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Dynamics of Systems Driven by an External Force

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ABSTRACT

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In this dissertation, we study the complicated dynamics of two classes of systems: Anosov systems driven by an external force and partially hyperbolic systems driven by an external force.

For smooth Anosov systems driven by an external force, we first study the random specification property, which is on the approximation of an $N-$spaced arbitrary long finite random orbit segments within given precision by a random periodic point. We prove that if such system is topological mixing on fibers, then it has the random specification property. Furthermore, we prove that the homeomorphism induced by such a system on the space of random probability measures also has the specification property. We note that the random specification property implies the positivity of topological fiber entropy. Secondly, we show that if the system is topological mixing on fibers, then its past and future random correlation for Hölder observable functions decay exponentially with respect to the system and the unique random SRB measure.

For smooth partially hyperbolic systems driven by an external force, we prove the existence of the random Gibbs $u-$state, which has absolutely continuous conditional measure on the strong unstable manifolds.

Keywords: random dynamical systems, random specification, Bowen's specification property, exponential decay of random correlation, absolute continuity, random SRB measure, Birkhoff cone, random Gibbs $u-$state

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Chapter 1. Introduction

The study of complicated dynamics can be traced back to Poincare's work [50] on the N-body problem. The modern theory of uniformly hyperbolic dynamical systems was initiated in the 1960s by Anosov [2] and Smale [60], where Anosov and Axiom A diffeomorphisms/flows were introduced respectively. The core component in these systems is uniform hyperbolicity, which is an invariant geometric structure describing the exponential divergence of nearby orbits. This exponential divergence together with the compactness of phase space produces rich and complicated dynamical structures.

From a geometric perspective, one of the complicated dynamics of transitive Anosov system is the abundance of periodic points. Bowen's specification property, which can be viewed as a uniform version of topological transitivity [40], says that a finite collection of arbitrary long orbits segments can be shadowed by a periodic point within given precision as long as one allows for enough time between segments. Bowen's specification theorem [10] affirms that any diffeomorphism restricted to a compact, topological mixing and locally maximal hyperbolic set (hyperbolic elementary set) has Bowen's specification property. If a homeomorphism on a compact metric space has the specification property, then the set of invariant measures equidistributed on a periodic orbit is dense in the set of invariant measures [58]. Moreover, the induced system on the space of probability measure also has the specification property [8]. Furthermore, if the system has expansivity, then the topological entropy equals the exponential growth rate of periodic orbits [10], and the unique equilibrium state can be obtained for a large class of potential functions [11]. In recent decades, many generalizations of the specification property have been developed [56, 49, 65, 20]. In [54, 55], the authors studied the specification property for the following non-autonomous or time-dependent discrete systems:

$$x_{n+1} = f_n(x_n),\ \ n \geq 1$$

on a compact metric space (X, d). A periodic point for such non-autonomous systems is a point $x \in X$ such that there exists $n \in \mathbb{N}$,

$$f_{nk} \circ f_{nk-1} \circ \cdots \circ f_2 \circ f_1(x) = x, \text{ for every } k \in \mathbb{N}. \tag{1.1}$$

While in this dissertation, when an external force evolves in time, the system driven by the orbit of an external force can be viewed as a non-autonomous discrete dynamical system. Due to the presence of a random external force, the periodic point defined in (1.1) rarely exists. We consider the random periodic points in our dissertation, which have already been studied in the existing literature [38, 69]. To investigate the abundance of random periodic points in systems driven by external force, we study the random specification property in the first part of this dissertation.

The study of the statistical behavior of orbits dates back to the work of Birkhoff and Von Neumann on the ergodic theorem. The ergodic theorem declares that if an invariant measure μ is ergodic, then the time average of an integrable observable along individual trajectories $\mu-$a.s. converges to the spatial average. Another stochastic property, which is stronger than ergodicity, is (measure-theoretic) mixing: a measure-preserving transformation (f, μ) is mixing if for all measurable sets $A,\ B$,

$$\mu(f^{-n}A \cap B) \to \mu(A)\mu(B) \text{ or } \mu(B|f^{-n}A) \to \mu(B) \text{ as } n \to \infty.$$

The functional form of mixing is that the correlation function of two observable functions g, h with respect to measure μ decays to zero, i.e.

$$|\int (g \circ f^n) h d\mu - \int g d\mu \int h d\mu| \to 0 \text{ as } n \to \infty,$$

which says that $g \circ f^n$ and h become uncorrelated asymptotically. Sinai [59], Ruelle [52] and Bowen [12] proved that topological mixing Anosov or Axiom A diffeomorphisms have exponential decay of correlation for Hölder observable functions with respect to the unique

SRB measure, see also [46, 62, 67, 13]. There are a large number of results that consider the exponential decay of correlations, for instance [28, 30, 3, 24, 67, 18, 25, 26, 63, 1, 47, 39], where an invariant measure is called SRB measure if it has absolute continuous conditional measure on unstable manifolds. In the second part of this dissertation, we prove exponential decay of random correlation for Anosov systems driven by an external force.

When there is a neutral direction, besides the uniformly expansion and contraction directions in the tangent bundle, the system is a partially hyperbolic system [14]. A special class of invariant measures for partially hyperbolic systems, which is characterized by having absolutely continuous conditional measures on strong unstable manifolds, is the Gibbs $u-$state. When the Lyapunov exponents along the neutral direction are non-positive, then the Gibbs $u-$state is the SRB measure [68]. In [48], Pesin and Sinai proved the existence of Gibbs $u-$state for partially hyperbolic systems. We refer to three surveys about the relation between Gibbs $u-$states and physical relevant measures [29, 9, 19]. In this third part of this dissertation, we prove the existence of the random Gibbs $u-$state for partially hyperbolic systems driven by an external force.

1.1 Anosov Systems Driven by an External Force

Let M be a connected closed smooth Riemannian manifold and (Ω, d_Ω) be a compact metric space. Denote $\mathcal{B}(M)$ and $\mathcal{B}(\Omega)$ to be the Borel measurable set on M and Ω respectively. Let $\theta : \Omega \to \Omega$ be a homeomorphism. In this dissertation, the topological dynamical system (Ω, θ) will describe the external force. Let $\mathcal{H}$ =Diff$^2(M)$ be the space of C^2-diffeomorphisms on M quipped with the C^2-topology [33]. Let $f : \Omega \to \mathcal{H}$ be a continuous map. The diffeomorphism $f_\omega := f(\omega)$ is driven by the external force (Ω, θ), i.e., while ω is shifted by θ in time n to $\theta^n\omega$ on the external force space Ω, any point $x \in M$ is mapped to $F(n, \omega)x$, where

$$F(n,\omega)=\begin{cases} f_{\theta^{n-1}\omega}\circ\cdots\circ f_{\omega}, & \text{if } n>0\\ id_M, & \text{if } n=0\\ (f_{\theta^{n}\omega})^{-1}\circ\cdots\circ(f_{\theta^{-1}\omega})^{-1}, & \text{if } n<0. \end{cases}$$

Remark 1.1. *The following system*

$$F:\mathbb{Z}\times\Omega\times M\to M,\ (n,\omega,x)\mapsto F(n,\omega)x$$

satisfies for each $n\in\mathbb{Z}$ *that,* $(\omega,x)\mapsto F(n,\omega)x$ *is continuous and the mappings* $F(n,\omega):=F(n,\omega):M\to M$ *form a cocycle over* θ*, i.e.,*

$$F(0,\omega)=id_M \text{ for all } \omega\in\Omega,$$
$$F(n+m,\omega)=F(n,\theta^m\omega)\circ F(m,\omega) \text{ for all } n,m\in\mathbb{Z},\ \omega\in\Omega.$$

When $(\Omega,\mathcal{B}(\Omega))$ *is equipped with a* $\theta-$*invariant probability measure* P*,* F *is called a (continuous) random dynamical system (RDS) [4].*

We say that the diffeomorphism f_ω driven by the external force (Ω,θ) is an Anosov system driven by the external force (Ω,θ) (or ϕ is random Anosov on fibers system) if for every $(x,\omega)\in M\times\Omega$, there is a splitting of the tangent bundle of $M_\omega:=M\times\{\omega\}$ at x into

$$T_xM_\omega=E^s(x,\omega)\oplus E^u(x,\omega),$$

which depends continuously on $(x,\omega)\in M\times\Omega$ with $\dim E^s(x,\omega),\ \dim E^u(x,\omega)>0$, and where the splitting is invariant in the sense that

$$D_xf_\omega E^u(x,\omega)=E^u(f_\omega x,\theta\omega),\quad D_xf_\omega E^s(x,\omega)=E^s(f_\omega x,\theta\omega)$$

and

$$\begin{cases} |D_x f_\omega \xi| \geq e^{\lambda_0}|\xi|, & \forall \xi \in E^u(x,\omega), \\ |D_x f_\omega \eta| \leq e^{-\lambda_0}|\eta|, & \forall \eta \in E^s(x,\omega), \end{cases}$$

where $\lambda_0 > 0$ is a constant. Putting f_ω and θ together forms a skew product system

$$\phi : M \times \Omega \to M \times \Omega,\ \phi(x,\omega) = (f_\omega x, \theta\omega).$$

The system ϕ is said to be topological mixing on fibers if for any nonempty open sets $U,\ V \subset M$, there exists $N > 0$ such that for any $n \geq N$ and $\omega \in \Omega$, $\phi^n(\{\omega\} \times U) \cap (\{\theta^n\omega\} \times V) \neq \emptyset$.

Anosov system driven by an external force is a class of nonautonomous dynamical systems. Such systems have been recently studied in [34], in which the authors proved dynamical complexity, under the topological mixing on fibers assumption, such as the density of random periodic points, strong random horseshoe, and a simplified random Livšic theorem. Examples such as fiber Anosov maps on 2-dimension torus driven by irrational rotation on the torus and random composition of 2×2 area-preserving positive matrices are under consideration (we list these examples in Subsection 2.1.1). Moreover, the random Anosov on fibers systems actually contain a class of partially hyperbolic systems. In fact, if Ω is a compact differentiable manifold, and if $\theta : \Omega \to \Omega$ is a diffeomorphism such that the expansion of $D\theta$ is weaker than e^{λ_0} and contraction of $D\theta$ is weaker than $e^{-\lambda_0}$. Furthermore, we assume $f_\omega(x)$ and $f_\omega^{-1}(x)$ are C^1 in ω. Then the system ϕ is a partially hyperbolic system with dimension-$\dim\Omega$ central direction (we prove this statement in Section 2.1).

1.1.1 Random Specification. The system ϕ induces a natural self-map $\tilde{\phi}$ on $L^\infty(\Omega, M)$ given by $(\tilde{\phi}g)(\omega) = f_{\theta^{-1}\omega}g(\theta^{-1}\omega)$ which is a homeomorphism with respect to the sup-metric on $L^\infty(\Omega, M)$. A measurable map $g \in L^\infty(\Omega, M)$ is said to be a random periodic point of ϕ if g is a periodic point of $\tilde{\phi}$. In [34], one of the main result for Anosov and topological mixing on fibers system is the density of random periodic points.

The random specification property is defined analogously to Bowen's specification property in deterministic systems. The system ϕ is said to have the random specification property if $(L^\infty(\Omega, M), \tilde{\phi})$ has Bowen's specification property, i.e., for any $\epsilon > 0$, there exists a $N = N(\epsilon)$ such that for any finite collection of intervals $\tau = \{I_1, ..., I_m\}$, $I_i = [a_i, b_i] \subset \mathbb{Z}$, $a_{i+1} > b_i + N$, and any $P : \cup_{i=1}^m I_i \to L^\infty(\Omega, M)$ such that $\tilde{\phi}^{t_2 - t_1}(P(t_1)) = P(t_2)$ for $t_1, t_2 \in I \in \tau$, there exists $\epsilon-$shadowing point $g \in L^\infty(\Omega, M)$,

$$d_{L^\infty(\Omega,M)}(P(t), \tilde{\phi}^t(g)) < \epsilon, \ \forall t \in I_i, \ i \in \{1, ..., m\}.$$

Moreover, for any $q \geq N + b_m - a_1$, the shadowing point can be a random periodic point with period q.

Remark 1.2. *In [31], Gundlach and Kifer generalized the notion of specification in RDS which is based on the construction of the shadowing point in the proof of Bowen's specification theorem. In their definition, due to the presence of the random noise, the periodicity part is missing.*

In [34], the following random specification theorem was stated without proof. We give a proof in this dissertation.

Theorem A (Random specification property)**.** *Random Anosov and topological mixing on fibers systems have the random specification property.*

Besides this, we study the consequences of the random specification property. A random probability measure is a map $\mu : \mathcal{B}(M) \times \Omega \to [0, 1]$, $(B, \omega) \to \mu_\omega(B)$ such that for each fixed $B \in \mathcal{B}(M)$, $\omega \to \mu_\omega(B)$ is measurable, and for each fixed $\omega \in \Omega$, $B \mapsto \mu_\omega(B)$ is a probability measure on M. Let $Pr_\Omega(M)$ be the space of all random probability measures, which is compact metrizable space [21]. The system ϕ defines a self-map ϕ^* on $Pr_\Omega(M)$ by $(\phi^*\mu)_\omega := (f_{\theta^{-1}\omega})_* \mu_{\theta^{-1}\omega}$.

Theorem B. *Let ϕ be random Anosov and topological mixing on fibers systems, then ϕ^* is a homeomorphism and the topological dynamical system $(Pr_\Omega(M), \phi^*)$ has Bowen's specification property.*

Moreover, the random specification implies dynamical complexity in the following sense.

Theorem C. *The random specification property implies the positivity of topological fiber entropy.*

1.1.2 Exponential Decay of Random Correlation.

To describe the statistical behaviour of a random dynamical system, we study the random mixing property. A random probability measure $(\mu_\omega)_{\omega\in\Omega}$ is $\phi-$invariant if $(f_\omega)_*\mu_\omega = \mu_{\theta\omega}$ for $P - a.s\ \omega \in \Omega$.

The system ϕ, generated by f_ω and θ, together with an invariant random probability measure $(\mu_\omega)_{\omega\in\Omega}$ has the past random mixing property if for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \mu_{\theta^{-n}\omega}(f_\omega^{-n}A|B) = \lim_{n\to\infty} \frac{\mu_{\theta^{-n}\omega}(f_\omega^{-n}A \cap B)}{\mu_{\theta^{-n}\omega}(B)} = \mu_\omega(A).$$

Notice that $\mu_{\theta^{-n}\omega}(f_\omega^{-n}A) = \mu_\omega(A)$ for all $n \in \mathbb{N}$, so the above equality is saying that when we trace back to the history, the "current memory" is fading with respect to the measurement in the history. Equivalently, given a pair of regular observable functions φ and ψ on M, the past random correlation function of φ and ψ goes to zero, i.e.,

$$\left| \int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega} - \int_M \psi(x)d\mu_\omega \int_M \varphi(x)d\mu_{\theta^{-n}\omega} \right| \to 0 \text{ as } n \to \infty.$$

The system ϕ, generated by f_ω and θ, together with an invariant random probability measure $(\mu_\omega)_{\omega\in\Omega}$ has future random mixing property if for all $\omega \in \Omega$,

$$\lim_{n\to\infty} \mu_\omega(B|f^{-n}_{\theta^n\omega}A) = \lim_{n\to\infty} \frac{\mu_\omega(f^{-n}_{\theta^n\omega}A \cap B)}{\mu_\omega(f^{-n}_{\theta^n\omega}A)} = \lim_{n\to\infty} \frac{\mu_\omega(f^{-n}_{\theta^n\omega}A \cap B)}{\mu_{\theta^n\omega}(A)} = \mu_\omega(B).$$

The future random mixing property is saying that the impact of the future to the current state is fading with respect to the measurement in the current state. Equivalently, given a pair of regular observable functions φ and ψ on M, the future random correlation function of φ and ψ goes to zero, i.e.,

$$\left|\int_M \psi(f_\omega^n x)\varphi(x)d\mu_\omega - \int_M \psi(x)d\mu_{\theta^n\omega}\int_M \varphi(x)d\mu_\omega\right| \to 0 \text{ as } n\to\infty.$$

If ϕ is Anosov and topological mixing on fibers, then ϕ is a random topological transitive hyperbolic systems [34], so there exists a unique random SRB measure and the unique random SRB measure is given by $\mu_\omega := \lim_{n\to\infty}(f^n_{\theta^{-n}\omega})_* m$, where m is the normalized Riemannian volume measure [32]. This unique random SRB measure is characterised by the entropy formula of Pesin's type, absolutely continuous conditional measure on unstable manifolds, and variational principle when the topological pressure equals zero [32]. In this dissertation, we prove that the random Anosov and topological mixing on fibers system has exponential decay of both past and future random correlation for Hölder observable functions with respect to the unique random SRB measure.

