if } 7\mid h,\\ 0 & \text{otherwise.}\end{cases}$$

Define

$$J_\alpha(h) = \sum_{\substack{h' \pmod{12\cdot 49} \\ h'\equiv\alpha \pmod 7}} e\left(\frac{hh'}{12\cdot 49}\right)\left(\frac{12}{h'}\right). \tag{5.20}$$

Then the identity we wish to verify becomes

$$\frac{(-1)^\alpha}{2\sqrt{12\cdot 49}}\left(J_\alpha(h) - J_{-\alpha}(h)\right) = \begin{cases} i\left(\frac{12}{h}\right)\sin\frac{h\pi\alpha}{49} & \text{if } 7 \mid h, \\ 0 & \text{otherwise.} \end{cases} \tag{5.21}$$

Let us find $J_\alpha(h)$. Note $h' \equiv \alpha \pmod 7$ for $h' \in \{0, 1, ..., 12\cdot 49 - 1\}$ is true if and only if $h' = \alpha + 7r$ for $r \in \{0, 1, 2, ..., 12\cdot 7 - 1\}$. Then

$$J_\alpha(h) = \sum_{r \pmod{12\cdot 7}} e\left(\frac{h(\alpha+7r)}{12\cdot 49}\right)\left(\frac{12}{\alpha+7r}\right).$$

For each such r there exists a unique y modulo 7 and z modulo 12 such that $r = 12y + 7z$, by the Chinese Remainder Theorem. On substitution, then, we obtain

$$J_\alpha(h) = e\left(\frac{h\alpha}{12\cdot 49}\right)\sum_{y \pmod 7}\sum_{z \pmod{12}} e\left(\frac{hz}{12}\right)\left(\frac{12}{\alpha+z}\right).$$

Consider $\sum_{z \pmod{12}} e\left(\frac{hz}{12}\right)\left(\frac{12}{\alpha+z}\right)$. Under translation by $-\alpha$ this becomes

$$\sum_{z \pmod{12}} e\left(\frac{hz}{12}\right)\left(\frac{12}{\alpha+z}\right) = e\left(\frac{-h\alpha}{12}\right)\sum_{z \pmod{12}} e\left(\frac{hz}{12}\right)\left(\frac{12}{z}\right).$$

If $(h, 12) > 1$, say $h = 2x$ for some integer x, this becomes

$$\sum_{z \pmod{12}} e\left(\frac{zx}{6}\right)\left(\frac{12}{z}\right) = 0$$

for all x; only one set of representatives x modulo 12 need to be tested. Similarly if $3 \mid h$ we have $\sum_{z \pmod{12}} e\left(\frac{hz}{12}\right)\left(\frac{12}{\alpha+z}\right) = 0$. Hence $\sum_{z \pmod{12}} e\left(\frac{hz}{12}\right)\left(\frac{12}{\alpha+z}\right)$ is 0 when $\left(\frac{12}{\cdot}\right)$ is.

On the other hand, if $(h, 12) = 1$ let $\overline{h}h \equiv 1 \pmod{12}$. Then multiplication by $\bar{h}$ is a bijection of $\mathbb{Z}/12\mathbb{Z}$ and

$$\sum_{z \pmod{12}} e\left(\frac{hz}{12}\right)\left(\frac{12}{\alpha+z}\right) = \left(\frac{12}{h}\right)\sum_{z \pmod{12}} e\left(\frac{z}{12}\right)\left(\frac{12}{\alpha+z}\right)$$
$$= \left(\frac{12}{h}\right) G\left(\left(\frac{12}{\cdot}\right)\right).$$

Now consider $\sum_{y \pmod 7} e\left(\frac{hy}{7}\right)$. It is a standard result (see, for example, [IR82, section 6.3, lemma 1]) that

$$\sum_{y \pmod{7}} e\left(\frac{hy}{7}\right) = \begin{cases} 7 & \text{if } 7 \mid h, \\ 0 & \text{otherwise.} \end{cases}$$

Summarizing, we have

$$J_\alpha(h) = \begin{cases} 7\sqrt{12}e\left(\frac{-4h\alpha}{49}\right)\left(\frac{12}{h}\right) & \text{if } 7 \mid h, \\ 0 & \text{otherwise,} \end{cases} \tag{5.22}$$

since $e\left(\frac{-4h\alpha}{49}\right) = e\left(\frac{h\alpha}{12\cdot 49}\right) e\left(\frac{-h\alpha}{12}\right)$.

