Sandwich Theorem and Calculation of the Theta Function for Several Graphs

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SANDWICH THEOREM AND CALCULATION OF THE
THETA FUNCTION FOR SEVERAL GRAPHS

by

Marcia Riddle

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics
Brigham Young University
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of a thesis submitted by

Marcia Riddle

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

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ABSTRACT

SANDWICH THEOREM AND CALCULATION OF THE THETA FUNCTION FOR SEVERAL GRAPHS

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Department of Mathematics
Master of Science

This paper includes some basic ideas about the computation of a function \( \vartheta(G) \), the theta number of a graph \( G \), which is known as the Lovász number of \( G \). \( \vartheta(G^c) \) lies between two hard-to-compute graph numbers \( \omega(G) \), the size of the largest clique in a graph \( G \), and \( \chi(G) \), the minimum number of colors need to properly color the vertices of \( G \). Lovász and Grötschel called this the “Sandwich Theorem”. Donald E. Knuth gives four additional definitions of \( \vartheta, \vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4 \) and proves that they are all equal.

First I am going to describe the proof of the equality of \( \vartheta, \vartheta_1 \) and \( \vartheta_2 \) and then I will show the calculation of the \( \vartheta \) function for some specific graphs: \( K_n \), graphs related to \( K_n \), and \( C_n \). This will help us understand the \( \vartheta \) function, an important function for graph theory. Some of the results are calculated in different ways. This will benefit students who have a basic knowledge of graph theory and want to learn more about the theta function.
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Terminology

Graph: Given a nonempty finite set $V$, let $V^{(2)}$ be the set of all 2-element subsets of $V$. A graph $G = (V, E)$ consists of two things, a nonempty finite set $V$ called the vertices of $G$ and a (possibly empty) subset $E$ of $V^{(2)}$ called the edges of $G$. Let $n = |V|$ be the number of vertices of $G$.

When more than one graph is under consideration, it may be useful to write $V(G)$ and $E(G)$ for the vertex and edge sets of $G$, respectively. We will use the notation of $ab$ instead of $\{a, b\}$ to represent an edge. For example, let $C_4$ be the graph with 4 vertices, $V = \{1, 2, 3, 4\}$ and $E = \{12, 13, 24, 34\}$, shown in Figure 1.

![Figure 1: Graph of $C_4$](image)

Subgraph: A subgraph of a graph $G$ is a graph $H$ such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$.

Induced subgraph: An induced subgraph of $G$ is a subgraph $H$ such that every edge of $G$ whose vertices are contained in $V(H)$ belongs to $E(H)$.

Complement $G^c$ of $G$: the graph which has $V(G)$ as its vertex set, and in which two vertices are adjacent if and only if they are not adjacent in $G$. If $G$ has $n$ vertices, then $G^c$ can be constructed by removing from $K_n$ all the edges of $G$. Note that the complement of a complete graph is a empty graph. An example of a graph and its complement is seen in Figure 2.

Positive definite (positive semidefinite) matrix: a real $n \times n$ symmetric matrix $A$ which satisfies $x^T A x > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$ $(x^T A x \geq 0$ for all $x \in \mathbb{R}^n$).

Negative definite (negative semidefinite) matrix: a real $n \times n$ symmetric matrix $A$
is negative definite if $-A$ is positive definite and is negative semidefinite if $-A$ is positive demidefinite.

*Clique*: a nonempty set of pairwise adjacent vertices. Some cliques in the graph $G$ in Figure 3 are $\{1, 2\}$, $\{3, 4, 5\}$, and $\{6, 7\}$.

*Clique number*: the maximum number of vertices of a clique in a graph. It is denoted by $\omega(G)$. The clique number of $G$ in Figure 3 is 3.

*Independent set*: a set of pairwise nonadjacent vertices of $G$. This is also called a *stable set*. Some independent sets of the graph $G$ in Figure 3 are: $V_1 = \{1, 3, 7\}$, $V_2 = \{2, 5, 6\}$, $V_3 = \{1, 4, 6\}$. So, $S \subset V(G)$ is an independent set if and only if no two of its vertices are adjacent, or if and only if $S$ is a clique in $G^c$.

*Independence number*, $\alpha(G) = \omega(G^c)$: the maximum number of vertices in an independent set of $G$.

*Chromatic number* $\chi(G)$: the minimum number of independent sets that partition $V(G)$. A graph $G$ is $m$-partite if $V(G)$ can be partitioned into $m$ or fewer independent sets. The independent sets in a specified partition are partite sets. The graph $G$ in Figure 3 has chromatic number 3 and is 3-partite.

*Clique cover number*: the smallest number of cliques that cover the vertices of $G$. 
It is denoted by $\chi(G)$. For example, $\chi(G) = 3$ for $G$ in Figure 3. We can also think of it as the minimum number of independent sets that partition $V(G^c)$. It is clear that $\overline{\chi}(G) = \chi(G^c)$.

*Perfect graph*: a graph $G$ such that each induced subgraph $G'$ of $G$ satisfies $\omega(G') = \chi(G')$. The graph $G$ in Figure 3 is perfect.

*Imperfect graph*: a graph that is not perfect. The graph $C_5$ (shown in Figure 4) is an imperfect graph because the clique number is $\omega(G) = 2$ but the chromatic number is $\chi(G) = 3$. It is the only imperfect graph for $n \leq 5$ vertices.

![Figure 4: $C_5$](image)

*Dot product*: The dot product of (column) vectors $a$ and $b$ is $a \cdot b = a^T b$.

*Orthogonal vectors*: the vectors $a, b \in \mathbb{R}^n$ are orthogonal if $a \cdot b = 0$.

*Orthogonal matrix*: a square matrix $Q$ such that $Q^T Q$ is the identity matrix $I$. In other words, $Q$ is orthogonal if and only if its columns are unit vectors perpendicular to each other.

*Symmetric matrix*: A matrix $A$ is symmetric if $A = A^T$. For any symmetric matrix $A$ we have that $A = QDQ^T$ where $Q$ is orthogonal and $D$ is a diagonal matrix.

*Orthogonal labeling*: an assignment of vectors $a_v \in \mathbb{R}^d$ to each vertex $v$ of a graph $G$ such that $a_u \cdot a_v = 0$ for each pair of nonadjacent vertices $u$ and $v$. In other words, whenever $a_u$ is not perpendicular to $a_v$ in the labeling, we will have $u$ and $v$ adjacent in the graph $G$.

*Convex set*: A set in Euclidean space $\mathbb{R}^d$ is a convex set if it contains all the line segments connecting any pair of its points.

*Convex hull*: Given a set of vectors $S \in \mathbb{R}^n$, the convex hull of $S$ is the smallest
convex set containing \( S \).

**STAB(\( G \)), TH(\( G \)) and QSTAB(\( G \))**

**Definition 1.** The incidence vector \( X^S \) of a set \( S \) of vertices of a graph \( G \) is defined by

\[
X^S_i = \begin{cases} 
1 & \text{if } i \in S \\
0 & \text{if } i \notin S
\end{cases}
\]

Typically, \( S \) will be a clique or independent set.

**Definition 2.** Given a graph \( G \), \( STAB(\( G \)) \) is the convex hull of incidence vectors of all stable sets of \( G \).

