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Zeros of a Family of Complex Harmonic Polynomials

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ABSTRACT

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In this thesis we study complex harmonic functions of the form $f = h + \bar{g}$ where h, g are analytic, nonconstant functions of one variable. The Fundamental Theorem of Algebra does not apply to such functions, so we ask how many zeros a complex harmonic function can have and where those zeros are located. This thesis focuses on the complex harmonic family of polynomials $p_c(z) = z + \frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n$. We first establish properties of the critical curve, which separates orientation preserving and reversing regions. These properties are then used to show the sum of the orders of the zeros of p_c is $-n$. In turn, we use this to show p_c has $n+2$ zeros when $0 < c < 1$, $n \geq 5$ and $n+4$ zeros when $c \geq 4$, $n \geq 6$. The total number of zeros of p_c changes when zeros interact with the critical curve, so we investigate where zeros occur on the critical curve to understand how the number of zeros of p_c changes for $1 \leq c \leq 4$.

Keywords: complex analysis, harmonic polynomials

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Chapter 1. Motivation

The Fundamental Theorem of Algebra states that every polynomial $f(z) \in \mathbb{C}[z]$ of degree n has precisely n zeros in $\mathbb{C}$, counted with multiplicity. As a simple illustration, consider $f_1(z) = z+z^3$. As a degree 3 polynomial, it has 3 zeros; because $f_1(z) = z+z^3 = z(z+i)(z-i)$, $z = 0, \pm i$ are the three zeros. These zeros are depicted in Figure 1.1.

The Fundamental Theorem of Algebra applies to polynomials in z, that is, to analytic polynomials of a single complex variable. What happens with polynomials in z and $\bar{z}$? As an example, consider $f_2(z) = z + \bar{z}^3$. To find the zeros of f_2, let $z = x + iy$. Then

$$f_2(x+iy) = (x+iy) + (x-iy)^3 = x + x^3 - 3xy^2 + i\left(y + y^3 - 3xy^2\right). \tag{1.1}$$

Setting the real and imaginary parts equal to zero, we find

$$x(1 + x^2 - 3y^2) = 0 \quad \text{and} \quad y(1 + y^2 - 3x^2) = 0. \tag{1.2}$$

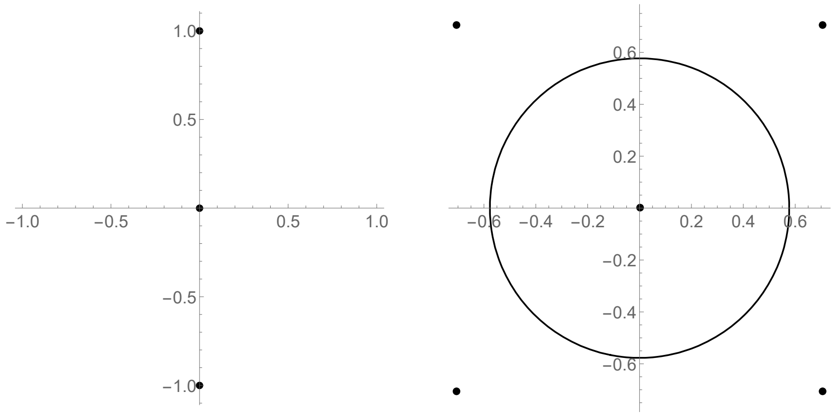
From (1.2), $x = 0$ or $x^2 = 3y^2 - 1$. If $x = 0$, then $y = 0$ or $y^2 = -1$. Because y is a real number, $y = 0$. If $x^2 = 3y^2 - 1$, then substituting into the second equation of (1.2) gives

$$0 = y + y^3 - 3x^2 y = -4y(2y^2 - 1), \tag{1.3}$$

so $y = \pm\frac{1}{\sqrt{2}}$ or $y = 0$. Notice that $y = 0$ leads to $x^2 = -1$ which is not possible because x is real. Thus $x^2 = 3y^2 - 1$ implies $x = \pm\frac{1}{\sqrt{2}}$, and there are five zeros of $f_2(z)$: $0, \pm\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$. They are shown in Figure 1.1.

This example shows how the Fundamental Theorem of Algebra does not extend directly to polynomials of the form $f = h + \bar{g}$ where h, g are nonzero polynomials in z; such polynomials are called *complex-valued harmonic polynomials* and are the subject of this thesis. We ask

Question: What can be said about the zeros of complex harmonic polynomials?



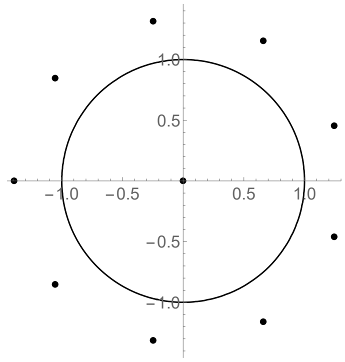
(a) $f_1(z) = z + z^3$ (b) $f_2(z) = z + \bar{z}^3$

Figure 1.1: Zeros of an analytic and a complex harmonic polynomial.

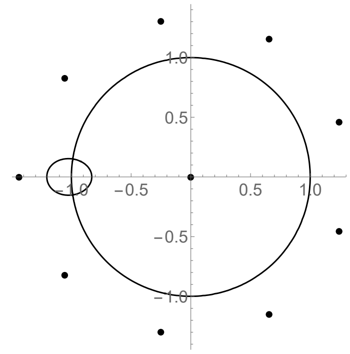
In response to the above question, mathematicians began investigating a bound on the total number of zeros. Sheil-Small [8] conjectured that for complex harmonic polynomials $f = h + \bar{g}$ where $\deg(h) = n$, $\deg(g) = m$, and $m \leq n$, the maximum number of zeros of f is n^2. Peretz and Schmid [7] and Wilmshurst [10] independently proved this conjecture. Wilmshurst also constructed a polynomial with n^2 zeros to show this bound is sharp, and Bshouty et al. [2] constructed another example illustrating that the bound is sharp. Wilmshurst then conjectured that in the particular case where $1 \leq m \leq n-1$, f has at most $m(m-1)+3n-2$ zeros; however, Lee et al. [6] constructed counterexamples that show the conjecture does not hold in general.

Other mathematicians considered particular families of polynomials. Khavinson and Swiatek [5] looked at complex harmonic polynomials of the form $f(z) = h(z) - \bar{z}$. They showed that for $n = \deg(h) > 1$, the number of zeros is bounded by $3n-2$. Brilleslyper et al. [1] investigated the family of complex harmonic trinomials $p_c(z) = z^n + c\bar{z}^k - 1$ where $1 \leq k \leq n-1$, $n \geq 3$, $c \in \mathbb{R}^+$, and $\gcd(n,k) = 1$. They discovered that as c increases, the number of zeros increases from n to $n+2k$.

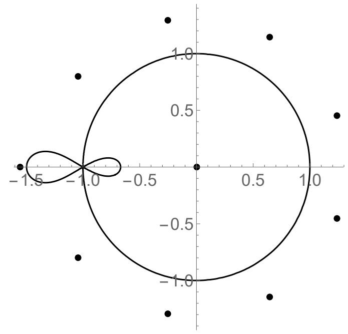
In this thesis we investigate the complex harmonic family $p_c(z) = z + \frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n$ for $c > 0$ and integers $n \geq 3$. In Figure 1.2, we graph the zeros and *critical curve* for $n = 8$ at several c values. The critical curve separates sense-preserving and sense-reversing regions;



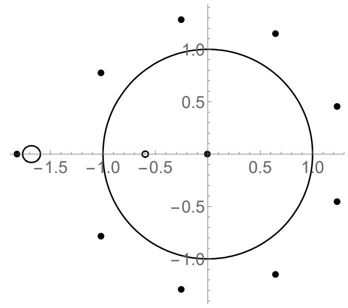
(a) $p_{0.8}(z)$



(b) $p_{1.1}(z)$



(c) $p_{1.4}(z)$



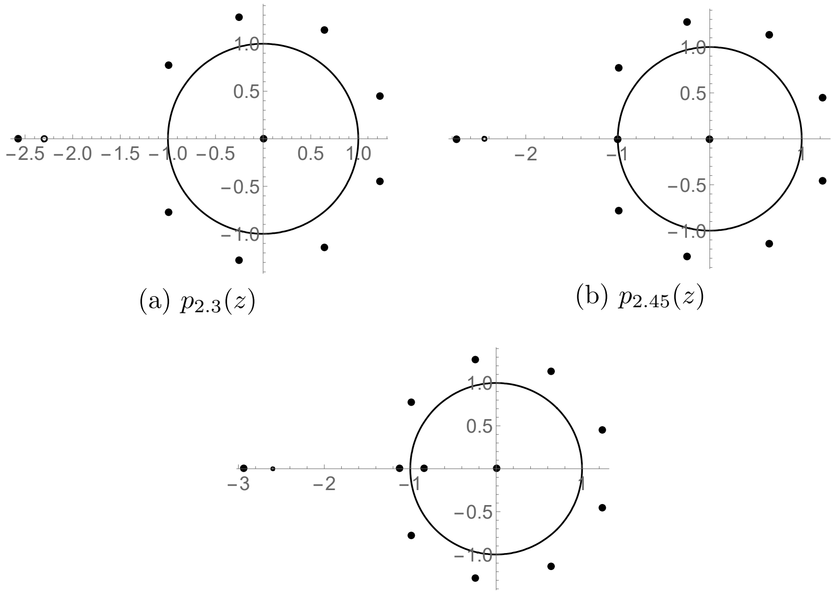
(d) $p_{1.7}(z)$

Figure 1.2: The zeros and critical curve of the eighth degree polynomial $p_c(z)$ when $c = 0.8$, $c = 1.1$, $c = 1.4$, and $c = 1.7$.

for a detailed discussion of these ideas see Chapter 2. In the case of $p_{0.8}$, the region inside the circle is sense-preserving and the region outside is sense-reversing. In the other graphs, the regions inside the shapes are sense-preserving except where they overlap; then they are sense-reversing. As shown in Figure 1.2, the unit circle is always part of the critical curve of p_c; we prove this in Chapter 3.

Proposition 3.3. *The unit circle* $|z| = 1$ *is always part of the critical curve of* $p_c(z)$*.*

For analytic functions, there is a notion of order of a zero which can be defined as the minimum degree of a term in the Taylor expansion of the function about that point. For complex harmonic functions, there is a similar notion, but now the order of a zero can be positive or negative depending on whether the zero lies in a sense-preserving or sense-reversing region. The order is undefined if the zero lies on the critical curve. We will show in Chapter 4 that all the zeros of p_c are simple. Thus the sum of the order of the zeros in



(a) $p_{2.3}(z)$

(b) $p_{2.45}(z)$

(c) $p_{2.6}(z)$

Figure 1.3: The eighth degree polynomial $p_c(z)$ when $c = 2.3$, $c = 2.45$, and $c = 2.6$.

Figure 1.3 is -8; in Chapter 4 we prove that for p_c the sum of the orders of the zeros is $-n$. In general, it is the sum of the orders of the zeros that is preserved in complex harmonic polynomials, not the total number of zeros; this gives a generalization of the Fundamental Theorem of Algebra that includes complex harmonic polynomials.

While the sum of the orders of the zeros is preserved, the total number of zeros may change. In Figure 1.3, we see that $p_{2.3}$ has ten zeros: one at 0, one on the negative real axis, and one near each of the numbers $8^{1/7}e^{i\frac{\pi+2\pi k}{9}}$ where $k = 0, 1, 2, 3, 5, 6, 7, 8$. (Note: $k = 4$ is not included in this list.) In the case of $p_{2.45}$, there are eleven zeros: approximately the same ten as $p_{2.3}$ as well as an additional one at $z = -1$ on the critical curve. The complex harmonic polynomial $p_{2.6}$ has twelve zeros. Again, ten of the zeros are similar in location to those of $p_{2.3}$, but there are two new zeros on the negative real axis to the left and right of the critical curve. This illustrates how the number of zeros changes as c changes and as the zeros interact with the critical curve. In Chapter 4, we prove the following two theorems:

Theorem 4.3. *For $n \geq 5$ and $0 < c < 1$, the complex harmonic polynomial $p_c(z)$ has $n + 2$ distinct zeros.*

Theorem 4.4. *For $n \geq 6$ and $c \geq 4$, $p_c(z)$ has $n+4$ distinct zeros.*

We now investigate what happens for values of c between 1 and 4. The number of zeros can change when a zero interacts with the critical curve and as mentioned previously the unit circle is always part of the critical curve; hence, we investigate when zeros of p_c occur on the unit circle. In Chapter 4, we prove

Theorem 4.11. *For even $n \geq 8$, the complex harmonic function $p_c(z)$ has no zeros on the unit circle except possibly at the point -1.*

The outline for the remainder of this thesis is as follows:

In Chapter 2 we introduce general results for complex harmonic functions of the form $f = h + \bar{g}$ where h, g are analytic. We will discuss properties of complex harmonic functions including orientation, orders of zeros, and a harmonic analog of Rouché's Theorem.

We use these results to analyze p_c in the following chapters. In Chapter 3, we analyze the critical curve for p_c. We show that the unit circle is always part of the critical curve; this is illustrated in Figure 1.3. For $0 < c < 1$, we show that the unit circle is the entire critical curve. For sufficiently large c, we show that the parts of the critical curve sans the unit circle are bounded away from any zeros of p_c.

In Chapter 4, we use Rouché's Theorem to establish that the sum of the orders of zeros of p_c is $-n$. We then prove Theorems 4.3 and 4.4. We even go a step further and use Rouché's Theorem to localize the zeros of p_c in annuli or sectors of annuli.

The above theorems treat the cases for sufficiently small and sufficiently large values of c; it remains to determine what happens for intermediate values of c. As illustrated in Figure 1.3, the number of zeros changes when zeros interact with the critical curve. We begin by analyzing when zeros occur on the unit circle. In Chapter 4, we prove Theorem 4.11.

Chapter 2. Background

Here we review the relevant complex analysis. The results in this chapter are developed from Duren [3] with details added but no original results.

2.1 Analytic and Complex Harmonic Functions

Let $u(x, y)$ and $v(x, y)$ be real-valued functions. A complex-valued function $f = u + iv$ is *analytic at the point* $z_0 \in \mathbb{C}$ if the limit

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

exists. We call this the derivative of f at z_0 and label it $f'(z_0)$. A function f is *analytic on* $D \subseteq \mathbb{C}$ if f is analytic at every point in D. An analytic function f satisfies the *Cauchy-Riemann equations*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Moreover, a function f satisfying the Cauchy-Riemann equations and having continuous first partial derivatives is analytic. In addition to the usual partial derivatives in x, y, we have the differential operators $\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$. It is common practice to use a subscript notation where $f_z = \frac{\partial f}{\partial z}$ and $f_{\bar{z}} = \frac{\partial f}{\partial \bar{z}}$. These operations relate to the Cauchy-Riemann equations:

Lemma 2.1. *The Cauchy-Riemann equations are equivalent to* $\frac{\partial f}{\partial \bar{z}} = 0$.

Proof. Let $f = u + iv$ be a complex function. Then the definition of $\frac{\partial f}{\partial \bar{z}}$ yields

$$\begin{aligned}\frac{\partial f}{\partial \bar{z}} &= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv)\\ &= \frac{1}{2}\left(\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + i\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)\right).\end{aligned}$$

Thus $\frac{\partial f}{\partial \bar{z}} = 0$ if and only if $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$. □

The focus of this thesis is complex harmonic polynomials, so we begin developing results for complex functions that are harmonic. Recall that a real-valued function $\phi(x, y)$ is *harmonic* if it is C^2 and satisfies Laplace's equation $\phi_{xx} + \phi_{yy} = 0$. A complex-valued harmonic function f has the form $f = u + iv$ for real harmonic functions u, v. Complex-valued harmonic functions have the following useful properties:

Lemma 2.2. *If f is harmonic with continuous second partial derivatives then f_z is analytic.*

Proof. Let f be harmonic, so $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$. By definition $f_z = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$. Applying the differential operator $\frac{\partial}{\partial \bar{z}}$ yields

$$f_{z\bar{z}} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} - i\frac{\partial^2 f}{\partial x \partial y} + i\frac{\partial^2 f}{\partial y \partial x} + \frac{\partial^2 f}{\partial y^2}\right).$$

Because partial derivatives commute, we have

$$f_{z\bar{z}} = \frac{1}{4}\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}\right).$$

Therefore $f_{z\bar{z}} = 0$ because f is harmonic. Then by Lemma 2.1 we know f_z is analytic. □

Using the above lemma, we now show every complex harmonic function can be written in the form $f = h + \bar{g}$.