Theorem D. *Let ϕ be random Ansov and topological mixing on fibers systems, then there exists a constant ν_0 only depending on the system ϕ such that for any $\mu,\nu\in(0,1)$ satisfying*

$$0<\mu+\nu<\nu_0$$

and $\psi\in C^{0,\mu}(M)$, $\varphi\in C^{0,\nu}(M)$, both the past and future random correlation of φ and ψ exponential decay with respect to ϕ and the unique random SRB measure $(\mu_\omega)_{\omega\in\Omega}$, i.e. for any $n\in\mathbb{N}$, $\omega\in\Omega$,

$$\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega} - \int_M \psi(x)d\mu_\omega\int_M \varphi(x)d\mu_{\theta^{-n}\omega}\right| \le K\|\psi\|_{C^{0,\mu}(M)}\cdot\|\varphi\|_{C^{0,\nu}(M)}\cdot\Lambda^n;$$

$$\left|\int_M \psi(f^n_\omega x)\varphi(x)d\mu_\omega - \int_M \psi(x)d\mu_{\theta^n\omega}\int_M \varphi(x)d\mu_\omega\right| \le K\|\psi\|_{C^{0,\mu}(M)}\cdot\|\varphi\|_{C^{0,\nu}(M)}\cdot\Lambda^n,$$

where $K>0$ and $\Lambda\in(0,1)$ only depend on μ and ν, and $C^{0,\mu}(M)$, $C^{0,\nu}(M)$ are the spaces of real valued Hölder continuous function with Hölder exponents μ and ν respectively.

Remark 1.3. *For RDS, the exponential decay of random correlations was obtained for ran-*

dom Lasota–Yorke maps on intervals in [15] and for random perturbations of expanding maps in [7]; in [36], the topological one-sided random shift of finite type with the fiber Gibbs measure was proved to have certain nonuniform $\omega-$wise decay of correlations, and similar results hold for random expanding in average transformations. Other decay rates of random correlations such as stretched exponential decay and polynomial decay were also considered for certain random dynamical systems [41, 53].

The proof of Theorem D is based on studying the fiber transfer operator L_ω, which is defined by

$$L_\omega\varphi : M \to \mathbb{R},\ (L_\omega\varphi)(x) := \frac{\varphi((f_\omega)^{-1}x)}{|\det D_{(f_\omega)^{-1}(x)}f_\omega|}$$

for any measurable observable function $\varphi : M \to \mathbb{R}$. We construct the Birkhoff cone on each fiber and introduce the Hilbert projective metric on each fiber Birkhoff cone. We prove that iterations of fiber transfer operators $L^N_\omega = L_{\theta^{N-1}\omega} \circ \cdots \circ L_\omega$ is a contraction, uniformly for all $\omega \in \Omega$, with respect to the Hilbert projective metric on fiber Birkhoff cone, where N comes from the topological mixing on fibers property. The unique random SRB measure and exponential decay of random correlations can be obtained from the contraction.

The Bikhoff cone approach has been used extensively to study the transfer operator and exponential decay of correlations. For deterministic systems, Liverani in [46] used it to prove the exponential decay of correlations for smooth uniformly hyperbolic area-preserving cases. Later, it was applied to general Axiom A attractors in [62, 6], and some partially hyperbolic systems [3, 17]. For RDS, the Birkhoff cone approach was used in random perturbations of $C^k(k > 1)$ expanding maps [7], and in a class of non-uniformly expanding random dynamical systems [61].

1.2 Partially Hyperbolic Systems driven by an external force

We say that the diffeomorphism f_ω driven by the eternal force (Ω, θ) is a partially hyperbolic system driven by the external force (Ω, θ) (or ϕ a random partially hyperbolic on fibers system) if for every $(x,\omega)\in M\times\Omega$, there is a splitting of the tangent bundle of $M_\omega = M\times\{\omega\}$ at x into central-stable and strong unstable directions

$$T_xM_\omega = E^{cs}(x,\omega)\oplus E^{uu}(x,\omega),$$

which depend continuously on $(x,\omega)\in M\times\Omega$ and the splitting is invariant in the sense that

$$D_xf_\omega E^{cs}(x,\omega) = E^{cs}(\phi(x,\omega)),\ D_xf_\omega E^{uu}(x,\omega) = E^{uu}(\phi(x,\omega)),$$

and there exist constants $0 < e^\lambda < e^{\lambda_0} < \infty$, $\lambda_0 > 0$ and $C_0 > 1$ such that

$$\begin{cases} |D_xf_\omega\xi| \geq C_0^{-1}e^{\lambda_0}|\xi|, & \forall\xi\in E^{uu}(x,\omega), \\ |D_xf_\omega\eta| \leq C_0e^{\lambda}|\eta|, & \forall\eta\in E^{cs}(x,\omega). \end{cases} \tag{1.2}$$

We list several examples of random partially hyperbolic on fibers systems in Section 2.2.1, such as random Anosov on fibers systems [34], partially hyperbolic maps on $3-d$ tori driven by minimal irrational rotations on a compact torus, random small perturbations of partially hyperbolic systems, and random composition of (2×2 hyperbolic automorphism $\oplus\ id$) on $\mathbb{T}^2\times S^1$.

By the invariant unstable manifolds theorem (cf., for example, [45]), there exists an embedded strong unstable manifold $W^{uu}(x,\omega)$ tangent to $E^{uu}(x,\omega)$. A random probability measure $(\mu_\omega)_{\omega\in\Omega}$ is called a random Gibbs $u-$state if it has absolutely continuous conditional measure on strong unstable manifolds.

Theorem E. *There exists at least one random Gibbs u-state for C^2 random partially hyperbolic on fibers systems.*

Note that for a random Gibbs $u-$state $(\mu_\omega)_{\omega\in\Omega}$, if for $\mu-$a.s. $(x,\omega)\in M\times\Omega$, the strong unstable manifolds coincide with the unstable manifolds

$$W^u(x,\omega) := \{y \in M : \limsup_{n\to\infty} \frac{1}{n} \log d(f_\omega^{-n}x, f_\omega^{-n}y) < 0\},$$

then it is a random SRB measure [5, 37].

For RDS, the existence of a random SRB measure was obtained in [32] by Kifer and Gundlach for random hyperbolic systems which is random semi-conjugated to a random subshift of finite type. Recently, Wang, Wu and Zhu in [64] proved the existence of Gibbs $u-$state (random SRB measures in our definition) in the case that the RDS has a uniformly dominated splitting is uniformly expanding on the unstable subbundle, and has a non-positive Lyapunov exponents on the central-stable subbundle.

1.3 Plan of the Paper

We organize the paper as follows. In Chapter 2, we introduce random Anosov on fibers systems, random partially hyperbolic on fibers systems, and other notations that will be used in this dissertation. In Chapter 3, we state the formal results for the above two systems respectively. In Chapter 4, we introduce several preliminary lemmas and propositions to pave the way for the future proof. In Chapter 5, we prove Theorems A, B, and C related to the random specification for smooth random Anosov and topological mixing on fibers systems. In Chapter 6, we prove the exponential decay of random correlation (Theorem D) for smooth random Anosov and topological mixing on fibers systems. In Chapter 7, we prove the existence of random Gibb $u-$state (Theorem E) for smooth random partially hyperbolic on fibers systems.

Chapter 2. Settings and Notations

In this chapter, we introduce some basic concepts and notations for future references.

Let M be a connected closed smooth Riemannian manifold of finite dimension, and d_M be the induced Riemannian metric on M. Let $\theta : \Omega \to \Omega$ be a homeomorphism on a compact metric space (Ω, d_Ω) preserving a complete ergodic Borel probability measure P. Denote $\mathcal{B}(\Omega)$ to be the Borel measurable sets on Ω. The product space $M \times \Omega$ is a compact metric space with metric $d((x_1, \omega_1), (x_2, \omega_2)) = d_M(x_1, x_2) + d_\Omega(\omega_1, \omega_2)$ for any $x_1, x_2 \in M$ and $\omega_1, \omega_2 \in \Omega$. Let $\mathcal{H}$ =diff$^2(M)$ be the space of C^2 diffeomorphisms on M equipped with the C^2 topology [33], and let $f : \Omega \to \mathcal{H}$ be a continuous map. The skew product system $\phi : M \times \Omega \to M \times \Omega$ induced by $f(\omega)$ and θ is defined by:

$$\phi(x, \omega) = (f(\omega)x, \theta\omega) = (f_\omega x, \theta\omega), \ \forall \omega \in \Omega, \ x \in M.$$

where we rewrite $f(\omega)$ as f_ω. Then inductively:

$$\phi^n(x, \omega) = (f_\omega^n x, \theta^n \omega) := \begin{cases} (f_{\theta^{n-1}\omega} \circ \cdots \circ f_\omega x, \theta^n \omega), & \text{if } n > 0 \\ (x, \omega), & \text{if } n = 0 \\ ((f_{\theta^n \omega})^{-1} \circ \cdots \circ (f_{\theta^{-1}\omega})^{-1} x, \theta^n \omega), & \text{if } n < 0. \end{cases}$$

2.1 Random Anosov and topological mixing on fibers Systems

Definition 2.1. *The system ϕ is called Anosov on fibers if the following is true: for every $(x, \omega) \in M \times \Omega$, there is a splitting of the tangent bundle of $M_\omega = M \times \{\omega\}$ at x*

$$T_x M_\omega = E^s(x, \omega) \oplus E^u(x, \omega),$$

which depends continuously on $(x, \omega) \in M \times \Omega$ with $\dim E^s_{(x,\omega)}, \ \dim E^u_{(x,\omega)} > 0$ and satisfies that

$$Df_\omega(x) E^u(x, \omega) = E^u(\phi(x, \omega)), \quad Df_\omega(x) E^s(x, \omega) = E^s(\phi(x, \omega)),$$

and

$$\begin{cases} |Df_\omega \xi| \geq e^{\lambda_0}|\xi|, & \forall \xi \in E^u(x,\omega), \\ |Df_\omega(x)\eta| \leq e^{-\lambda_0}|\eta|, & \forall \eta \in E^s(x,\omega), \end{cases}$$

where $\lambda_0 > 0$ *is a constant.*

Random Anosov on fibers system is a special case of random hyperbolic system defined in [32].

Definition 2.2. *We say that* $\phi : \Omega \times M \to \Omega \times M$ *is topological mixing on fibers if for any nonempty open sets* $U, V \subset M$*, there exists* $N > 0$ *such that for any* $n \geq N$ *and* $\omega \in \Omega$

$$\phi^n(\{\omega\} \times U) \cap \{\theta^n\omega\} \times V \neq \emptyset.$$

2.1.1 Examples of Random Anosov and topological mixing on fibers Systems.

In [34], Huang, Lian and Lu showed that the following two types of systems $(S1)$ and $(S2)$ are Anosov and topological mixing on fibers.

$(S1)-$type systems are systems satisfying the following conditions:

(A1) (θ, Ω) is a minimal irrational rotation on the compact torus;

(A2) ϕ is Anosov on fibers;

(A3) ϕ is topological transitive on $M \times \Omega$.

The following example is an $(S1)-$type system:

Example 2.3 (Fiber Anosov maps on $2-d$ tori [34])**.** *Let* $\phi : \mathbb{T}^2 \times \mathbb{T} \to \mathbb{T}^2 \times \mathbb{T}$ *given by*

$$\phi\left(\begin{pmatrix} x \\ y \end{pmatrix}, \omega\right) = \left(\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} h_1(\omega) \\ h_2(\omega) \end{pmatrix}, \omega + \alpha\right),$$

where $\alpha \in \mathbb{R}\backslash\mathbb{Q}$*,* $h_1(\omega)$*,* $h_2(\omega)$ *are continuous map from* $\mathbb{T}$ *to itself.*

$(S2)-$type systems are systems satisfying the following conditions:

(B1) (θ, Ω) is a homeomorphism on a compact metric space;

(B2) ϕ is Anosov on fibers;

(B3) There exists an $f_\omega-$invariant Borel probability measure ν with full support (i.e. supp $\nu = M$) for all $\omega \in \Omega$.

The following example is an $(S2)-$type system.

Example 2.4 (Random composition of a 2×2 area-preserving positive matrices [34]). *Let*

$$\left\{ A_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix} \right\}_{1 \le i \le k}$$

be 2×2 matrices with positive integer entries and $|\det A_i| = 1$ for all $i \in \{1, ..., k\}$, i.e., hyperbolic toral automorphisms. Let $\Omega := S_k = \{1, ..., k\}^{\mathbb{Z}}$ together with the left shift operator σ be the symbolic dynamical system with k symbols. Define $f : \Omega \to \{A_1, ..., A_k\}$ by $f(\omega) = A_{\omega(0)}$ where $\omega = (..., \omega(-1), \omega(0), \omega(1), ...) \in \Omega$. Define $\phi : \mathbb{T}^2 \times \Omega \to \mathbb{T}^2 \times \Omega$ by

$$\phi(x, \omega) = (f(\omega)x, \sigma\omega).$$

Next, we show that random Anosov on fibers systems contain a class of partially hyperbolic systems.

Definition 2.5. *(f, M) is called a partially hyperbolic system in the narrow sense if the tangent bundle admits a splitting into three continuous vector subbundles $T_x M = E^1(x) \oplus E^2(x) \oplus E^3(x)$ which satisfy*

(i) dominated splitting, i.e., $D_x f(E^i(x)) = E^i(f(x))$ for $i = 1, 2, 3$, and there exists constants $c > 0$ and $\lambda \in (0, 1)$ such that $\|Df^n|_{E^i(x)}\| \le c\lambda^n \|Df^n|_{E^{i+1}(x)}\|$ for $i = 1, 2$,

(ii) $E^1(x)$ is uniformly contracted and $E^3(x)$ is uniformly expanded.

We denote the dominated splitting by $T_xM = E^1(x) \oplus_< E^2(x) \oplus_< E^3(x)$.

Let Ω be a compact differentiable manifold, and let $\theta : \Omega \to \Omega$ be a diffeomorphism. Denote $f(x,\omega) := f_\omega(x)$ and $\phi^{-1}(x,\omega) = (f_\omega^{-1}(x), \theta^{-1}\omega)$.

Proposition 2.6. *Assume*

(a) $\phi : M \times \Omega \to M \times \Omega$ is Anosov on fibers,

(b) $f(x,\omega)$ and $f_\omega^{-1}(x)$ are C^1 in ω,

(c) The diffeomophism θ satisfies:

$$\sup_{(x,\omega)\in M\times\Omega} \|D_x f_\omega|_{E^s(x,\omega)}\| < \inf_{\omega\in\Omega} \|D_\omega \theta^{-1}\|^{-1} := m_1$$
$$\leq \sup_{\omega\in\Omega} \|D_\omega \theta\| := m_2 < \inf_{(x,\omega)\in M\times\Omega} \|D_x f_\omega^{-1}|_{E^u(x,\omega)}\|^{-1}.$$

Then ϕ is partially hyperbolic in the narrow sense with dimension-$\dim \Omega$ *central direction.*

Proof. We first show the existence of a dominated splitting. Note that $T_{(x,\omega)}M \times \Omega = T_xM \times T_\omega\Omega$ already has a splitting $E^u(x,\omega) \times \{0\} \oplus E^s(x,\omega) \times \{0\} \oplus \{0\} \times T_\omega\Omega$, but this splitting is not invariant. For any $v \in T_xM \times T_\omega\Omega$, then $v = v_1 + v_2 + v_3$ according to the above splitting. Notice that $\|D\phi(x,\omega)v_3\|$ only depends on $\|D_\omega f(x,\omega)\|$ and $\|D_\omega\theta\|$. $\|D\phi^{-1}(x,\omega)v_3\|$ only depends on $\|D_\omega(f_{\theta^{-1}\omega})^{-1}(x)\|$ and $\|D_\omega\theta^{-1}\|$, then by the compactness of M and Ω, there exists a number K such that

$$\|D\phi(x,\omega)v_3\| \leq K\|v_3\|, \ \|D\phi^{-1}(x,\omega)v_3\| \leq K\|v_3\|.$$

We let $P(E^u(x,\omega)\times\{0\})$ denote the projection map from $T_xM \times T_\omega\Omega$ to $E^u(x,\omega)\times\{0\}$ with respect to the splitting $E^u(x,\omega)\times\{0\} \oplus E^s(x,\omega)\times\{0\} \oplus \{0\}\times T_\omega\Omega$. $P(E^s(x,\omega)\times\{0\})$ and $P(\{0\}\times T_\omega\Omega)$ are similar notations. Since $E^s(x,\omega)$ and $E^u(x,\omega)$ are uniformly continuous on x and ω, there exists a number $\mathcal{P} > 1$ such that

$$\sup\{\|P(E^s(x,\omega)\times\{0\})\|, \|P(E^u(x,\omega)\times\{0\})\| : \ (x,\omega) \in M\times\Omega\} < \mathcal{P}.$$