**Example 1.** Consider \( K_3 \) (a clique with 3 vertices): the independent sets are \( \emptyset \), \{1\}, \{2\}, \{3\} and the corresponding incidence vectors are

\[
\begin{bmatrix} 
0 \\
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix} 
1 \\
0 \\
0
\end{bmatrix}, \quad \begin{bmatrix} 
0 \\
1 \\
0
\end{bmatrix}, \quad \begin{bmatrix} 
0 \\
0 \\
1
\end{bmatrix}.
\]

![Figure 5: \( K_3 \) and STAB(\( K_3 \))]({})

**Definition 3.** \( QSTAB(\( G \)) \) is the set in \( \mathbb{R}^n \) defined by

\[
x_i \geq 0 \text{ for each } i \in V, \quad \sum_{i \in K} x_i \leq 1 \text{ for each clique } K \text{ of } G.
\]

It suffices to consider the maximal cliques of \( G \). In the example of \( K_3 \), these become \( x_1 \geq 0, x_2 \geq 0 \) and \( x_3 \geq 0, x_1 + x_2 + x_3 \leq 1 \). It is clear that the last inequality implies \( x_1 + x_2 \leq 1 \), corresponding to the nonmaximal clique \{1, 2\}. For \( K_3 \), \( STAB(\( K_3 \)) = QSTAB(\( K_3 \)) \). These can be seen in Figure 5.
**Definition 4.** The cost $c(u_i)$ of a vector $u_i$ in an orthogonal labeling of $G$ is defined to be 0 if $u_i = 0$, otherwise

$$c(u_i) = \frac{u_{1i}^2}{\|u_i\|^2} = \frac{u_{1i}^2}{u_{1i}^2 + \ldots + u_{di}^2}.$$  

Because $c(u)$ is a homogenous function, we may whenever convenient take all nonzero vectors in an orthogonal labeling to be unit vectors.

**Definition 5.** Given a graph $G = (V, E)$ on $n$ vertices,

$$\text{TH}(G) = \{x \geq 0 \mid \sum_{i \in V} c(u_i) x_i \leq 1 \text{ for all orthogonal labelings } u \text{ of } G\}.$$  

Notice that $\text{TH}(G)$ is the intersection of infinitely many halfspaces, so $\text{TH}(G)$ is a convex set.

**Lemma 1.** If $S$ is a stable set in $G$ and $u$ is an orthogonal labeling of $G$, then $\sum_{i \in S} c(u_i) \leq 1$.

**Proof.** Let $S = \{i_1, i_2, \ldots, i_k\}$. It suffices to consider the case when $\{u_{i_1}, u_{i_2}, \ldots, u_{i_k}\}$ is a set of $k$ orthonormal vectors. There exists a $d \times d$ orthogonal matrix $Q$ such that the first $k$ columns are $u_{i_1}, u_{i_2}, \ldots, u_{i_k}$. Since the length of the first row of $Q$ is 1,

$$1 \geq u_{1i_1}^2 + u_{1i_2}^2 + \ldots + u_{1i_k}^2 = c(u_{i_1}) + c(u_{i_2}) + \ldots + c(u_{i_k}) = \sum_{i \in S} c(u_i).$$

Hence, $\sum_{i \in S} c(u_i) \leq 1$. \hfill \qed

**Lemma 2.** $\text{STAB}(G) \subseteq \text{TH}(G) \subseteq \text{QSTAB}(G)$.

**Proof.** If $S$ is a stable set, $X^S$ is its incidence vector and $\{u_1, u_2, \ldots, u_n\}$ is an orthogonal labeling of $G$, by the previous lemma,

$$\sum_{i \in V} c(u_i) \cdot X_i^S = \sum_{i \in S} c(u_i) \cdot 1 + \sum_{i \notin S} c(u_i) \cdot 0 \leq 1.$$  

Since $\text{TH}(G)$ contains each incidence vector, $\text{TH}(G)$ contains the convex hull of incidence vectors of stable sets. Therefore, $\text{TH}(G) \supseteq \text{STAB}(G)$. 

5
For each clique $K$ of $G$, define the map from $V$ to $\mathbb{R}^1$ by $i \rightarrow X^K_i$. If $ij \notin E$, either $i \notin K$ or $j \notin K$. So we know that $X^K_i X^K_j = 0$. So this map is an orthogonal labeling. Therefore, if $x \in TH(G),

1 \geq \sum_{i \in V} c(X^K_i) x_i = \sum_{i \in V} X^K_i \cdot x_i = \sum_{i \in K} 1 \cdot x_i + \sum_{i \notin K} 0 \cdot x_i = \sum_{i \in K} x_i,

so $x \in QSTAB(G).$ Hence $TH(G) \subseteq QSTAB(G).$ \hfill $\Box$

1 The Theta Function

**Definition 6.** Given a graph $G$, we define

$$\vartheta(G) = \max \{ \sum_{i \in V} x_i \mid x \in TH(G) \}.$$ 

Similar definitions can be given for $STAB$ and $QSTAB$:

$$\alpha(G) = \max \{ \sum_{i \in V} x_i \mid x \in STAB(G) \},$$

$$\kappa(G) = \max \{ \sum_{i \in V} x_i \mid x \in QSTAB(G) \}.$$

First, we need to show that this definition of $\alpha(G)$ is in fact the size of the largest stable set in $G$.

Let $G$ be a graph with $\alpha(G) = m$. Let $S$ be an independent set of size $m$ and let $X^S = [x_1, x_2, \ldots, x_n]$ be the incidence vector of $S$. Then,

$$\sum_{i=1}^{n} x_i = m.$$

So,

$$\alpha(G) = m \leq \max \{ \sum_{i \in V} x_i \mid x \in STAB(G) \}.$$ 

For example consider the graph $G'$ shown in Figure 6. For this graph we have, $\alpha(G') = 2$. The incidence vector of the stable set $\{1, 4\}$ is $x = [1, 0, 0, 1]^T \in STAB(G')$, so we have $\sum_{i=1}^{4} x_i = 2$. It is clear that $2 \leq \max \{ \sum_{i \in V} x_i \mid x \in STAB(G') \}$. So

$$\alpha(G') \leq \max \{ \sum_{i \in V} x_i \mid x \in STAB(G') \}.$$
We now prove the reverse inequality.

We know that $\text{STAB}(G)$ is convex set. Let $\{X^{S_1}, X^{S_2}, \ldots, X^{S_m}\}$ be the set of incidence vectors of the stable sets of $G$. Then any vector $x \in \text{STAB}(G)$ is a convex combination of $X^{S_1}, X^{S_2}, \ldots, X^{S_m}$, so

$$x = \sum_{i=1}^{m} a_i X^{S_i}, \text{ where } \sum_{i=1}^{m} a_i = 1 \text{ and } a_i \geq 0 \text{ for } i = 1, 2, \ldots, m.$$ 

Since $\alpha(G)$ is the maximum number of vertices in an independent set of $G$, for each vector $X^{S_i}, \alpha(G) \geq \sum_{j=1}^{n} X^{S_i}_j$, so we have,

$$\sum_{j=1}^{n} x_j = \sum_{j=1}^{n} (\sum_{i=1}^{m} a_i X^{S_i}_j) = \sum_{i=1}^{m} a_i \sum_{j=1}^{n} X^{S_i}_j \leq \sum_{i=1}^{m} a_i \cdot \alpha(G) = \alpha(G) \cdot \sum_{i=1}^{m} a_i = \alpha(G),$$

so

$$\alpha(G) \geq \max\{\sum_{i \in V} x_i | x \in \text{STAB}(G)\}.$$ 

Therefore,

$$\alpha(G) = \max\{\sum_{i \in V} x_i | x \in \text{STAB}(G)\}.$$ 

This shows that our two different definitions of $\alpha(G)$ are consistent.

We want to prove the sandwich theorem that we have mentioned at the beginning of the paper which is that for any graph $G$, $\vartheta(G^c)$ lies between the clique number $\omega(G)$ and the chromatic number $\chi(G)$.

**Theorem 1 (Sandwich Theorem).** $\omega(G) \leq \vartheta(G^c) \leq \chi(G).$

**Proof.** Since $\alpha(G^c) = \omega(G)$ and $\bar{\chi}(G^c) = \chi(G)$, this is equivalent to proving that $\alpha(G^c) \leq \vartheta(G^c) \leq \bar{\chi}(G^c)$ or $\alpha(G) \leq \vartheta(G) \leq \bar{\chi}(G)$.
Since \( \vartheta(G) = \max\{\sum_{i \in V} x_i \mid x \in \text{TH}(G)\} \), \( \alpha(G) = \max\{\sum_{x \in V} x_i \mid x \in \text{STAB}(G)\} \), and \( \text{STAB}(G) \subseteq \text{TH}(G) \), we have \( \alpha(G) \leq \vartheta(G) \).