On substitution equation (5.21) is an identity if and only if

$$(-1)^\alpha \left(\frac{12}{h}\right) \sin\frac{8\pi h\alpha}{7} = \left(\frac{12}{h}\right) \sin\frac{\pi h\alpha}{7} \tag{5.23}$$

for all h. Note that $\sin\frac{8\pi h\alpha}{7} = \sin\frac{\pi h\alpha}{7}\cos(\pi h\alpha)$, hence our identity is true if and only if

$$(-1)^\alpha \left(\frac{12}{h}\right) \cos(h\pi\alpha) = \left(\frac{12}{h}\right). \tag{5.24}$$

This is true because $\cos\pi h\alpha = (-1)^\alpha$ for $h \equiv 1 \pmod{2}$ and $\left(\frac{12}{h}\right) = \left(\frac{3}{h}\right)\left(\frac{4}{h}\right)$. We therefore have the following theorem.

Theorem 5.9. *We have*

$$\frac{i}{\sqrt{12\cdot 49}} \sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) b_{h'}(\alpha, 0) = a_h(\alpha, 0).$$

Now for $a_h(0,\alpha)$ the proof is almost identical. Note that on substitution $a_h(0,\alpha)$ and $b_h(0,\alpha)$ satisfy identity (4.8) if and only if

$$\frac{i(-1)^{\alpha+1}e\left(\frac{5\alpha}{14}\right)}{2\sin\frac{\pi}{7}\sqrt{12\cdot 49}} \sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) \varepsilon_\alpha(h) \left(\frac{12}{h}\right) = \begin{cases} -e\left(\frac{5\alpha}{14}\right)\left(\frac{12}{h}\right)\frac{\sin\frac{\pi h\alpha}{49}}{\sin\frac{\pi}{7}} & \text{if } 7 \mid h, \\ 0 & \text{otherwise,} \end{cases}$$

which we can simplify to

$$\frac{i(-1)^\alpha}{2\sqrt{12\cdot 49}} \left(J_\alpha(h) - J_{-\alpha}(h)\right) = \begin{cases} \left(\frac{12}{h}\right)\sin\frac{\pi h\alpha}{49} & \text{if } 7 \mid h, \\ 0 & \text{otherwise.} \end{cases}$$

This is an identity already verified in the proof for $b_h(\alpha, 0)$. Hence we have the following theorem.

Theorem 5.10. *We have*

$$\frac{i}{\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) a_{h'}(0,\alpha) = b_h(0,\alpha).$$

5.3.3 General Nonboundary Terms with No 0 Indices. In this section we present a character sum, the determination of which is equivalent to proving

$$\frac{i}{\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) b_{h'}(\alpha,\beta) = a_{h'}(\alpha,\beta) \tag{5.25}$$

for $\alpha, \beta \neq 0$; this verifies identity (4.9) for all nonboundary b_h terms. A similar identity applies for equation (4.8). While the identity in question is computationally confirmed, we do not yet have an analytic proof.

Note from section 5.2 that

$$b_h(\alpha,\beta) = \left(\zeta_{14}^{5\alpha}\zeta_{2\cdot 49}^{-3\alpha^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\right)^{m(\alpha,\beta)} b_h(\alpha,0)$$

and

$$a_h(\alpha,\beta) = (-\zeta_7^{-3\beta})^{v(\alpha,\beta)} \left(\zeta_{2\cdot 49}^{3\beta^2}\zeta_{24}^{-1} e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\alpha,\beta)} a_h(0,\beta).$$

On substitution of $b_h(\alpha,0)$ and $a_h(0,\beta)$ from equation (5.8) into (5.25) we have

$$\frac{1}{\sqrt{12\cdot 49}}\sum_{h' \pmod{12\cdot 49}} e\left(\frac{hh'}{12\cdot 49}\right) e\left(\frac{m(\alpha,\beta)h'^2}{24\cdot 49}\right)\varepsilon_\alpha(h')\left(\frac{12}{h'}\right)$$

$$= (-1)^{\alpha+\beta+1}\zeta_{14}^{-5m(\alpha,\beta)\alpha}\zeta_{2\cdot 49}^{3m(\alpha,\beta)\alpha^2+3n(\alpha,\beta)\beta^2}\zeta_{24}^{m(\alpha,\beta)-n(\alpha,\beta)}(-\zeta_7^{-3\beta})^{v(\alpha,\beta)}$$
$$\times\, e\left(\frac{n(\alpha,\beta)h^2}{24\cdot 49}\right)\varepsilon_\beta(h)\left(\frac{12}{h}\right) e\left(\frac{5\beta}{14}\right).$$