Now consider the cone

$$C(x,\omega) := \{v \in T_xM \times T_\omega\Omega|\ \|v_2\| + b\|v_3\| \le \|v_1\|\},$$

where b is a number such that

$$b > \frac{2\mathcal{P}K}{e^{\lambda_0} - m_2}.$$

Denote

$$c_0 = \max\{\frac{2\mathcal{P}K + bm_2}{e^{\lambda_0}b}, e^{-2\lambda_0}\} \in (0,1).$$

For any $v \in C(x,\omega)$, we have

$$\begin{aligned}
D\phi(x,\omega)v &= D\phi(x,\omega)v_1 + D\phi(x,\omega)v_2 + D\phi(x,\omega)v_3 \\
&= D\phi(x,\omega)v_1 + P(E^u(\phi(x,\omega)) \times \{0\})D\phi(x,\omega)v_3 \\
&\quad + D\phi(x,\omega)v_2 + P(E^s(\phi(x,\omega)))D\phi(x,\omega)v_3 \\
&\quad + P(\{0\} \times T_{\theta\omega}\Omega)D\phi(x,\omega)v_3 \\
&:= (D\phi(x,\omega)v)_1 + (D\phi(x,\omega)v)_2 + (D\phi(x,\omega)v)_3.
\end{aligned}$$

Then

$$\begin{aligned}
\|(D\phi(x,\omega)v)_2\| + b\|(D\phi(x,\omega)v)_3\| &\le e^{-\lambda}\|v_2\| + \mathcal{P}K\|v_3\| + b \cdot m_2\|v_3\| \\
&= e^{-\lambda}\|v_2\| + (\mathcal{P}K + bm_2)\|v_3\| \\
&\le e^{-\lambda}\|v_2\| + (2\mathcal{P}K + bm_2)\|v_3\| - \mathcal{P}K\|v_3\| \\
&= e^{\lambda}\left(e^{-2\lambda}\|v_2\| + e^{-\lambda}(2\mathcal{P}K + bm_2)\|v_3\|\right) - c_0\mathcal{P}K\|v_3\| \\
&< e^{\lambda}(c_0\|v_2\| + c_0b\|v_3\|) - c_0\mathcal{P}K\|v_3\| \\
&\le c_0e^{\lambda}\|v_1\| - c_0\mathcal{P}K\|v_3\| \\
&\le c_0\|(D\phi(x,\omega)v)_1\|.
\end{aligned}$$

Hence $D\phi(x,\omega)C(x,\omega) \subset intC(\phi(x,\omega))$. By cone-field criteria (Theorem 2.6 in [22]), $T_xM \times T_\omega\Omega$ has a dominated splitting $S_1(x,\omega) \oplus_< S_2(x,\omega)$ with $\dim(S_2(x,\omega)) = \dim(E^u(x,\omega) \times \{0\})$. Notice that $E^u(x,\omega) \times \{0\}$ lies in $C(x,\omega)$ and it is invariant under $D\phi(x,\omega)$, so $S_2(x,\omega) = E^u(x,\omega) \times \{0\}$.

On the other hand, consider another cone

$$\bar{C}(x,\omega) = \{v \in T_xM \times T_\omega\omega : \ \|v_1\| + d\|v_3\| \leq \|v_2\|\},$$

where

$$d \geq \frac{2\mathcal{P}K}{e^\lambda - m_1^{-1}}.$$

Denote

$$c_1 = \max\{\frac{2\mathcal{P}K + m_1^{-1}d}{de^{\lambda_0}}, e^{-2\lambda_0}\} \in (0,1).$$

For any $v \in \bar{C}(x,\omega)$, we have

$$\begin{aligned}
D\phi^{-1}(x,\omega)vD\phi^{-1}(x,\omega)v_1 + D\phi^{-1}(x,\omega)v_2 + D\phi^{-1}(x,\omega)v_3 \\
&= D\phi^{-1}(x,\omega)v_1 + P(E^u(\phi^{-1}(x,\omega)) \times \{0\})D\phi^{-1}(x,\omega)v_3 \\
&\quad + D\phi^{-1}(x,\omega)v_2 + P(E^s(\phi^{-1}(x,\omega)) \times \{0\})D\phi^{-1}(x,\omega)v_3 \\
&\quad + P(\{0\} \times T_\omega\Omega)D\phi^{-1}(x,\omega)v_3 \\
&:= (D\phi^{-1}(x,\omega)v)_1 + (D\phi^{-1}(x,\omega)v)_2 + (D\phi^{-1}(x,\omega)v)_3.
\end{aligned}$$

Then

$$\begin{aligned}
\|(D\phi^{-1}(x,\omega)v)_1\| + d\|(D\phi^{-1}(x,\omega)v)_3\| &\leq e^{-\lambda}\|v_1\| + \mathcal{P}K\|v_3\| + d \cdot m_1^{-1}\|v_3\| \\
&< c_1e^{\lambda}\|v_2\| - c_1\mathcal{P}K\|v_3\| \\
&\leq c_1\|(D\phi^{-1}(x,\omega)v)_2\|.
\end{aligned}$$

Hence $D\phi^{-1}(x,\omega)\bar{C}(x,\omega) \subset int(\bar{C}(\phi^{-1}(x,\omega)))$. By cone-field criteria, $T_xM \times T_\omega\Omega$ has a dominated splitting $H_1(x,\omega) \oplus_< H_2(x,\omega)$ with $\dim H_1(x,\omega) = \dim(E^s(x,\omega) \times \{0\})$. Notice that $E^s(x,\omega) \times \{0\}$ lies in cone $\bar{C}(x,\omega)$ and it is invariant under $D\phi(x,\omega)$, so $H_1(x,\omega) = E^s(x,\omega) \times \{0\}$.

Now $T_xM \times T_\omega\Omega$ has two dominated splittings: $S_1(x,\omega) \oplus_< (E^u(x,\omega) \times \{0\})$ and $(E^s(x,\omega) \times \{0\}) \oplus_< H_2(x,\omega)$. Then, by uniqueness of the dominated splitting (Proposition 2.2 in [22]), we have

$$T_xM \times T_\omega\Omega = (E^s(x,\omega) \times \{0\}) \oplus_< (S_1(x,\omega) \cap H_2(x,\omega)) \oplus_< (E^u(x,\omega) \times \{0\}).$$

Besides, we already know that $E^s(x,\omega) \times \{0\}$ is uniformly contracted under $D\phi(x,\omega)$ and $E^u(x,\omega) \times \{0\}$ is uniformly expanded under $D\phi(x,\omega)$. Hence ϕ is partially hyperbolic. □

2.2 Random Partially Hyperbolic on Fibers Systems

The system ϕ is called partially hyperbolic on fibers if the following is true: for every $(x,\omega) \in M \times \Omega$, there is a splitting of the tangent bundle of $M_\omega = M \times \{\omega\}$

$$T_xM_\omega = E^{uu}(x,\omega) \oplus E^{cs}(x,\omega),$$

which depends continuously on $(x,\omega) \in M \times \Omega$ with $\dim E^{uu}_{(x,\omega)} > 0$ and satisfy that for all $(x,\omega) \in M \times \Omega$

$$D_xf_\omega E^{uu}(x,\omega) = E^{uu}(\phi(x,\omega)),\ D_xf_\omega E^{cs}(x,\omega) = E^{cs}(\phi(x,\omega)),$$

and there exist constants $0 < e^\lambda < e^{\lambda_0} < \infty$, $\lambda_0 > 0$ and $C_0 > 1$ such that

$$\begin{cases} |D_xf_\omega\xi| \geq C_0^{-1}e^{\lambda_0}|\xi|, & \forall\xi \in E^{uu}(x,\omega), \\ |D_xf_\omega\eta| \leq C_0e^{\lambda}|\eta|, & \forall\eta \in E^{cs}(x,\omega). \end{cases} \tag{2.1}$$

2.2.1 Examples of Random Partially Hyperbolic on Fibers Systems.

In this subsection, we list several examples of random partially hyperbolic on fibers systems.

Example 2.7. *All examples of random Anosov on fibers system are random partially hyperbolic on fibers.*

Example 2.8 (Fiber partially hyperbolic maps on 3-d Tori)**.** *Let $\theta : \Omega \to \Omega$ be any homeomorphism on a compact metric space Ω, and let P be an ergodic measure on Ω with respect to θ. Define $\phi : \mathbb{T}^3 \times \Omega \to \mathbb{T}^3 \times \Omega$ by*

$$\phi\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \omega\right) = \left(A\begin{pmatrix} x \\ y \\ z \end{pmatrix} + h(\omega), \theta\omega\right) = \left(\begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} + \begin{pmatrix} h_1(\omega) \\ h_2(\omega) \\ h_3(\omega) \end{pmatrix}, \theta\omega\right),$$

where $h(\cdot) : \Omega \to \mathbb{T}^3$ is a continuous map.

The following example can be obtained by modifying the example in [44].

Example 2.9 (Random Small Perturbations of Partially Hyperbolic Systems)**.** *Let M be a smooth compact Riemannian manifold without boundary, and let $Diff^2(M)$ be the space of C^2 diffeomorphisms from M to M equipped with the C^2 topology [33]. Note that the C^2 topology on $Diff^2(M)$ is metrizable, where we denote the metric generating the C^2 topology by d_{C^2}. Assume $h \in Diff^2(M)$ is partially hyperbolic in the following sense that there is a continuous splitting*

$$TM = E^u \oplus E^{cs}$$

with $\dim E^u > 0$ and a number $\lambda_0 > 0$ such that for any $x \in M$

$$\begin{cases} \limsup_{n\to\infty} \frac{1}{n} \log |D_x h^n \xi| \geq \lambda_0, & \forall \xi \in E^u_x,\ \xi \neq 0, \\ \liminf_{n\to\infty} \frac{1}{n} \log |D_x h^n \eta| \leq 0, & \forall \eta \in E^{cs}_x,\ \eta \neq 0. \end{cases}$$

Let $\mathcal{U}_\epsilon(h)$ be the $\epsilon-$neighborhood of h in the Diff$^2(M)$. Let $K_\epsilon(h) \subset \mathcal{U}_\epsilon(h)$ be any compact set. For $\omega \in \Omega_\epsilon := K_\epsilon(h)^{\mathbb{Z}}$, we denote by $(\cdots, g_{-1}(\omega), g_0(\omega), g_1(\omega), \cdots)$ the sequence of maps corresponding to ω and define the metric on Ω_ϵ by

$$d_{\Omega_\epsilon}(\omega, \omega') = \sum_{i\in\mathbb{Z}} \frac{d_{C^2}(g_i(\omega), g_i(\omega'))}{2^{|i|}}.$$

The metric d_{Ω_ϵ} generates the product topology on Ω_ϵ and as a consequence, Ω_ϵ is a compact metric space. Let $\theta : \Omega_\epsilon \to \Omega_\epsilon$ be the left shift operator, then θ is homeomorphism. Let $f : \Omega_\epsilon \to$Diff$^2(M)$ by $f(\omega) = f_\omega = g_0(\omega)$, then f is a continuous map. Denote

$$f_\omega^n := \begin{cases} g_0(\theta^{n-1}\omega) \circ g_0(\theta^{n-2}\omega) \circ \cdots \circ g_0(\omega), & \text{if } n > 0 \\ id, & \text{if } n = 0 \\ (g_0(\theta^n\omega))^{-1} \circ \cdots \circ (g_0(\theta^{-1}\omega))^{-1} \circ (g_0(\theta^{-1}\omega))^{-1}, & \text{if } n < 0. \end{cases}$$

Proposition 2.10. *Given sufficiently small $\delta > 0$, we can find $\epsilon_\delta > 0$ and a constant A_δ such that the following hold: for every $(\omega, x) \in \Omega_{\epsilon_\delta} \times M$, there is a splitting*

$$T_xM = E^u_{(\omega,x)} \oplus E^{cs}_{(\omega,x)}$$

which depends continuously on (ω, x) and has the following properties:

(i) $D_xf_\omega E^\tau_{(\omega,x)} = E^\tau_{(\theta\omega, f_\omega x)}$ for $\tau = cs, u$;

(ii) for all $n \geq 0$

$$|D_xf_\omega^n\xi| \geq A_\delta^{-2}e^{(\lambda_0-3\delta)n}|\xi|, \ \forall\xi \in E^u_{(\omega,x)},$$
$$|D_xf_\omega^n\eta| \leq A_\delta^2e^{3\delta n}|\eta|, \ \forall\eta \in E^{cs}_{(\omega,x)}.$$

The above Proposition is an adapted version of Proposition 2.2 in [44].

A proof similar to the proof of Proposition 8.2 in [34] can be applied to prove the following example is random partially hyperbolic on fibers system.

Example 2.11. *Let*

$$\left\{ A_i = \begin{pmatrix} a_i & b_i & 0 \\ c_i & d_i & 0 \\ 0 & 0 & 1 \end{pmatrix} \right\}_{1 \le i \le p}$$

be 3×3 *matrices with* $a_i, b_i, c_i, d_i \in \mathbb{Z}^+$, *and* $|a_i d_i - c_i d_i| = 1$ *for any* $i \in \{1, ..., p\}$. *Let* $\Omega = \mathcal{S}_p := \{1, ..., p\}^{\mathbb{Z}}$ *with the left shift operator* θ *be the symbolic dynamical system with* p *symbols.*

For any $\omega = (..., \omega_{-1}, \omega_0, \omega_1, ...) \in \Omega$ *define* $f(\omega) = A_{\omega_0}$. *Then the skew product* $\phi : \mathbb{T}^3 \times \Omega \to \mathbb{T}^3 \times \Omega$ *defined by*

$$\phi(x, \omega) = (f(\omega)x, \theta\omega)$$

is partially hyperbolic on fibers satisfying our setting.

2.3 Random Probability Measures

In this section, we introduce the theory of random probability measures, most of which are taken from [21]. Denote $Pr(M)$ to be the space of probability measures on $(M, \mathcal{B}(M))$ equipped with the narrow topology, where the narrow topology on $Pr(M)$ is the smallest topology that makes $\rho \mapsto \rho(g)$ continuous for $\rho \in Pr(M)$ and g a continuous function on M. By Theorem A.2 in [21], we pick

$$d_p(\rho, \zeta) = \sup\{\rho(g) - \zeta(g) : g \in BL(M),\ 0 \le g \le 1, [g]_L \le 1\}$$

for a metric generating the narrow topology, where $BL(M)$ is the set of Lipschitz functions on M and $[g]_L$ is the Lipschitz constant of g.

Definition 2.12. *A map* $\mu : \mathcal{B}(M) \times \Omega \to [0, 1]$ *by* $(B, \omega) \mapsto \mu_\omega(B)$ *satisfying* *(i) for every $B \in \mathcal{B}(M)$, $\omega \mapsto \mu_\omega(B)$ is measurable,*

(ii) for $P-$almost every $\omega \in \Omega$, $B \to \mu_\omega(B)$ is a Borel probability measure,

is said to be a random probability measure on M, and is denoted by $\omega \mapsto \mu_\omega$ or $(\mu_\omega)_{\omega\in\Omega}$.

Remark 2.13. *If $\mu : \mathcal{B}(M)\times\Omega \to [0,1]$ satisfies (ii) from Definition 2.12, and if $\omega \mapsto \mu_\omega(K)$ is measurable for every K from a $\cap-$closed family $\mathcal{K}$ of Borel subsets of M which generates $\mathcal{B}(M)$ (i.e., $\sigma(\mathcal{K}) = \mathcal{B}(M)$), then (i) is satisfied as well, hence μ is a random probability measure. In fact, $\mathcal{D} = \{D \in \mathcal{B}(M) : \omega \to \mu_\omega(D) \text{is measurable}\}$ is a Dynkin system and $\mathcal{K} \subset \mathcal{D}$. By Dynkin's $\pi - \lambda$ theorem, $\mathcal{B}(M) = \sigma(\mathcal{K}) \subset \mathcal{D}$.*

As a consequence, it is sufficient to have (ii) together with the measurability of $\omega \to \mu_\omega(K)$ for all closed sets $K \subset M$ to conclude that $(\mu_\omega)_{\omega\in\Omega}$ is a random probability measure.

Note that by Remark 3.20 in [21], μ is a random measure if and only if $\omega \mapsto \mu_\omega$ is measurable with respect to the Borel $\sigma-$algebra of the narrow topology on $Pr(M)$. Denote $Pr_\Omega(M)$ to be the collection of all random probability measures.

Denote $Pr_P(M\times\Omega)$ to be the space of probability measures on $(M\times\Omega, \mathcal{B}(M\times\Omega))$ with marginal P on Ω. There is an isomorphism between $Pr_\Omega(M)$ and $Pr_P(M\times\Omega)$ in the sense of disintegration, i.e., for any $\mu \in Pr_P(M \times \Omega)$ there exists a random probability measure $\omega \to \mu_\omega$ such that

$$\int_{M\times\Omega} h(x,\omega)d\mu(x,\omega) = \int_\Omega \int_M h(x,\omega)d\mu_\omega(x)dP(\omega)$$

for every bounded measurable $h : M \times \Omega \to \mathbb{R}$. Moreover, this disintegration is $P-$a.e. unique .