Proposition 2.3. *Complex-valued harmonic functions defined on a simply-connected domain can be written in the form $f = h + \bar{g}$ for analytic functions h and g. This representation is unique up to an additive constant.*

Proof. Let f be harmonic, so f_z is analytic by Lemma 2.2. Let $h' = f_z$ and $g = \bar{f} - \bar{h}$. Then

$$g_{\bar{z}} = \frac{d}{d\bar{z}}\left(\bar{f} - \bar{h}\right) = \bar{f}_{\bar{z}} - \bar{h}_{\bar{z}} = \bar{f}_z - \bar{h_z} = \bar{f}_z - \bar{f}_z = 0$$

Therefore $g_{\bar{z}} = 0$. Then g is analytic because g has continuous first partial derivatives and satisfies Lemma 2.1. Re-arranging $g_{\bar{z}} = \bar{f}_z - \bar{h}_z$ gives

$$f_z = h_z + \bar{g}_z.$$

Using the fact that analytic functions have analytic antiderivatives,

$$f = h + \bar{g} + c.$$

Thus, $f = h + \bar{g} + c$ for some constant c. Because constants are the only analytic and anti-analytic function, h and g are therefore only determined up to a constant. □

For a moment we return to considering analytic functions. Recall that the Jacobian of a function $f = u + iv$ viewed as a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$ is

$$J_f(z) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = u_x v_y - u_y v_x.$$

For f analytic, this simplifies to

$$J_f(x) = (u_x)^2 + (v_x)^2 = \left|f'(z)\right|^2.$$

Notice that this quantity is always non-negative. When $J_f(z)$ is positive, f is univalent and hence conformal, meaning f is orientation preserving (or sense-preserving) and f preserves angles between curves. Similarly, $\bar{f}$ will be an anti-analytic function satisfying $J_{\bar{f}}(z) \leq 0$ that is orientation reversing (or sense-reversing) when $J_{\bar{f}}(z) < 0$. Because complex harmonic polynomials are the sum of an analytic and an anti-analytic function, some portions of the complex plane will be sense-preserving and some will be sense-reversing. We give the details of these ideas below.

We claim that for $f\colon \mathbb{C} \to \mathbb{C}$, $J_f(z)$ can be written as $|f_z|^2 - |f_{\bar{z}}|^2$. First, we compute $|f_z|^2$

and $|f_{\bar{z}}|^2$:

$$|f_z|^2 = \left|\frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv)\right|^2 = \frac{1}{4}\left|\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y}\right|^2$$
$$= \frac{1}{4}\left(\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 - 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right),$$

and

$$|f_{\bar{z}}|^2 = \left|\frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv)\right|^2 = \frac{1}{4}\left|\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y}\right|^2$$
$$= \frac{1}{4}\left(\left(\frac{\partial u}{\partial x}\right)^2 - 2\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial u}{\partial y}\frac{\partial v}{\partial x} + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2\right).$$

Taking their difference yields

$$|f_z|^2 - |f_{\bar{z}}|^2 = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}.$$

Therefore, $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2$ for any $f\colon \mathbb{C} \to \mathbb{C}$. As before, we are interested in knowing when $J_f(z)$ is positive or negative. To this end, we write $J_f(z) = |f_z|^2 - |f_{\bar{z}}|^2 = \big(|f_z| - |f_{\bar{z}}|\big)\big(|f_z| + |f_{\bar{z}}|\big)$. Because $|f_z| + |f_{\bar{z}}| \geq 0$, the sign of $J_f(z)$ is determined by $|f_z| - |f_{\bar{z}}|$; i.e., $J_f(z) > 0$ if and only if $|f_z| > |f_{\bar{z}}|$, which occurs if and only if $|f_{\bar{z}}|/|f_z| < 1$. Similarly, $J_f(z) < 0$ if and only if $|f_{\bar{z}}|/|f_z| > 1$. We use this to define a function $\omega(z) = \overline{f_{\bar{z}}}/f_z$ and call it the complex dilatation.

Definition 2.4. The *complex dilatation* of a complex function f is $\omega(z) = \overline{f_{\bar{z}}}/f_z$.

We have thus proved the following proposition:

Proposition 2.5. *A complex function f is sense-preserving when $|\omega(z)| < 1$ and sense-reversing when $|\omega(z)| > 1$.*

We call the curve separating the sense-preserving and sense-reversing regions of a function f the critical curve, and it is the set of all points in the complex plane such that $|\omega(z)| = 1$.

Definition 2.6. The *critical curve* of a complex function f is the set of all points $z \in \mathbb{C}$ such that $\left|\omega(z)\right| = 1$.

Because complex harmonic functions have the form $f = h + \bar{g}$, we can re-write the function $\omega(z)$ for complex harmonic functions as follows:

$$\omega(z) = \frac{\overline{f_{\bar{z}}}}{f_z} = \frac{\overline{\frac{d}{d\bar{z}}\,(h+\bar{g})}}{\frac{d}{dz}\,(h+\bar{g})} = \frac{\overline{\bar{g}_{\bar{z}}(z)}}{h_z(z)} = \frac{g'(z)}{h'(z)}.$$

2.2 Order of a Zero

We also need to understand the definition for the order of a zero of a complex harmonic function. Recall that for an analytic function F a point z_0 is called a zero of order m if its first $m-1$ derivatives vanish at z_0 but $F^{(m)}(z_0) \neq 0$; equivalently, the Taylor series for F around z_0 takes the form $F(z) = \sum_{k=m}^{\infty} a_k(z-z_0)^k$ where $a_m \neq 0$. Now consider a complex harmonic function in the form $f = h + \bar{g}$ where h, g are analytic. Suppose f has a zero at some $z_0 \in \mathbb{C}$. As we did above, write h and g as Taylor series centered at z_0:

$$h(z) = a_0 + \sum_{j=r}^{\infty} a_j(z-z_0)^j, \quad g(z) = b_0 + \sum_{j=s}^{\infty} b_j(z-z_0)^j$$

where $r > 0$, $s > 0$, $a_r \neq 0$, and $b_s \neq 0$. Because $f(z_0) = 0$, $b_0 = -\bar{a}_0$. Then we consider the order of z_0 to be r if z_0 is in a sense-preserving region or $-s$ if z_0 is in a sense-reversing region; i.e., the notion of order for a zero of a complex harmonic function is analogous to the definition of order for a zero of an analytic function but now we include the added information about the region in which it lies. We comment that zeros in a sense-preserving or sense-reversing region are called *nonsingular zeros*. Zeros that lie on the critical curve are called *singular zeros*, and their order is not defined.

More rigorously, we consider cases to determine whether the zero z_0 is in a sense-preserving or sense-reversing region. We then know the order of the zero from the Taylor series. Let z_0 be a zero of the complex harmonic function $f = h + \bar{g}$.

Case 1: Suppose $s > r$. Then

$$\begin{aligned}
\omega(z_0) &= \lim_{z\to z_0} \frac{g'(z)}{h'(z)} \\
&= \lim_{z\to z_0} \frac{\sum_{j=s}^{\infty} jb_j(z-z_0)^{j-1}}{\sum_{j=r}^{\infty} ja_j(z-z_0)^{j-1}} \\
&= \lim_{z\to z_0} \frac{\sum_{j=s}^{\infty} jb_j(z-z_0)^{j-r}}{\sum_{j=r}^{\infty} ja_j(z-z_0)^{j-r}} \\
&= 0
\end{aligned}$$

Thus z_0 is always in a sense-preserving region when $s > r$ and z_0 has order r.

Case 2: Suppose $s < r$. Then

$$\begin{aligned}
\omega(z_0) &= \lim_{z\to z_0} \frac{g'(z)}{h'(z)} \\
&= \lim_{z\to z_0} \frac{\sum_{j=s}^{\infty} jb_j(z-z_0)^{j-1}}{\sum_{j=r}^{\infty} ja_j(z-z_0)^{j-1}} \\
&= \lim_{z\to z_0} \frac{\sum_{j=s}^{\infty} jb_j(z-z_0)^{j-s}}{\sum_{j=r}^{\infty} ja_j(z-z_0)^{j-s}} \\
&= \infty
\end{aligned}$$

Therefore, z_0 is always in a sense-reversing region when $s < r$ and z_0 has order $-s$.

Case 3: Lastly, suppose $s = r$. Then

$$\begin{aligned}
\omega(z_0) &= \lim_{z\to z_0} \frac{g'(z)}{h'(z)} \\
&= \lim_{z\to z_0} \frac{\sum_{j=s}^{\infty} b_j j(z-z_0)^{j-1}}{\sum_{j=s}^{\infty} a_j j(z-z_0)^{j-1}} \\
&= \lim_{z\to z_0} \frac{\sum_{j=s}^{\infty} b_j j(z-z_0)^{j-s}}{\sum_{j=s}^{\infty} a_j j(z-z_0)^{j-s}} \\
&= \frac{b_s}{a_s}
\end{aligned}$$

Then z_0 is in a sense-preserving region if $|b_s| < |a_s|$; in this case, z_0 has order s. If $|b_s| > |a_s|$

then z_0 is a zero of order $-s$ in a sense-reversing region.

2.3 The Argument Principle for Complex Harmonic Functions

Lastly, we consider the Argument Principle for analytic functions and its analog for complex harmonic functions. First recall the Argument Principle for analytic functions. Let f be an analytic function defined on a domain D bounded by a Jordan curve C oriented in the positive direction. Suppose that f is analytic in D, continuous in $\bar{D}$, and $f(z) \neq 0$ on C. The *index* or *winding number* of the curve $f(C)$ about the origin is the total change in argument of $f(z)$ as z goes once around C divided by 2π. We write it as $I = (1/2\pi)\Delta_C \arg f(z)$. Let N be the total number of zeros of f in D counted according to multiplicity. The Argument Principle states that $N = I$, and the proof of it utilizes the observation that f'/f has a simple pole with residue n when f has a zero of order n; written symbolically,

$$N = \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\Delta_C \log f(z) = I.$$

We develop an analogous Argument Principle for complex harmonic functions, but first we need the following lemma:

Lemma 2.7. *Nonsingular zeros of harmonic functions are isolated.*

Proof. Let f be a harmonic polynomial and let z_0 be a nonsingular zero. Without loss of generality, suppose that z_0 is in a sense-preserving region. Then as before, we write $f = h + \bar{g}$ where

$$h(z) = a_0 + \sum_{j=r}^{\infty} a_j(z-z_0)^j, \quad g(z) = b_0 + \sum_{j=s}^{\infty} b_j(z-z_0)^j,$$

are the Taylor series of h, g centered at z_0. Because $f(z_0) = 0$, we have $b_0 = -\bar{a}_0$. Since z_0 is in a sense-preserving region, we know $r < s$. Let $b_j = 0$ for $r \leq j < s$, so

$$h(z) = a_0 + \sum_{j=r}^{\infty} a_j(z-z_0)^j, \quad g(z) = b_0 + \sum_{j=r}^{\infty} b_j(z-z_0)^j.$$

$$f(z) = h(z) + \overline{g(z)} = a_r(z - z_0)^r \left(1 + \psi(z)\right),$$

where

$$\psi(z) = (\bar{b}_r/a_r)(\bar{z} - \bar{z}_0)^r(z - z_0)^{-r} + O(z - z_0).$$

Because $\left|\bar{b}_r/a_r\right| < 1$, there exists a $\delta > 0$ such that $\left|\psi(z)\right| < 1$ for all z satisfying $0 < |z - z_0| < \delta$. Therefore, $f(z) \neq 0$ near z_0.

A similar argument applies for zeros in sense-reversing regions, so we conclude that nonsingular zeros are isolated. □

We now prove the analog of the Argument Principle for harmonic functions. This result, and its proof, are due to Duren et al. [4].

Theorem 2.8. *(Argument Principle for Harmonic Functions) Let f be a harmonic function in a Jordan domain D with boundary C. Suppose f is continuous in $\bar{D}$ and $f(z) \neq 0$ on C. Suppose f has no singular zeros, and let N be the sum of the orders of the zeros of f in D. Then $\Delta_C \arg f(z) = 2\pi N$.*

Proof. First, suppose f has no zeros in D; consequently, $N = 0$. We then need to show $\Delta_C \arg f(z) = 0$. Let ϕ be a homeomorphism from the closed unit square S onto $D \cup C$ where $\phi\colon \partial S \to C$ is a homeomorphism. Then $F = f \circ \phi$ is a continuous map of S into the complex plane with no zeros. We wish to show $\Delta_{\partial S} \arg F(z) = 0$. To this end, subdivide S into finitely many squares S_j. Choose them to be sufficiently small such that the argument of $F(z)$ varies by at most $\pi/2$. Consequently, $\Delta_{\partial S_j} \arg F(z) = 0$ and

$$\Delta_{\partial S} \arg F(z) = \sum_j \Delta_{\partial S_j} \arg F(z) = 0.$$

Because ϕ is a homeomorphism of the boundary, we also get $\Delta_C f(z) = 0$.

Now suppose that f does have zeros in D. By Lemma 2.7, the zeros are isolated. Because the zeros are isolated and f does not vanish on C, there can only be a finite number of distinct zeros in D; call them z_j for $j = 1, 2, \cdots, \nu$. At each zero z_j, take a circle γ_j of radius

δ centered at z_j. Because there are a finite number of zeros, take δ small enough so that the γ_j all lie in D and do not intersect. Because there are finitely many γ_j, we can take a curve λ_j connecting γ_j to C such that each λ_j does not intersect any other λ_k or γ_ℓ. We now consider the closed contour Γ formed by traveling along C in the positive direction and making detours along each λ_j to γ_j back along λ_j. Notice that Γ contains no zeros of f, so $\Delta_\Gamma f(z) = 0$ by the above paragraph. We also have the contributions of the λ_j cancelling out because we traverse them in both directions. We are then left with

$$\Delta_C \arg f(z) = \sum_{j=1}^{\nu} \Delta_{\gamma_j} \arg f(z),$$

where the circles γ_j are now traversed in the positive direction. We now only need to consider what happens at each z_j.

Suppose that f has a zero of order $n_j > 0$ at z_j. Then by Lemma 2.7 we know $f(z) = a_{n_j}(z - z_j)^{n_j}\left(1 + \psi(z)\right)$ where $a_{n_j} \neq 0$ and $\left|\psi(z)\right| < 1$ on a sufficiently small circle γ_j defined by $\left|z - z_j\right| = \delta$. This gives us

$$\Delta_{\gamma_j} \arg f(z) = n_j \Delta_{\gamma_j} \arg(z - z_j) + \Delta_{\gamma_j} \arg(1 + \psi(z)) = 2\pi n_j.$$

Similarly, $\Delta_{\gamma_j} f(z) = 2\pi n_j$ for a zero of order $n_j < 0$. Therefore,

$$\Delta_C \arg f(z) = \sum_{j=1}^{\nu} \Delta_{\gamma_j} \arg f(z) = 2\pi \sum_{j=1}^{\nu} n_j = 2\pi N,$$

where N is the sum of the orders of the zeros of f in D. □

As in the analytic case, we get a version of Rouché's Theorem for harmonic functions as a corollary:

Corollary 2.9. *(Rouché's Theorem for Complex Harmonic Functions) Let p and $p + q$ be harmonic functions in D, continuous in $\bar{D}$, with no singular zeros in $\bar{D}$. If $\left|q(z)\right| < \left|p(z)\right|$*

on C, then the sum of the orders of zeros of p and the sum of the orders of zeros of $p+q$ are the same in D.

We comment that the above results are proved in generality, and in the following chapters we apply it to complex harmonic functions.

Chapter 3. Critical Curve

It is well known that analytic functions are conformal when they have non-zero derivative which means in particular they are sense (or orientation) preserving. Because complex harmonic polynomials are the sum of an analytic function and the conjugate of an analytic function, they have regions of the plane in which they are sense-preserving and regions in which they are sense-reversing. The critical curve is the curve separating these regions. In this chapter, we explore the critical curve for the polynomial $p_c(z) = z + \frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n$ for $c > 0$.

3.1 Properties of the Critical Curve

Using the notation of Chapter 2, $p_c = h + \bar{g}$ where $h(z) = z + \frac{c}{2}z^2$ and $g(z) = \frac{c}{n-1}z^{n-1} + \frac{1}{n}z^n$. The polynomial p_c is constructed so that the dilatation function is the product of Möbius transformations. In particular, the complex dilatation function for p_c is

$$\omega(z) = \frac{g'(z)}{h'(z)} = z^{n-2}\frac{z+c}{cz+1}. \tag{3.1}$$

There are advantages to this construction, and we begin by analyzing properties of the Möbius transformation $\psi(z) = \frac{z+c}{cz+1}$.