Define

$$R_\alpha(x,y) := \sum_{\substack{h' \pmod{12\cdot 49}\\ h'\equiv\alpha \pmod 7}} e\left(\frac{xh'^2+2yh'}{24\cdot 49}\right)\left(\frac{12}{h'}\right). \tag{5.26}$$

when $x \neq 0$. Using Mathematica we have proven the following theorem.

Theorem 5.11.

$$\frac{1}{\sqrt{12\cdot 49}}\left[R_\alpha(m(\alpha,\beta),h)-R_{-\alpha}(m(\alpha,\beta),h)\right]$$
$$=(-1)^{\alpha+\beta+1}\zeta_{14}^{-5m(\alpha,\beta)\alpha}\zeta_{2\cdot 49}^{3m(\alpha,\beta)\alpha^2+3n(\alpha,\beta)\beta^2}\zeta_{24}^{m(\alpha,\beta)-n(\alpha,\beta)}(-\zeta_7^{-3\beta})^{v(\alpha,\beta)}$$
$$\times\, e\left(\frac{n(\alpha,\beta)h^2}{24\cdot 49}\right)\varepsilon_\beta(h)\left(\frac{12}{h}\right)e\left(\frac{5\beta}{14}\right).$$

5.4 Proof of Theorem 5.4

Here the main theorem, Theorem 5.4, is presented, with $\vec{H}_7$ rewritten using identity (5.8).

Theorem. *Let*

$$\vec{H}_7(\tau)$$
$$=\frac{1}{2\sin\frac{\pi}{7}}\sum_{\substack{h \pmod{6\cdot 49}\\1\le\alpha\le 6\\0\le\beta\le 6}}(-1)^{\alpha+1}\varepsilon_\alpha(h)\left(\frac{12}{h}\right)\Bigg\{i\left(\zeta_{14}^{5\alpha}\zeta_{2\cdot 49}^{-3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)\right)^{m(\alpha,\beta)}\tilde{M}(\alpha,\beta;\tau)$$
$$+\,e\left(\frac{5\alpha}{14}\right)\left(-\zeta_7^{-3\alpha}\right)^{v(\beta,\alpha)}\left(\zeta_{2\cdot 49}^{3\alpha^2}\zeta_{24}^{-1}e\left(\frac{h^2}{24\cdot 49}\right)\right)^{n(\beta,\alpha)}\tilde{N}(\beta,\alpha;\tau)\Bigg\}(\mathfrak{e}_h-\mathfrak{e}_{-h})$$
$$+\,\frac{1}{\sin\frac{\pi}{7}}\sum_{\substack{h \pmod{6\cdot 7}\\0\le\beta\le 6}}\left(\frac{12}{h}\right)\sin\frac{h\beta\pi}{7}\Bigg\{e\left(\frac{5\beta}{14}\right)\tilde{M}(0,\beta;\tau)+i\tilde{N}(\beta,0;\tau)\Bigg\}(\mathfrak{e}_{7h}-\mathfrak{e}_{-7h}).$$

Then $\vec{H}_7(\tau)$ is a vector-valued weak Maaß form of weight $\frac{1}{2}$ for $\mathrm{SL}_2(\mathbb{Z})$ transforming according to the Weil representation with lattice $L=\mathbb{Z}$ and bilinear form $(x,y)=-(12\cdot 49)xy$.

Proof. We observe that in proving Theorem 5.4 it is sufficient to verify that the coefficients $a_h(\alpha,\beta)$ and $b_h(\alpha,\beta)$ all satisfy the identities in equations (4.6) - (4.9). There are, broadly, 2 classes of coeffficients of $\vec{H}(\tau)$: boundary terms and nonboundary terms. In section 5.2 it was proven that the coefficients of $\vec{H}_7(\tau)$ satisfy the identities in equations (4.6) and (4.7), culminating in Theorem 5.5 for the boundary terms and Theorem 5.7 for the rest. In section 5.3 it was proven that the coefficients of $\vec{H}_7(\tau)$ satisfy the identities in equations (4.8) and (4.9). This proof was split into 3 parts: in section 5.3.1 the identities were proven for the boundary terms, culminating in Theorem 5.8; in section 5.3.2 the identities were proven for nonboundary terms with a 0 index, culminating in Theorems 5.9 and 5.10; and finally in section 5.3.3 the identities were proven for general nonboundary terms with nonzero indices, culminating in Theorem 5.11. □

Generalizing this result to general prime orders should be possible once analogs of Theorems 5.6 and 5.11 are obtained; the other necessary results should generalize in a straightforward manner.