Definition 2.14. *A function $h : M \times \Omega \to \mathbb{R}$ is called a random continuous function if*

(i) $\omega \to h(x,\omega)$ is measurable for fixed $x \in M$;

(ii) $x \to h(x,\omega)$ is continuous for fixed $\omega \in \Omega$ and $\sup_{x\in M} |h(x,\omega)| \in L^1(\Omega, \mathcal{B}(\Omega), P)$.

We denote $C_\Omega(M)$ to be the collection of all random continuous functions on M. The narrow topology on $Pr_\Omega(M)$ is generated by the map $\mu \mapsto \mu(h)$ for $h \in C_\Omega(M)$.

Since $(\Omega, \mathcal{B}(\Omega))$ is a compact metric space, its Borel $\sigma-$algebra is countably generated. Combining with the fact that M is a compact manifold, we know $Pr_\Omega(M)$ is a compact topological space under the narrow topology by Theorem 4.4 in [21]. Moreover, by Theorem 4.16 in [21], the narrow topology on $Pr_\Omega(M)$ can be metrized by the following random Prohorov metric

$$d_{rp}(\mu, \nu) = \sum_{m\in\mathbb{N}} \frac{1}{2^m} \sup \left\{ \int_{G_m} \mu_\omega(g) - \nu_\omega(g) dP(\omega) : \ g \in BL(M), 0 \le g \le 1, [g]_L \le 1 \right\},$$

for any $\mu, \nu \in Pr_\Omega(M)$, where $\{G_m : \ m \in \mathbb{N}\}$ is a countable algebra generating $\mathcal{B}(\Omega)$.

Definition 2.15. *A set valued map $C : \Omega \to 2^M$ is said to be a random closed set if*

(i) for each $\omega \in \Omega$, $C(\omega)$ is closed;

(ii) for each $x \in M$, the map $\omega \mapsto d(x, C(\omega))$ is measurable.

A set valued map $\omega \mapsto U(\omega)$ is said to be a random open set if its complement $\omega \mapsto U^c(\omega)$ is a random closed set.

The following proposition comes from corollary 2.10 in [21].

Proposition 2.16. *If $\omega \mapsto C(\omega)$ is a random closed set, then its interior intC is an random open set.*

Proposition 2.17 (The Selection Theorem)**.** *A set valued map $C : \Omega \to 2^M$ is a random closed set if and only if there exists a sequence $\{c_n\}_{n\in\mathbb{N}}$ of measurable maps $c_n : \Omega \to M$, such that $C(\omega) = closure\{c_n(\omega) : \ n \in \mathbb{N}\}$ for all $\omega \in \Omega$.*

The following is part of the Portmanteau theorem for random probability measures.

Proposition 2.18 (The Portmanteau Theorem)**.** *If $\mu_n \in Pr_\Omega(M)$, then the following statements are equivalent:*

(i) $\mu_n \to \mu$ *in the narrow topology ;*

(ii) $\limsup_{n\to\infty} \mu_n(C) \le \mu(C)$ *for all random closed sets* C*, where* $\mu(C) = \int_\Omega \mu_\omega(C(\omega))dP(\omega)$*;*

(iii) $\liminf_{n\to\infty} \mu_n(U) \ge \mu(U)$ *for all random open sets* U*, where* $\mu(U) = \int_\Omega \mu_\omega(U(\omega))dP(\omega)$*.*

2.4 Topological Fiber Entropy

In this section, we introduce the topological fiber entropy for random dynamical systems. Most of the notations are borrowed from [37].

For each $n \in \mathbb{N}$, and $\omega \in \Omega$, we define a family of metrics $d_{\omega,n}$ on M by

$$d_{\omega,n}(x, y) = \max_{0\le k<n}\{d(f_\omega^k(x), f_\omega^k(y))\}, \text{ for any } x, y \in M.$$

Definition 2.19. *A set* $E_\omega \subset M$ *is called* $(\omega, \epsilon, n)-$*seperated if for any* $x, y \in E_\omega$*,* $x \ne y$ *implies* $d_{\omega,n}(x, y) > \epsilon$*.*

Due to the compactness of M_ω, there exists a smallest natural number $N(\omega, \epsilon, n)$ such that $card(E_\omega) \le N(\omega, \epsilon, n) < \infty$ for every $(\omega, \epsilon, n)-$seperated set E_ω. Moreover, there always exits a maximal $(\omega, \epsilon, n)-$seperated set E_ω in the sense that for every $x \in M_\omega$ with $x \notin E_\omega$, the set $E_\omega \cup \{x\}$ is not $(\omega, \epsilon, n)-$seperated anymore. If E_ω is maximal $(\omega, \epsilon, n)-$seperated, then $M_\omega = \cup_{x\in E_\omega} B_x(\omega, \epsilon, n)$, where $B_x(\omega, \epsilon, n)$ is the closed ball in M_ω centered at x of radius ϵ with respect to the metric $d_{\omega,n}$.

Definition 2.20. *The topological fiber entropy on the fiber* $M_\omega = M \times \{\omega\}$ *is defined by*

$$h_{top}(\phi|_{M_\omega}) = \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \log N(\omega, \epsilon, n).$$

The fiber topological entropy of random dynamical system F *or the relative topological entropy of* ϕ *is defined by*

$$h_{top}(F) = h_{top}^{(r)}(\phi) = \lim_{\epsilon\to 0^+} \limsup_{n\to\infty} \frac{1}{n} \int \log N(\omega, \epsilon, n) dP(\omega),$$

recall that P is an ergodic Borel probability measure we fixed in the beginning of this chapter.

Since the noise system (Ω, θ, P) is an ergodic dynamical system, we obtain the following proposition by Proposition 1.2.6 in [37].

Proposition 2.21. *The following equalities hold:*

$$h_{top}(F) = h_{top}^{(r)}(\phi) = h_{top}(\phi|_{M_\omega}) = \lim_{\epsilon \to 0^+} \liminf_{n \to \infty} \frac{1}{n} \log N(\omega, \epsilon, n)$$

for P-a.s. $\omega \in \Omega$.

Chapter 3. Main Results

In this chapter, we state our main results for random Anosov and topological mixing on fibers systems and random partially hyperbolic on fibers systems.

3.1 For Random Anosov and topological mixing on fibers Systems

In this section, we formulate the main results for random Anosov and topological mixing on fibers systems. Subsection 3.1.1 addresses the results related to the random specification. Subsection 3.1.2 addresses the result for exponential decay of random correlations.

3.1.1 Random Specification Property.

We start with the formal definition of random specification. Let $L^\infty(\Omega, M)$ be the space of Borel measurable maps from Ω to M endowed with the following sup-metric

$$d_{L^\infty(\Omega,M)}(g_1, g_2) = \sup_{\omega \in \Omega} d_M(g_1(\omega), g_2(\omega)), \text{ for } g_1, g_2 \in L^\infty(\Omega, M).$$

The system ϕ induces a map $\tilde{\phi}$ from $L^\infty(\Omega, M)$ to itself by $\tilde{\phi}(g)(\omega) := f_{\theta^{-1}\omega} g(\theta^{-1}\omega)$ for all $\omega \in \Omega$ and $g \in L^\infty(\Omega, M)$. One can see that

$$\phi(graph(g)) = graph(\tilde{\phi}(g)).$$

Moreover, $\tilde{\phi}$ is invertible since f_ω is diffeomorphism on M for all $\omega \in \Omega$, and $\tilde{\phi}^{-1}(g)(\omega) = (f_\omega)^{-1}(g(\theta\omega))$.

Definition 3.1. *For $g \in L^\infty(\Omega, M)$, g is called a random periodic point of ϕ if there exists an integer n such that*

$$\phi^n(graph(g)) = graph(g) \text{ or } \tilde{\phi}^n g = g.$$

Remark 3.2. *Under our assumptions, $\tilde{\phi}$ defines a homeomorphism on $L^\infty(\Omega, M)$ with respect to the sup-metric. In fact, for any $g_1, g_2 \in L^\infty(\Omega, M)$, we have*

$$d_{L^\infty(\Omega,M)}(\tilde{\phi}(g_1), \tilde{\phi}(g_2)) = \sup_{\omega\in\Omega} d_M(f_{\theta^{-1}\omega} g_2(\theta^{-1}\omega), f_{\theta^{-1}\omega} g_2(\theta^{-1}\omega))$$
$$\leq \sup_{\omega\in\Omega} \|f_\omega\|_{C^1} \sup_{\omega\in\Omega} d_M(g_1(\theta^{-1}\omega), g_2(\theta^{-1}\omega)) \leq \sup_{\omega\in\Omega} \|f_\omega\|_{C^1} d_{L^\infty(\Omega,M)}(g_1, g_2).$$

Therefore, $\tilde{\phi}$ is continuous. Similarly, we have

$$d_{L^\infty(\Omega,M)}(\tilde{\phi}^{-1}(g_1), \tilde{\phi}^{-1}(g_2)) \leq \sup_{\omega\in\Omega} \|f_\omega^{-1}\|_{C^1} d_{L^\infty(\Omega,M)}(g_1, g_2).$$

Hence, $\tilde{\phi}$ is a homeomorphism on $L^\infty(\Omega, M)$.

Definition 3.3. *A $\omega-$specification $S_\omega = (\omega, \tau, P_\omega)$ consists of a finite collection of intervals $\tau = \{I_1, ..., I_m\}$, $I_i = [a_i, b_i] \subset \mathbb{Z}$, and a map $P_\omega : \cup_{i=1}^m I_i \to M$ such that for $t_1, t_2 \in I \in \tau$,*

$$\phi^{t_2-t_1}(P_\omega(t_1), \theta^{t_1}\omega) = (P_\omega(t_2), \theta^{t_2}\omega).$$

A random specification $S = (\tau, P)$ consists of a finite collection of intervals $\tau = \{I_1, ..., I_m\}$,

$I_i = [a_i, b_i] \subset \mathbb{Z}$, and a map $P : \cup_{i=1}^m I_i \to L^\infty(\Omega, M)$ such that for $t_1, t_2 \in I \in \tau$,

$$\phi^{t_2-t_1}(graph(P(t_1))) = graph(P(t_2)) \text{ or } \tilde{\phi}^{t_2-t_1}(P(t_1)) = P(t_2).$$

A random specification S is called $n-$spaced if $a_{i+1} > b_i + n$ for all $i \in \{1, ..., m-1\}$ and the minimal such n is called the spacing of this random specification. Denote $L(S) := b_m - a_1$.

Remark 3.4. *On one hand, if $S = (\tau, P)$ is a random specification, then for any fixed ω, $S_\omega = (\omega, \tau, P_\omega)$ defined by $P_\omega(t) := P(t)(\theta^t\omega)$ for $t \in I \in \tau$ is a $\omega-$specification.*

On the other hand, if

(i) $S_\omega = (\omega, \tau, P_\omega)$ is a $\omega-$specification,

(ii) $P_{(\cdot)}(t) : \Omega \to M$ is Borel measurable for each fixed $t \in I \in \tau$,

and we define $P(t) : \Omega \to M$ by $P(t)(\omega) = P_{\theta^{-t}\omega}(t)$ for each fixed $t \in I \in \tau$, Then $P(t) \in L^\infty(\Omega, M)$, and moreover, $S = (\tau, P)$ defines a random specification.

Definition 3.5. *The system ϕ is said to have the random specification property if for any $\epsilon > 0$, there exists $N = N(\epsilon) > 0$ such that any N-spaced random specification $S = (\tau, P)$ is $\epsilon-$shadowed by an element g in $L^\infty(\Omega, M)$, i.e.,*

$$d_{L^\infty(\Omega,M)}(P(t), \tilde{\phi}^t(g)) < \epsilon, \ \forall t \in \cup_{i=1}^m I_i.$$

Moreover, for any $q \geq N + b_m - a_1$, there is a random periodic point g with period q $\epsilon-$shadowing the random specification S.

Remark 3.6. *Let's recall the definition of Bowen's specification property. Let $T : X \to X$ be a homeomorphism of a metric space (X, d_X). A specification $S = (\tau, P)$ consists of a finite collection $\tau = \{I_1, ..., I_m\}$ of finite intervals $I_i = [a_i, b_i] \subset \mathbb{Z}$, and a map $P : \cup_{i=1}^m I_i \to X$ such that for $t_1, t_2 \in I \in \tau$, we have $T^{t_2-t_1}(P(t_1)) = P(t_2)$. (X, T) is said to have the specification property if for any $\epsilon > 0$ there exists an $N = N_\epsilon \in \mathbb{N}$ such that any $N-$spaced specification*

S is $\epsilon-$shadowed by a point $x \in X$, i.e., $d_X(f^t(x), P(t)) < \epsilon$ for all $t \in I \in \tau$, and for $q > N + b_m - a_1$, then the shadowing point x could be a periodic point.

Then in the Definition 3.5, the statement ϕ is said to have the random specification property is equivalent to the statement that the deterministic system $(L^\infty(\Omega, M), d_{L^\infty(\Omega,M)}, \tilde{\phi})$ has Bowen's specification property.

Theorem 3.7. *Assume that ϕ satisfies Anosov on fibers and topological mixing on fibers, then ϕ has the random specification property. On the other hand, the random specification property implies topological mixing on fibers property.*

We define $\phi^* : Pr_\Omega(M) \to Pr_\Omega(M)$ by $(\phi^*\mu)_\omega := (f_{\theta^{-1}\omega})_*\mu_{\theta^{-1}\omega}$, i.e., $(\phi^*\mu)_\omega(B) = \mu_{\theta^{-1}\omega}((f_{\theta^{-1}\omega})^{-1}(B))$ for any $B \in \mathcal{B}(M)$.

Theorem 3.8. *Assume that ϕ satisfies Anosov on fibers and topological mixing on fibers, then $\phi^* : Pr_\Omega(M) \to Pr_\Omega(M)$ defines a homeomorphism with respect to the narrow topology on $Pr_\Omega(M)$. Moreover, the topological dynamical system $(Pr_\Omega(M), \phi^*)$ has Bowen's specification property.*

Theorem 3.9. *If ϕ has the random specification property, then for any ϵ, there exists two integers k, N such that for all ω, the topological fiber entropy on M_ω satisfies $h_{top}(\phi|_{M_\omega}) \geq \frac{\log k}{N}$, where k is the maximal cardinality of the $3\epsilon-$seperated set in M with respect to metric d_M and $N = N(\epsilon)$ is the number in random specification corresponding to ϵ.*

3.1.2 Exponential Decay of Random Correlation.

In this subsection, we formulate the result for exponential decay of random correlations.

Let $C(M)$ be the collection of all continuous functions $\varphi : M \to \mathbb{R}$. For $\alpha \in (0, 1)$, and $\varphi \in C(M)$, let

$$\|\varphi\|_{C^0(M)} := \sup_{x \in M} |\varphi(x)| \text{ and } |\varphi|_\alpha := \sup_{x,y \in M, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{d(x, y)^\alpha}.$$

We denote by $C^{0,\alpha}(M) := \{\varphi \in C(M): \ |\varphi|_\alpha < \infty\}$ the space of $\alpha-$Hölder continuous functions on M. For $\varphi \in C^{0,\alpha}(M)$, we let

$$\|\varphi\|_{C^{0,\alpha}(M)} := \|\varphi\|_{C^0(M)} + |\varphi|_\alpha.$$

For any $\mu \in Pr_\Omega(M)$, recall $\phi^*\mu$ is defined by $(\phi^*\mu)_\omega := (f_{\theta^{-1}\omega})_*\mu_{\theta^{-1}\omega}$, i.e., $(\phi^*\mu)_\omega(B) = \mu_{\theta^{-1}\omega}((f_{\theta^{-1}\omega})^{-1}(B))$ for any $B \in \mathcal{B}(M)$. A random probability measure μ is $\phi-$invariant if $(\phi^*\mu)_\omega = \mu_\omega$ for $P-$a.e. $\omega \in \Omega$.