Lemma 3.1. *Let $\psi(z) = \frac{z+c}{cz+1}$ for $c \neq \pm 1$. Then $\left|\psi(z)\right| = 1$ if and only if $|z| = 1$.*

Proof. This is a standard fact about Möbius transformations, and we include its proof for completeness.

Suppose $\left|\psi(z)\right| = 1$. Then

$$1 = \left|\frac{z+c}{cz+1}\right|^2 = \frac{(z+c)(\bar{z}+c)}{(cz+1)(c\bar{z}+1)}.$$

This equation is equivalent to

$$c^2|z|^2 + cz + c\bar{z} + 1 = |z|^2 + cz + c\bar{z} + c^2.$$

Simplifying yields

$$(c^2-1)|z|^2 = c^2 - 1.$$

Because $c \neq \pm 1$, this is equivalent to $|z| = 1$. Thus $\left|\psi(z)\right| = 1$ implies $|z| = 1$. Assuming $|z| = 1$, the above set of equivalent equalities similarly gives $\left|\psi(z)\right| = 1$; therefore, $\left|\psi(z)\right| = 1$ if and only if $|z| = 1$. □

Lemma 3.2. *When $0 < c < 1$, the function $\psi(z) = \frac{z+c}{cz+1}$ is an automorphism of the unit disc with inverse $\psi^{-1}(z) = \frac{z-c}{-cz+1}$.*

Proof. This is a standard fact about Möbius transformations, and we include its proof for completeness.

First, notice that ψ is holomorphic in the unit disc because $0 < c < 1$ gives $1 < 1/c$.

By Lemma 3.1, we know $\left|\psi(z)\right| = 1$ if and only if $|z| = 1$. Therefore, if $|z| = 1$ we have $\left|\psi(z)\right| = 1$ which means $\left|\psi(z)\right| < 1$ for $|z| < 1$ by the Maximum Modulus Principle. Thus ψ maps the unit disc into the unit disc. Because ψ^{-1} is the same form as ψ, the above argument also gives that ψ^{-1} maps the unit disc into the unit disc.

Now observe that ψ^{-1} is in fact the inverse to ψ:

$$\psi(\psi^{-1}(z)) = \frac{\frac{z-c}{-cz+1} + c}{c\frac{z-c}{-cz+1} + 1} = \frac{z - c - c^2 z + c}{cz - c^2 - cz + 1} = \frac{(1-c^2)z}{1-c^2} = z.$$

A similar computation gives that $\psi^{-1}(\psi(z)) = z$ for all z. Therefore, ψ and ψ^{-1} are inverses and ψ is an automorphism of the unit disc. □

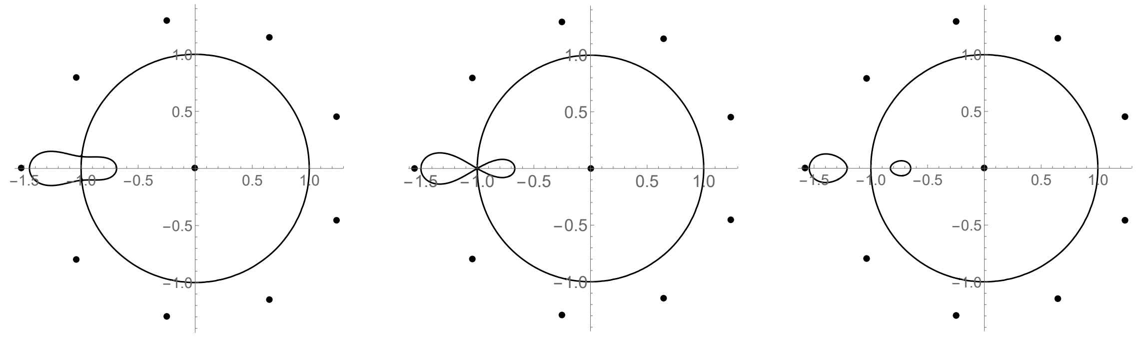


Figure 3.1: The critical curve of the eighth degree polynomial $p_c(z)$ when $c = 1.35$, $c = 1.4$, and $c = 1.45$.

We can now utilize the properties of ψ to prove properties of our critical curve. Recall from Chapter 2 that the critical curve is the set of all points $z \in \mathbb{C}$ such that $\big|\omega(z)\big| = 1$; we call this collection of points Ω. The set Ω changes as c changes, as illustrated in Figure 3.1. Moreover, for sufficiently large values of c (i.e., $c > \frac{n-1}{n-3}$), the critical curve splits into three distinct curves: the unit circle, a curve outside the unit circle, and a curve inside the unit circle. For these large values of c, we call the curve outside the unit circle Ω_1 and the curve inside the unit circle Ω_2. These figures suggest, however, that the unit circle is part of the critical curve Ω for any value of c.

Proposition 3.3. *The unit circle* $|z| = 1$ *is always part of the critical curve of* p_c*.*

Proof. If $|z| = 1$, then $|z|^{n-2} = 1$ and $\big|\psi(z)\big| = 1$ by Lemma 3.1. Consequently, $\big|\omega(z)\big| = |z|^{n-2}\big|\psi(z)\big| = 1$ when $|z| = 1$. Thus the unit circle $|z| = 1$ is always part of p_c's critical curve. □

While the above proof is sufficient, it will be convenient to have an equation describing the critical curve Ω, so we also provide an algebraic proof of Proposition 3.3.

Proof. Let $\big|\omega(z)\big| = 1$. Then

$$1 = \big|z^{n-2}\big| \frac{|z+c|}{|1+cz|}.$$

Squaring both sides of the equation yields

$$1 = \left|z^{n-2}\right|^2 \frac{|z+c|^2}{|1+cz|^2} = |z|^{2n-4} \frac{(z+c)(\overline{z+c})}{(1+cz)(\overline{1+cz})}.$$

Multiplying and simplifying then gives

$$1 = |z|^{2n-4} \frac{z\bar{z} + c(z+\bar{z}) + c^2}{1 + c(z+\bar{z}) + c^2 z\bar{z}}.$$

Letting $z = re^{i\theta}$ for some $r \geq 0$ and $\theta \in [0, 2\pi)$, the above equation becomes

$$1 = r^{2n-4} \frac{r^2 + 2cr\cos\theta + c^2}{1 + 2cr\cos\theta + c^2 r^2}.$$

A simple calculation then yields

$$0 = r^{2n-2} + 2cr^{2n-3}\cos\theta + c^2 r^{2n-4} - c^2 r^2 - 2cr\cos\theta - 1. \tag{3.2}$$

Rearranging and factoring out $r^2 - 1$ leaves us with

$$0 = (r^2 - 1)\left[\sum_{k=0}^{n-2} r^{2k} + 2cr\cos\theta \sum_{k=0}^{n-3} r^{2k} + c^2 r^2 \sum_{k=0}^{n-4} r^{2k}\right]. \tag{3.3}$$

Thus the above equation is satisfied when $r = 1$, i.e., when z is on the unit circle. □

While the unit circle is always part of the critical curve, for sufficiently small c the critical curve consists only of the unit circle.

Proposition 3.4. *For $0 < c < 1$, the critical curve of p_c consists only of the unit circle.*

Again, we will provide two proofs. The first proof utilizes the properties of the Möbius function ψ. The second is an algebraic proof utilizing the equation from the algebraic proof of Proposition 3.3.

Proof. Let $0 < c < 1$. If $|z| < 1$, then $|z|^{n-2} < 1$ and $|\psi|(z) < 1$ by the proof of Lemma 3.1, so $\left|\omega(z)\right| = |z|^{n-2}\left|\psi(z)\right| < 1$ when $|z| < 1$. If $|z| = 1$, $|z|^{n-2} = 1$ and $\left|\psi(z)\right| = 1$; consequently, $\left|\omega(z)\right| = 1$ when $|z| = 1$. If $|z| > 1$, then $|z|^{n-2} > 1$ and $\left|\psi(z)\right| > 1$ because ψ is an automorphism of the unit disc when $0 < c < 1$ by Lemma 3.2. Then for $|z| > 1$, $\left|\omega(z)\right| > 1$. Therefore, $\left|\omega(z)\right| = 1$ if and only if $|z| = 1$. □

We now give the algebraic proof.

Proof. Let $0 < c < 1$. Recall Equation 3.3 which gives a formulation of the critical curve:

$$0 = (r^2 - 1)\left[\sum_{k=0}^{n-2} r^{2k} + 2cr\cos\theta \sum_{k=0}^{n-3} r^{2k} + c^2r^2 \sum_{k=0}^{n-4} r^{2k}\right],$$

so either $r^2 - 1 = 0$ or $\sum_{k=0}^{n-2} r^{2k} + 2cr\cos\theta \sum_{k=0}^{n-3} r^{2k} + c^2r^2 \sum_{k=0}^{n-4} r^{2k} = 0$. The former equation is satisfied by $r = \pm 1$; we show that the latter equation cannot be satisfied when $0 < c < 1$.

First, notice that the following are equivalent:

$$\begin{aligned}
&\sum_{k=0}^{n-2} r^{2k} + 2cr\cos\theta \sum_{k=0}^{n-3} r^{2k} + c^2r^2 \sum_{k=0}^{n-4} r^{2k} \\
&= \sum_{k=0}^{n-4} r^{2k} + r^{2n-6} + r^{2n-4} + 2cr\cos\theta \sum_{k=0}^{n-4} r^{2k} + 2cr\cos\theta\, r^{2n-6} + c^2r^2 \sum_{k=0}^{n-4} r^{2k} \\
&= (1 + 2cr\cos\theta + c^2r^2) \sum_{k=0}^{n-4} r^{2k} + r^{2n-6} + r^{2n-4} + 2c\cos\theta r^{2n-5} \\
&= (1 + 2cr\cos\theta + c^2r^2) \sum_{k=0}^{n-4} r^{2k} + (1 + r^2 + 2cr\cos\theta) r^{2n-6}
\end{aligned}$$

By assumption $0 < c < 1$, so $0 < c^2 < 1$. Then

$$\begin{aligned}
&(1 + 2cr\cos\theta + c^2r^2) \sum_{k=0}^{n-4} r^{2k} + (1 + r^2 + 2cr\cos\theta) r^{2n-6} \\
&> (1 + 2cr\cos\theta + c^2r^2) \sum_{k=0}^{n-4} r^{2k} + (1 + c^2r^2 + 2cr\cos\theta) r^{2n-6}
\end{aligned}$$

Factoring out $c^2r^2 + 2cr\cos\theta + 1$ yields

$$\begin{aligned}(c^2r^2 + 2cr\cos\theta + 1)\left(\sum_{k=0}^{n-3} r^{2k}\right) &\geq (c^2r^2 - 2cr + 1)\left(\sum_{k=0}^{n-3} r^{2k}\right) \\ &= (cr-1)^2\left(\sum_{k=0}^{n-3} r^{2k}\right) \\ &\geq 0.\end{aligned}$$

Thus when $0 < c < 1$

$$\sum_{k=0}^{n-2} r^{2k} + 2cr\cos\theta \sum_{k=0}^{n-3} r^{2k} + c^2r^2 \sum_{k=0}^{n-4} r^{2k} > 0.$$

Consequently, Equation 3.3 is only satisfied when $r^2 - 1 = 0$; therefore, the critical curve of p_c consists only of the unit circle when $0 < c < 1$. □

3.2 Annuli for the Critical Curve

The results that follow in this section are due to South and Woodall [9].

For sufficiently small c, the critical curve Ω is the unit circle and determining sense-preserving and sense-reversing regions is simple. For sufficiently large c, the critical curve Ω is composed of the unit circle, Ω_1, and Ω_2, so determining sense-preserving and sense-reversing regions will naturally become more difficult. However, recall that we only use the orientation of a region to determine the order of a zero. If we can show that there are no zeros contained in Ω_1 or Ω_2, then we only need to consider the location of a zero relative to the unit circle in order to determine its order. The rest of this section provides the details of such an argument.

Recall that the set of points satisfying Equation 3.2,

$$0 = r^{2n-2} + 2cr^{2n-3}\cos\theta + c^2r^{2n-4} - c^2r^2 - 2cr\cos\theta - 1,$$

is the critical curve. Consider the function

$$G(re^{i\theta}) = c^2r^2(r^{2n-6} - 1) + 2cr\cos(\theta)(r^{2n-4} - 1) + r^{2n-2} - 1. \tag{3.4}$$

The critical curve is the set of points satisfying $G(re^{i\theta}) = 0$, the sense-preserving region(s) are the sets of points satisfying $G(re^{i\theta}) < 0$, and the sense-reversing region(s) are the sets of points satisfying $G(re^{i\theta}) > 0$. We show that Ω_1 and Ω_2 are contained inside the respective annuli

$$A_1 = \left\{ z \in \mathbb{C} \mid 2 \leq |z| \leq c\left(\frac{n+1}{n}\right) \right\} \text{ and } A_2 = \left\{ z \in \mathbb{C} \mid 0 < |z| \leq \frac{3}{2c} \right\},$$

by showing that G is strictly positive (respectively negative) along the inner and outer circles bounding the annuli A_1 (respectively A_2) and by showing that G attains a negative (respectively positive) value inside the annuli. Because G is a continuous function equaling zero if and only if $re^{i\theta}$ is part of the critical curve, this shows $\Omega_1 \subseteq A_1$ and $\Omega_2 \subseteq A_2$.

To show G is strictly positive or negative along the inner and outer circles of our annuli, we create the following equivalencies for $r < 1$ and $r > 1$:

Case 1: Fix $r < 1$. Then $r^k - 1 < 0$ for any positive integer k; consequently, G has a maximum value when $\theta = \pi$. Then to prove $G < 0$ on a given circle of radius $r < 1$, it suffices to show

$$\begin{aligned} G(re^{i\pi}) &= c^2r^{2n-4} - c^2r^2 - 2cr^{2n-3} + 2cr + r^{2n-2} - 1 \\ &= (cr^{n-2} - r^{n-1})^2 - (cr - 1)^2 \\ &< 0. \end{aligned}$$

This is equivalent to

$$\left|cr^{n-2} - r^{n-1}\right| < |cr - 1|. \tag{3.5}$$

Case 2: A similar result holds for $r > 1$: For fixed $r > 1$, $r^k - 1 > 0$; consequently, the minimum occurs at $\theta = \pi$. To prove $G > 0$ on a given circle of radius $r > 1$, it suffices to show

$$G(re^{i\pi}) = (cr^{n-2} - r^{n-1})^2 - (cr-1)^2 > 0.$$

Equivalently,

$$|cr - 1| < \left| cr^{n-2} - r^{n-1} \right|. \tag{3.6}$$

Lemma 3.5. *For $n \geq 4$ and $c \geq 4$, the curve Ω_1 is contained in A_1.*

Proof. Let $n \geq 4$ and $c \geq 4$. First, notice that $-c$ is contained in Ω_1 because

$$G(ce^{i\pi}) = -(c^2 - 1)^2 < 0.$$

Therefore, $-c$ is in a sense-preserving region outside the unit circle; hence, $-c$ is inside Ω_1. Because $2 < c < c\left(\frac{n+1}{n}\right)$, we also have $-c \in A_1$. It remains to show G is strictly positive on the inner and outer circles bounding A_1.

Let $r = 2$. Because $2 > 1$, we know $G > 0$ on the circle of radius 2 if and only if Equation 3.6 is satisfied for $r = 2$; i.e.,

$$|2c - 1| < \left| 2^{n-2}c - 2^{n-1} \right|$$
$$2c - 1 < 2^{n-2}c - 2^{n-1}.$$

Solving for c, the above is equivalent to $c > \frac{2^{n-1}-1}{2^{n-2}-2}$. Because $c \geq 4$ by assumption and

$$\frac{2^{n-1} - 1}{2^{n-2} - 2} \leq \frac{2^{n-1}}{2^{n-2} - 2^{n-3}} = 4 \leq c,$$

Equation 3.6 is satisfied. Therefore, $G > 0$ on the circle of radius 2.