Similarly, since \( \text{TH}(G) \subseteq \text{QSTAB}(G) \), we have \( \vartheta(G) \leq \kappa(G) \).

Now, we need to show that \( \kappa(G) \leq \chi(G) \).

Suppose \( K_1, \cdots, K_p \) is a smallest set of cliques that cover the vertices of \( G \), and let \( x \in \text{QSTAB}(G) \). Then

\[
\sum_{i=1}^{n} x_i = \sum_{j=1}^{p} \sum_{i \in K_j} x_i \leq \sum_{j=1}^{p} 1 = p = \chi(G).
\]

Therefore,

\[
\kappa(G) = \max\{\sum_{i \in V} x_i \mid x \in \text{QSTAB}(G)\} \leq \chi(G).
\]

Combining, we have \( \alpha(G) \leq \vartheta(G) \leq \kappa(G) \leq \chi(G) \), which completes the proof.

For example, consider the cyclic graph \( C_5 \) in Figure 4 with vertices \( \{1, 2, 3, 4, 5\} \). For \( x \in \text{QSTAB}(G) \), \( x_1 + x_2 \leq 1, x_2 + x_3 \leq 1, x_3 + x_4 \leq 1, x_4 + x_5 \leq 1, x_5 + x_1 \leq 1 \). Then \( 2(x_1 + x_2 + x_3 + x_4 + x_5) \leq 5 \). Hence \( \kappa(C_5) \leq \frac{5}{2} \). Since \( (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) \in \text{QSTAB}(C_5) \), \( \kappa(C_5) = \frac{5}{2} \). But \( \chi(C_5) = 3 > \kappa(C_5) \).

**Corollary 1.** If \( G \) is a perfect graph, then \( \vartheta(G) = \alpha(G) \).

**Proof.** By the Perfect Graph Theorem, \( G^c \) is a perfect graph. Therefore, we have \( \omega(G^c) = \chi(G^c) \).

By the Sandwich Theorem, \( \omega(G^c) \leq \vartheta(G) \leq \chi(G^c) \), so \( \vartheta(G) = \omega(G^c) \). Since \( \omega(G^c) = \omega(G^c) \), therefore \( \vartheta(G) = \alpha(G) \).

### 2 Two Additional Functions \( \vartheta_1(G) \) and \( \vartheta_2(G) \)

We have just shown that \( \vartheta(G) \) lies between \( \chi(G) \) and \( \alpha(G) \). In this part, we are going to introduce the other two functions \( \vartheta_1(G) \) and \( \vartheta_2(G) \). They are different ways of defining \( \vartheta(G) \). This will not only help us to understand \( \vartheta(G) \) but also help us to compute \( \vartheta(G) \). We will prove that \( \vartheta(G) \leq \vartheta_1(G) \leq \vartheta_2(G) \). In [3], Donald Knuth introduced two additional functions \( \vartheta_3, \vartheta_4 \) in order to prove \( \vartheta_2 \leq \vartheta \). Therefore \( \vartheta, \vartheta_1, \vartheta_2 \) are all equivalent.
2.1 The $\vartheta_1(G)$ function

If $G$ is a graph,

$$\vartheta_1(G) = \min_{a} \left( \max_{v} \left( \frac{1}{c(a_v)} \right) \right),$$

over all orthogonal labelings $a$.

The max is $\infty$ if there is some $v$ such that $c(a_v) = 0$.

Lemma 3. $\vartheta(G) \leq \vartheta_1(G)$.

Proof. If $G$ is a graph, suppose $x \in \text{TH}(G)$ and $a$ is an orthogonal labeling of $G$. Then

$$\vartheta(G) = \max \left\{ \sum_v x_v \mid x \in \text{TH}(G) \right\}$$

$$= \max \left\{ \sum_v \frac{1}{c(a_v)} \cdot c(a_v) \cdot x_v \mid x \in \text{TH}(G) \right\}$$

$$\leq \max_v \frac{1}{c(a_v)} \max \left\{ \sum_v c(a_v) \cdot x_v \mid x \in \text{TH}(G) \right\}.$$ 

By the definition of $\text{TH}(G)$, $\sum_v c(a_v)x_v \leq 1$. Therefore,

$$\vartheta(G) \leq \max_v \frac{1}{c(a_v)} \cdot 1.$$ 

Since this inequality holds for each orthogonal labeling,

$$\vartheta(G) \leq \min_{a} \left( \max_{v} \frac{1}{c(a_v)} \right) = \vartheta_1(G).$$

\[ \Box \]

2.2 The $\vartheta_2(G)$ function

Definition 7. Matrix $A$ is a feasible matrix for the graph $G$ if $A$ is indexed by the vertices of $G$ and

(i) $A$ is real and symmetric,

(ii) $a_{ii} = 1$, for all $i \in V$,

(iii) $a_{ij} = 1$, whenever $i$ and $j$ are nonadjacent in $G$. 

The rank 1 matrix

\[ A_0 = [1, 1, \cdots, 1]^T[1, 1, \cdots, 1] \]

is feasible for any graph \( G \) and has one nonzero eigenvalue

\[ \lambda_1(A_0) = \text{tr}(A) = \sum_{i=1}^{n} 1 = n. \]

If \( A \) is any real symmetric matrix which has eigenvalues \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \), by the Spectral Theorem, \( A = QDQ^T \) for some orthogonal matrix \( Q \) and diagonal matrix \( D \) where \( D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) \). Since all the eigenvalues of \( A \) are real, by the Rayleigh-Ritz Theorem, \( A \) has the maximum eigenvalue,

\[ \lambda_1(A) = \max \{ x^T Ax \mid \|x\|_2 = 1, \ x \in \mathbb{R}^n \} \]

**Lemma 4.** The set of feasible \( A \) with \( \lambda_1(A) \leq n \) is compact.

**Proof.** Starting with

\[ \lambda_1(A) + \lambda_2(A) + \cdots + \lambda_n(A) = n, \]

we use what we know about the ordering to get

\[ (n - 1)\lambda_1(A) + \lambda_n(A) \geq n \quad \text{so} \quad (n - 1)n + \lambda_n(A) \geq n. \]

In particular, we have that

\[ \lambda_n(A) \geq -(n - 2)n \quad \text{so that} \quad -(n - 2)n \leq \lambda_i(A) \leq n \quad \text{for} \quad i = 1, \cdots, n. \]

This shows that \( \lambda_i^2(A) \leq n^2(n - 2)^2 \) and so

\[ \sum_{i,j=1}^{n} |a_{ij}|^2 = \sum_{i=1}^{n} \lambda_i^2(A) \leq n^3(n - 2)^2. \]

Therefore, the set is bounded. The set is clearly closed, so it is compact.

From Lemma 4 and the fact that \( \lambda_1(A) \) is a continuous function of the entries of \( A \), we know that the minimum value of \( \lambda_1(A) \) exists.
Definition 8. Given a graph $G$, then

$$\vartheta_2(G) = \min \{ \lambda_1(A) \mid A \text{ is a feasible matrix for } G \}.$$  

Lemma 5. $\vartheta_1(G) \leq \vartheta_2(G)$.

Proof. Let $A$ be a feasible matrix such that $\vartheta_2(G) = \lambda_1 \geq 0$ and let $B = \lambda_1 I - A$. Let $\lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_n$ be the remaining eigenvalues of $A$.

The eigenvalues of $B$ are $\lambda_1 - \lambda_i$. It is clear that all the eigenvalues of $B$ are nonnegative, so $B$ is positive semidefinite.