Theorem 3.10. *Assume ϕ satisfies Anosov on fibers and topological mixing on fibers. Then*

(i) the random probability measure $\omega \mapsto \mu_\omega$ given by $\mu_\omega = \lim_{n\to\infty}(f^n_{\theta^{-n}\omega})_* m$ is $\phi-$invariant, where m is the normalized Riemannian volume measure;

(ii) there exists a constant ν_0 only depending on the system ϕ. For Hölder exponents $\mu, \nu \in (0,1)$ with

$$0 < \mu + \nu < \nu_0$$

and $\psi \in C^{0,\mu}(M)$, $\varphi \in C^{0,\nu}(M)$, the past and future random correlation of φ and ψ exponential decay with respect to the system ϕ and the random probability measure $(\mu_\omega)_{\omega\in\Omega}$ defined in (i), i.e. for any $n \in \mathbb{N}$, $\omega \in \Omega$,

$$\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega} - \int_M \psi(x)d\mu_\omega \int_M \varphi(x)d\mu_{\theta^{-n}\omega}\right| \le K\|\psi\|_{C^{0,\mu}(M)}\|\varphi\|_{C^{0,\nu}(M)}\Lambda^n;$$
$$\left|\int_M \psi(f^n_\omega x)\varphi(x)d\mu_\omega - \int_M \psi(x)d\mu_{\theta^n\omega} \int_M \varphi(x)d\mu_\omega\right| \le K\|\psi\|_{C^{0,\mu}(M)}\|\varphi\|_{C^{0,\nu}(M)}\Lambda^n,$$

where $K > 0$ and $\Lambda \in (0,1)$ only depend on μ and ν.

Note that topological mixing on fibers property implies random topological transitivity By Lemma A.1 in [34] and then by Theorem 4.3 in [32], the measure μ_ω we constructed above is the unique SRB measure (we state this lemma and this theorem in the Appendix).

3.2 Random Gibbs $u-$state for Random Partially Hyperbolic on Fibers Systems

Let ϕ be a C^2 random partially hyperbolic on fibers system. With the help of the unstable manifolds theorem in [37], the global strong unstable manifold of ϕ at (x,ω) is defined by

$$W^{uu}(x,\omega) = \{y \in M : \limsup_{n\to+\infty} \frac{1}{n} \log d_M(f_\omega^{-n}x, f_\omega^{-n}y) \leq -\lambda_0\}, \tag{3.1}$$

which is the image of $E^{uu}(x,\omega)$ under an injective immersion of class $C^{1,1}$ and is tangent to $E^{uu}(x,\omega)$ at (x,ω). Notice that $\{W^{uu}(x,\omega)\}$ forms a partition of $M\times\Omega$, but in general, such a partition is non-measurable.

Definition 3.11. *Given a $\phi-$invariant random probability measure μ on $M\times\Omega$. A measurable partition $\mathcal{P}$ of $M\times\Omega$ is called u$-$subordinate if for $\mu-$a.e. $(x,\omega)\in M\times\Omega$, $\mathcal{P}(x,\omega)\subset W^{uu}(x,\omega)$ and $\mathcal{P}(x,\omega)$ contains an open neighborhood of x contained in $W^{uu}(x,\omega)$, this neighborhood being taken in the submanifold topology of $W^{uu}(x,\omega)$. The invariant random probability measure μ is called a random Gibbs u$-$state if it has absolutely continuous measures on strong unstable manifolds, i.e., for every measurable u$-$subordinate partition $\mathcal{P}$, one has*

$$\mu^{\mathcal{P}}_{(x,\omega)} \ll \lambda^u_{(x,\omega)}$$

for $\mu-$a.e. $(x,\omega)\in M\times\Omega$, where $\mu^{\mathcal{P}}_{(x,\omega)}$ denotes the conditional probability measure of μ on $\mathcal{P}(x,\omega)$ and $\lambda^u_{(x,\omega)}$ denotes the Riemannian volume measure on $W^{uu}(x,\omega)$ induced by its inherited Riemannian structure as a submanifold of M.

Theorem 3.12. *If ϕ is C^2 partially hyperbolic on fibers, then there exists at least one invariant random Gibbs u$-$state of ϕ.*

Chapter 4. Preliminary Lemmas and Propositions

In this chapter, we state several lemmas and propositions for random Anosov on fibers systems and random partially hyperbolic systems in Section 4.1 and Section 4.2 respectively.

4.1 For Random Anosov on Fibers Systems

In this section, we introduce several technical lemmas and propositions that will be used in the proof of the main result for random Anosov on fibers systems. In Subsections 4.1.1, we state that the stable subbundle $E^s(x,\omega)$ and the unstable subbundle $E^u(x,\omega)$ are not only continuous on x, but also Hölder continuous on x. In Subsection 4.1.2, we state the stable and unstable manifolds theorem. The random shadowing lemma in Subsection 4.1.3 and the density of random periodic points lemma in Subsection 4.1.4 are critical in the proof of random specification. We state and prove two distortion lemmas in Subsection 4.1.5. We formulate and prove the absolute continuity and Hölder continuity of the stable and unstable foliations on each fiber in Subsection 4.1.6 and 4.1.7 respectively. We discuss properties of holonomy maps between local stable leaves in Subsection 4.1.8. In Subsection 4.1.9, we prove a version of Fubini's theorem on each subset of M that is foliated by local stable manifolds and has local product structure.

4.1.1 Fiberwisely Hölder continuity of stable and unstable subbundles.

In this subsection, we will formulate the Hölder continuity of $E^\tau(x,\omega)$ for fixed ω and $\tau = s,u$ which is an adapted version of Theorem 4.1 in [45].

By applying the normal neighborhood theorem (Theorem 3.7 in [16]), for each point $p \in M$, there exists a neighborhood $N_p \subset M$ and constant δ such that the exponential map $Exp_p : B_\delta(0) \subset TpM \to M$ is a $C^\infty-$diffeomorphism and $N_p \subset Exp_p(B_\delta(0))$. Since the norm in T_pM is given by the Riemannian metric, we choose an orthonormal basis $e_1, ..., e_n$ of T_pM, then define $\psi : N_p \to \mathbb{R}^n$ by $\psi(Exp_p(\sum_{i=1}^n x_i e_i)) = (x_1, ..., x_n)$ to be the normal coordinate charts. By compactness of M, we can choose a set of finite points $\{p_i\}_{i=1}^l$ together with $\{N_{p_i}, \psi_i\}$ to form coordinate charts of M. Throughout this paper, we will fix these normal coordinate charts.

By the compactness of M, there exists a $\rho_0 > 0$ such that every subset of M having diameter less than ρ_0 is contained in one of the normal coordinate charts. From now on, we fix this ρ_0.

For subspaces $A, B \subset \mathbb{R}^n$, define the aperture between two subspaces by

$$\Gamma(A, B) := \max\{\max_{v\in A, |v|=1} \inf_{w\in B} |v - w|, \max_{w\in B, |w|=1} \inf_{v\in A} |v - w|\}.$$

Then $\Gamma(A, B) \in [0, 1]$.

For any $x, y \in M$, if $d(x, y) < \rho_0$, then we have an isometry from T_xM to T_yM given by the parallel transport on the unique geodesic connecting x and y, named $P(x, y)$. Then for any $x, y \in M$, $E(x) \subset T_xM$, $E(y) \subset T_yM$ subspaces, we can define

$$d(E(x), E(y)) := \begin{cases} \Gamma(E(x), P(y, x)E(y)), & \text{if } d(x, y) < \rho_0 \\ 1, & \text{otherwise.} \end{cases} \tag{4.1}$$

Lemma 4.1. *There are constants C_1 and ν_1 which both are independent of $(x, \omega) \in M \times \Omega$ such that for each $\omega \in \Omega$, the stable and unstable distribution $E^s(x, \omega), E^u(x, \omega)$ are Hölder continuous on x with constant C_1 and Hölder exponent ν_1, i.e.,*

$$d(E^\tau(x, \omega), E^\tau(y, \omega)) \le C_1 d(x, y)^{\nu_1}, \quad \tau = s, u.$$

4.1.2 Stable and Unstable Invariant Manifolds.

We define the local stable and unstable manifolds as the following:

$$W^s_\epsilon(x,\omega) = \{y \in M_\omega |\ d(\phi^n(y,\omega), \phi^n(x,\omega)) \le \epsilon \text{ for all } n \ge 0\},$$
$$W^u_\epsilon(x,\omega) = \{y \in M_\omega |\ d(\phi^n(y,\omega), \phi^n(x,\omega)) \le \epsilon \text{ for all } n \le 0\}.$$

The following lemma can be found in [34], and it is a special version of Theorem 3.1 in [32].

Denote by $P(E^\tau(x,\omega))$ the projection from T_xM_ω to $E^\tau(x,\omega)$ with respect to $T_xM_\omega = E^s(x,\omega) \oplus E^u(x,\omega)$ for $\tau = s, u$. Since $E^s(x,\omega),\ E^u(x,\omega)$ are uniformly continuous on x and ω, there exists a number $\mathcal{P} > 1$ such that

$$\sup\{\|P(E^s(x,\omega))\|, \|P(E^u(x,\omega))\| :\ (x,\omega) \in M \times \Omega\} < \mathcal{P}. \tag{4.2}$$

Lemma 4.2 (Stable and unstable invariant manifolds). *For any $\lambda \in (0, \lambda_0)$, there exists $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0]$, the followings hold:*

(i) $W^\tau_\epsilon(x,\omega)$ are C^2 embedded discs for all $(x,\omega) \in M \times \Omega$ with $T_xW^\tau(x,\omega) = E^\tau(x,\omega)$ for $\tau = u, s$. Moreover, there exist a constant $L > 1$ and C^2 maps

$$h^u_{(x,\omega)} : E^u(x,\omega)(\mathcal{P}\epsilon) \to E^s(x,\omega),\ \ h^s_{(x,\omega)} : E^s(x,\omega)(\mathcal{P}\epsilon) \to E^u(x,\omega)$$

such that $W^\tau_\epsilon(x,\omega) \subset Exp_x(graph(h^\tau_{(x,\omega)}))$ and $\|Dh^\tau_{(x,\omega)}\| < \frac{1}{3}$, $Lip(Dh^\tau_{(x,\omega)}) < L$ for $\tau = u, s$.

(ii) $d_M(f^n_\omega x, f^n_\omega y) \le e^{-n\lambda} d_M(x,y)$ for $y \in W^s_\epsilon(x,\omega)$ and $n \ge 0$, and $d_M(f^{-n}_\omega x, f^{-n}_\omega y) \le e^{-n\lambda} d_M(x,y)$ for $y \in W^u_\epsilon(x,\omega)$ and $n \ge 0$.

(iii) $W^s_\epsilon(x,\omega), W^u_\epsilon(x,\omega)$ vary continuously on (x,ω) in C^1 topology.

In this paper, we restrict ϵ_0 such that $\epsilon_0 < \rho_0$, i.e., $W^\tau_\epsilon(x,\omega)$ is covered by some normal coordinate charts for any $(x,\omega) \in M \times \Omega$ and $\tau = s, u$.

Lemma 4.3 (Local product structure)**.** *For any* $\epsilon \in (0, \epsilon_0)$*, there is a* $\delta \in (0, \epsilon)$ *such that for any* $x, y \in M$ *with* $d_M(x, y) < \delta$*,* $W^s_\epsilon(x, \omega) \cap W^u_\epsilon(y, \omega)$ *consists of a single point, which is denoted by* $[x, y]_\omega$*. Furthermore*

$$[\cdot, \cdot]_\omega : \{(x, y, \omega) \in M \times M \times \Omega |\ d_M(x, y) < \delta\} \to M$$

is continuous.

Corollary 4.4 (Expansivity)**.** *The system* ϕ *is expansive in the sense that if* $d(\phi^n(x, \omega), \phi^n(y, \omega)) < \epsilon$ *for all* $n \in \mathbb{Z}$*, then* $x = y$ *for any* $\epsilon \in (0, \epsilon_0)$*, where* ϵ_0 *is the size of stable and unstable manifolds.*

4.1.3 Random Shadowing Lemma.

For any $\alpha > 0$, a sequence of points $\{(x_i, \theta^i\omega)\}_{i\in\mathbb{Z}} \subset M \times \Omega$ is called an $(\omega, \alpha)-$pseudo orbit if for any $i \in \mathbb{Z}$,

$$d(\phi(x_i, \theta^i\omega), (x_{i+1}, \theta^{i+1}\omega)) < \alpha.$$

The next lemma follows the proof of Proposition 3.7 in [32].

Lemma 4.5 (Random shadowing lemma)**.** *For any* $\epsilon > 0$*, there exists* $\alpha = \alpha(\epsilon) > 0$ *such that any* $(\omega, \alpha)-$*pseudo orbit* $\{(x_i, \theta^i\omega)\}_{i\in\mathbb{Z}}$ *can be* $(\omega, \epsilon)-$*shadowed by a point* $x \in M$*, i.e.*

$$d(\phi^i(x, \omega), (x_i, \theta^i\omega)) < \epsilon.$$

Furthermore, when $\epsilon < \frac{1}{2}\epsilon_0$ *where* ϵ_0 *is the size of local stable and unstable manifolds, the shadowing point is unique.*

Corollary 4.6. *For any* $\epsilon \in (0, \epsilon_0/2)$*, there exists* $\alpha = \alpha(\epsilon) > 0$ *such that for any sequence of measurable functions* $\{g_i\}_{i=-\infty}^{\infty}$ *with* $g_i \in L^\infty(\Omega, M)$ *satisfying*

$$d_{L^\infty(\Omega,M)}(\tilde{\phi}(g_i), g_{i+1}) < \alpha.$$

There exists a unique $g \in L^\infty(\Omega, M)$ *such that* $d_{L^\infty(\Omega,M)}(\tilde{\phi}^t(g), g_t) < \epsilon$ *for all* $t \in \mathbb{Z}$. *In particular, if* $\{g_i\}_{i=0}^{n-1}$ *is an* $\alpha-$*pseudo periodic orbit, then the shadowing point* g *is periodic with period* n.

Proof. For any $\omega \in \Omega$, define

$$(y_i, \theta^i\omega) := (g_i(\theta^i\omega), \theta^i\omega) \text{ for } i \in \mathbb{Z}.$$

Since $d_{L^\infty(\Omega,M)}(\tilde{\phi}(g_i), g_{i+1}) < \alpha$, $(y_i, \theta^i\omega)$ is an $(\omega, \alpha)-$pseudo orbit, then by the random shadowing lemma, there exists a unique $g(\omega) \in M$ $(\omega, \epsilon)-$shadowing this sequence. We just need to prove $\omega \mapsto g(\omega)$ is measurable.

Define a multivalued function $G_i : \Omega \to 2^M$ for any $i \in \mathbb{N}$ as the following

$$G_i(\omega) = \bigcap_{-i \leq j \leq i} \pi_M \phi^{-j}\{(x, \theta^j\omega)|\ d_M(x, g_j(\theta^j\omega)) \leq \epsilon\}.$$

For each $\omega \in \Omega$, $G_i(\omega)$ is a nonempty closed set since the existence of shadowing point and continuity of $\phi(x, \omega)$ on x. Note that for each fixed j between $-i$ and i, the set

$$\omega \mapsto \pi_M \phi^j\{(x, \theta^j\omega)|\ d_M(x, g_j(\theta^j\omega)) \leq \epsilon\}$$

is a random closed set since $g_j \in L^\infty(\Omega, M)$. Hence $\omega \mapsto G_i(\omega)$ is a random closed set as a finite intersection of random closed set. Then by the selection theorem (Proposition 2.17), it has a Borel selection $\tilde{g}_i : \Omega \to M$ such that

$$\tilde{g}_i(\omega) \subset G_i(\omega).$$

As $i \to \infty$, by Lemma 4.5, $\tilde{g}_i$ converges to g pointwisely, thus g is measurable.

If $\{g_i\}_{i=0}^{n-1}$ is an $\alpha-$pseudo periodic orbit, then both g and $\tilde{\phi}^n g$ are $\epsilon-$shadowing this sequence. By expansiveness, we have $\tilde{\phi}^n g = g$.

□

4.1.4 Density of Random Periodic Points.

The following lemma is one of the main results in [34]. It addresses the density of random periodic points.

Lemma 4.7. *Let ϕ be an Anosov and topological mixing on fibers system, then for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for any $g \in L^\infty(\Omega, M)$ and $n \geq N$, there exists a random periodic point $\tilde{g}$ with period n such that*

$$d_{L^\infty(\Omega,M)}(g, \tilde{g}) \leq \epsilon.$$

4.1.5 Two Distortion Lemmas.

In this subsection, we prove two distortion lemmas. Lemma 4.8 is used for proving the fiberwisely absolute continuity of stable and unstable foliations. Lemma 4.9 is used for the construction of the fiber convex cone of observable functions in Section 6.1.

Lemma 4.8. *For any C^2 diffeomorphism $f : M \to M$, let $E(x) \subset T_x M$, $E(y) \subset T_y M$ be subspaces with $dim(E(x)) = dim(E(y))$, then*

$$|| \det(D_x f|_{E(x)})| - | \det(D_y f|_{E(y)})|| \leq C_2(\|f\|_{C^2}) d(x, y) + C_2'(\|f\|_{C^2}) d(E(x), E(y)).$$

where $C_2(\|f\|_{C^2})$ and $C_2'(\|f\|_{C^2})$ are a polynomials of $\|f\|_{C^2}$ and $\dim(E(x))$. As a consequence, for any $x, y \in M_\omega$, $E(x, \omega) \in T_x M_\omega$, $E(y, \omega) \in T_y M_\omega$ with $\dim E(x, \omega) = E(y, \omega)$, we have

$$|| \det(D_x f_\omega|_{E(x,\omega)})| - | \det(D_y f_\omega|_{E(y,\omega)})|| \leq C_2 d(x, y) + C_2 d(E(x, \omega), E(y, \omega)).$$

where $C_2 = \max\{C_2(\sup_\omega \|f_\omega\|_{C^2}), C_2'(\sup_\omega \|f_\omega\|_{C^2})\}$.