We now show $G > 0$ on the circle of radius $c\left(\frac{n+1}{n}\right)$. By assumption, $c \geq 4$. Because we also assume $n \geq 4$,

$$n^{\frac{1}{n-3}}\left(\frac{n}{n+1}\right) \leq n^{\frac{1}{n-3}} \leq 4^{\frac{1}{n-3}} \leq 4.$$

Therefore $c > n^{\frac{1}{n-3}}\left(\frac{n}{n+1}\right)$. We now work backwards to show Equation 3.6 is satisfied. First, notice $c > n^{\frac{1}{n-3}}\left(\frac{n}{n+1}\right)$ is equivalent to the following:

$$\begin{aligned} n^{\frac{1}{n-3}}\frac{n}{n+1} &< c \\ n\left(\frac{n}{n+1}\right)^{n-3} &< c^{n-3} \\ c^2\left(\frac{n+1}{n}\right) &< c^{n-1}\left(\frac{n+1}{n}\right)^{n-2}\left(\frac{1}{n}\right). \end{aligned}$$

Because $c^2\left(\frac{n+1}{n}\right) - 1 < c^2\left(\frac{n+1}{n}\right)$, we have $c^2\left(\frac{n+1}{n}\right) - 1 < c^{n-1}\left(\frac{n+1}{n}\right)^{n-2}\left(\frac{1}{n}\right)$. This gives the following equivalent statements:

$$\begin{aligned} c^2\left(\frac{n+1}{n}\right) - 1 &< c^{n-1}\left(\frac{n+1}{n}\right)^{n-2}\left(\frac{1}{n}\right) \\ c^2\left(\frac{n+1}{n}\right) - 1 &< c^{n-1}\left(\frac{n+1}{n}\right)^{n-2}\left(\frac{n+1}{n} - 1\right) \\ c^2\left(\frac{n+1}{n}\right) - 1 &< c^{n-1}\left(\frac{n+1}{n}\right)^{n-1} - c^{n-1}\left(\frac{n+1}{n}\right)^{n-2} \\ \left|c^2\left(\frac{n+1}{n}\right) - 1\right| &< \left|c^{n-1}\left(\frac{n+1}{n}\right)^{n-2} - c^{n-1}\left(\frac{n+1}{n}\right)^{n-1}\right| \\ |cr - 1| &< \left|cr^{n-2} - r^{n-1}\right|. \end{aligned}$$

Therefore, Equation 3.6 is satisfied by $r = c\left(\frac{n+1}{n}\right)$ and all $c > n^{\frac{1}{n-3}}\frac{n}{n+1}$; hence, it is satisfied by $c \geq 4$. Therefore, $G\left(c\frac{n+1}{n}e^{i\theta}\right) > 0$ for all θ.

Because $-c$ is contained in Ω_1 and G is strictly positive on the circles of radius 2 and radius $c\left(\frac{n+1}{n}\right)$, we know Ω_1 is contained inside the annulus A_1. □

We now show there are no zeros of p_c inside A_1. This then allows us to conclude that there are no zeros of p_c inside Ω_1.

Lemma 3.6. *Let $n \geq 6$ and $c \geq 4$. Then there are no zeros of p_c inside A_1.*

Proof. Let $n \geq 6$ and $c \geq 4$. We wish to show $p_c(z) \neq 0$ for any $z \in A_1$. This claim will follow if we can show

$$\left|p_c(z)\right| \geq \left|\frac{c}{n-1}\bar{z}^{n-1}\right| - \left|z + \frac{c}{2}z^2 + \frac{1}{n}\bar{z}^n\right| \geq \frac{c}{n-1}|z|^{n-1} - |z| - \frac{c}{2}|z|^2 - \frac{1}{n}|z|^n > 0$$

for all $z \in A_1$. Thus we will show $p(x) = \frac{c}{n-1}x^{n-1} - x - \frac{c}{2}x^2 - \frac{1}{n}x^n$ is positive on the interval $\left[2, c\left(\frac{n+1}{n}\right)\right]$. Notice that dividing by x does not impact the sign for positive x, so without loss of generality consider $q(x) = \frac{c}{n-1}x^{n-2} - 1 - \frac{c}{2}x - \frac{1}{n}x^{n-1}$. By Descartes' Rule of Signs, $q(x) = -\frac{1}{n}x^{n-1} + \frac{c}{n-1}x^{n-2} - \frac{c}{2}x - 1$ has at most two positive real roots. We use the Intermediate Value Theorem to show the zeros of q happen in the intervals $(0, 2)$ and $\left(c\left(\frac{n+1}{n}\right), \infty\right)$. We first show $q(2) > 0$ and $q\left(c\left(\frac{n+1}{n}\right)\right) > 0$. Observe the following:

$$\begin{aligned} q(2) &= -\frac{1}{n}2^{n-1} + \frac{c}{n-1}2^{n-2} - \frac{c}{2}2 - 1 = 2^{n-1}\left(\frac{c}{2(n-1)} - \frac{1}{n}\right) - c - 1 \\ &> \frac{2^{n-1}}{n}\left(\frac{c}{2} - 1\right) - c - 1 \geq \frac{16}{3}\left(\frac{c}{2} - 1\right) - c - 1 = \frac{5}{3}c - \frac{19}{3}. \end{aligned}$$

Because $c \geq 4$, $q(2) > \frac{5}{3}c - \frac{19}{3} \geq \frac{1}{3} > 0$.

We now show $q\left(c\left(\frac{n+1}{n}\right)\right) > 0$. Evaluating q at $c\left(\frac{n+1}{n}\right)$ yields

$$\begin{aligned} q\left(c\left(\frac{n+1}{n}\right)\right) &= \frac{1}{n-1}c^{n-1}\left(\frac{n+1}{n}\right)^{n-2} - \frac{1}{n}c^{n-1}\left(\frac{n+1}{n}\right)^{n-1} - \frac{1}{2}c^2\left(\frac{n+1}{n}\right) - 1 \\ &= c^{n-1}\left(\frac{n+1}{n}\right)^{n-2}\left(\frac{1}{n^2(n-1)}\right) - \frac{1}{2}\left(\frac{n+1}{n}\right)c^2 - 1 \\ &= c^2\left(\frac{n+1}{n}\right)\left(c^{n-3}\left(\frac{n+1}{n}\right)^{n-3}\left(\frac{1}{n^2(n-1)}\right) - \frac{1}{2}\right) - 1 \\ &> c^2\left(\frac{n+1}{n}\right)\left(4^{n-3}\left(\frac{n+1}{n}\right)^{n-3}\left(\frac{1}{n^2(n-1)}\right) - \frac{1}{2}\right) - 1 \\ &= c^2\left(\frac{n+1}{n}\right)\left(\lambda(n) - \frac{1}{2}\right) - 1, \end{aligned} \tag{3.7}$$

where $\lambda(n) = 4^{n-3}\left(\frac{n+1}{n}\right)^{n-3}\left(\frac{1}{n^2(n-1)}\right)$; we will show $\lambda(n+1) > \lambda(n)$. Equivalently, we show $\lambda(n+1)/\lambda(n) > 1$:

$$\begin{aligned}
\frac{\lambda(n+1)}{\lambda(n)} &= \frac{2\cdot 4^{n-2}\left(\frac{n+2}{n+1}\right)^{n-2}\frac{1}{(n+1)^2 n}}{2\cdot 4^{n-3}\left(\frac{n+1}{n}\right)^{n-3}\frac{1}{n^2(n-1)}} \\
&= 4\left(\frac{n+2}{n+1}\right)^{n-2}\left(\frac{n}{n+1}\right)^{n-3}\frac{n(n-1)}{(n+1)^2} \\
&= 4\left(\frac{n^2+2n}{n^2+2n+1}\right)^{n-3}\frac{(n+2)n(n-1)}{(n+1)^3} \\
&= 4\left(1-\frac{1}{n^2+2n+1}\right)^{n-3}\frac{n^3+n^2-2n}{(n+1)^3} \\
&\geq 4\left(1-\frac{1}{2n}\right)^{n}\frac{n^3+n^2-2n}{(n+1)^3}.
\end{aligned}$$

Let $A(n) = \left(1-\frac{1}{2n}\right)^n$ and $B(n) = \frac{n^3+n^2-2n}{(n+1)^3}$. We will show that A and B are both increasing functions by taking their derivatives. First,

$$A'(n) = \left(1-\frac{1}{2n}\right)^n\left(\ln\left(1-\frac{1}{2n}\right)+\frac{1}{2n-1}\right).$$

Then $A(n)$ will be increasing whenever $\ln\left(1-\frac{1}{2n}\right)+\frac{1}{2n-1} > 0$. Notice $1+x < e^x$ for all $x \neq 0$. Then $1+\frac{1}{2n-1} \leq e^{\frac{1}{2n-1}}$, so $\frac{2n}{2n-1} \leq e^{\frac{1}{2n-1}}$. Taking the natural log of both sides, $\ln\left(\frac{2n}{2n-1}\right) \leq \frac{1}{2n-1}$ which simplifies to $-\ln\left(1-\frac{1}{2n}\right) < \frac{1}{2n-1}$, and we conclude that $A(n)$ is increasing. For $B(n)$,

$$B'(n) = 2\frac{n^2+3n-1}{(n+1)^4}$$

which is clearly positive for $n \geq 1$. Therefore $B(n)$ is increasing for $n \geq 1$. Because A and B are both increasing,

$$\frac{\lambda(n+1)}{\lambda(n)} \geq 4\left(1-\frac{1}{2n}\right)^n\frac{n^3+n^2-2n}{(n+1)^3} \geq 1.66 > 1.$$

Therefore $\lambda(n+1) > \lambda(n)$ for $n \geq 6$; consequently, $\lambda(n) \geq \lambda(6)$ for all n. Calculating $\lambda(6)$

yields

$$\lambda(6) = 4^3 \left(\frac{7}{6}\right)^3 \left(\frac{1}{180}\right) \geq 0.564.$$

We now show $q\left(c\left(\frac{n+1}{n}\right)\right) > 0$. Equation 3.7 becomes

$$\begin{aligned} q\left(c\left(\frac{n+1}{n}\right)\right) &> c^2\left(\frac{n+1}{n}\right)\left(\lambda(n) - \frac{1}{2}\right) - 1 \geq c^2\left(\frac{n+1}{n}\right)\left(0.564 - \frac{1}{2}\right) - 1 \\ &> 4^2\,(0.064) - 1 = 0.024 > 0. \end{aligned}$$

We now return to our IVT argument. Observe that $q(0) = -1$ and $q(2) > 0$; therefore, q has a zero in the interval $(0, 2)$. Similarly $q\left(c\left(\frac{n+1}{n}\right)\right) > 0$ and $\lim_{x\to\infty} q(x) = -\infty$; therefore, q has a zero in the interval $\left(c\left(\frac{n+1}{n}\right), \infty\right)$. Because all the positive zeros of q are accounted for, we know $q(x) > 0$ in the interval $\left(2, c\left(\frac{n+1}{n}\right)\right)$; consequently, $p(x) > 0$ for all such x and p_c cannot have a zero in the annulus A_1. □

Our desired result then follows as a corollary.

Corollary 3.7. *Let $n \geq 6$ and $c \geq 4$. There are no zeros of p_c inside Ω_1.*

Proof. There are no zeros of p_c in A_1 by Lemma 3.6 and Ω_1 is contained in A_1 by Lemma 3.6; therefore, there are no zeros of p_c inside Ω_1. □

We now make a similar argument to show $\Omega_2 \subseteq A_2$.

Lemma 3.8. *For $n \geq 4$ and $c \geq 2\left(\frac{3}{2}\right)^{\frac{n-2}{n-3}}$, the curve Ω_2 is contained inside A_2.*

Proof. Let $n \geq 4$ and $c \geq 2\left(\frac{3}{2}\right)^{\frac{n-2}{n-3}}$. First, we show $-1/c$ is in Ω_2. Notice that

$$G\left(\frac{1}{c}e^{i\pi}\right) = \frac{1}{c^{2n-2}}(c^2-1)^2 > 0,$$

so $-1/c$ is in a sense-reversing region inside the unit circle; therefore, $-1/c$ is inside Ω_1. Because $0 < 1/c < 3/(2c)$, $-1/c \in A_2$. It remains to show no part of the critical curve lies on the boundary of A_2; i.e., G is strictly negative on the outer circle of A_2.

We now work backwards to show Equation 3.5 is satisfied on the circle of radius $\frac{3}{2c}$. By assumption, $c \geq 2\left(\frac{3}{2}\right)^{\frac{n-2}{n-3}}$. This is equivalent to

$$\begin{aligned} 2\left(\frac{3}{2}\right)^{n-2} &\leq c^{n-3} \\ \left(\frac{3}{2}\right)^{n-2}\frac{c}{c^{n-2}} &\leq \frac{1}{2} \\ \left(\frac{3}{2c}\right)^{n-2} c &\leq \frac{1}{2}. \end{aligned}$$

Then $\left(\frac{3}{2c}\right)^{n-2}\left(c - \frac{3}{2c}\right) < \left(\frac{3}{2c}\right)^{n-2} c \leq \frac{1}{2}$, which gives us the following set of equivalences:

$$\begin{aligned} \left(\frac{3}{2c}\right)^{n-2}\left(c - \frac{3}{2c}\right) &< \frac{1}{2} \\ \left(\frac{3}{2c}\right)^{n-2}\left|c - \frac{3}{2c}\right| &< \frac{1}{2} \\ \left|c\left(\frac{3}{2c}\right)^{n-2} - \left(\frac{3}{2c}\right)^{n-1}\right| &< \left|c\frac{3}{2c} - 1\right| \\ \left|cr^{n-2} - r^{n-1}\right| &< \left|cr - 1\right|. \end{aligned}$$

where $r = \frac{3}{2c}$. Therefore, Equation 3.5 is satisfied and $G < 0$ on the circle of radius $r = \frac{3}{2c}$.

Additionally, note that $G(0) = -1 < 0$. We then conclude that Ω_2 is contained inside the punctured disc A_2. □

Similar to Ω_1 and A_1, we have $\Omega_2 \subseteq A_2$ and we now show there are no zeros of p_c inside the punctured disc A_2. We then conclude there are no zeros of p_c inside Ω_2.

Lemma 3.9. *Let $n \geq 6$ and $c \geq 4$. Then there are no zeros of p_c inside A_2.*

Proof. Let $n \geq 6$ and $c \geq 4$. We want to show $p_c(z) \neq 0$ for any $z \in A_2$. This claim will follow if we can show

$$\left|p_c(z)\right| \geq |z| - \left|\frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n\right| \geq |z| - \frac{c}{2}|z|^2 - \frac{c}{n-1}|z|^{n-1} - \frac{1}{n}|z|^n > 0.$$

for all $z \in A_2$. This simplifies to showing $p(x) = x - \frac{c}{2}x^2 - \frac{c}{n-1}x^{n-1} - \frac{1}{n}x^n$ is positive for all x satisfying $0 < x < \frac{3}{2c}$. Notice that dividing by x does not impact the sign for such x values, so without loss of generality consider $q(x) = 1 - \frac{c}{2}x - \frac{c}{n-1}x^{n-2} - \frac{1}{n}x^{n-1}$. Notice that

$$\begin{aligned} q(x) &\geq 1 - \frac{c}{2}\left(\frac{3}{2c}\right) - \frac{c}{n-1}\left(\frac{3}{2c}\right)^{n-2} - \frac{1}{n}\left(\frac{3}{2c}\right)^{n-1} \\ &= \frac{1}{4} - \frac{c}{n-1}\left(\frac{3}{2c}\right)^{n-2} - \frac{1}{n}\left(\frac{3}{2c}\right)^{n-1} \qquad (3.8) \\ &= \frac{1}{4} - \frac{3^{n-2}}{(n-1)2^{n-2} \cdot c^{n-2}} - \frac{3^{n-1}}{n2^{n-1} \cdot c^{n-1}}. \qquad (3.9) \end{aligned}$$

Because $\frac{3}{2c} < 1$, Equation 3.8 illustrates that for fixed c the latter terms are decreasing as n increases. Thus the expression $\frac{1}{4} - \frac{c}{n-1}\left(\frac{3}{2c}\right)^{n-2} - \frac{1}{n}\left(\frac{3}{2c}\right)^{n-1}$ increases as n increases. Equation 3.9 shows how for fixed n the latter terms are decreasing as c increases; consequently, the expression $\frac{1}{4} - \frac{c}{n-1}\left(\frac{3}{2c}\right)^{n-2} - \frac{1}{n}\left(\frac{3}{2c}\right)^{n-1}$ is increasing as c increases. Because the above expression is increasing in n, c and $n \geq 6$, $c \geq 4$, we know it attains a minimum when $n = 6$ and $c = 4$. This yields

$$\begin{aligned} q(x) &\geq \frac{1}{4} - \frac{c}{n-1}\left(\frac{3}{2c}\right)^{n-2} - \frac{1}{n}\left(\frac{3}{2c}\right)^{n-1} \\ &\geq \frac{1}{4} - \frac{4}{6-1}\left(\frac{3}{2 \cdot 4}\right)^{6-2} - \frac{1}{6}\left(\frac{3}{2 \cdot 4}\right)^{6-1} \\ &\geq 0.23 \\ &> 0. \end{aligned}$$

Therefore $q(x) > 0$ for all $0 < x \leq \frac{3}{2c}$; hence $p(x) > 0$ for all $0 < x \leq \frac{3}{2c}$. We conclude $p_c(z) \neq 0$ for any $z \in A_2$. □

Again, our desired result follows as a corollary.