If 

$$A = Q \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} Q^T$$

then

$$B = Q \begin{bmatrix} \lambda_1 - \lambda_1 & & & \\ & \lambda_1 - \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_1 - \lambda_n \end{bmatrix} Q^T = Q \begin{bmatrix} (\sqrt{\lambda_1 - \lambda_1})^2 & & & \\ & (\sqrt{\lambda_1 - \lambda_2})^2 & & \\ & & \ddots & \\ & & & (\sqrt{\lambda_1 - \lambda_n})^2 \end{bmatrix} Q^T = X^T X,$$

where

$$X = Q \begin{bmatrix} \sqrt{\lambda_1 - \lambda_1} & & & \\ & \sqrt{\lambda_1 - \lambda_2} & & \\ & & \ddots & \\ & & & \sqrt{\lambda_1 - \lambda_n} \end{bmatrix} Q^T.$$

Let $X = [x_1, x_2, \cdots, x_n]$ (i.e., the $x_i$ are the columns of $X$), so that we have $b_{ij} = x_i^T x_j$.

Let $u_i = \begin{bmatrix} 1 \\ x_i \end{bmatrix} \in \mathbb{R}^{n+1}$, $x_i \in \mathbb{R}^n$. 

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If $i \neq j$ and $ij$ is not in $E$, 

$$u_i \cdot u_j = 1 + x_i^T x_j = 1 + b_{ij} = a_{ij} + b_{ij}.$$ 

Since $B = \lambda_1 I - A$, we have $b_{ij} = -a_{ij}$, so $u_i \cdot u_j = a_{ij} - a_{ij} = 0$.

Therefore, 

$$u_1, u_2, \cdots, u_n$$ 

is an orthogonal labeling of $G$.

Note that appealing to the definition of cost that 

$$c(u_i) = \frac{1}{1 + \|x_i\|^2}.$$ 

So for each $i = 1, 2, \cdots, n$, 

$$\frac{1}{c(u_i)} = 1 + \|x_i\|^2 = 1 + x_i \cdot x_i = 1 + b_{ii} = a_{ii} + \lambda_1 - a_{ii} = \lambda_1.$$ 

This implies 

$$\lambda_1 = \max_{i \in V} \frac{1}{c(u_i)}.$$ 

Hence, 

$$\vartheta_1(G) = \min_{a} \max_{i} \left( \frac{1}{c(a_i)} \right) \leq \max_{i} \frac{1}{c(u_i)} = \lambda_1 = \vartheta_2(G).$$ 

This completes the proof. 

In [3], Donald E. Knuth has proved that \( \vartheta_2(G) \leq \vartheta(G) \) by proving \( \vartheta_2(G) \leq \vartheta_3(G) \leq \vartheta_4(G) \leq \vartheta(G) \). So, we can conclude that \( \vartheta(G) = \vartheta_2(G) \).

3 \quad \vartheta_2(G) \text{ and } K_n

Now we are going to use \( \vartheta_2(G) \) to calculate \( \vartheta(G) \) for the two simplest graphs, \( K_n \) and \( K_n^c \).

3.1 \quad \vartheta_2(K_n) \text{ and } \vartheta_2(K_n^c)

Lemma 6. \( \vartheta(K_n) = \vartheta_2(K_n) = 1. \)
Consider the graph $K_n$. We know that $K_n$ is a perfect graph. By the Sandwich Theorem, $\vartheta(K_n) = \alpha(K_n) = 1$.

We also obtain this conclusion in two different ways.

There are no missing edges in the graph $K_n$, so any set of $n$ vectors is an orthogonal labeling. Let $u_i = [\sqrt{a}, \sqrt{b}]^T$ for $i = 1, 2, \ldots, n$, then $c(u_i) = a/(a + b)$. Now consider the following,

$$\sum c(u_i) \cdot x_i = \sum \frac{a}{a + b} \cdot x_i \leq 1,$$

rearranging we get $\sum x_i \leq 1 + \frac{b}{a}$.

From this last statement, it is clear that if we let $b = 0$ then $\sum x_i \leq 1$. It follows that

$$\text{TH}(K_n) = \{ x \geq 0 | \sum x_i \leq 1 \}.$$

Therefore,

$$\vartheta(K_n) = \max \{ \sum x_i | x \in \text{TH}(K_n) \} = \max \{ \sum x_i | \sum x_i \leq 1, \forall x \geq 0 \} = 1.$$

We should get the same result if we look at $\vartheta_2(K_n)$.

Let $A_n$ be the feasible matrix of $K_n$, and let $\lambda_1$ be the maximum eigenvalue of $A_n$, then the matrix $A_n$ has the form

$$A_n = \begin{bmatrix}
1 & x_{12} & \cdots & x_{1(n-1)} & x_{1n} \\
x_{12} & 1 & \cdots & x_{2(n-1)} & x_{2n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
x_{1(n-1)} & x_{2(n-1)} & \cdots & 1 & x_{(n-1)n} \\
x_{1n} & x_{2n} & \cdots & x_{(n-1)n} & 1
\end{bmatrix}.$$ 

Since for any graph $G$

$$n\lambda_1 = \lambda_1 + \lambda_1 + \cdots + \lambda_1 \geq \lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{tr} A_n = n,$$

we have that $\lambda_1 \geq 1$. On the other hand, $A_n$ becomes $I$ when $x_{ij} = 0$ for any $i \neq j$, and in that case $\lambda_1 = 1$.

So $\vartheta_2(K_n) = \min \lambda_1 = 1$.

**Lemma 7.** $\vartheta_2(K_n^c) = n$. 

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The graph of $K^c_n$ is very simple. It just contains $n$ vertices without any edges, which means there are no pairwise adjacent vertices. Since $K^c_n$ is a perfect graph, by the Sandwich Theorem, $\vartheta(K^c_n) = \alpha(K^c_n) = n$.

We can also see this result very easily from the definition of the $\vartheta_2$ function. Since the feasible matrix $A_n$ of $K^c_n$ is a $n \times n$ matrix with all the entries equal to 1, the rank of $A_n$ is 1. Therefore, there is only one nonzero eigenvalue and it equals $\text{tr}(A_n)$ which is $n$. Hence $\vartheta(K^c_n) = \vartheta_2(K^c_n) = n$.

### 3.2 $\vartheta_2(G)$ for complete multipartite graphs

**Definition 9.** The union of graphs $G$ and $H$ with $V(G) \cap V(H) = \emptyset$, written $G \cup H$, has vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Similarly, the join of graphs $G$ and $H$ with $V(G) \cap V(H) = \emptyset$, written $G \vee H$, is obtained from $G \cup H$ by adding the edges $\{xy : x \in V(G), y \in V(H)\}$.

Let $G = K_{n_1,n_2,\ldots,n_k}$ be a complete multipartite graph, then

$$K_{n_1,n_2,\ldots,n_k} = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k})^c$$

$$= K^c_{n_1} \vee K^c_{n_2} \vee \cdots \vee K^c_{n_k}.$$  

For example, in Figure 7 we have the union and the join of the two graphs $P_3$ and $K_3$. The bold edge in the join indicates that we join all vertices between these two sets.

![Figure 7:](image)

**Lemma 8.** $\vartheta(K_{n_1,n_2,\ldots,n_k}) = \max\{n_1, n_2, \ldots, n_k\}$.

We will prove this lemma in two different ways.
Proof. For our first proof, consider the following.

First, let $G = K_{n_1, n_2, \ldots, n_k}$. We show that $G$ is a perfect graph. This follows since,

$$\omega(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}) = \max\{n_1, n_2, \ldots, n_k\} = \chi(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}),$$

with a similar equality for each induced subgraph. So $K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}$ is a perfect graph. By the Perfect Graph Theorem, a graph is perfect if and only if its complement is perfect. Therefore, $K_{n_1, n_2, \ldots, n_k} = (K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k})^c$ is perfect. So by Corollary 1,

$$\vartheta(K_{n_1, n_2, \ldots, n_k}) = \alpha(K_{n_1, n_2, \ldots, n_k}) = \omega(K_{n_1} \cup K_{n_2} \cup \cdots \cup K_{n_k}) = \max\{n_1, n_2, \ldots, n_k\}.$$

Before we use the second method of proving the lemma, we need to state a useful result.