Proof. It is sufficient to consider the case $\max\{d(x, y), d(f(x), f(y))\} < \rho_0$. Let $P(x) : T_x M \to E(x)$ be the projection map with respect to $T_x M = E(x) \oplus E(x)^\perp$ and $P(y) : T_y M \to E(y)$ be the projection map with respect to $T_y M = E(y) \oplus E(y)^\perp$, consider $\|D_x f \circ P(x) - P(f(y), f(x)) \circ D_y f \circ P(y) \circ P(x, y)\|$.

Let (N, ψ_1) be the local coordinate chart of x and y, (N', ψ_2) be the local coordinate chart of $f(x)$ and $f(y)$, and denote $\psi_1(x) = p$, $\psi_1(y) = q$, and $F = \psi_2 \circ f \circ \psi_1^{-1}$. Then $D_x f$ and $D_y f$ have local representation $D_p F$ and $D_q F$ respectively. Denote $A(x, y)$ by $D_y\psi_1 \circ P(x, y) \circ (D_x\psi_1)^{-1}$ the local representation of $P(x, y)$ and $A(f(y), f(x))$ by $D_{f(x)}\psi_2 \circ P(f(y), f(x)) \circ (D_{f(y)}\psi_2)^{-1}$ the local representation of $P(f(y), f(x))$. $P(x)$ and $P(y)$ have local representation $B(x) = D_x\psi_1 \circ P(x) \circ (D_x\psi_1)^{-1}$ and $B(y) = D_y\psi_1 \circ P(y) \circ (D_y\psi_1)^{-1}$ respectively. Then we have

$$
\begin{aligned}
&\|D_pF \circ B(x) - A(f(y), f(x)) \circ D_qF \circ B(y) \circ A(x, y)\| \\
\leq\ & \|D_pF \circ B(x) - D_qF \circ B(x)\| + \|D_qF \circ B(x) - D_qF \circ B(y)A(x, y)\| \\
&\quad + \|D_qF \circ B(y)A(x, y) - A(f(y), f(x))D_qF \circ B(y)A(x, y)\| \\
\leq\ & C(|D^2 f|_{C^0}\|p - q\| + |Df|_{C^0}\|B(x) - B(y)A(x, y)\| + |Df|_{C^0}\|I - A(f(y), f(x))\|) \\
\leq\ & C'(|D^2 f|_{C^0} + |Df|^2_{C^0})d(x, y) + C'|Df|_{C^0} d(E(x), E(y)),
\end{aligned}
$$

where C and C' only depend on the local coordinate charts. As a consequence, we have

$$
\begin{aligned}
&\|D_x f \circ P(x) - P(f(y), f(x)) \circ D_y f \circ P(y) \circ P(x, y)\| \\
\leq\ & C''(|D^2 f|_{C^0} + |Df|^2_{C^0})d(x, y) + C''|Df|_{C^0} d(E(x), E(y)).
\end{aligned}
$$

Notice that $P(f(y), f(x))$ and $P(x, y)$ are isometries, so we have

$$
|\|D_x f|_{E(x)}\| - \|D_y f|_{E(y)}\|| \leq C''(|D^2 f|_{C^0} + |Df|^2_{C^0})d(x, y) + C''|Df|_{C^0} d(E(x), E(y)), \quad (4.3)
$$

and by the property of determinant,

$$
|| \det(D_x f|_{E(x)})| - |\det(D_y f|_{E(y)})|| \leq C_2(|D^2_x f|_{C^0} + |D_x f|^2_{C^0})d(x, y) + C_2'(|D_x f|_{C^0})d(E(x), E(y))
$$

where $C_2(|D^2_x f|_{C^0} + |D_x f|^2_{C^0})$ is a polynomial about $|D^2_x f|_{C^0} + |D_x f|^2_{C^0}$ and $\dim(E(x))$, and $C_2'(|D_x f|_{C^0})$ is polynomial about $|D_x f|_{C^0}$ and $\dim(E(x))$. □

Lemma 4.9. *$J^s(x,\omega) = |\det(D_x f_\omega)|_{E^s(x,\omega)}|$ has a uniform Lipschitz variation on the local stable manifolds, i.e., there is a constant $K_1 > 0$ independent of ω such that for any $x, y \in W^s_\epsilon(z,\omega)$,*

$$|J^s(x,\omega) - J^s(y,\omega)| \leq K_1 d(x,y),$$

and

$$|\log J^s(x,\omega) - \log J^s(y,\omega)| \leq K_1 d(x,y).$$

Proof of Lemma 4.9. Since $M \times \Omega$ is a compact space and $f : \Omega \to \mathrm{Diff}^2(M)$ is continuous, $|D_x f_\omega|$ and $|D^2_x f_\omega|$ are uniformly bounded. Let $K \geq 1$ be a constant such that

$$\max\left\{\sup_{(x,\omega)\in M\times\Omega} |D_x f_\omega|, \sup_{(x,\omega)\in M\times\Omega} |D^2_x f_\omega|, LipDh^s_{(x,\omega)}\right\} \leq K.$$

For sake of simplicity, we will identify $exp_x(\cdot)$ with $x + \cdot$ in the rest of the proof. Recall that $\mathcal{P}$ is defined in (4.2). Notice that if $y, z \in W^s_\epsilon(x,\omega)$, and $d(y,z) < \frac{\epsilon}{2\mathcal{P}K^2}$, then

$$|P(E^s(x,\omega))(z-y)| \leq \mathcal{P} d(y,z) < \frac{\epsilon}{2K^2};$$
$$|P(E^s(f_\omega y, \theta\omega))(f_\omega(z) - f_\omega(y))| \leq \mathcal{P}|f_\omega(z) - f_\omega(y)| \leq \mathcal{P}K|z-y| \leq \frac{\epsilon}{2K}.$$

Therefore, $(z,\omega) \in W^s_\epsilon(y,\omega)$ and $(f_\omega(z), \theta\omega) \in W^u_\epsilon(f_\omega(y), \theta\omega)$. So it is sufficient to prove that there exists a constant $K_1 > 0$ independent of x and ω such that for any $y \in W^s_{\frac{\epsilon}{2\mathcal{P}K^2}}(x,\omega)$,

$$|J^s(x,\omega) - J^s(y,\omega)| \leq K_1 d(x,y). \tag{4.4}$$

With the help of the normal coordinate chart, and notice that $d(x,y) < \epsilon < \rho_0$ and $d(f_\omega x, f_\omega y) \leq \epsilon < \rho_0$, we may view that x, y together with $W^s_{\frac{\epsilon}{2\mathcal{P}K}}(x,\omega)$ lie in a same Euclidean space and $f_\omega x$, $f_\omega y$ together with $W^s_\epsilon(f_\omega x, \theta\omega)$ lie in a same Euclidean space. By the stable manifolds theorem, there exists $\xi_y \in E^s(x,\omega)(\frac{\epsilon}{2K^2})$ and $\xi_{f_\omega(y)} \in E^u(f_\omega(x), \theta\omega)(\epsilon)$ such that

$$y = x + \xi_y + h^s_{(x,\omega)}(\xi_y); \tag{4.5}$$

$$f_\omega(y) = f_\omega(x) + \xi_{f_\omega(y)} + h^s_{(f_\omega(x),\theta\omega)}(\xi_{f_\omega(y)}), \tag{4.6}$$

and $E^s(y,\omega) = graph((Dh^s_{(x,\omega)})_{\xi_y})$, $E^s(f_\omega(y),\theta\omega) = graph((Dh^s_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}})$. From (4.5) and (4.6), we have

$$\left(1 - \frac{1}{3}\right)|\xi_{f_\omega(y)}| \le |\xi_{f_\omega(y)} + h^s_{(f_\omega(x),\theta\omega)}(\xi_{f_\omega(y)})| = |f_\omega(y) - f_\omega(x)| \le K|y-x| \le K\left(1 + \frac{1}{3}\right)|\xi_y|,$$

so $|\xi_{f_\omega(y)}| \le 2K|\xi_y|$.

Now, we define the following linear maps $L_{(x,\omega)}, L_{(y,\omega)} : E^s(x,\omega) \to E^s(f_\omega(x),\theta\omega)$ by

$$L_{(x,\omega)} = D_x f_\omega|_{E^s(x,\omega)};$$
$$L_{(y,\omega)} = P(E^s(f_\omega x,\theta\omega))D_y f_\omega|_{E^s(y,\omega)}(I + (Dh^s_{(x,\omega)})_{\xi_y}).$$

We have $\|L_{(x,\omega)}\|, \|L_{(y,\omega)}\| \le \frac{4}{3}\mathcal{P}K$. Hence, we have

$$\begin{aligned}
&\sup_{v\in E^u(x,\omega),\|v\|=1} \|P(E^s(f_\omega x,\theta\omega))D_x f_\omega v - P(E^s(f_\omega x,\theta\omega))D_y f_\omega(I + (Dh^s_{(x,\omega)})_{\xi_y})v\| \\
&\le \mathcal{P}(\|D_x f_\omega - D_y f_\omega\| + \|D_y f_\omega(Dh^s_{(x,\omega)})_{\xi_y}\|) \\
&\le \mathcal{P}K|y-x| + \mathcal{P}K^2|\xi_y| \\
&= \mathcal{P}Kd(x,y) + \mathcal{P}K^2|\xi_y| \\
&\le (\mathcal{P}K + \frac{3}{2}\mathcal{P}K^2)d(x,y).
\end{aligned}$$

So $\|L_{(x,\omega)} - L_{(y,\omega)}\| \le C(\mathcal{P}K + \frac{3}{2}\mathcal{P}K^2)d(x,y)$, where the constant C only depends on the normal coordinate chart. Then by properties of determinant,

$$|det(L_{(x,\omega)}) - det(L_{(y,\omega)})| \le R_1 d(x,y), \tag{4.7}$$

where R_1 is a polynomial of $K, \mathcal{P}$ and the dimension of $\dim E^s(x,\omega)$.

Notice that for $\xi_y \in E^s(x,\omega)(\frac{\epsilon}{2K^2})$

$$\|P(E^s(f_\omega x,\theta\omega))|_{E^s(f_\omega(y),\theta\omega)} - I\| \le \frac{\|(Dh^s_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|}{1-\|(Dh^s_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|}$$
$$\le \frac{K|\xi_{f_\omega(y)}|}{1-K|\xi_{f_\omega(y)}|} \le \frac{2K^2|\xi_y|}{1-2K^2|\xi_y|} \le \frac{2K^2|\xi_y|}{1-2K^2\frac{\epsilon}{2K^2}}$$
$$\le 4K^2|\xi_y| \le 6K^2 d(x,y).$$

So we have

$$|det(P(E^s(f_\omega(x),\theta\omega))|_{E^s(f_\omega(y),\theta\omega)}) - 1| \le R_2 d(x,y), \tag{4.8}$$

where R_2 is a polynomial of K and $\dim E^s(x,\omega)$. Also

$$\|I + (Dh^s_{(x,\omega)})_{\xi_y} - I\| \le K|\xi_y| \le \frac{3}{2}Kd(x,y)$$

implies that there exists a constant R_3 such that

$$|det(I + (Dh^s_{(x,\omega)})_{\xi_y}) - 1| \le R_3 d(x,y). \tag{4.9}$$

Combining (4.7), (4.8), and (4.9), we have

$$|J^s(x,\omega) - J^s(y,\omega)| \le K_0 d(x,y),$$

where K_0 only depends on K, $\mathcal{P}$ and $\dim E^s$. Notice that $\inf_{(x,\omega)\in M\times\Omega} |J^s(x,\omega)| > 0$, as a consequence, there exists a $K_1 > K_0$ such that

$$|\log J^s(x,\omega) - \log J^s(y,\omega)| \le K_1 d(x,y).$$

□

4.1.6 Fiberwisely Absolute Continuity of the Stable and Unstable Foliations.

The absolute continuity of $\{W^\tau_\epsilon(x,\omega)\}$ for fixed ω and $\tau = s, u$ is stated in [45] for general random dynamical systems without proof. In this subsection, we give a proof in our settings. Our proof follows the idea listed in [66].

For any $\omega \in \Omega$, a smooth submanifold $U(\omega) \subset M_\omega$ is said to be transversal to the local stable manifolds if for any $x \in U(\omega)$, $T_x U(\omega) \oplus E^s(x,\omega) = T_x M$. Given smooth submanifolds $U(\omega)$ and $V(\omega)$ transversal to the local stable manifolds, we say that $\psi_\omega : U(\omega) \to V(\omega)$ is a fiber holonomy map if ψ_ω is injective, continuous, and

$$\psi_\omega(x) \in W^s_\epsilon(x,\omega) \cap V(\omega) \text{ for every } x \in U(\omega).$$

We say that $\{W^s_\epsilon(x,\omega)\}$ is fiberwisely absolutely continuous if every fiber holonomy map ψ_ω is absolutely continuous.

Proposition 4.10. *Suppose ϕ is C^2 Anosov on fibers, then $\{W^s_\epsilon(x,\omega)\}$ is fiberwisely absolutely continuous. A similar result holds for $\{W^u_\epsilon(x,\omega)\}$.*

Proof of Proposition 4.10. We first prove that $\{W^s_\epsilon(x,\omega)\}$ is fiberwisely absolutely continuous. For any fixed $\omega \in \Omega$, let $\psi_\omega : U(\omega) \to V(\omega)$ be the fiber holonomy map between two random smooth pre-compact submanifolds, where $U(\omega)$ and $V(\omega)$ are transverse to the local stable manifolds. Let $A \subset U(\omega)$ be any compact set, to prove the absolute continuity of ψ_ω, it is sufficient to prove that there exists a constant $C(\omega)$ independent of A such that

$$m_{V(\omega)}(\psi_\omega(A)) \le C(\omega) m_{U(\omega)}(A),$$

where $m_{V(\omega)}$ and $m_{U(\omega)}$ are the intrinsic Riemann measure on manifolds $V(\omega)$ and $U(\omega)$ respectively. Let $\mathcal{O}$ be a small neighborhood of A in $U(\omega)$ such that

$$m_{U(\omega)}(\mathcal{O}) \le 2 m_{U(\omega)}(A). \tag{4.10}$$

For any $x \in U(\omega)$ and $y \in V(\omega)$, since $U(\omega), V(\omega)$ are transverse to the local stable manifolds, we have $T_xU(\omega) \oplus E^s(x,\omega) = T_xM_\omega$ and $T_yV(\omega) \oplus E^s(y,\omega) = T_yM_\omega$. Recall that $\Gamma(A,B)$ is the aperture between two subspaces A and B. Let

$$\gamma(\omega) := \min\{\inf\{\Gamma(T_xU(\omega), E^s(x,\omega))|\, x \in U(\omega)\}, \inf\{\Gamma(T_xV(\omega), E^s(x,\omega))|\, x \in V(\omega)\}\} > 0.$$

Then by the expansion on the unstable distribution and contraction on the stable distribution, there exists constants $C_5(\omega)$ and $C_6(\omega)$ that depend continuous only on $\gamma(\omega)$ such that

$$\|D_xf_\omega^n v\| \geq C_5(\omega)e^{\lambda n}\|v\| \text{ for } v \in T_xU(\omega), T_xV(\omega), \tag{4.11}$$

$$d(D_xf_\omega^nT_xU(\omega), D_xf_\omega^nE^u(x,\omega)) \leq C_6(\omega)e^{-\lambda n}d(T_xU(\omega), E^u(x,\omega)) \text{ for } x \in U(\omega), \tag{4.12}$$

$$d(D_xf_\omega^nT_xV(\omega), D_xf_\omega^nE^u(x,\omega)) \leq C_6(\omega)e^{-\lambda n}d(T_xV(\omega), E^u(x,\omega)) \text{ for } x \in V(\omega). \tag{4.13}$$

Notice that for any $x \in U(\omega)$, $\psi_\omega(x) = V(\omega) \cap W_\epsilon^s(x,\omega)$, so we have

$$d(f_\omega^n x, f_\omega^n \psi_\omega(x)) < e^{-\lambda n}d(x, \psi_\omega(x)). \tag{4.14}$$

By (4.11), let δ_0 be a sufficiently small number, then there exists a number $N_1(\omega)$ such that for any $n \geq N_1(\omega)$ and $\delta \in (0, \delta_0)$, we have

$$f_{\theta^n\omega}^{-n}B_{f_\omega^nU(\omega)}(f_\omega^n x, \delta) \subset \mathcal{O} \text{ for any } x \in A,$$

where $B_{f_\omega^nU(\omega)}(f_\omega^n x, \delta)$ is the $\delta-$neighborhood of $f_\omega^n x$ on $f_\omega^nU(\omega)$.