Corollary 3.10. *Let $c \geq 4$ and $n \geq 6$. Then there are no zeros of p_c inside Ω_2.*

Proof. There are no zeros of p_c in A_2 by Lemma 3.9 and Ω_2 is contained in A_2 by Lemma 3.8; therefore, there are no zeros of p_c inside Ω_2. □

We now combine Corollaries 3.7 and 3.10 into one theorem.

Theorem 3.11. *For $n \geq 6$ and $c \geq 4$, p_c has no zeros inside Ω_1 or Ω_2.*

This theorem allows us to state that for sufficiently large c, the critical curve is composed of the unit circle, Ω_1, and Ω_2, but we only need to consider the unit circle when determining the order of a zero. We will use this theorem extensively in Chapter 4.

Chapter 4. Zeros

With Rouché's Theorem for Complex Harmonic Functions from Chapter 2 and the results about the critical curve of p_c from Chapter 3, we are now able to prove results about the total number of zeros of p_c and their locations. We first prove that the nonsingular zeros of p_c are simple. We then go into Section 4.1 to prove results for sufficiently small and sufficiently large values of c. In Section 4.2, we begin investigating what happens for intermediate values of c.

Proposition 4.1. *Let $z_0 \in \mathbb{C}$ be a nonsingular zero of p_c. Then z_0 has order 1 if it is in a sense-preserving region and order -1 if it is in a sense-reversing region.*

Proof. Let $z_0 \in \mathbb{C}$ be a nonsingular zero of p_c. First, suppose that z_0 is in a sense-preserving region, so $\left|\omega(z_0)\right| < 1$ and the order of z_0 can be determined by considering the order of vanishing of h at z_0. Notice that $h'(z_0) = 1 + cz_0$. Therefore, $h'(z_0) \neq 0$ whenever $z_0 \neq -1/c$ and z_0 is a simple zero. If $z_0 = -1/c$, $\left|\omega(-1/c)\right| = \infty > 1$, a contradiction. Therefore every nonsingular zero of p_c in a sense-preserving region has order 1.

Now let $z_0 \in \mathbb{C} \setminus \{0\}$ be a nonsingular zero of p_c in a sense-reversing region, so $\left|\omega(z_0)\right| > 1$. Then the order of z_0 is determined by considering the order of vanishing of g at z_0. We have $g'(z_0) = z_0^{n-2}(c + z_0)$. Therefore z_0 has order -1 unless $z_0 = 0$ or $-c$. If $z_0 = 0$, then $\left|\omega(0)\right| = 0 < 1$, a contradiction. Similarly if $z_0 = -c$ then $\left|\omega(-c)\right| = 0 < 1$, a contradiction. Therefore every zero in a sense-reversing region has order -1. □

4.1 Rouché Theorem Arguments

We begin with a standard result to find the sum of the orders of the zeros of p_c.

Proposition 4.2. *For $n \geq 3$ and any $c \in \mathbb{C}$, the sum of the orders of the nonsingular zeros of $p_c(z) = z + \frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n$ is $-n$.*

Proof. Let R be sufficiently large so that $R + \frac{|c|}{2}R^2 + \frac{|c|}{n-1}R^{n-1} < \frac{1}{n}R^n$, and let $f(z) = \frac{1}{n}\bar{z}^n$. Then on the circle $|z| = R$,

$$\begin{aligned}\left|p_c(z) - f(z)\right| &= \left|z + \frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1}\right| \leq |z| + \frac{|c|}{2}|z|^2 + \frac{|c|}{n-1}|\bar{z}|^{n-1} \\ &= R + \frac{|c|}{2}R^2 + \frac{|c|}{n-1}R^{n-1} < \frac{1}{n}R^n = \left|f(z)\right|.\end{aligned}$$

Because $|p_c - f| < |f|$ where $|z| = R$, Rouché's Theorem for Complex Harmonic Functions gives that $p_c(z)$ and $f(z)$ have the same sum of orders of zeros. Notice that f has one zero at $z = 0$ of order $-n$; hence, the orders of the zeros of p_c sum to $-n$. □

Now that we know the sum of the orders of zeros of p_c is $-n$, we can prove how many distinct zeros p_c must have for certain values of c. First, we consider sufficiently small values of c.

Theorem 4.3. *For $n \geq 5$ and $0 < c < 1$, the complex harmonic polynomial $p_c(z)$ has $n+2$ distinct zeros.*

Proof. Let $f(z) = z$. We apply Rouché's Theorem for Complex Harmonic Functions to $p_c - f$ on the unit circle $|z| = 1$:

$$\begin{aligned}\left|p_c(z) - f(z)\right| &= \left|\frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n\right| \leq \frac{c}{2} + \frac{c}{n-1} + \frac{1}{n} \\ &< \frac{1}{2} + \frac{1}{n-1} + \frac{1}{n} \leq \frac{1}{2} + \frac{1}{4} + \frac{1}{5} < 1 = \left|f(z)\right|.\end{aligned}$$

Thus $\left|p_c(z) - f(z)\right| < \left|f(z)\right|$ on $|z| = 1$. Because $f(z) = z$ has one zero at $z = 0$ of order 1 in the unit circle, the sum of the orders of zeros of p_c in the unit circle is 1.

Recall that for $0 < c < 1$ the critical curve of p_c consists only of the unit circle and $\left|\omega(z)\right| < 1$ if and only if $|z| < 1$ by Proposition 3.4. Therefore the zeros of p_c in the unit circle must have positive order, so p_c has one zero of order 1 inside the unit circle. Since all our zeros are simple, there must be $n+1$ distinct zeros in the sense-reversing region by Proposition 4.2. Therefore, $p_c(z)$ has $n+2$ distinct zeros when $0 < c < 1$. □

Now, we consider sufficiently large values of c.

Theorem 4.4. *For $n \geq 6$ and $c \geq 4$, $p_c(z)$ has $n+4$ distinct zeros.*

Proof. Let $f(z) = \frac{c}{2}z^2$. We apply Rouché's Theorem for Complex Harmonic Functions to $p_c - f$ on the unit circle $|z| = 1$:

$$\begin{aligned}\left|p_c(z) - f(z)\right| &= \left|z + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n\right| \leq 1 + \frac{c}{n-1} + \frac{1}{n} \\ &\leq 1 + \frac{c}{5} + \frac{1}{6} = \frac{7}{6} + \frac{c}{5} < \frac{c}{2} = \left|f(z)\right|.\end{aligned}$$

Thus $\left|p_c(z) - f(z)\right| < \left|f(z)\right|$ on $|z| = 1$. Because f only has one zero of order 2 in $|z| = 1$, we know by Rouché's Theorem for Complex Harmonic Functions that the sum of the orders of zeros of p_c inside $|z| = 1$ is 2. Because p_c is sense-preserving inside the unit circle and Ω_2 is bounded away from any zeros inside the unit circle by Theorem 3.11, all the zeros of p_c inside the unit circle must have positive order. Therefore, p_c has two simple zeros of positive order inside the unit circle. Because there are no zeros in Ω_1 by Theorem 3.11, there must be $n+2$ zeros in the sense-reversing region by Proposition 4.2. Because all these zeros are simple, p_c has $n+4$ distinct zeros when $c \geq 4$. □

4.1.1 Location of Zeros for Small Values of c. By Theorem 4.3, p_c has $n+2$ distinct zeros for $0 < c < 1$. As shown in Figure 4.1, $n+1$ of those zeros are arranged in a circle about the origin. It then makes sense to use Rouché's Theorem for Complex Harmonic Functions to pin down the locations of these zeros to annuli and to sectors of annuli. Because Rouché's Theorem compares functions, we first construct a candidate function by taking

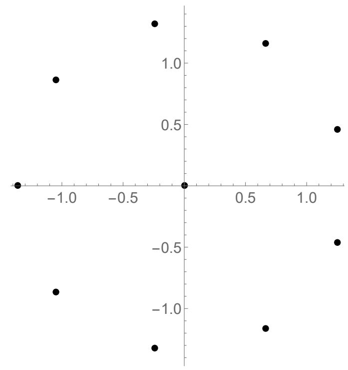


Figure 4.1: The zeros of the eighth degree polynomial $p_{0.4}(z)$.

$f_0(z) = \lim_{c\to 0} p_c(z)$. We locate the zeros and sense-preserving and sense-reversing regions of f_0. Then we compare p_c to f_0 in order to determine where the zeros of p_c are for sufficiently small values of c.

Lemma 4.5. *Let $f_0(z) = \lim_{c\to 0} p_c(z) = z + \frac{1}{n}\bar{z}^n$ for any $n \geq 2$. Then $f_0(z)$ has $n+2$ zeros: $z = 0$ and $n+1$ of the form $z = n^{\frac{1}{n-1}} e^{i\frac{\pi+2\pi k}{n+1}}$ for $0 \leq k \leq n$.*

Proof. The dilatation function of f_0 is $\omega_{f_0}(z) = z^{n-1}$. Then the critical curve is $\left|\omega_{f_0}(z)\right| = 1$, which trivially simplifies to the unit circle $|z| = 1$, and the sense-preserving region of f_0 is the set of all z such that $|z| < 1$ and the sense-reversing region is the set of all z such that $|z| > 1$. By Proposition 4.2, the sum of the orders of the zeros of f_0 is $-n$.

We explicitly calculate the zeros of f_0. Clearly, $f_0(0) = 0$. Moreover, the dilatation curve of f_0 is $\omega_{f_0}(z) = z^{n-1}$, so $\omega_{f_0}(0) = 0 < 1$ which means 0 is in the sense-preserving region of f_0. Because $\frac{d}{dz}(z) = 1$, 0 is a zero of order 1.

Now let $z = re^{it}$ such that $f(z) = 0$ and $z \neq 0$. Then $f_0(z) = 0$ becomes $re^{it} + \frac{1}{n}r^n e^{-int} = 0$. This simplifies to $e^{i(n+1)t} = -\frac{1}{n}r^{n-1}$. Because $\left|e^{i(n+1)t}\right| = 1$, $\left|-\frac{1}{n}r^{n-1}\right| = 1$, so $\frac{1}{n}r^{n-1} = 1$ which means $r = n^{\frac{1}{n-1}}$. Then $e^{i(n+1)t} = -1$, so $t = \frac{\pi+2\pi k}{n+1}$ for integers k satisfying $0 \leq k \leq n$. Then $f_0(z_k) = 0$ where $z_k = n^{\frac{1}{n-1}} e^{i\frac{\pi+2\pi k}{n+1}}$ for $0 \leq k \leq n$. Notice that each z_k is in a sense-reversing region. Moreover, $\frac{d}{dz}\left(\frac{1}{n}z^n\right) = z^{n-1}$ but $z_k^{n-1} \neq 0$. Therefore, each z_k is a zero of f_0 of order -1.

Thus f_0 has $n+1$ zeros. $\square$

Now that we have our candidate function f_0, we can compare f_0 and p_c using Rouché's Theorem for Complex Harmonic Functions. Let $\arg_0(z)$ be the branch of $\arg(z)$ taking values in $[0, 2\pi)$.

Proposition 4.6. *Let $n \geq 5$. For r_1, r_2 chosen such that $0 < r_1 < n^{\frac{1}{n-1}} < r_2$ there exists a $0 < c_0 < 1$ such that for all $0 < c \leq c_0$ there are $n+1$ zeros of p_c in the annulus $A_0 = \{z \in \mathbb{C} \colon r_1 < |z| < r_2\}$. Moreover, for these same r_1, r_2, there exists a $c_S > 0$ such that for all $0 < c \leq c_S$ each sector $S_k = \{z \in \mathbb{C} \colon r_1 < |z| < r_2, \frac{\pi(1+4k)}{2(n-1)} < \arg_0(z) < \frac{\pi(3+4k)}{2(n+1)}\}$ contains one zero of p_c.*

Proof. Let $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 < n^{\frac{1}{n-1}} < r_2$. Let $|z| = r$ be arbitrary. Then $\left|f_0(z)\right| = \left|z + \frac{1}{n}\bar{z}^n\right| \geq \left|r - \frac{1}{n}r^n\right|$. We will use Rouché's Theorem for Complex Harmonic Functions to compare $p_c - f_0$ to f_0 on $C_{r_1} = \{z \in \mathbb{C} \colon |z| = r_1\}$, $C_{r_2} = \{z \in \mathbb{C} \colon |z| = r_2\}$, and line segments connecting these two circles.

Case 1: Let $r = r_1 < n^{\frac{1}{n-1}}$. Then $r_1 - \frac{1}{n}r_1^n > 0$, so $(r_1 - \frac{1}{n}r_1^n)/(\frac{1}{2}r_1^2 + \frac{1}{n-1}r_1^{n-1}) > 0$. There exists a $0 < c_1 < 1$ such that $(r_1 - \frac{1}{n}r_1^n)/(\frac{1}{2}r_1^2 + \frac{1}{n-1}r_1^{n-1}) > c_1 > 0$; thus, $c_1(\frac{1}{2}r_1^2 + \frac{1}{n-1}r_1^{n-1}) < r_1 - \frac{1}{n}r_1^n$. Then on the circle of radius r_1 and all $0 < c \leq c_1$,

$$\begin{aligned}\left|p_c(z) - f_0(z)\right| &= \left|\frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1}\right| \leq \frac{c_1}{2}r_1^2 + \frac{c_1}{n-1}r_1^{n-1} \\ &= c_1\left(\frac{1}{2}r_1^2 + \frac{1}{n-1}r_1^{n-1}\right) < r_1 - \frac{1}{n}r_1^n \leq \left|f_0(z)\right|.\end{aligned}$$

Case 2: Let $r = r_2 > n^{\frac{1}{n-1}}$. Then $\frac{1}{n}r_2^n - r_2 > 0$, so $(\frac{1}{n}r_2^n - r_2)/(\frac{1}{2}r_2^2 + \frac{1}{n-1}r_2^{n-1}) > 0$. There exists a $0 < c_2 < 1$ such that $(\frac{1}{n}r_2^n - r_2)/(\frac{1}{2}r_2^2 + \frac{1}{n-1}r_2^{n-1}) > c_2 > 0$; thus $c_2(\frac{1}{2}r_2^2 + \frac{1}{n-1}r_2^{n-1}) < \frac{1}{n}r_2^n - r_2$. Then when $|z| = r_2$ and $0 < c \leq c_2$,

$$\begin{aligned}\left|p_c(z) - f_0(z)\right| &= \left|\frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1}\right| \leq \frac{c_2}{2}r_2^2 + \frac{c_2}{n-1}r_2^{n-1} \\ &= c_2\left(\frac{1}{2}r_2^2 + \frac{1}{n-1}r_2^{n-1}\right) < \frac{1}{n}r_2^n - r_2 \leq \left|f_0(z)\right|.\end{aligned}$$

Case 3: Let r_1 and r_2 be positive real numbers such that $0 < r_1 < n^{\frac{1}{n-1}} < r_2$. Consider the pair of line segments $\ell_{k,1} = \{z = re^{i\theta} \mid 0 < r_1 \le r \le r_2, \theta = \frac{\pi}{2(n+1)}(1+4k)\}$ and $\ell_{k,2} = \{z = re^{i\theta} \mid 0 < r_1 \le r \le r_2, \theta = \frac{\pi}{2(n+1)}(3+4k)\}$ for some integer k. For z on $\ell_{k,1}$ and $\ell_{k,2}$,

$$\begin{aligned}\left|f_0(z)\right|^2 &= \left|z + \frac{1}{n}\bar{z}^n\right|^2 = (z + \frac{1}{n}\bar{z}^n)(\bar{z} + \frac{1}{n}z^n) = z\bar{z} + \frac{1}{n}(z^{n+1} + \bar{z}^{n+1}) + \frac{1}{n^2}z^n\bar{z}^n \\ &= r^2 + \frac{2r^{n+1}}{n}\cos\left(\pi \pm \frac{\pi}{2} + 2\pi k\right) + \frac{1}{n^2}r^{2n} = r^2 + \frac{1}{n^2}r^{2n}.\end{aligned}$$

Then for $0 < c_3 < \sqrt{\frac{r^2 + \frac{1}{n^2}r^{2n}}{\frac{1}{4}r^4 + \frac{1}{(n-1)^2}r^{2n-2}}}$, along the pair of line segments $\ell_{k,1}$ and $\ell_{k,2}$ and all $0 < c \le c_3$,

$$\begin{aligned}\left|p_c(z) - f_0(z)\right|^2 &= \left|\frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1}\right|^2 = \left(\frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1}\right)\left(\frac{c}{2}\bar{z}^2 + \frac{c}{n-1}z^{n-1}\right) \\ &= \frac{c^2}{4}z^2\bar{z}^2 + \frac{c^2}{2(n-1)}(z^{n+1} + \bar{z}^{n+1}) + \left(\frac{c}{n-1}\right)^2 z^{n-1}\bar{z}^{n-1} \\ &= \frac{c^2}{4}r^4 + \frac{c^2}{n-1}r^{n+1}\cos(\pi \pm \frac{\pi}{2} + 2\pi k) + \left(\frac{c}{n-1}\right)^2 r^{2n-2} \\ &= \frac{c^2}{4}r^4 + \left(\frac{c}{n-1}\right)^2 r^{2n-2} = c^2\left(\frac{1}{4}r^4 + \left(\frac{1}{n-1}\right)^2 r^{2n-2}\right) \\ &< r^2 + \frac{1}{n^2}r^{2n} = \left|f_0(z)\right|^2.\end{aligned}$$

Therefore, $\left|p_c(z) - f_0(z)\right| < \left|f_0(z)\right|$.