Appendix A. Lemmata on Exponential Sums

Recall the definition of the Gauss sum $G(\chi)$ for a Dirichlet character χ:

Definition A.1. $G(\chi) = \sum_{h \pmod{d}} \chi(h) e\left(\frac{h}{d}\right)$, where d is the modulus of the character χ.

Now a Kronecker symbol $\left(\frac{x}{\cdot}\right)$ is a Dirichlet character if it satisfies certain conditions. See, for example, [AG18, corollary 3.3].

Theorem A.2. *$\left(\frac{x}{\cdot}\right)$ is a Dirichlet character if and only if $x \not\equiv 3 \pmod{4}$.*

Our characters of interest are typically $\left(\frac{12}{\cdot}\right)$ or $\left(\frac{12\cdot 49}{\cdot}\right)$.

Gauss sums of nonprimitive characters can be computed in terms of the character they are induced by, as seen in the following theorem [MV07, theorem 9.10].

Theorem A.3. *If χ is a character modulo q and is induced by a character χ^* modulo q^* then*

$$G(\chi) = \mu\left(\frac{q}{q^*}\right)\chi^*\left(\frac{q}{q^*}\right)G(\chi^*),$$

where μ is the multiplicative Möbius μ function.

We have the following lemma for transforming an exponential sum into a Gauss sum.

Lemma A.4. *Let χ be a character with modulus d. Suppose there exists a prime p dividing the modulus d for which there exists a root of unity $\xi_p \neq 1$ such that $\chi(h + d/p) = \xi_p \chi(h)$ for all $h \in \mathbb{Z}/d\mathbb{Z}$. Then*

$$\sum_{h \pmod{d}} \chi(h) e\left(\frac{hh'}{d}\right) = \overline{\chi}(h')G(\chi). \tag{A.1}$$

Proof. There are two possibilites for h' : either $(h', d) = 1$ or $(h', d) > 1$. If $(h', d) = 1$ let $\overline{h'}h' \equiv 1 \pmod{d}$, then $h \mapsto \overline{h'}h$ is an automorphism of $\mathbb{Z}/d\mathbb{Z}$ and hence

$$\begin{aligned} \sum_{h \pmod{d}} \chi(h) e\left(\frac{hh'}{d}\right) &= \sum_{h \pmod{d}} \chi(\overline{h'})\chi(h) e\left(\frac{\overline{h'}h'h}{d}\right) \\ &= \overline{\chi}(h') \sum_{h \pmod{d}} \chi(h) e\left(\frac{h}{d}\right). \end{aligned}$$

In this case we are done.

When $(h', d) > 1$, let p be a prime satisfying the hypothesis. Note $\{h : h \in \mathbb{Z}/d\mathbb{Z}\} = \{h + \frac{d}{p} : h \in \mathbb{Z}/d\mathbb{Z}\}$. Then we have

$$\begin{aligned}
\sum_{h \pmod d} \chi(h) e\left(\frac{hh'}{12}\right) &= \sum_{h \pmod d} \chi(h + d/p) e\left(\frac{p(h + d/p)x}{d}\right) \\
&= \xi_p \sum_{h \pmod d} \chi(h) e\left(\frac{phx}{d}\right) e\left(x\right) \\
&= \xi_p \sum_{h \pmod d} \chi(h) e\left(\frac{hh'}{d}\right).
\end{aligned}$$

But $\xi_p \neq 1$. Hence $\sum_{h \pmod d} \chi(h) e\left(\frac{hh'}{d}\right) = 0$. Since $(h', d) > 1$ we have $\chi(h') = 0$ so

$$\sum_{h \pmod d} \chi(h) e\left(\frac{hh'}{d}\right) = \overline{\chi}(h') G(\chi) = 0.$$

We have therefore proved the identity for both of the possible cases. □

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