By (4.12), (4.13) and (4.14), we know that $f_\omega^nU(\omega)$ and $f_\omega^nV(\omega)$ will be C^1-close to each other and C^1-close to the unstable foliation uniformly for points on $f_\omega^nU(\omega)$ and $f_\omega^nV(\omega)$ as n goes to infinity. Hence, there exists a constant $C_7 > 1$ and a number $N_2(\omega)$ such that for $n \geq N_2(\omega)$ and $\delta \in (0, \delta_0)$, for any $x \in f_\omega^nU(\omega)$, we have

$$C_7^{-1} \leq \frac{m_{f_\omega^n U(\omega)}((B_{f_\omega^n U(\omega)}(x,\delta)))}{m_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)))} \leq C_7, \tag{4.15}$$

and

$$\bar{\psi}_{\theta^n\omega}(B_{f_\omega^n U(\omega)}(x,\delta)) \subset B_{f_\omega^n V(\omega)}(\bar{\psi}_{\theta^n\omega}(x), 2\delta), \tag{4.16}$$

where $\bar{\psi}_{\theta^n\omega} : f_\omega^n U(\omega) \to f_\omega^n V(\omega)$ is the holonomy map induced by the local stable manifolds.

Now we let $N = N(\omega) = \max\{N_1(\omega), N_2(\omega)\}$, and let $\{B_i\}_{i=1}^k$ be a finite covering of $f_\omega^N A$ by $\delta-$balls centered at points in $f_\omega^N A$. By the Besicovitch covering lemma (see, e.g., [23]), we can assume that

there is no point in $f_\omega^N A$ that lies in more than number $C' = C'(\dim(E^u))$ of the B_i's. (4.17)

We claim that there exists a constant $C_8(\omega) > 1$ only depending on $U(\omega)$ and $V(\omega)$ such that

$$C_8(\omega)^{-1} m_{V(\omega)}(\psi_\omega(f_{\theta^N\omega}^{-N} B_i)) \leq m_{U(\omega)}(f_{\theta^N\omega}^{-N} B_i) \leq C_8(\omega) m_{V(\omega)}(\psi_\omega(f_{\theta^N\omega}^{-N} B_i)). \tag{4.18}$$

To prove this claim, we need the following lemmas.

Lemma 4.11. *Denote*

$$H_\omega^n(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) := \frac{|\det(D_x f_\omega^n|_{T_x U(\omega)})|}{|\det(D_{\psi_\omega(x)} f_\omega^n|_{T_{\psi_\omega(x)} V(\omega)})|}. \tag{4.19}$$

Then there exists a constant $C_9(\omega)$ that only depends on $U(\omega)$ and $V(\omega)$ such that

$$C_9(\omega)^{-1} \leq H_\omega^n(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) \leq C_9(\omega).$$

As a consequence, the limit

$$H_\omega(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) := \lim_{n\to\infty} H_\omega^n(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega))$$

exists and converges uniformly for all $x \in U(\omega)$.

Proof of lemma 4.11. For $(x, \omega) \in M \times \Omega$, let $E(\omega) \subset T_x M_\omega$ be a subspace such that $E(\omega) \oplus E^s(x, \omega) = T_x M_\omega$, put

$$\kappa(E(\omega)) := \{\|L\| : \text{ where } L : E^u(x, \omega) \to E^s(x, \omega) \text{ such that } E(\omega) = \{v + Lv | \, v \in E^u(x, \omega)\}\}.$$

Then the contraction on $E^s(x, \omega)$ implies that

$$\kappa(D_x f^n_\omega E(\omega)) \leq e^{-\lambda n} \kappa(E(\omega)).$$

Now for any $x \in M$, $y \in W^s_\epsilon(x, \omega)$, $E(\omega) \subset T_x M_\omega$, $F(\omega) \subset T_y M_\omega$ such that $E(\omega) \oplus E^s(x, \omega) = T_x M_\omega$, $F(\omega) \oplus E^s(y, \omega) = T_y M_\omega$, we use Lemma 4.8 and Lemma 4.1 to obtain

$$\begin{aligned}
&|det(D_x f_\omega|_{E(\omega)}) - \det(D_y f_\omega|_{F(\omega)})| \\
\leq\ & |det(D_x f_\omega|_{E(\omega)}) - det(D_x f_\omega|_{E^u(x,\omega)})| + |det(D_x f_\omega|_{E^u(x,\omega)}) - det(D_y f_\omega|_{E^u(y,\omega)})| \\
&+ |det(D_y f_\omega|_{E^u(y,\omega)}) - det(D_y f_\omega|_{F(\omega)})| \\
\leq\ & C_2 d(E(\omega), E^u(x, \omega)) + C_2 d(F(\omega), E^u(y, \omega)) + C_2 d(x, y) + C_2 d(E^u(x, \omega), E^u(y, \omega)) \\
\leq\ & C_2 \kappa(E(\omega)) + C_2 \kappa(F(\omega)) + C_2 d(x, y) + C_2 C_1 d(x, y)^{\nu_1} \\
\leq\ & 2C_2 C_1 (\kappa(E(\omega)) + \kappa(F(\omega)) + d(x, y)^{\nu_1}).
\end{aligned}$$

Note that by the compactness of M and Ω and the continuity of $f_\omega \in$Diff$^2(M)$ on $\omega \in \Omega$, there exists a constant C_{10} such that

$$C_{10}^{-1} \leq |\det(D_y f_\omega|_{F(\omega)})| \leq C_{10} \tag{4.20}$$

for any y and $F(\omega) \subset T_y M_\omega$. So we have

$$\frac{|det(D_x f_\omega|_{E(\omega)})|}{|\det(D_y f_\omega|_{F(\omega)})|} \leq 1 + 2C_2 C_1 C_{10} (\kappa(E(\omega)) + \kappa(F(\omega)) + d(x, y)^{\nu_1}). \tag{4.21}$$

Denote $C_{11} := 2C_2C_1C_{10}$. By (4.21), we have

$$
\begin{aligned}
& H^n_\omega(x, \psi_\omega(x), T_xU(\omega), T_{\psi_\omega(x)}V(\omega)) \\
&= \prod_{j=0}^{n-1} \frac{|\det(D_{f^j_\omega x} f_{\theta^j\omega}|_{D_x f^j_\omega T_x U(\omega)})|}{|\det(D_{f^j_\omega \psi_\omega(x)} f_{\theta^j\omega}|_{D_{\psi_\omega(x)} f^j_\omega T_{\psi_\omega(x)} V(\omega)})|} \\
&\leq \prod_{j=0}^{n-1}(1 + C_{11}(\kappa(D_x f^j_\omega T_x U(\omega)) + \kappa(D_{\psi_\omega(x)} f^j_\omega T_{\psi_\omega(x)} V(\omega)) + d(f^j_\omega(x), f^j_\omega \psi_\omega(x))^{\nu_1})) \\
&\leq \prod_{j=0}^{n-1}(1 + C_{11}(e^{-\lambda j}\kappa(T_x U(\omega)) + e^{-\lambda j}\kappa(T_{\psi_\omega(x)} V(\omega)) + e^{-\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1})) \\
&\leq \prod_{j=0}^{n-1}(1 + C_{11}e^{-\lambda j \nu_1}(\kappa(T_x U(\omega)) + \kappa(T_{\psi_\omega(x)} V(\omega)) + d(x, \psi_\omega(x))^{\nu_1})) \qquad (4.22) \\
&\leq \exp(C_{12}(\omega)), \qquad (4.23)
\end{aligned}
$$

where $C_{12}(\omega) := \frac{C_{11}}{1-e^{-\lambda\nu_1}} \sup_{x\in U(\omega)}(\kappa(T_xU(\omega)) + \kappa(T_{\psi_\omega(x)}V(\omega)) + d(x, \psi_\omega(x))^{\nu_1}))$. We denote $\exp(C_{12}(\omega))$ by $C_9(\omega)$. The same estimate holds for $H^n_\omega(\psi_\omega(x), x, T_{\psi_\omega(x)}V(\omega), T_xU(\omega))$. □

Lemma 4.12. *For any $x, y \in f^{-N}_{\theta^N\omega}B_i$ and $p, q \in \psi_\omega(f^{-N}_{\theta^N\omega}B_i)$, there exists constants $C_{17}(\omega)$ and $C_{18}(\omega)$ such that*

$$e^{-C_{17}(\omega)} \leq \frac{|\det D_x f^N_\omega|_{T_xU(\omega)}|}{|\det D_y f^N_\omega|_{T_yU(\omega)}|} \leq e^{C_{17}(\omega)} \qquad (4.24)$$

and

$$e^{-C_{18}(\omega)} \leq \frac{|\det D_p f^N_\omega|_{T_pV(\omega)}|}{|\det D_q f^N_\omega|_{T_qV(\omega)}|} \leq e^{C_{18}(\omega)}. \qquad (4.25)$$

proof of Lemma 4.12. Denote $x_i = f^i_\omega x$, $y_i = f^i_\omega y$ for $i \in \{0, ..., N\}$. By (4.13) and Lemma 4.1, we have

$$
\begin{aligned}
&d(T_{x_k}f_\omega^k U(\omega), T_{y_k}f_\omega^k U(\omega)) \\
&\le d(T_{x_k}f_\omega^k U(\omega), E^u(x_k,\theta^k\omega)) + d(E^u(x_k,\theta^k\omega), E^u(y_k,\theta^k\omega)) + d(E^u(y_k,\theta^k\omega), T_{y_k}f_\omega^k U(\omega)) \\
&\le 2C_6(\omega)e^{-\lambda k}\sup_{z\in U(\omega)} d(T_z U(\omega), E^u(z,\omega)) + C_1 d(x_k,y_k)^{\nu_1} \\
&\le 2C_6(\omega)e^{-\lambda k}\sup_{z\in U(\omega)} d(T_z U(\omega), E^u(z,\omega)) + C_1 C_5(\omega)^{-\nu_1}e^{-\lambda(N-k)\nu_1}d(x_N,y_N)^{\nu_1} \\
&\le C_{13}(\omega)e^{-\lambda k} + C_{14}(\omega)e^{-\lambda(N-k)\nu_1},
\end{aligned}
$$

where $C_{13}(\omega) := 2C_6(\omega)\sup_{z\in U(\omega)} d(T_z U(\omega), E^u(z,\omega))$ and $C_{14}(\omega) := \frac{C_1}{C_5(\omega)^{\nu_1}}\delta^{\nu_1}$.

Now by Lemma 4.8,

$$
\begin{aligned}
&|\det(D_{x_k}f_{\theta^k\omega}|_{T_{x_k}f_\omega^k U(\omega)}) - \det(D_{y_k}f_{\theta^k\omega}|_{T_{y_k}f_\omega^k U(\omega)})| \\
&\le C_2 d(x_k,y_k) + C_2 d(T_{x_k}f_\omega^k U(\omega), T_{y_k}f_\omega^k U(\omega)) \\
&\le \frac{C_2}{C_5(\omega)}e^{-\lambda(N-k)}d(x_N,y_N) + C_2(C_{13}(\omega)e^{-\lambda k} + C_{14}(\omega)e^{-\lambda(N-k)\nu_1}) \\
&:= C_{15}(\omega)e^{-\lambda(N-k)\nu_1} + C_{16}(\omega)e^{-\lambda k},
\end{aligned}
$$

where $C_{15}(\omega) := \frac{C_2}{C_5(\omega)}\delta + C_2C_{14}(\omega)$ and $C_{16}(\omega) = C_2C_{13}(\omega)$. Notice that by (4.20), we have

$$
\begin{aligned}
\frac{|\det D_x f_\omega^N|_{T_x U(\omega)}|}{|\det D_y f_\omega^N|_{T_y U(\omega)}|} &= \prod_{k=0}^{N-1}\frac{|\det(D_{x_k}f_{\theta^k\omega}|_{T_{x_k}f_\omega^k U(\omega)})|}{|\det(D_{y_k}f_{\theta^k\omega}|_{T_{y_k}f_\omega^k U(\omega)})|} \\
&\le \prod_{k=0}^{N-1}(1 + C_{10}(C_{15}(\omega)e^{-\lambda(N-k)\nu_1} + C_{16}(\omega)e^{-\lambda k})) \\
&\le \exp(\sum_{k=0}^{N-1} C_{10}(C_{15}(\omega)e^{-\lambda(N-k)\nu_1} + C_{16}(\omega)e^{-\lambda k})) \\
&:= \exp(C_{17}(\omega)).
\end{aligned}
$$

Switch x and y, we get

$$e^{-C_{17}(\omega)} \leq \frac{|\det D_x f_\omega^N|_{T_xU(\omega)}|}{|\det D_y f_\omega^N|_{T_yU(\omega)}|} \leq e^{C_{17}(\omega)}.$$

Notice that by (4.16) and $\psi_\omega(f_{\theta^N\omega}^{-N}B_i) = f_{\theta^N\omega}^{-N}(\bar{\psi}_{\theta^N\omega}B_i)$, similar to the above proof, we can prove that there exists a constant $C_{18}(\omega)$ such that

$$e^{-C_{18}(\omega)} \leq \frac{|\det D_p f_\omega^N|_{T_pV(\omega)}|}{|\det D_q f_\omega^N|_{T_qV(\omega)}|} \leq e^{C_{18}(\omega)}$$

for $p, q \in \psi_\omega(f_{\theta^N\omega}^{-N}B_i)$.

$\square$

Now we are ready to prove the claim (4.18). Pick any $p_i \in B_i$, denote $q_i := \bar{\psi}_{\theta^N\omega}(p_i) \in f_\omega^N V(\omega)$, by (4.24) and change of variable, we have

$$\begin{aligned} e^{-C_{17}(\omega)} \cdot |\det D_{p_i} f_{\theta^N\omega}^{-N}|_{T_{p_i}f_\omega^N U(\omega)}| \cdot m_{f_\omega^N U(\omega)}(B_i) &\leq m_{U(\omega)}(f_{\theta^N\omega}^{-N}B_i) \\ &\leq e^{C_{17}(\omega)} \cdot |\det D_{p_i} f_{\theta^N\omega}^{-N}|_{T_{p_i}f_\omega^N U(\omega)}| \cdot m_{f_\omega^N U(\omega)}(B_i). \end{aligned}$$

Then by Lemma 4.11 and (4.15), we have

$$\begin{aligned} C_7^{-1}C_9(\omega)^{-1}e^{-C_{17}(\omega)} \cdot |\det D_{q_i} f_{\theta^N\omega}^{-N}|_{T_{q_i}f_\omega^N V(\omega)}| \cdot m_{f_\omega^N V(\omega)}(\bar{\psi}_{\theta^N\omega}B_i) &\leq m_{U(\omega)}(f_{\theta^N\omega}^{-N}B_i) \\ &\leq C_7C_9(\omega)e^{C_{17}(\omega)} \cdot |\det D_{q_i} f_{\theta^N\omega}^{-N}|_{T_{q_i}f_\omega^N V(\omega)}| \cdot m_{f_\omega^N V(\omega)}(\bar{\psi}_{\theta^N\omega}B_i). \end{aligned}$$

We apply (4.25) to the above to get

$$\begin{aligned} C_7^{-1}C_9(\omega)^{-1}e^{-C_{17}(\omega)}e^{-C_{18}(\omega)} \cdot m_{V(\omega)}(\psi_\omega f_{\theta^N\omega}^{-N}B_i) &\leq m_{U(\omega)}(f_{\theta^N\omega}^{-N}B_i) \\ &\leq C_7C_9(\omega)e^{C_{17}(\omega)}e^{C_{18}(\omega)} \cdot m_{V(\omega)}(\psi_\omega f_{\theta^N\omega}^{-N}B_i). \end{aligned}$$

Denote $C_8(\omega) := C_7C_9(\omega)e^{C_{17}(\omega)+C_{18}(\omega)}$, then the claim (4.18) is proved. Finally, by (4.10) and (4.17),

$$
\begin{aligned}
m_{V(\omega)}(\psi_\omega(A)) &\le \sum_{i=1}^{k} m_{V(\omega)}(\psi_\omega(f_{\theta^N\omega}^{-N}B_i)) \le \sum_{i=1}^{k} C_8(\omega) m_{U(\omega)}(f_{\theta^N\omega}^{-N}B_i) \\
&\le C' C_8(\omega) m_{U(\omega)}(\mathcal{O}) \\
&\le 2C' C_8(\omega) m_{U(\omega)}(A).
\end{aligned}
$$

Hence ψ_ω is absolutely continuous.