Applying Rouché's Theorem for Complex Harmonic Functions to Case 1, the sum of the orders of the zeros of f_0 and the sum of the orders of the zeros of p_c are the same inside the circle $|z| = r_1$. By Lemma 4.5, f_0 has one zero of order 1 inside any circle of radius $r_1 < n^{\frac{1}{n-1}}$. Therefore, the sum of the orders of the zeros of p_c is 1 inside the circle of radius r_1 for all c satisfying $0 < c \le c_1$.

Now recall that when $0 < c < 1$, p_c is sense-preserving if and only if $|z| < 1$. Also recall that all the zeros of p_c have order 1 or -1. First, suppose $r_1 \le 1$. Because the sum of the orders of zeros of p_c is 1 in a sense-preserving region, p_c has one zero of order 1 inside C_{r_1}. Second, suppose $r_1 > 1$. The sum of the orders of the zeros of p_c in C_{r_1} is still 1 and p_c still

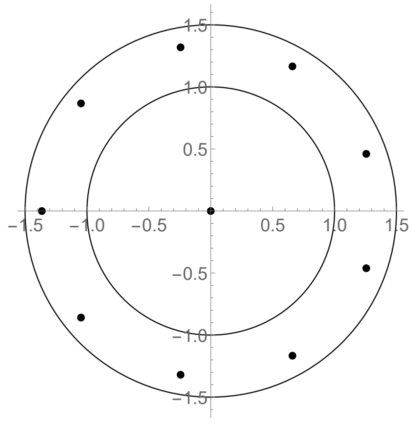


Figure 4.2: The zeros of the eighth degree polynomial $p_{0.4}(z)$ with 9 of the zeros inside the annulus defined by all z such that $1 < |z| < 1.5$.

has one zero of order 1 inside the unit circle by Theorem 4.3. Moreover, any zeros outside the unit circle have order -1 which would cause the sum of the orders of the zeros of p_c inside C_{r_1} to be less than 1, a contradiction. Therefore, p_c has one zero of order 1 inside the circle of radius r_1 for any $r_1 < n^{\frac{1}{n-1}}$ and $0 < c \leq c_1$.

Applying Rouché's Theorem for Complex Harmonic Functions to Case 2, the sum of the orders of the zeros of f_0 and the sum of the orders of the zeros of p_c are the same inside the circle of radius r_2. By Lemma 4.5, f_0 has one zero of order 1 and $n+1$ zeros of order -1 inside C_{r_2}; thus, the sum of the orders of the zeros of f_0, and consequently p_c, is $-n$. By the above work, p_c has one zero of order 1 inside C_{r_1}. Extending the radius of this circle extends it into a sense-reversing region; consequently, p_c has $n+1$ zeros of order -1 in the annulus $A_0 = \{z \in \mathbb{C} \colon r_1 < |z| < r_2\}$ for all c such that $0 < c \leq c_0$ where $c_0 = \min\{c_1, c_2\}$.

Let c be a value such that $0 < c \leq c_S$ where $c_S = \min\{c_1, c_2, c_3\}$. Then Cases 1, 2, and 3 give the sum of the orders of the zeros of f_0 and the sum of the orders of the zeros of p_c are equal in each sector $S_k = \{z \in \mathbb{C} \colon r_1 < |z| < r_2, \frac{\pi(1+4k)}{2(n-1)} < \arg_0(z) < \frac{\pi(3+4k)}{2(n+1)}\}$. By Lemma 4.5 f_0 has one zero of order -1 inside each S_k, so the sum of the orders of the zeros of p_c in each S_k is -1. Because there are no zeros of positive order in A_0, each S_k contains one zero of p_c of order -1. □

The above proposition states that there is a range of c values such that $n+1$ zeros of

p_c are located in an annulus or a zero of p_c is located in a sector of an annulus; Figure 4.2 illustrates this for the $n = 8$ case. It is natural to ask what those ranges of c values are. As expected, the closer r_1 and r_2 are to $n^{\frac{1}{n-1}}$, the smaller c must be in order to guarantee the zeros are inside the annulus. Due to the decreasing nature of $n^{\frac{1}{n-1}}$, we do not provide a range of c values for Cases 1 and 2 of Proposition 4.6 (though we comment that for $n \geq 9$, $r_1 = 1$ and $r_2 = 3/2$ allow for all values $0 < c < 1/2$). However, the range of valid c values for Case 3 can be considered quite nicely:

Lemma 4.7. *For any $r > 0$ and any $n \geq 4$,*

$$\frac{r^2 + \frac{1}{n^2} r^{2n}}{\frac{1}{4} r^4 + \left(\frac{1}{n-1}\right)^2 r^{2n-2}} > \frac{9}{32}.$$

Proof. First, notice that

$$\frac{r^2 + \frac{1}{n^2} r^{2n}}{\frac{1}{4} r^4 + \left(\frac{1}{n-1}\right)^2 r^{2n-2}} = \frac{4r^2 \frac{1}{n^2}(n^2 + r^{2n-2})}{r^4 \frac{1}{(n-1)^2}((n-1)^2 + 4r^{2n-6})} = \frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2 + r^{2n-2}}{(n-1)^2 + 4r^{2n-6}}.$$

From here we consider two cases.

Case 1: Suppose that $(n-1)^2 \leq 4r^{2n-6}$; hence, $r \geq \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$, so $r > 1$ for all $n \geq 4$. Then

$$\begin{aligned}
\frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2 + r^{2n-2}}{(n-1)^2 + 4r^{2n-6}} &\geq \frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2 + r^{2n-2}}{8r^{2n-6}} \\
&= \left(\frac{n-1}{n}\right)^2 \frac{n^2}{2r^{2n-4}} + \left(\frac{n-1}{n}\right)^2 \frac{r^2}{2} \\
&> \left(\frac{n-1}{n}\right)^2 \frac{r^2}{2} \\
&> \left(\frac{3}{4}\right)^2 \frac{1}{2} \\
&= \frac{9}{32}.
\end{aligned}$$

Case 2: Suppose that $(n-1)^2 \geq 4r^{2n-6}$; equivalently, $r \leq \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$. Then

$$\begin{aligned}
\frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2+r^{2n-2}}{(n-1)^2+4r^{2n-6}} &\geq \frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2+r^{2n-2}}{2(n-1)^2} \\
&= \frac{2}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2}{(n-1)^2} + \frac{2}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{r^{2n-2}}{(n-1)^2} \\
&> \frac{2}{r^2} \\
&\geq \frac{2}{\left(\frac{n-1}{2}\right)^{\frac{2}{n-3}}}
\end{aligned}$$

Now notice that $\left(\frac{n-1}{2}\right)^{\frac{2}{n-3}} \leq \left(\frac{3}{2}\right)^2$ if and only if $\frac{n-1}{2} \leq \left(\frac{3}{2}\right)^{n-3}$. This latter inequality is an equality at $n = 4$, and clearly $\left(\frac{3}{2}\right)^{n-3}$ increases at a faster rate than $\frac{n-1}{2}$. Therefore, the latter inequality holds and we also get $\left(\frac{n-1}{2}\right)^{\frac{2}{n-3}} \leq \left(\frac{3}{2}\right)^2$. Applying this to the above set of inequalities yields

$$\frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2+r^{2n-2}}{(n-1)^2+4r^{2n-6}} \geq \frac{2}{\left(\frac{n-1}{2}\right)^{\frac{2}{n-3}}} \geq \frac{2}{\left(\frac{3}{2}\right)^2} = \frac{8}{9}.$$

Thus for any $r > 0$ and $n \geq 4$,

$$\frac{r^2 + \frac{1}{n^2}r^{2n}}{\frac{1}{4}r^4 + \left(\frac{1}{n-1}\right)^2 r^{2n-2}} > \min\left\{\frac{9}{32}, \frac{8}{9}\right\} = \frac{9}{32}. \qquad \square$$

Therefore, Case 3 of Proposition 4.6 is always satisfied by $0 < c \leq \frac{3}{4\sqrt{2}} \approx 0.53033$.

4.1.2 Locations of Zeros for Large Values of c. This section closely follows Section 4.1.1 except we now consider sufficiently large values of c. In particular, recall that p_c has $n+4$ distinct zeros for $c \geq 4$, $n \geq 6$ by Theorem 4.4. As shown in Figure 4.1, $n+1$ of those zeros are arranged in a circle about the origin, so we can use Rouché's Theorem for Complex Harmonic Functions to localize these zeros to annuli and sectors of annuli. Rouché's Theorem compares functions, so we first construct a candidate function by taking $f_\infty(z) = \lim_{c\to\infty} p_c(z)/c$. Then we locate the zeros and sense-preserving and sense-reversing

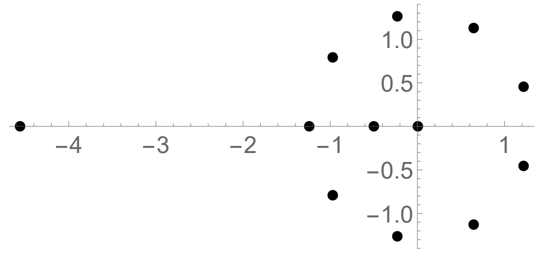


Figure 4.3: The zeros of the eighth degree polynomial $p_4(z)$.

regions of f_∞. Lastly, we compare p_c/c to f_∞ to determine where the zeros of p_c/c are located for sufficiently large values of c.

Lemma 4.8. *Let $f_\infty(z) = \lim_{c\to\infty} p_c(z)/c = \frac{1}{2}z^2 + \frac{1}{n-1}\bar{z}^{n-1}$ for $n > 3$. Then $f_\infty(z)$ has $n+3$ zeros: $n+1$ of the form $z = \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}} e^{i\frac{\pi+2\pi k}{n+1}}$ for $0 \le k \le n$ and a zero at $z = 0$ of order 2.*

Proof. The dilatation of f_∞ is $\omega_{f_\infty}(z) = z^{n-3}$. Then the critical curve is the set of points such that $\left|\omega_{f_\infty}(z)\right| = 1$, which trivially simplifies to the unit circle $|z| = 1$. Then the sense-preserving region of f_∞ is the set of all z such that $|z| < 1$ and the sense-reversing region is the set of all z such that $|z| > 1$.

We now explicitly calculate the zeros not at the origin: Let $z = re^{it}$. Then $f_\infty(z) = 0$ gives $\frac{1}{2}r^2e^{i2t} + \frac{1}{n-1}r^{n-1}e^{-i(n-1)t} = 0$ which simplifies to $e^{i(n+1)t} = -\frac{2}{n-1}r^{n-3}$. Because $\left|e^{i(n+1)t}\right| = 1$, $\left|-\frac{2}{n-1}r^{n-3}\right| = 1$. Thus $\frac{2}{n-1}r^{n-3} = 1$ which means $r = \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$. Then $e^{i(n+1)t} = -1$, so $t = \frac{\pi+2\pi k}{n+1}$ for integers k satisfying $0 \le k \le n$. Therefore, $f_\infty(z_k) = 0$ where $z_k = \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}} e^{i\frac{\pi+2\pi k}{n+1}}$ for $0 \le k \le n$. Notice that all z_k are in sense-reversing regions and $\frac{d}{dz}\left(\frac{1}{n-1}z^{n-1}\right) = z^{n-2}$ but $z_k^{n-2} \neq 0$. Therefore, each z_k is a zero of order -1 and there are $n+1$ nonzero zeros.

We will now consider $z = 0$. Because $\frac{d}{dz}(z^2) = 2z$, $\frac{d}{dz}(2z) = 2$, we know that 0 is a zero of order 2.

Therefore, f_∞ has $n+3$ zeros. □

We now have our candidate function f_∞, and we compare f_∞ and p_c/c using Rouché's Theorem for Complex Harmonic Functions to locate the zeros of p_c.

Proposition 4.9. *Let $n \ge 6$. For r_1, r_2 chosen such that $0 < r_1 < \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}} < r_2$ there exists a $c_\infty \ge 4$ such that for all $c \ge c_\infty$ there are $n+1$ zeros of p_c in the annulus $A_\infty =$*

$\{z \in \mathbb{C}\colon r_1 < |z| < r_2\}$. *Moreover, for these same* r_1, r_2, *there exists a* $c_S \geq 4$ *such that for all* $c \geq c_S$ *each sector* $S_k = \{z \in \mathbb{C} \mid r_1 < |z| < r_2, \frac{\pi(1+4k)}{2(n-1)} < \arg_0(z)\frac{\pi(3+4k)}{2(n+1)}\}$ *contains one zero of* p_c.

Proof. Let $r_1, r_2 \in \mathbb{R}$ such that $0 < r_1 < \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}} < r_2$. Let $|z| = r$ be arbitrary. Then $\left|f_\infty(z)\right| = \left|\frac{1}{2}z^2 + \frac{1}{n-1}\bar{z}^{n-1}\right| \geq \left|\frac{1}{2}r^2 - \frac{1}{n-1}r^{n-1}\right|$. We will use Rouché's Theorem for Complex Harmonic Functions to compare $p_c - f_\infty$ to f_∞ on $C_{r_1} = \{z \in \mathbb{C}\colon |z| = r_1\}$, $C_{r_2} = \{z \in \mathbb{C}\colon |z| = r_2\}$, and line segments connecting these two circles. For clarity, we write these as three separate cases.