Theorem 2 (Interlacing Inequalities). Let $A \in M_n$ be Hermitian and for any $i \in N$, let $A(i)$ be the principal submatrix obtained from $A$ by deleting the $i^{th}$ row and column. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of $A$ and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1}$ be the eigenvalues of $A(i)$, then

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n.$$

Corollary 2. Let $A \in M_n$ be a Hermitian matrix and $B \in M_m$ be any principal submatrix of $A$, then $\lambda_k(A) \geq \lambda_k(B), \ k = 1, 2, \cdots, m$.

Now we are ready prove Lemma 8 by using the definition of the $\vartheta_2$ function. First, consider an example, namely let $G = K_{3,2,1} = (K_3 \cup K_2 \cup K_1)^c$. This graph is shown in Figure 8.

Let $A$ be a feasible matrix of $K_{3,2,1}$. Then $A$ has the form

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$
Figure 8:

where the remaining entries are any real numbers for which $A$ is symmetric. There are three blocks in $A$, they are $A_3$, $A_2$, $A_1$, the feasible matrices for $K_3^c$, $K_2^c$ and $K_1^c$. The eigenvalues for these matrices are:

$$
\lambda(A_3) = 3, 0, 0; \quad \lambda(A_2) = 2, 0; \quad \lambda(A_1) = 1.
$$

Let $\lambda_1$ be the maximum eigenvalue of $A$. Then by Corollary 2,

$$
\vartheta_2(K_{3,2,1}) = \lambda_1 \geq \max\{\lambda_1(A_3), \lambda_1(A_2), \lambda_1(A_1)\} = 3.
$$

However if we set all remaining entries in $A$ equal to 0, $A = A_3 \oplus A_2 \oplus A_1$ which has eigenvalues $3, 0, 0, 2, 0, 1$ and so $\lambda_1 = 3$. Therefore $\vartheta_2(K_{3,2,1}) = 3$.

We are now ready to generalize this for the proof of Lemma 8.

**Proof.** For our second proof, consider the following.

Consider the feasible matrix $A_n$ of the graph $G = K_{n_1, n_2, \ldots, n_k}$. Then

$$
A_n = \begin{bmatrix}
J_{n_1} & \cdots & \\
\vdots & \ddots & \\
& & J_{n_k}
\end{bmatrix},
$$

where all the off diagonal entries are some real numbers which preserve symmetry and $J_{n_i}$ are the feasible matrices for $K^n_{n_i}$, so all the entries equal 1. Let $\lambda_1$ be the maximum eigenvalue. By Corollary 2, $\lambda_1(A_n) \geq \lambda_1(J_{n_i})$ for $i = 1, 2, \ldots, k$. Since the eigenvalues of the matrix $J_{n_i}$ are $n_i$ and 0, $\lambda_1(J_{n_i}) = n_i$. Therefore $\lambda_1(K_{n_1, n_2, \ldots, n_k}) \geq \max\{n_1, n_2, \ldots, n_k\}$. Equality holds when all the off diagonal entries of $A_n$ equal 0 which reaches the minimum value of $\lambda_1$. Hence $\vartheta_2(K_{n_1, n_2, \ldots, n_k}) = \max\{n_1, n_2, \ldots, n_k\}$. \qed
3.3 $\vartheta_2(G)$ for the union of two complete graphs

We have proved earlier that $\vartheta(K_n) = 1$ (Lemma 6). Now we want to show the calculation of the $\vartheta$ function for the union of two complete graphs.

**Lemma 9.** $\vartheta(K_n \cup K_m) = 2$.

**Proof.** Let $G = K_m \cup K_n$, and $A$, $A_m$ and $A_n$ be feasible matrices of $G$, $K_m$, and $K_n$ respectively. Let $\lambda_1(A)$, $\lambda_1(A_m)$, $\lambda_1(A_n)$ represent the maximum eigenvalues of each matrix. Then $A_m$ is a $m \times m$ symmetric matrix with all the diagonal entries equal 1 and all the off diagonal entries are real numbers. Similarly, $A_n$ is a $n \times n$ symmetric matrix with all the diagonal entries equal 1 and all the off diagonal entries are real numbers. So $A$ has the form

$$A = \begin{bmatrix} A_n & 1 \\ 1 & A_m \end{bmatrix}.$$  

Now suppose each off diagonal entry in $A_n$ and $A_m$ equals $x$ and define

$$B = A + (x - 1)I.$$  

Then we will obtain an $(m + n) \times (m + n)$ symmetric matrix

$$\begin{bmatrix} x & 1 \\ 1 & x \end{bmatrix}. $$

Since the rank of $B$ is at most two, we have at most two nonzero eigenvalues. So, at least $(m + n - 2)$ eigenvalues are zero. The characteristic polynomial for $B$ is

$$t^{m+n} - (m + n)xt^{m+n-1} + mn(x^2 - 1)t^{m+n-2}.$$  

Using the quadratic formula to solve for the possible nonzero eigenvalues we find they are

$$t = \frac{(m + n)x \pm \sqrt{(m - n)^2x^2 + 4mn}}{2}.$$  

Hence the eigenvalues of $B$ are:

$$\frac{(m + n)x \pm \sqrt{(m - n)^2x^2 + 4mn}}{2},$$  

and 0 with multiplicity $m + n - 2$.  

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It follows that the eigenvalues of $A$ are $\lambda(B) - (x - 1)$:

$$
\frac{(m + n)x \pm \sqrt{(m - n)^2x^2 + 4mn}}{2} - (x - 1), \quad \text{and} \quad -(x - 1) \text{ with multiplicity } m + n - 2.
$$

When $x = -1$ the eigenvalues are $0$, $-(m+n)$ and $2$. We have the largest eigenvalue $\lambda_1 = 2$. This shows $\vartheta_2(K_n \cup K_m) \leq 2$. By the Sandwich Theorem, $\vartheta(K_n \cup K_m) \geq \alpha(K_n \cup K_m) = 2$, therefore, $\vartheta(K_n \cup K_m) = 2$. 

\section{4 $\vartheta_2(G)$ and $C_n$}

In this part, we are going to utilize the $\vartheta_2(G)$ function to find $\vartheta(G)$ for the $n$-cycle, $C_n$, which is an imperfect graph for each odd integer greater than three.

First we will illustrate how to use $\vartheta_2$ to calculate $\vartheta(C_5)$.

We first need to prove one important fact. Given a symmetric matrix $A$, let $\lambda_1(A)$ denote the maximum eigenvalue of $A$.

\textbf{Lemma 10.} Let $A_i, i = 1, 2, \cdots, k$ be symmetric matrices. Then

$$
\lambda_1 \left( \sum_{i=1}^{k} A_i \right) \leq \sum_{i=1}^{k} \lambda_1(A_i).
$$

\textbf{Proof.} Let $A$ and $B$ be symmetric matrices.

By the Rayleigh-Ritz theorem

$$
\lambda_1(A) = \max_{x \neq 0} \frac{x^T A x}{x^T x}, \quad \lambda_1(B) = \max_{x \neq 0} \frac{x^T B x}{x^T x}.
$$

Since

$$
\frac{x^T (A + B) x}{x^T x} = \frac{x^T A x}{x^T x} + \frac{x^T B x}{x^T x} \leq \max_{x \neq 0} \frac{x^T A x}{x^T x} + \max_{x \neq 0} \frac{x^T B x}{x^T x} = \lambda_1(A) + \lambda_1(B),
$$

we have

$$
\lambda_1(A + B) = \max_{x \neq 0} \frac{x^T (A + B) x}{x^T x} \leq \lambda_1(A) + \lambda_1(B).
$$

The result follows by mathematical induction. \hfill \Box
Notice that the Lemma is false if $A$ is not a symmetric matrix. For example, if $A = \begin{bmatrix} 0 & \frac{1}{m} \\ m & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & m \\ \frac{1}{m} & 0 \end{bmatrix}$, then the eigenvalues of $A$ and $B$ are $1$ and $-1$. Then $\lambda_1(A) + \lambda_1(B) = 2$, but $A + B = \begin{bmatrix} 0 & m + \frac{1}{m} \\ m + \frac{1}{m} & 0 \end{bmatrix}$, and the eigenvalues of $A + B$ are $m + \frac{1}{m}, -(m + \frac{1}{m})$. It is clear that $m + \frac{1}{m} > 2$ for any $m > 0$ but $m \neq 1$.

Now we compute $\vartheta_2(C_5)$. Consider the graph $G = C_5$ with $V = \{1, 2, 3, 4, 5\}$, $E = \{x_1, x_2, x_3, x_4, x_5\}$. The relationship of the vertices with the edges is shown in Figure 9.

![Figure 9](image)

The feasible matrix for $C_5$ is:

$$A_1 = \begin{bmatrix} 1 & x_1 & 1 & 1 & x_5 \\ x_1 & 1 & x_2 & 1 & 1 \\ 1 & x_2 & 1 & x_3 & 1 \\ 1 & 1 & x_3 & 1 & x_4 \\ x_5 & 1 & 1 & x_4 & 1 \end{bmatrix}$$

where now $x_1, x_2, x_3, x_4, x_5$ represent real numbers. We want to show that the minimum value of $\lambda_1$ occurs when $x_1 = x_2 = x_3 = x_4 = x_5$. To do this consider the matrix $P$ where,

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix},$$

i.e., $P$ is a permutation matrix. Now define

$$A_2 = P^T A P = \begin{bmatrix} 1 & x_5 & 1 & 1 & x_4 \\ x_5 & 1 & x_1 & 1 & 1 \\ 1 & x_1 & 1 & x_2 & 1 \\ 1 & 1 & x_2 & 1 & x_3 \\ x_4 & 1 & 1 & x_3 & 1 \end{bmatrix}. $$

We may view the action of $P$ as rotating each vertex of the graph $C_5$ one position counterclockwise, or equivalently, each edge one position clockwise. This is seen in Figure 10.
We similarly define $A_3 = (P^2)^T AP^2$, $A_4 = (P^3)^T AP^3$ and $A_5 = (P^5)^T AP^5$,

$$
A_3 = \begin{bmatrix} 1 & x_4 & 1 & 1 & x_3 \\ x_4 & 1 & x_5 & 1 & 1 \\ 1 & x_5 & 1 & x_1 & 1 \\ 1 & 1 & x_1 & 1 & x_2 \\ x_3 & 1 & 1 & x_2 & 1 \end{bmatrix}, \quad A_4 = \begin{bmatrix} 1 & x_3 & 1 & 1 & x_2 \\ x_3 & 1 & x_4 & 1 & 1 \\ 1 & x_4 & 1 & x_5 & 1 \\ 1 & 1 & x_5 & 1 & x_1 \\ x_2 & 1 & 1 & x_1 & 1 \end{bmatrix},
$$

$$
A_5 = \begin{bmatrix} 1 & x_2 & 1 & 1 & x_1 \\ x_2 & 1 & x_3 & 1 & 1 \\ 1 & x_3 & 1 & x_4 & 1 \\ 1 & 1 & x_4 & 1 & x_5 \\ x_1 & 1 & 1 & x_5 & 1 \end{bmatrix}.
$$

Since $A_1, A_2, A_3, A_4$ and $A_5$ are similar, they have the same eigenvalues. Now let

$$
A_0 = \frac{1}{5}(A_1 + A_2 + A_3 + A_4 + A_5) = \begin{bmatrix} 1 & x & 1 & 1 & x \\ x & 1 & x & 1 & 1 \\ 1 & x & 1 & x & 1 \\ 1 & 1 & x & 1 & x \\ x & 1 & 1 & x & 1 \end{bmatrix}
$$

where $x = \frac{1}{5}(x_1 + x_2 + x_3 + x_4 + x_5)$. By Lemma 10 we have

$$
\lambda_1(A_0) = \lambda_1\left[\frac{1}{5}(A_1 + A_2 + A_3 + A_4 + A_5)\right]
\leq \lambda_1\left(\frac{1}{5}A_1\right) + \lambda_1\left(\frac{1}{5}A_2\right) + \lambda_1\left(\frac{1}{5}A_3\right) + \lambda_1\left(\frac{1}{5}A_4\right) + \lambda_1\left(\frac{1}{5}A_5\right)
= \frac{1}{5}[\lambda_1(A_1) + \lambda_1(A_2) + \lambda_1(A_3) + \lambda_1(A_4) + \lambda_1(A_5)]
= \lambda_1(A_1).
$$

Therefore, the minimum eigenvalue $\lambda_1$ occurs at $A_0$, which means when $x_1 = x_2 = x_3 = x_4 = x_5$.

I used Maple to compute the eigenvalues of $A_0$ as follows:

$$
3 + 2x, \quad (-\frac{1}{2} + \frac{1}{2}\sqrt{5})(-1 + x), \quad (-\frac{1}{2} - \frac{1}{2}\sqrt{5})(-1 + x),
(-\frac{1}{2} + \frac{1}{2}\sqrt{5})(-1 + x), \quad (-\frac{1}{2} - \frac{1}{2}\sqrt{5})(-1 + x).
$$
Notice that there are only three distinct eigenvalues, namely,

$$\lambda_{11} = 3 + 2x, \quad \lambda_{12} = (-\frac{1}{2} + \frac{1}{2}\sqrt{5})(-1 + x), \quad \lambda_{13} = (-\frac{1}{2} - \frac{1}{2}\sqrt{5})(-1 + x).$$

We know that $\vartheta_2(C_5) = \min \lambda_1$ over all feasible matrices. Figure 11 shows plots of the lines $\lambda_{11}$, $\lambda_{12}$, $\lambda_{13}$ as functions of $x$.

![Graph showing plots of $\lambda_{11}$, $\lambda_{12}$, $\lambda_{13}$ as functions of $x$.]

From the graph (Figure 11) we see that the minimum $\lambda_1$ occurs at the intersection of $\lambda_{11}$ and $\lambda_{13}$, which occurs at $x = (-3 + \sqrt{5})/2$, and gives $\lambda_1 = \sqrt{5}$.

Therefore, $\vartheta_2(C_5) = \min \lambda_1(A) = \sqrt{5}$. Since $C_5^c = C_5$, the Sandwich Theorem says that $\omega(C_5) \leq \vartheta(C_5) \leq \chi(C_5)$ which is $2 < \sqrt{5} < 3$. This is our first example of the sandwich theorem in which the inequalities are strict.

Now, we will look at another important property of the $\vartheta_2$ function which enables us to compute $\vartheta_2(C_n)$.

First, we need to introduce some related definitions and prove one lemma.

**Definition 10.** Let $G = (V, E)$ be a graph with vertex set $V = \{1, 2, 3, \cdots, n\}$. The n-by-n adjacency matrix $A_n(G) = (a_{ij})$ is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } ij \in E, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 11.** The eigenvalues of a graph are the eigenvalues of its adjacency matrix.
Lemma 11. Let $A_n$ be the adjacency matrix of $C_n$. Then the eigenvalues for $A_n$ are
\[ \lambda_j = 2 \cos \frac{2\pi j}{n}, \quad j = 0, 1, \ldots, n - 1. \]

Proof. Assume the vertices of $C_n$ are labeled 1 through $n$. Then the adjacency matrix $A_n$ of $C_n$ is
\[
A_n = \begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 1 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 1 \\
1 & 0 & 0 & \cdots & 1 & 0
\end{bmatrix}.
\]
Assume $\lambda$ is an eigenvalue of $A_n$ and $x = [x_1, x_2, \ldots, x_n]^T$ is a corresponding eigenvector. The equation $Ax = \lambda x$ written in components is
\[
x_n + x_2 = \lambda x_1 \\
x_1 + x_3 = \lambda x_2 \\
x_2 + x_4 = \lambda x_3 \\
\vdots \\
x_{n-1} + x_1 = \lambda x_n
\]
We can rewrite this as a single equation
\[ x_{k-1} + x_{k+1} = \lambda x_k \quad (1) \]
with boundary conditions $x_1 = x_{n+1}$ and $x_0 = x_n$.