Case 1: Let $r = r_1 < \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$. Then $r_1^2 - \frac{2}{n-1}r_1^{n-1} > 0$, so $\frac{1}{2}r_1^2 - \frac{1}{n-1}r_1^{n-1} > 0$, and we choose $c_1 > (r_1 + \frac{1}{n}r_1^n)/(\frac{1}{2}r_1^2 - \frac{1}{n-1}r_1^{n-1}) > 0$. Thus, $\frac{1}{c_1}(r_1 + \frac{1}{n}r_1^n) < \frac{1}{2}r_1^2 - \frac{1}{n-1}r_1^{n-1}$. Then for all $c \geq c_1$ on $|z| = r_1$ yields

$$\begin{aligned}\left|p_c(z)/c - f_\infty(z)\right| &= \left|\frac{1}{c}z + \frac{1}{cn}\bar{z}^n\right| \leq \frac{1}{c_1}\left(r_1 + \frac{1}{n}r_1^n\right) \\ &< \frac{1}{2}r_1^2 - \frac{1}{n-1}r_1^{n-1} \leq \left|f_\infty(z)\right|.\end{aligned}$$

Case 2: Let $r = r_2 > \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$. Then $\frac{2}{n-1}r_2^{n-1} - r_2^2 > 0$, so $\frac{1}{n-1}r_2^{n-1} - \frac{1}{2}r_2^2 > 0$. We choose $c_2 > (r_2 + \frac{1}{n}r_2^n)/(\frac{1}{n-1}r_2^{n-1} - \frac{1}{2}r_2^2) > 0$; consequently, $\frac{1}{c_2}(r_2 + \frac{1}{n}r_2^n) < \frac{1}{n-1}r_2^{n-1} - \frac{1}{2}r_2^2$. Then for all $c \geq c_2$ on the circle $|z| = r_2$,

$$\begin{aligned}\left|p_c(z)/c - f_\infty(z)\right| &= \left|\frac{1}{c}z + \frac{1}{cn}\bar{z}^n\right| \leq \frac{1}{c_2}r_2 + \frac{1}{c_2 n}r_2^n = \frac{1}{c_2}\left(r_2 + \frac{1}{n}r_2^n\right) \\ &< \frac{1}{n-1}r_2^{n-1} - \frac{1}{2}r_2^2 \leq \left|f_\infty(z)\right|.\end{aligned}$$

Case 3: Let r_1 and r_2 be positive real numbers such that $0 < r_1 < \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}} < r_2$. Consider the pair of line segments $\ell_{k,1} = \{z = re^{i\theta} \mid 0 < r_1 \leq r \leq r_2, \theta = \frac{\pi}{2(n+1)}(1 + 4k)\}$ and $\ell_{k,2} = \{z = re^{i\theta} \mid 0 < r_1 \leq r \leq r_2, \theta = \frac{\pi}{2(n+1)}(3 + 4k)\}$ for some integer k. For z on $\ell_{k,1}$

or $\ell_{k,2}$,

$$\begin{aligned}\left|f_\infty(z)\right|^2 &= \left|\frac{1}{2}z^2 + \frac{1}{n-1}\bar{z}^{n-1}\right|^2 = \left(\frac{1}{2}z^2 + \frac{1}{n-1}\bar{z}^{n-1}\right)\left(\frac{1}{2}\bar{z}^2 + \frac{1}{n-1}z^{n-1}\right)\\ &= \frac{1}{4}z^2\bar{z}^2 + \frac{1}{2(n-1)}(z^{n+1} + \bar{z}^{n+1}) + \left(\frac{1}{n-1}\right)^2 z^{n-1}\bar{z}^{n-1}\\ &= \frac{1}{4}r^4 + \frac{r^{n+1}}{n-1}\cos(\pi \pm \frac{\pi}{2} + 2\pi k) + \left(\frac{1}{n-1}\right)^2 r^{2n-2}\\ &= \frac{1}{4}r^4 + \left(\frac{1}{n-1}\right)^2 r^{2n-2}.\end{aligned}$$

Then for $c_3 > \sqrt{\frac{r^2+\frac{1}{n^2}r^{2n}}{\frac{1}{4}r^4+\left(\frac{1}{n-1}\right)^2 r^{2n-2}}}$ along the pair of line segments $\ell_{k,1}$ and $\ell_{k,2}$ and all $c \geq c_3$,

$$\begin{aligned}\left|p_c(z)/c - f_\infty(z)\right|^2 &= \left|\frac{1}{c}z + \frac{1}{cn}\bar{z}^n\right|^2 = \frac{1}{c^2}z\bar{z} + \frac{1}{c^2 n}z^{n+1} + \frac{1}{c^2 n}\bar{z}^{n+1} + \frac{1}{c^2 n^2}z^n\bar{z}^n\\ &= \frac{1}{c^2}r^2 + \frac{1}{c^2 n}r^{n+1}2\cos\left(\pi \pm \frac{\pi}{2} + 2\pi k\right) + \frac{1}{c^2 n^2}r^{2n}\\ &\leq \frac{1}{c_3^2}(r^2 + \frac{1}{n^2}r^{2n}) < \frac{1}{4}r^4 + \left(\frac{1}{n-1}\right)^2 r^{2n-2} = \left|f_\infty(z)\right|^2.\end{aligned}$$

Therefore, $\left|p_c(z)/c - f_\infty(z)\right| < \left|f_\infty(z)\right|$.

Applying Rouché's Theorem for Complex Harmonic Functions to Case 1, the sum of the orders of the zeros of f_∞ and the sum of the orders of the zeros of p_c are the same inside the circle of radius r_1. By Lemma 4.8, f_∞ has one zero of order 2 inside any circle of radius $r_1 < \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$. Therefore, the sum of the orders of the zeros of p_c/c is 2 inside the circle of radius r_1 for all c satisfying $c \geq c_1$.

Now recall Theorem 3.11: for $c \geq 4$ and $n \geq 6$, the portions of the critical curve distinct from the unit circle (i.e., Ω_1 and Ω_2) do not contain any zeros of p_c; consequently, they do not contain any zeros of p_c/c. Then a zero of p_c/c is in a sense-preserving region if and only $|z| < 1$. Recall also that every zero of p_c, and consequently p_c/c, has order 1 or -1.

Now suppose $r_1 \leq 1$. Because the sum of the orders of zeros of p_c/c is 2 in a sense-preserving region, p_c/c has two zeros of order 1 inside C_{r_1}. Now suppose $r_1 > 1$. The sum of

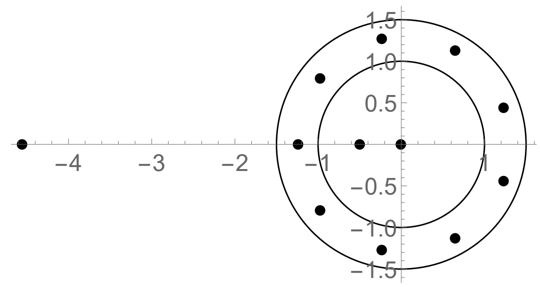


Figure 4.4: The zeros of the eighth degree polynomial $p_4(z)$ with 9 of the zeros inside the annulus defined by all z such that $1 < |z| < 1.5$.

the orders of zeros of p_c/c in C_{r_1} is still 2, and p_c/c still has two zeros of order 1 inside the unit circle. Moreover, any zeros outside the unit circle have order -1 which would cause the sum of the orders of the zeros of p_c/c inside C_{r_1} to be less than 2, a contradiction. Therefore, p_c/c has two zeros of order 1 inside the circle of radius r_1 for any $r_1 < \left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$ and $c \geq c_1$.

Applying Rouché's Theorem for Complex Harmonic Functions to Case 2, the sum of the orders of the zeros of f_∞ and the sum of the orders of the zeros of p_c are the same inside the circle of Radius r_2. By Lemma 4.8, f_∞ has one zero of order 2 and $n+1$ zeros of order -1 inside C_{r_2}. Hence, the sum of the orders of the zeros of f_∞ is $-n+1$; consequently the sum of the orders of the zeros of p_c/c is also $-n+1$. By the above work, p_c/c has two zeros of order 1 inside C_{r_1}. Extending the radius of this circle extends it into a sense-reversing region; consequently, p_c has $n+1$ zeros of order -1 in the annulus $A_\infty = \{z \in \mathbb{C} \colon r_1 < |z| < r_2\}$ for all c such that $c \geq c_\infty$ where $c_\infty = \max\{c_1, c_2\}$.

Now let c be a value such that $c \geq c_S$ where $c_S = \max\{c_1, c_2, c_3\}$. Then Cases 1, 2, and 3 give the sum of the orders of the zeros of f_∞ and the sum of the orders of p_c/c are equal in each sector $S_k = \{z \in \mathbb{C} \mid r_1 < |z| < r_2, \frac{\pi(1+4k)}{2(n-1)} < \arg_0(z) \frac{\pi(3+4k)}{2(n+1)}\}$. By Lemma 4.8, f_∞ has one zero of order -1 in each S_k; hence, the sum of the orders of the zeros of p_c/c in each S_k is -1. Because there are no zeros of positive order in A_∞, each S_k contains one zero of p_c/c.

Because p_c/c and p_c have the same zeros, our desired result(s) hold. $\square$

Proposition 4.9 states that for given r_1, r_2, there exists a range of c values such that $n+1$ zeros of p_c are located in an annulus and a zero of p_c is located in a sector of an annulus;

Figure 4.4 illustrates this for the $n = 8$ case. We ask what range of c values give these results. As before, the closer r_1 and r_2 are to $\left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$, the larger c must become in order to guarantee the zeros stay inside the annulus. Due to the decreasing nature of $\left(\frac{n-1}{2}\right)^{\frac{1}{n-3}}$, we do not provide a range of c values for Cases 1 and 2 of Proposition 4.9 (though we comment that for $n \geq 8$, $r_1 = 1$ and $r_2 = 3/2$ allow for all $c \geq 4$). However, the range of valid c values for Case 3 is simple to determine:

Lemma 4.10. *For $\frac{1}{2} \leq r \leq 2$ and $n \geq 4$,*

$$\frac{r^2 + \frac{1}{n^2} r^{2n}}{\frac{1}{4} r^4 + \left(\frac{1}{n-1}\right)^2 r^{2n-2}} < 20.$$

Proof. First, recall that

$$\frac{r^2 + \frac{1}{n^2} r^{2n}}{\frac{1}{4} r^4 + \left(\frac{1}{n-1}\right)^2 r^{2n-2}} = \frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2 + r^{2n-2}}{(n-1)^2 + 4r^{2n-6}}.$$

Then

$$\begin{aligned}
\frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \frac{n^2 + r^{2n-2}}{(n-1)^2 + 4r^{2n-6}} &= \frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \left[\frac{n^2}{(n-1)^2 + 4r^{2n-6}} + \frac{r^{2n-2}}{(n-1)^2 + 4r^{2n-6}}\right] \\
&< \frac{4}{r^2}\left(\frac{n-1}{n}\right)^2 \left[\frac{n^2}{(n-1)^2} + \frac{r^{2n-2}}{4r^{2n-6}}\right] \\
&= \frac{4}{r^2} + \left(\frac{n-1}{n}\right)^2 r^2 \\
&< \frac{4}{r^2} + r^2 \\
&\leq 20.
\end{aligned}$$

□

Therefore, Case 3 of Proposition 4.9 is always satisfied by $c \geq 2\sqrt{5} \approx 4.472136$.

4.2 Zeros on the Unit Circle

Extensive numerical experimentation leads us to conjecture that the only zeros on the critical curve are real. Because the critical curve always contains the unit circle, we first show:

Theorem 4.11. *For even $n \geq 8$, the complex harmonic function $p_c(z)$ has no zeros on the unit circle except possibly at the point -1.*

To prove this, we need several lemmas. First, we will show that the real part of any such zero must lie between -1 and $-\frac{\sqrt{(n-1)(n-3)}}{n-2}$. Equivalently, the angle θ of our zero must lie on or between $\pi - \sin^{-1}\left(\frac{1}{n-2}\right)$ and $\pi + \sin^{-1}\left(\frac{1}{n-2}\right)$. Second, we will show that the only valid angle in that interval is π.

Lemma 4.12. *If $z \in \mathbb{C}$ such that $p_c(z) = 0$, $\left|\omega(z)\right| = 1$, and $|z| = 1$ then $-1 \leq \operatorname{Re}(z) \leq -\frac{\sqrt{(n-1)(n-3)}}{n-2}$.*

Proof. Let z be such that $\left|\omega(z)\right| = 1$ and $p_c(z) = 0$. Then $\left|z^{n-2}\frac{c+z}{1+cz}\right| = 1$, so $\left|z^{n-2}\right| \cdot |c+z| = |1+cz|$. Then for some $\alpha \in \mathbb{C}$, $|\alpha| = 1$, $\bar{z}^{n-2}(c+z) = \alpha(1+cz)$; thus, $\bar{z}^{n-2} = \alpha\frac{1+cz}{c+z}$. Substituting into $p_c(z) = 0$ gives

$$\begin{aligned} 0 &= p_c(z) \\ &= z + \frac{c}{2}z^2 + \frac{c}{n-1}\bar{z}^{n-1} + \frac{1}{n}\bar{z}^n \\ &= z + \frac{c}{2}z^2 + \alpha\frac{c}{n-1} \cdot \frac{1+cz}{c+z}\bar{z} + \alpha\frac{1}{n} \cdot \frac{1+cz}{c+z}\bar{z}^2. \end{aligned}$$

Thus,

$$-z - \frac{c}{2}z^2 = \alpha\frac{1+cz}{c+z}\left(\frac{c}{n-1}\bar{z} + \frac{1}{n}\bar{z}^2\right).$$

Taking the squared modulus of both sides yields

$$\left(z + \frac{c}{2}z^2\right)\left(\bar{z} + \frac{c}{2}\bar{z}^2\right) = \frac{1+cz}{c+z} \cdot \frac{1+c\bar{z}}{c+\bar{z}}\left(\frac{c}{n-1}\bar{z} + \frac{1}{n}\bar{z}^2\right)\left(\frac{c}{n-1}z + \frac{1}{n}z^2\right).$$

which simplifies to

$$\left(z\bar{z} + \frac{c}{2}(z\bar{z}^2 + z^2\bar{z}) + \frac{c^2}{4}z^2\bar{z}^2\right)(c^2 + c(z+\bar{z}) + z\bar{z})$$
$$= (1 + c(z+\bar{z}) + c^2 z\bar{z})\left(\frac{c^2}{(n-1)^2}z\bar{z} + \frac{c}{n(n-1)}(z\bar{z}^2 + z^2\bar{z}) + \frac{1}{n^2}z^2\bar{z}^2\right).$$

If we let $z = re^{i\theta}$, then

$$\left(r^2 + cr^3\cos(\theta) + \frac{c^2}{4}r^4\right)(c^2 + 2cr\cos(\theta) + r^2)$$
$$= (1 + 2cr\cos(\theta) + c^2r^2)\left(\frac{c^2}{(n-1)^2}r^2 + 2\frac{c}{n(n-1)}r^3\cos(\theta) + \frac{1}{n^2}r^4\right).$$

Then in the case where $r = 1$,

$$\left(1 + c\cos(\theta) + \frac{c^2}{4}\right)(c^2 + 2c\cos(\theta) + 1)$$
$$= (1 + 2c\cos(\theta) + c^2)\left(\frac{c^2}{(n-1)^2} + 2\frac{c}{n(n-1)}\cos(\theta) + \frac{1}{n^2}\right).$$

If $c^2+2c\cos(\theta)+1 = 0$, then $\cos(\theta) = -\frac{c^2+1}{2c}$. By the Arithmetic-Geometric Mean Inequality, $2c \leq c^2 + 1$; hence, $\frac{c^2+1}{2c} \geq 1$ with equality if and only if $c = 1$. Hence, $\cos(\theta) = -\frac{c^2+1}{2c} \leq -1$ with equality if and only if $c = 1$, and hence $\theta = \pi$. Assuming $c \neq 1$ and $\theta \neq \pi$, we can divide both sides by $c^2 + 2c\cos(\theta) + 1$. This results in

$$1 + c\cos(\theta) + \frac{c^2}{4} = \frac{c^2}{(n-1)^2} + \frac{2c}{n(n-1)}\cos(\theta) + \frac{1}{n^2}.$$

Solving the above for $\cos(\theta)$ yields

$$\cos(\theta) = -\left(\frac{1}{c}\cdot\frac{1-\frac{1}{n^2}}{1-\frac{2}{n(n-1)}} + c\cdot\frac{\frac{1}{4}-\frac{1}{(n-1)^2}}{1-\frac{2}{n(n-1)}}\right),$$

which simplifies to

$$\cos(\theta) = -\left(\frac{1}{c}\cdot\frac{(n-1)^2}{n(n-2)} + c\cdot\frac{n(n-3)}{4(n-1)(n-2)}\right).$$

The AGM inequality then gives

$$\cos(\theta) \le -\frac{\sqrt{(n-1)(n-3)}}{n-2},$$

with equality if and only if $c = \pm\frac{2(n-1)}{n}\sqrt{\frac{n-1}{n-3}}$. Since we are only concerned about $c > 0$, we have $c = \frac{2(n-1)}{n}\sqrt{\frac{n-1}{n-3}}$. Thus we see that the real part of any zero on the unit circle lies between -1 and $-\frac{\sqrt{(n-1)(n-3)}}{n-2}$. □

Now that we have a restriction on the real part of any zero on the unit circle, when we view z in polar coordinates $e^{i\theta}$, we have a restriction on the value of θ. This gives us the following corollary:

Corollary 4.13. *Let $z = e^{i\theta}$ be a zero of p_c. Then $\pi - \sin^{-1}\left(\frac{1}{n-2}\right) \le \theta \le \pi + \sin^{-1}\left(\frac{1}{n-2}\right)$ for all $n > 2$.*

Proof. To find the interval of possible θ's that give zeros on the unit circle, we can solve the equation $\cos^2(\theta) + \sin^2(\theta) = 1$ with $\cos(\theta) = -\frac{\sqrt{(n-1)(n-3)}}{n-2}$. This gives us that

$$\theta = \pi \pm \sin^{-1}\left(\sqrt{1 - \frac{(n-1)(n-3)}{(n-2)^2}}\right) = \pi \pm \sin^{-1}\left(\frac{1}{n-2}\right). \tag{4.1}$$

Thus $\theta = \pi \pm \sin^{-1}\left(\frac{1}{n-2}\right)$; consequently, the only places where a zero of p_c could happen on the unit circle are for values of θ satisfying $\pi - \sin^{-1}\left(\frac{1}{n-2}\right) \le \theta \le \pi + \sin^{-1}\left(\frac{1}{n-2}\right)$. □

To handle the case where $\pi - \sin^{-1}\left(\frac{1}{n-2}\right) \le \theta \le \pi + \sin^{-1}\left(\frac{1}{n-2}\right)$, we will set the real and imaginary parts of $p_c(z)$ equal to zero and let $z = e^{i\theta}$. We obtain an equation that must be satisfied for any zero on the unit circle. This equation will be derived in Lemma 4.14. We will then show that the only θ that satisfies the equation is $\theta = \pi$.