This is a second degree linear difference equation with constant coefficients. For a difference equation, the fundamental solutions have the form $r^k$, $r \neq 0$.

Let $x_k = r^k$ in equation (1). Then we have $r^{k-1} + r^{k+1} = \lambda r^k$. Dividing by $r^{k-1}$, we have $r^2 - \lambda r + 1 = 0$. Let $r_1, r_2$ be the two roots of this equation. Then
\[ r_1 + r_2 = \lambda, \quad r_1 \cdot r_2 = 1. \]
First assume that $r_1 = r_2$. So $r_1 = r_2 = 1$ or $r_1 = r_2 = -1$.

Consider $r_1 = r_2 = 1$, then $\lambda = 2$. The solution for equation (1) is $x_k = 1^k = 1$. Also $x_k = k \cdot 1^k = k$ is a solution since

$$x_{k-1} + x_{k+1} - \lambda x_k = (k-1) + (k+1) - 2k = 0.$$ 

So if $r_1 = r_2 = 1$, the general solution for equation (1) is $x_k = c_1 + c_2 \cdot k$. From the boundary conditions we have

$$c_1 = c_1 + nc_2 \quad \text{and} \quad c_1 + c_2 = c_1 + (n+1)c_2.$$ 

Therefore $c_2 = 0$. Hence $x_1 = x_2 = \cdots = x_n = c_1$, i.e., $c_1[1,1,\ldots,1]^T$ is an eigenvector corresponding to $\lambda = 2$.

Similarly, if $r_1 = r_2 = -1$, $\lambda = -2$, the solutions for the equation (1) are $x_k = (-1)^k$ and $x_k = k \cdot (-1)^k$. The general solution for equation (1) is also

$$x_k = c_1(-1)^k + c_2 \cdot k(-1)^k.$$ 

From the boundary conditions we have

$$x = c_1[-1,1,-1,1,\cdots,1]^T \text{ if } n \text{ is even, and } x = 0 \text{ if } n \text{ is odd.}$$ 

So $\lambda = -2$ is an eigenvalue if $n$ is even.

Now we assume that $r_1 \neq r_2$. Then the general solution for equation (1) is

$$x_k = c_1 r_1^k + c_2 r_2^k.$$ 

From the boundary conditions we have

$$c_1 + c_2 = c_1 r_1^n + c_2 r_2^n \quad \text{and} \quad c_1 r_1 + c_2 r_2 = c_1 r_1^{n+1} + c_2 r_2^{n+1}.$$ 

Equivalently,

$$c_1(1 - r_1^n) + c_2(1 - r_2^n) = 0 \quad \text{and} \quad c_1 r_1(1 - r_1^n) + c_2 r_2(1 - r_2^n) = 0. \quad (2)$$
From equation (2) we have
\[
\begin{bmatrix}
1 - r_1^n & 1 - r_2^n \\
1 & r_2(1 - r_2^n)
\end{bmatrix}
\begin{bmatrix}
c_1 \\
c_2
\end{bmatrix} = 0.
\]

Since \([c_1, c_2]^T\) is nonzero, therefore \(B = \begin{bmatrix}
1 - r_1^n & 1 - r_2^n \\
1 & r_2(1 - r_2^n)
\end{bmatrix}\) is a singular matrix, which means \(\det(B) = 0\). That is,
\[
(1 - r_1^n)r_2(1 - r_2^n) - (1 - r_2^n)r_1(1 - r_1^n) = 0,
\]
so,
\[
(1 - r_1^n)(1 - r_2^n)(r_2 - r_1) = 0.
\]
For \(r_1^n = 1\), \(r_1 = e^{2\pi ij/n} , j = 0, 1, \ldots, n - 1\). Then \(r_2 = \frac{1}{r_1} = e^{-2\pi ij/n}\) which gives possible eigenvalues
\[
\lambda_j = r_1 + r_2
= e^{2\pi ij/n} + e^{-2\pi ij/n}
= 2 \cos \frac{2\pi j}{n} , j = 0, 1, \ldots, n - 1.
\]
For \(r_2^n = 1\), we obtain the same result.

Now we see the only solutions are
\[
\lambda_j = 2 \cos \frac{2\pi j}{n}, \text{ for } j = 0, 1, \ldots, n - 1.
\]
Note that \(j = 0\) gives the solution for the case \(r_1 = r_2 = 1\) and when \(n\) is even, \(j = n/2\) gives the solution for \(r_1 = r_2 = -1\), Therefore, these two previous cases can be combined with this case.

Next, we need to show that the eigenvalues of \(A\) and the \(\lambda_j\) are in one-to-one correspondence.

Consider the matrix
\[
B = A - \lambda I = \begin{bmatrix}
-\lambda & 1 & 0 & \cdots & 0 & 1 \\
1 & -\lambda & 1 & \cdots & 0 & 0 \\
0 & 1 & -\lambda & \ddots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & 0 & \ddots & -\lambda & 1 \\
1 & 0 & 0 & \cdots & 1 & -\lambda
\end{bmatrix}
\]
If we delete the first and last rows and first two columns, we obtain an upper triangular matrix, which is invertible. Therefore \( \text{rank}(B) \geq n - 2 \), so nullity\( (B) \leq 2 \). So

\[
\text{geometric multiplicity}(\lambda) \leq 2,
\]

\[
\text{algebraic multiplicity}(\lambda) \leq 2.
\]

For \( \lambda_0 = 2 \), we consider the matrix \( B_1 = A - 2I \),

\[
B_1 = \begin{bmatrix}
-2 & 1 & 1 \\
1 & -2 & 1 \\
& & \ddots & 1 \\
1 & & & -2
\end{bmatrix}.
\]

If we delete the first row and first column, we obtain the matrix

\[
B_{11} = \begin{bmatrix}
-2 & 1 \\
1 & -2 \\
& & \ddots & 1 \\
1 & & & -2
\end{bmatrix}.
\]

For any \( x \in \mathbb{R}^{n-1} \),

\[
x^T B_{11} x = -2x_1^2 + 2x_1x_2 - 2x_2^2 + 2x_2x_3 - 2x_3^2 + \cdots + 2x_{n-2}x_{n-1} - 2x_{n-1}^2
\]

\[
= -x_1^2 - (x_1 - x_2)^2 - (x_2 - x_3)^2 - \cdots - (x_{n-2} - x_{n-1})^2 - x_{n-1}^2 \leq 0.
\]

So, \( B_{11} \) is negative semidefinite.

If \( x^T B_{11} x = 0 \), then \( x_1 = 0, x_1 - x_2 = 0, x_2 - x_3 = 0, \ldots, x_{n-2} - x_{n-1} = 0 \). Therefore \( x_1 = x_2 = \cdots = x_{n-1} = 0 \). So if \( x \neq 0 \), then \( x^T B_{11} x < 0 \) and \( B_{11} \) is negative definite. We know that \(-B_{11}\) is invertible because it is positive definite. Therefore \( B_{11} \) is also invertible.

So the rank of \( B_1 \) is at least \( n - 1 \). Also, since \( B_1 \) is a singular matrix, we have

\[
n - 1 \geq \text{rank}(B_1) \geq n - 1.
\]

Therefore \( \text{rank}(B_1) = n - 1 \) and nullity\( (B_1) = 1 \). Therefore,

\[
\text{geometric multiplicity}(2) = 1, \quad \text{and} \quad \text{algebraic multiplicity}(2) = 1.
\]
If \( n \) is an odd number there are \((n + 1)/2\) distinct values in \( \lambda_j = 2 \cos(2\pi j/n) \). We let \( \lambda^{(2)} \) be the number of eigenvalues with algebraic multiplicity equal 2, and \( \lambda^{(1)} \) be the number of eigenvalues with algebraic multiplicity equal 1. Since \( B \) has \( n \) eigenvalues

\[
n = \lambda^{(2)} + \lambda^{(1)} \\
\leq 2 \left( \frac{n+1}{2} - 1 \right) + 1 \\
= n.
\]

So each eigenvalue \( 2 \cos(2\pi j/n) \), \( j = 1, \ldots, \frac{n-1}{2} \) has multiplicity 2.

If \( n \) is an even number, \( \lambda = -2 \) is also one of the eigenvalues of matrix \( A \) since for eigenvector \( x = [-1, 1, -1, 1, \ldots, 1]^{T} \)

\[
\begin{bmatrix}
0 & 1 & 0 & \cdots & 0 & 1 \\
1 & 0 & 1 & \cdots & 0 & 0 \\
0 & 1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 1 \\
1 & 0 & \cdots & 0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
-1 \\
1 \\
-1 \\
\vdots \\
1 \\
1
\end{bmatrix}
= (-2)
\begin{bmatrix}
-1 \\
1 \\
-1 \\
\vdots \\
1 \\
1
\end{bmatrix}.
\]

Consider the matrix \( B_{2} = A - \lambda I \)

\[
B_{2} = \begin{bmatrix}
2 & 1 & 1 \\
1 & 2 & 1 \\
1 & 1 & 2 \\
\vdots & \vdots & \vdots \\
1 & 1 & 2
\end{bmatrix}.
\]

If we delete the first row and the first column, we obtain the matrix

\[
B_{21} = \begin{bmatrix}
2 & 1 \\
1 & 2 \\
\vdots & \vdots \\
1 & 1 \\
1 & 2
\end{bmatrix}.
\]

For any \( x \in \mathbb{R}^{n-1} \), by an argument similar to the case when \( \lambda_0 = 2 \), \( B_{21} \) is positive definite. Therefore \( B_{21} \) is invertible.

So the rank of \( B_{2} \) is at least \( n - 1 \). On the other hand, \( B_{2} \) is a singular matrix because \(-2\) is an eigenvalue. So

\[
n - 1 \geq \text{rank}(B_{2}) \geq n - 1.
\]
Therefore \( \text{rank}(B_2) = n - 1 \) and \( \text{nullity}(B_2) = 1 \). So we have

\[
\text{geometric multiplicity}(-2) = 1, \quad \text{and} \quad \text{algebraic multiplicity}(-2) = 1.
\]

So the dimension of the eigenspace of \( B_2 \) corresponding to the eigenvalue \(-2\) is one. By a similar argument as for when \( n \) is odd, we see that 2 and \(-2\) are eigenvalues of multiplicity one and \( 2 \cos(2\pi j/n), j = 1, 2 \cdots, (n/2) - 1 \) are eigenvalues of multiplicity two.

Equivalently, the eigenvalues of \( A \) in all cases are

\[
\lambda_j = 2 \cos \frac{2\pi j}{n}, \quad j = 0, 1, \ldots, n - 1.
\]

\[\square\]

**Theorem 3.** For any odd number \( n \)

\[
\vartheta(C_n) = \begin{cases} 
\frac{n \cos(\pi/n)}{1 + \cos(\pi/n)} & \text{if } n \text{ is odd}, \\
\frac{n}{2} & \text{if } n \text{ is even}.
\end{cases}
\]

**Proof.** Let \( A_n \) be the adjacency matrix of \( C_n \) and \( J_n \) be a \( n \)-by-\( n \) matrix with all entries equal to 1. As in the case \( n = 5 \), it is sufficient to consider a feasible matrix for \( C_n \) to be \( J_n - (1 - x)A_n \). We have just proved that the eigenvalues of the graph \( C_n \) are

\[
2 \cos \frac{2\pi j}{n}, \text{ for } j = 0, 1, \ldots, n - 1.
\]

Let \( u_j \) be the corresponding eigenvector. Note that \( u_0 = [1, 1, \ldots, 1]^T \) is an eigenvector corresponding to \( \lambda_0 = 2 \). Then we have

\[
[J_n - (1 - x)A_n]u_0 = [n - (1 - x)2]u_0 = (n - 2 + 2x)u_0.
\]

Since all the eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal, then for \( j > 0 \) we have,

\[
J_n u_j = \begin{bmatrix} u_0^T \\
\vdots \\
u_j^T \\
\vdots \\
u_0^T \end{bmatrix} u_j = \begin{bmatrix} u_0^T u_j \\
\vdots \\
u_j^T u_j \\
\vdots \\
u_0^T u_j \end{bmatrix} = 0.
\]

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so then,

$$[J_n - (1 - x)A_n]u_j = 0u_j - (1 - x)A_n u_j$$

$$= -(1 - x)2\cos \frac{2\pi j}{n} u_j.$$ 

So the eigenvalues of $J_n - (1 - x)A_n$ are

$$\mu_0 = n - 2 + 2x$$

$$\mu_j = -2(1 - x)\cos \frac{2\pi j}{n}, \quad j = 1, 2, \ldots, n - 1.$$ 

Let $\lambda_1$ be the maximum eigenvalue.

If $n$ is even, we consider the graph of $\mu_0$ and $\mu_i$, and consider the intersection of these lines. We know that all the lines of the form $\mu_i = -2(1 - x)\cos(2\pi j/n)$ for $j = 1, 2, \ldots, n - 1$ pass through the point $(1, 0)$. Since line $\mu_0 = n - 2 + 2x$ has biggest positive slope and biggest $y$ intercept, it crosses each of the lines $\mu_1, \mu_2, \ldots, \mu_{n-1}$ to the left of the vertical axis. The line $\mu_{n/2} = -2(1 - x)\cos(\pi) = 2 - 2x$ has the most negative slope so is the first to meet $\mu_0$.

It is easy to see this result if we consider the example of $n = 6$, which is illustrated in Figure 12. The distinct eigenvalues are

$$\mu_0 = 4 + 2x$$

$$\mu_1 = -2(1 - x)\cos \frac{\pi}{3}$$

$$\mu_2 = -2(1 - x)\cos \frac{2\pi}{3}$$

$$\mu_3 = -2(1 - x)\cos \pi.$$ 

By graphing these eigenvalues, we can see that the maximum eigenvalue occurs at the intersection of $\mu_0$ and $\mu_3$ which occurs at $(-1/2, 3)$ and gives $\lambda_1 = 3 = n/2$. 

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So, for any even $n$, we consider the intersection of the lines $2 - 2x$ and $n - 2 + 2x$, which occurs when $x = 1 - (n/4)$. Then

$$\lambda_1 = n - 2 + 2\left(1 - \frac{n}{4}\right) = \frac{n}{2} = \alpha(C_n).$$

If $n$ is odd and $j = (n - 1)/2$,

$$\mu_j = -2(1 - x) \cos\left(\frac{2\pi(n-1)}{n}\right) = -2(1 - x) \cos\left(\pi - \frac{\pi}{n}\right) = 2(1 - x) \cos\frac{\pi}{n}.$$  

Similarly, we can find the maximum eigenvalue $\lambda_1$ by finding the intersection of $\mu_0$ and $\mu_j$. This occurs when

$$n - 2 + 2x = 2\cos\frac{\pi}{n} - 2x \cos\frac{\pi}{n}.$$  

Rearranging we get,

$$2\left(1 + \cos\frac{\pi}{n}\right)x = 2\cos\frac{\pi}{n} - n + 2,$$

so that the intersection will occur when

$$x = \frac{2\cos(\pi/n) + 2 - n}{2(1 + \cos(\pi/n))} = 1 - \frac{n}{2(1 + \cos(\pi/n))}.$$
Therefore

\[
\lambda_1 = n - 2 + 2x = n - 2 + 2 \left(1 - \frac{n}{2(1 + \cos(\pi/n))}\right) \\
= n - 2 + 2 - \frac{2n}{2(1 + \cos(\pi/n))} = n - \frac{n}{1 + \cos(\pi/n)} \\
= \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)}.
\]

Hence

\[
\varphi_2(C_n) = \frac{n \cos(\pi/n)}{1 + \cos(\pi/n)}, \text{ for any odd } n.
\]
References