Lemma 4.14. *Let $z \in \mathbb{C}$ and let $0 \leq \theta < 2\pi$ such that $z = e^{i\theta}$. If $p_c(z) = 0$, then*

$$\frac{1}{2n}\sin((n+2)\theta) + \left(\frac{1}{2} + \frac{1}{n(n-1)}\right)\sin(\theta) - \frac{1}{n-1}\sin(n\theta) = 0.$$

Proof. Suppose $p_c(z) = 0$. Then the real and imaginary parts of $p_c(z)$ also equal zero. Observe,

$$\mathrm{Re}(p_n(z)) = r\cos(\theta) + \frac{c}{2}r^2\cos(2\theta) + \frac{c}{n-1}r^{n-1}\cos((n-1)\theta) + \frac{1}{n}r^n\cos(n\theta),$$

and

$$\mathrm{Im}(p_n(z)) = r\sin(\theta) + \frac{c}{2}r^2\sin(2\theta) - \frac{c}{n-1}r^{n-1}\sin((n-1)\theta) - \frac{1}{n}r^n\sin(n\theta).$$

Solving these equations for c yields

$$c = \frac{\frac{1}{n}r^n\sin(n\theta) - r\sin(\theta)}{\frac{1}{2}r^2\sin(2\theta) - \frac{1}{n-1}r^{n-1}\sin((n-1)\theta)} = \frac{-\frac{1}{n}r^n\cos(n\theta) - r\cos(\theta)}{\frac{1}{2}r^2\cos(2\theta) + \frac{1}{n-1}r^{n-1}\cos((n-1)\theta)}.$$

Eliminating the denominators gives

$$\begin{aligned}&\left(\frac{1}{2}r^2\cos(2\theta) + \frac{1}{n-1}r^{n-1}\cos((n-1)\theta)\right)\left(\frac{1}{n}r^n\sin(n\theta) - r\sin(\theta)\right)\\ &= \left(-\frac{1}{n}r^n\cos(n\theta) - r\cos(\theta)\right)\left(\frac{1}{2}r^2\sin(2\theta) - \frac{1}{n-1}r^{n-1}\sin((n-1)\theta)\right).\end{aligned}$$

Simplifying the LHS gives

$$\begin{aligned}&\left(\frac{1}{2}r^2\cos(2\theta) + \frac{1}{n-1}r^{n-1}\cos((n-1)\theta)\right)\left(\frac{1}{n}r^n\sin(n\theta) - r\sin(\theta)\right)\\ &= \frac{1}{2n}r^{n+2}\cos(2\theta)\sin(n\theta) - \frac{1}{2}r^3\cos(2\theta)\sin(\theta) + \frac{1}{n(n-1)}r^{2n-1}\cos((n-1)\theta)\sin(n\theta)\\ &\quad - \frac{1}{n-1}r^n\cos((n-1)\theta)\sin(\theta).\end{aligned}$$

Similarly, for the RHS we have

$$\begin{aligned}&\left(-\frac{1}{n}r^n\cos(n\theta)-r\cos(\theta)\right)\left(\frac{1}{2}r^2\sin(2\theta)-\frac{1}{n-1}r^{n-1}\sin((n-1)\theta)\right)\\&=-\frac{1}{2}r^3\cos(\theta)\sin(2\theta)+\frac{1}{n-1}r^n\cos(\theta)\sin((n-1)\theta)-\frac{1}{2n}r^{n+2}\cos(n\theta)\sin(2\theta)\\&\quad+\frac{1}{n(n-1)}r^{2n-1}\cos(n\theta)\sin((n-1)\theta).\end{aligned}$$

Subtracting the RHS, we are left with

$$\begin{aligned}&\frac{1}{2n}r^{n+2}\big(\cos(2\theta)\sin(n\theta)+\cos(n\theta)\sin(2\theta)\big)+\frac{1}{2}r^3\big(-\cos(2\theta)\sin(\theta)+\cos(\theta)\sin(2\theta)\big)\\&+\frac{1}{n(n-1)}r^{2n-1}\big(\cos((n-1)\theta)\sin(n\theta)-\cos(n\theta)\sin((n-1)\theta)\big)\\&-\frac{1}{n-1}r^n\big(\cos((n-1)\theta)\sin(\theta)+\cos(n\theta)\sin((n-1)\theta)\big)=0,\end{aligned}$$

which simplifies to

$$s_{n,r}(\theta)=\frac{1}{2n}r^{n+2}\sin((n+2)\theta)+\left(\frac{1}{2}r^3+\frac{1}{n(n-1)}r^{2n-1}\right)\sin(\theta)-\frac{1}{n-1}r^n\sin(n\theta)=0.$$

Then on the unit circle $r=1$,

$$s_{n,1}(\theta)=\frac{1}{2n}\sin((n+2)\theta)+\left(\frac{1}{2}+\frac{1}{n(n-1)}\right)\sin(\theta)-\frac{1}{n-1}\sin(n\theta)=0,$$

must be satisfied. □

Thus to prove Theorem 4.11, it suffices to show that $s_{n,1}(\theta)$ is strictly decreasing on $\left(\pi-\sin^{-1}\left(\frac{1}{n-2}\right),\pi+\sin^{-1}\left(\frac{1}{n-2}\right)\right)$ and hence is only 0 at π. To prove this, we need the following lemma:

Lemma 4.15. *Let $0<a\le 8\pi$. If $x\ge 8$, then $I(x)=\frac{x}{a}\sin\left(\frac{a}{x}\right)$ is increasing.*

Proof. Let $I(x)=\frac{x}{a}\sin\left(\frac{a}{x}\right)=\int_0^1\cos\left(\frac{a}{x}t\right)\,dt$ where $a>0$, so $I'(x)=\frac{a}{x^2}\int_0^1 t\sin\left(\frac{a}{x}t\right)\,dt$. Because $\sin\left(\frac{a}{x}t\right)$ has period $\frac{2\pi x}{a}$, $\sin\left(\frac{a}{x}t\right)$ is positive on $\left(0,\frac{\pi x}{a}\right)$. If $\frac{\pi x}{a}\ge 1$, then $\sin\left(\frac{a}{x}t\right)$ will be positive on $(0,1)$. Notice that $\frac{\pi x}{a} \geq \frac{8\pi}{a}$ holds because $x \geq 8$. Then assume $\frac{8\pi}{a} \geq 1$, so $8\pi \geq a$. Then for all $0 < a < 8\pi$ we have $\frac{\pi x}{a} \geq \frac{8\pi}{a} \geq 1$. Therefore, $\sin\left(\frac{a}{x}t\right)$ will be positive on $(0,1)$ for all $x \geq 8$, $0 < a \leq 8\pi$. Since t is also positive on $(0,1)$, we have that $\int_0^1 t \sin\left(\frac{x}{a}t\right)\, dt$ will be positive. As $\frac{a}{x^2} > 0$ for $x \geq 8$, we have that $I'(x) > 0$ for $x \geq 8$. □

Utilizing the above lemma, we now prove that $s_{n,1}(\theta)$ has only $t = \pi$ as a zero on the interval $\left[\pi - \sin^{-1}\left(\frac{1}{n-2}\right), \pi + \sin^{-1}\left(\frac{1}{n-2}\right)\right]$.

Proposition 4.16. *The function $s_{n,1}(\theta)$ has only one zero, $\theta = \pi$, on the interval $\left[\pi - \sin^{-1}\left(\frac{1}{n-2}\right), \pi + \sin^{-1}\left(\frac{1}{n-2}\right)\right]$ for even $n \geq 8$.*

Proof. Let $\mathcal{S} = \sin^{-1}\left(\frac{1}{n-2}\right)$. We will shift $s_{n,1}(\theta)$ by π so that we can consider the interval $[-\mathcal{S}, \mathcal{S}]$. Since n is even,

$$\begin{aligned} s_{n,1}(\theta - \pi) &= \frac{1}{2n}\sin((n+2)(\theta-\pi)) + \left(\frac{1}{2} + \frac{1}{n(n-1)}\right)\sin(\theta - \pi) - \frac{1}{n-1}\sin(n(\theta-\pi)) \\ &= \frac{1}{2n}\sin\big((n+2)\theta\big) - \left(\frac{1}{2} + \frac{1}{n(n-1)}\right)\sin(\theta) - \frac{1}{n-1}\sin(n\theta). \end{aligned}$$

Thus,

$$s'_{n,1}(\theta - \pi) = \frac{n+2}{2n}\cos\big((n+2)\theta\big) - \left(\frac{1}{2} + \frac{1}{n(n-1)}\right)\cos(\theta) - \frac{n}{n-1}\cos(n\theta).$$

Notice that $s_{n,1}(\theta - \pi)$ is odd because $\sin(\theta)$ is odd, so we only need to consider the interval $[0, \mathcal{S}]$. We will show that the derivative $s'_{n,1}(\theta - \pi)$ is strictly negative on $[0, \mathcal{S}]$ by finding an upper bound for each summand.

The first summand is simple:

$$\frac{n+2}{2n}\cos\big((n+2)\theta\big) \leq \frac{n+2}{2n} = \frac{1}{2} + \frac{1}{n} \leq \frac{5}{8}.$$

For the second summand, notice that $-\cos(\theta)$ is increasing on $(0, \pi)$ and $\mathcal{S} \leq \frac{\pi}{2} < \pi$;

consequently, the maximum value of $-\left(\frac{1}{2}+\frac{1}{n(n-1)}\right)\cos(\theta)$ is

$$-\left(\frac{1}{2}+\frac{1}{n(n-1)}\right)\cos(\mathcal{S}) = -\left(\frac{1}{2}+\frac{1}{n(n-1)}\right)\frac{\sqrt{(n-3)(n-1)}}{n-2}.$$

Because $\left(\frac{\sqrt{(x-3)(x-1)}}{x-2}\right)' = \frac{1}{(x-2)^2\sqrt{x^2-4x+3}} > 0$ for $x \geq 8$, for $n \geq 8$ we have

$$-\left(\frac{1}{2}+\frac{1}{n(n-1)}\right)\frac{\sqrt{(n-3)(n-1)}}{n-2} \leq -\frac{1}{2}\frac{\sqrt{(n-3)(n-1)}}{n-2} \leq -\frac{1}{2}\frac{\sqrt{35}}{6} = -\frac{\sqrt{35}}{12}.$$

The last summand will take some work. First, notice that $-\frac{n}{n-1}\cos(n\theta)$ increases on $(0, \pi/n)$. If we can show that $\mathcal{S} \leq \frac{\pi}{n}$, then we know that $-\frac{n}{n-1}\cos(n\theta) \leq -\frac{n}{n-1}\cos(n\mathcal{S})$ on $[0, \mathcal{S}]$. This statement is equivalent to each of the following:

$$\begin{aligned} \sin^{-1}\left(\frac{1}{n-2}\right) &\leq \frac{\pi}{n} \\ \frac{1}{n-2} &\leq \sin\left(\frac{\pi}{n}\right) \\ 1 &\leq (n-2)\sin\left(\frac{\pi}{n}\right). \end{aligned}$$

Equivalently, we want to prove

$$1 \leq \pi \cdot \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \cdot \frac{n-2}{n}.$$

Because $0 < \pi \leq 8\pi$, $\frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}}$ is increasing by Lemma 4.15. We also have that $\frac{n-2}{n} = 1 - \frac{2}{n}$ is increasing. Then for $n \geq 8$,

$$\pi \cdot \frac{\sin\left(\frac{\pi}{n}\right)}{\frac{\pi}{n}} \cdot \frac{n-2}{n} \geq \pi \cdot \frac{\sin\left(\frac{\pi}{8}\right)}{\frac{\pi}{8}} \cdot \frac{8-2}{8} \geq 2.2961 > 1,$$

and our inequality holds. Therefore $\mathcal{S} \leq \frac{\pi}{n}$.

As desired we have that $-\frac{n}{n-1}\cos(n\theta) \leq -\frac{n}{n-1}\cos(n\mathcal{S})$ for $\theta \in [0, \mathcal{S}]$, and we now need to find an upper bound for $-\frac{n}{n-1}\cos(n\mathcal{S})$. We start by finding a bound for $(n\mathcal{S})$: For $x \geq 8$, we claim that $x\sin^{-1}\left(\frac{1}{x-2}\right) \leq 1.34$. This statement is equivalent to

$$\begin{aligned}
\sin^{-1}\left(\frac{1}{x-2}\right) &\leq \frac{1.34}{x} \\
\frac{1}{x-2} &\leq \sin\left(\frac{1.34}{x}\right) \\
1 &\leq (x-2)\sin\left(\frac{1.34}{x}\right).
\end{aligned}$$

Similar to the above, we equivalently want to prove

$$1 \leq 1.34 \cdot \frac{\sin\left(\frac{1.34}{x}\right)}{\frac{1.34}{x}} \cdot \frac{x-2}{x}.$$

Because $0 < 1.34 \leq 8\pi$, $\frac{\sin\left(\frac{1.34}{x}\right)}{\frac{1.34}{x}}$ is increasing by Lemma 4.15. Also, $\frac{x-2}{x} = 1 - \frac{2}{x}$ is increasing, so for $x \geq 8$,

$$1 < 1.000307\cdots \leq 1.34 \cdot \frac{\sin\left(\frac{1.34}{8}\right)}{\frac{1.34}{8}} \cdot \frac{6}{8} \leq 1.34 \cdot \frac{\sin\left(\frac{1.34}{x}\right)}{\frac{1.34}{x}} \cdot \frac{x-2}{x}.$$

Thus, $0 \leq n\sin^{-1}\left(\frac{1}{n-2}\right) \leq 1.34$ for $n \geq 8$. Then we have $\cos\left(n\sin^{-1}\left(\frac{1}{n-2}\right)\right) \geq \cos(1.34)$; hence, $-\cos\left(n\sin^{-1}\left(\frac{1}{n-2}\right)\right) \leq -\cos(1.34)$ so $-\frac{n}{n-1}\cos\left(n\sin^{-1}\left(\frac{1}{n-2}\right)\right) \leq -\frac{n}{n-1}\cos(1.34)$. Because $\frac{x}{x-1}$ decreases to 1,

$$-\frac{n}{n-1}\cos\left(n\sin^{-1}\left(\frac{1}{n-2}\right)\right) \leq -\frac{n}{n-1}\cos(1.34) \leq -\cos(1.34),$$

for $n \geq 8$.

Combining the results for each of the summands yields

$$s'_{n,1}(\theta-\pi) \leq \frac{5}{8} - \frac{\sqrt{35}}{12} - \cos(1.34) = -0.09675945\cdots < 0.$$

Therefore, $s_{n,1}(\theta-\pi)$ is strictly decreasing on $[0,\mathcal{S}]$. Because $s_{n,1}(\theta-\pi)$ is odd, we also know that $s_{n,1}(\theta-\pi)$ is strictly decreasing on $[-\mathcal{S},0]$. Moreover, since $s_{n,1}(0)=0$ we see that π is the only root of $s_{n,1}(\theta-\pi)$ in $[-\mathcal{S},\mathcal{S}]$. $\square$

We now prove Theorem 4.11:

Proof. (*Theorem 4.11.*) Let $n \geq 8$ be even. Let z be a zero of $p_c(z)$ on the unit circle; hence, $z = e^{i\theta}$ for $\theta \in [0, 2\pi]$. By Corollary 4.13, θ must lie between $\pi - \sin^{-1}\left(\frac{1}{n-2}\right)$ and $\pi + \sin^{-1}\left(\frac{1}{n-2}\right)$. By Proposition 4.16, the only θ in $\left[\pi - \sin^{-1}\left(\frac{1}{n-2}\right), \pi + \sin^{-1}\left(\frac{1}{n-2}\right)\right]$ that satisfies $p(e^{i\theta}) = 0$ is $\theta = \pi$. Therefore, the only zero of $p_c(z)$ on the unit circle is $z = e^{i\pi} = -1$. $\square$

Chapter 5. Directions for Future Research

(1) In this thesis, we showed there are no zeros of p_c on the unit circle except at $z = -1$. What can be said about the other portions of the critical curve?

(2) What can be proved about the total number of zeros of the family p_c?

(3) What can be proved about the number and location of zeros of other families of harmonic polynomials?

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