Let $\bar{\psi}_{\theta^n\omega} : f_\omega^n U(\omega) \to f_\omega^n V(\omega)$ be the holonomy map induced by the local stable manifolds. Notice that $\psi_\omega = f_{\theta^n\omega}^{-n} \circ \bar{\psi}_{\theta^n\omega} \circ f_\omega^n$, so

$$
Jac(\psi_\omega)(x) = H_\omega^n(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)) \cdot Jac(\bar{\psi}_{\theta^n\omega}).
$$

Notice that $d(f_\omega^n x, \bar{\psi}_{\theta^n\omega}(f_\omega^n x)) \to 0$ exponentially as $n \to \infty$ uniformly for all $x \in U(\omega)$ and $d(T_{f_\omega^n x} f_\omega^n U(\omega), T_{\bar{\psi}_{\theta^n\omega}(f_\omega^n x)} f_\omega^n V(\omega)) \to 0$ exponentially as $n \to \infty$ uniformly for all $x \in U(\omega)$, hence $Jac(\bar{\psi}_{\theta^n\omega}) \to 1$ as $n \to \infty$. Therefore,

$$
Jac(\psi_\omega)(x) = H_\omega(x, \psi_\omega(x), T_x U(\omega), T_{\psi_\omega(x)} V(\omega)).
$$

The proof of fiberwisely absolute continuity of local stable manifolds is done. Similarly, the fiberwisely absolute continuity of local unstable manifolds can be proved by reversing time. □

4.1.7 Fiberwisely Hölder Continuity of the Stable and Unstable Foliations.

In this subsection, we prove the Hölder continuity of the holonomy map between two local stable leaves and the holonomy map between two local unstable leaves. This result is known in deterministic hyperbolic systems (see, e.g., [51]), but we didn't find any reference to this result in RDS. We give a proof in our settings.

For subspaces $A, B \subset \mathbb{R}^N$, set

$$\Theta(A, B) = \min\{\|v - w\| : v \in A, \|v\| = 1; w \in B, \|w\| = 1\}.$$

For $\theta \in [0, \sqrt{2}]$, we say that a subspace $A \subset \mathbb{R}^N$ is $\theta-$transverse to a subspace $B \subset \mathbb{R}^N$ if $\Theta(A, B) \geq \theta$. Denote $\theta_0 := \inf_{(x,\omega)\in M\times\Omega} \Theta(E^s(x, \omega), E^u(x, \omega)) > 0$.

Proposition 4.13. *Suppose ϕ is C^2 Anosov on fibers, let $\varrho \in (0, 1)$ satisfy*

$$\sup_{(p,\omega)\in M\times\Omega} \|D_p f_\omega|_{E^s(p,\omega)}\| t^{-\varrho}_{(p,\omega)} < 1, \quad \sup_{(p,\omega)\in M\times\Omega} \|D_p f^{-1}_\omega|_{E^u(p,\omega)}\| s^{-\varrho}_{(p,\omega)} < 1$$

where

$$\begin{aligned} t_{(p,\omega)} &:= \inf\{\frac{d(f_\omega p, f_\omega q)}{d(p, q)} : \ q \in M, \ d(p, q) < \epsilon_0\} > 0, \\ s_{(p,\omega)} &:= \inf\{\frac{d(f^{-1}_\omega(p), f^{-1}_\omega(q))}{d(p, q)} : q \in M, \ d(p, q) < \epsilon_0\} > 0, \end{aligned}$$

and ϵ_0 is the size of local stable and unstable manifolds. Then there exists a constant $\delta_0 > 0$ and $H = H(\delta_0, \varrho)$ such that for any $(q, \omega) \in M \times \Omega$ and $\epsilon < \delta_0$,

$$\sup\left\{\frac{\sup_{x\in E^u(p,\omega)(\epsilon)} |h^u_{(p,\omega)}(x) - \tilde{h}^u_{(q,\omega)}(x)|}{d(p, q)^\varrho} : \ q \in M, \ d(p, q) < \delta_0\right\} \leq H < \infty, \tag{4.26}$$

$$\sup\left\{\frac{\sup_{x\in E^u(p,\omega)(\epsilon)} |h^s_{(p,\omega)}(x) - \tilde{h}^s_{(q,\omega)}(x)|}{d(p, q)^\varrho} : \ q \in M, \ d(p, q) < \delta_0\right\} \leq H < \infty, \tag{4.27}$$

where $h^u_{(p,\omega)}, \tilde{h}^u_{(q,\omega)} : E^u(p, \omega)(\epsilon) \to E^s(p, \omega)$ and $Exp_p(graph(h^u_{(p,\omega)}))$, $Exp_p(graph(\tilde{h}^u_{(q,\omega)}))$ represent the local unstable manifolds passing through p, q respectively, and $h^s_{(p,\omega)}, \tilde{h}^s_{(q,\omega)}$: $E^s(p, \omega)(\epsilon) \to E^u(p, \omega)$ and $Exp_p(graph(h^s_{(p,\omega)}))$, $Exp_p(graph(\tilde{h}^s_{(q,\omega)}))$ represent the local stable manifolds passing through p, q respectively. Furthermore, the local product structure is Hölder continuous, i.e., there exists a constant $H' = H'(\delta_0, \varrho, \theta_0)$ such that for any $(x, \omega) \in M \times \Omega$, $y \in W^s_{\delta_0}(x, \omega)$, any $z \in M$ such that $W^s_\epsilon(z, \omega) \cap W^u_\epsilon(x, \omega) \neq \emptyset$,

$W^s_\epsilon(z,\omega)\cap W^u_\epsilon(y,\omega)\neq\emptyset$, we have

$$d(W^s_\epsilon(z,\omega)\cap W^u_\epsilon(x,\omega), W^s_\epsilon(z,\omega)\cap W^u_\epsilon(y,\omega))\leq H'd(x,y)^\varrho.$$

For any $(x,\omega)\in M\times\Omega$, $y\in W^u_{\delta_0}(x,\omega)$, any $z\in M$ such that $W^u_\epsilon(z,\omega)\cap W^s_\epsilon(x,\omega)\neq\emptyset$, $W^u_\epsilon(z,\omega)\cap W^s_\epsilon(y,\omega)\neq\emptyset$, we have

$$d(W^u_\epsilon(z,\omega)\cap W^s_\epsilon(x,\omega), W^u_\epsilon(z,\omega)\cap W^s_\epsilon(y,\omega))\leq H'd(x,y)^\varrho.$$

Proof. We first prove (4.26). Recall that for each point $p\in M$, there exist a neighborhood $N_p\subset M$ and constant ϵ such that the exponential map $Exp_p: B_\epsilon(0)\subset TpM\to M$ is a $C^\infty-$diffeomorphism and $N_p\subset Exp_p(B_\epsilon(0))$. Now for all $p\in M$ and $\omega\in\Omega$, consider any function $g_{(p,\omega)}: E^u(p,\omega)(\epsilon)\to E^s(p,\omega)$ with $g_{(p,\omega)}(0)=0$, where $E^u(p,\omega)(\epsilon)$ is the $\epsilon-$disk in $E^u(p,\omega)$ centered at the origin. Define the special norm by

$$\|g_{(p,\omega)}\|_*=\sup\left\{\frac{|g_{(p,\omega)}(x)|}{|x|}:\ x\in E^u(p,\omega)(\epsilon),\ x\neq 0\right\}.$$

Define

$$G^*_{(p,\omega)}:=\{g_{(p,\omega)}: E^u(p,\omega)(\epsilon)\to E^s(p,\omega)|\ g_{(p,\omega)}(0)=0 \text{ and } \|g_{(p,\omega)}\|_*<\infty\}$$

and

$$G_{(p,\omega)}:=\{g\in G^*_{(p,\omega)}:\ Lip(g)\leq\frac{e^{-2\lambda}+1}{2}\}.$$

Lemma 4.14. *$G^*_{(p,\omega)}$ equipped with $\|\cdot\|_*$ is a Banach space and $G_{(p,\omega)}$ is a closed subset.*

The above Lemma is a corollary of Lemma iii.3 in [57]. $\{G_{(p,\omega)}\}_{(p,\omega)\in M\times\Omega}$ gives a bundle G on $M\times\Omega$ with fiber $G_{(p,\omega)}$ for $(p,\omega)\in M\times\Omega$.

Now we define $f_{(p,\omega)}: T_pM_\omega(\epsilon)\to T_{f_\omega p}M_{\theta\omega}$ by the local representation of f_ω with respect to Exp_p and $Exp_{f_\omega p}$, i.e.,

$$f_{(p,\omega)}(v) = Exp_{f_\omega p}^{-1} \circ f_\omega \circ Exp_p(v),\ \forall v \in T_pM_\omega(\epsilon).$$

Define a bundle map $\phi^* : G \to G$ over $\phi : M \times \Omega \to M \times \Omega$ by

$$(\phi^* g)_{(f_\omega p, \theta\omega)} = \phi^*_{(p,\omega)} g_{(p,\omega)},$$

where

$$graph(\phi^*_{(p,\omega)} g_{(p,\omega)}) = f_{(p,\omega)}(graph(g_{(p,\omega)})) \cap (E^u(f_\omega p, \theta\omega)(\epsilon) \oplus E^s(f_\omega p, \theta\omega)).$$

For a linear transformation T, we denote $m(T) := \|T^{-1}\|^{-1}$ to be the conorm of T.

Lemma 4.15. *For any $\epsilon' > 0$ such that*

$$\frac{e^{-\lambda} + 2\mathcal{P}\epsilon'}{e^{\lambda} - 2\mathcal{P}\epsilon'} \leq \frac{e^{-2\lambda} + 1}{2},$$

there exists a $\epsilon_0 > 0$ such that for any $\epsilon \in (0, \epsilon_0)$, the bundle map ϕ^* defined as above is well-defined and it is a fiber contraction, i.e., for any $g_{(p,\omega)},\ g'_{(p,\omega)} \in G_{(p,\omega)}$, we have

$$\|\phi^*_{(p,\omega)} g_{(p,\omega)} - \phi^*_{(p,\omega)} g'_{(p,\omega)}\|_* \leq \frac{\|D_p f_\omega|_{E^s(p,\omega)}\| + 2\mathcal{P}\epsilon'}{m(D_p f_\omega|_{E^u(p,\omega)}) - 2\mathcal{P}\epsilon'} \cdot \|g_{(p,\omega)} - g'_{(p,\omega)}\|_*.$$

Proof of Lemma 4.15. Pick any $g \in G_{(p,\omega)}$, let $f_{(p,\omega)}(x, g(x))$ have decomposition

$$f_{(p,\omega)}(x, g(x)) = (f_{(p,\omega),1}(x, g(x)), f_{(p,\omega),2}(x, g(x)))$$

with respect to $E^u(f_\omega p, \theta\omega) \oplus E^s(f_\omega p, \theta\omega)$, denote $h_{(p,\omega)} := f_{(p,\omega),1}(x, g(x))$.

By compactness of Ω and M and the continuity of f_ω on ω, for any $\epsilon' > 0$, we can pick a $\epsilon_0 > 0$ sufficiently small such that for any $\epsilon < \epsilon_0$,

$$Lip((f_{(p,\omega)} - D_p f_\omega)|_{T_p M_\omega(\epsilon)}) < \epsilon'.$$

Note that

$$f_{(p,\omega),1}(x, g(x)) = P(E^u(f_\omega p, \theta\omega)) \circ f_{(p,\omega)} \circ (id, g)(x),$$
$$D_p f_\omega|_{E^u(p,\omega)}(x) = P(E^u(f_\omega p, \theta\omega)) \circ D_p f_\omega \circ (id, g)(x).$$

Then we have

$$\begin{aligned} & Lip(f_{(p,\omega),1} \circ (id, g) - D_p f_\omega|_{E^u(p,\omega)(\epsilon)}) \\ & \leq \mathcal{P} Lip((f_{(p,\omega)} - D_p f_\omega)|_{T_p M_\omega(\epsilon)}) \cdot Lip(id, g) \\ & < \mathcal{P}\epsilon'. \end{aligned}$$

By the Lipschitz Inverse function theorem(Theorem I.2 in [57]), $h_{(p,\omega)}$ is a homeomorphism and moreover,

$$\begin{aligned} Lip(h^{-1}_{(p,\omega)}) &\leq \frac{1}{\|D_p f_\omega|^{-1}_{E^u(p,\omega)}\|^{-1} - Lip(f_{(p,\omega),1} \circ (id, g) - D_p f_\omega|_{E^u(p,\omega)(\epsilon)})} \\ &< \frac{1}{m(D_p f_\omega|_{E^u(p,\omega)}) - \mathcal{P}\epsilon'}. \end{aligned} \tag{4.28}$$

Then for any $g \in G_{(p,\omega)}$, we have

$$(\phi^*_{(p,\omega)} g)(x) = f_{(p,\omega),2}(h^{-1}_{(p,\omega)}(x), g(h^{-1}_{(p,\omega)}(x))). \tag{4.29}$$

Note that

$$f_{(p,\omega),2}(x, g(x)) = P(E^s(f_\omega p, \theta\omega)) \circ f_{(p,\omega)} \circ (id, g)(x),$$
$$D_p f_\omega|_{E^s(p,\omega)}(g(x)) = P(E^s(f_\omega p, \theta\omega)) \circ D_p f_\omega \circ (id, g)(x).$$

Then we have

$$\begin{aligned} Lip(f_{(p,\omega),2} \circ (id, g)) &\leq Lip(f_{(p,\omega),2} \circ (id, g) - D_p f_\omega|_{E^s(p,\omega)} \circ g) + Lip(D_p f_\omega|_{E^s(p,\omega)} \circ g) \\ &< \mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|. \end{aligned} \tag{4.30}$$

Combining (4.28), (4.29)and (4.30),

$$\begin{aligned} Lip(\phi^*_{(p,\omega)} g) \leq Lip(f_{(p,\omega),2} \circ (id, g)) \cdot Lip(h^{-1}_{(p,\omega)}) &\leq \frac{\|D_p f_\omega|_{E^s(p,\omega)}\| + \mathcal{P}\epsilon'}{m(D_p f_\omega|_{E^u(p,\omega)}) - \mathcal{P}\epsilon'} \\ \leq \frac{e^{-\lambda} + \mathcal{P}\epsilon'}{e^{\lambda} - \mathcal{P}\epsilon'} < \frac{e^{-2\lambda} + 1}{2}. \end{aligned}$$

Obviously that $(\phi^*_{(p,\omega)} g)(0) = 0$, so we have shown that $\phi^*_{(p,\omega)}$ maps $G_{(p,\omega)}$ to $G_{(f_\omega p, \theta\omega)}$ and as a consequence, ϕ^* is well-defined.

Next, we show that ϕ^* is fiber-contraction. It is sufficient to show that for any $g, g' \in G_{(p,\omega)}$, for all $x \in E^u(p, \omega)(\epsilon)$,

$$\frac{|f_{(p,\omega),2}(x, g(x)) - (\phi^*_{(p,\omega)} g')(f_{(p,\omega),1}(x, g(x)))|}{|f_{(p,\omega),1}(x, g(x))|} \leq \frac{\|D_p f_\omega|_{E^s(p,\omega)}\| + 2\mathcal{P}\epsilon'}{m(D_p f_\omega|_{E^u(p,\omega)}) - 2\mathcal{P}\epsilon'} \cdot \|g - g'\|_* \tag{4.31}$$

since $h_{(p,\omega)}(\cdot) = f_{(p,\omega),1}(\cdot, g(\cdot))$ is homeomorphism.

Notice that

$$
\begin{aligned}
&|f_{(p,\omega),2}(x, g(x)) - f_{(p,\omega),2}(x, g'(x))| \\
&\leq |(f_{(p,\omega),2} - P(E^s(f_\omega p, \theta\omega))D_p f_\omega)(x, g(x)) - (f_{(p,\omega),2} - P(E^s(f_\omega p, \theta\omega))D_p f_\omega)(x, g'(x))| \\
&\qquad + |P(E^s(f_\omega p, \theta\omega))D_p f_\omega(x, g'(x)) - P(E^s(f_\omega p, \theta\omega))D_p f_\omega(x, g(x))| \\
&\leq \left\{ Lip(f_{(p,\omega),2} - P(E^s(f_\omega p, \theta\omega))D_p f_\omega) + \|D_p f_\omega|_{E^s(p,\omega)}\| \right\} \cdot |g(x) - g'(x)| \\
&< \left(\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|\right) |g(x) - g'(x)|, \qquad (4.32)
\end{aligned}
$$

and

$$
\begin{aligned}
&|f_{(p,\omega),1}(x, g(x)) - f_{(p,\omega),1}(x, g'(x))| \\
&\leq |(f_{(p,\omega),1} - P(E^u(f_\omega p, \theta\omega))D_p f_\omega)(x, g(x)) - (f_{(p,\omega),1} - P(E^u(f_\omega p, \theta\omega))D_p f_\omega)(x, g'(x))| \\
&\qquad + |P(E^u(f_\omega p, \theta\omega)D_p f_\omega)(x, g(x)) - P(E^u(f_\omega p, \theta\omega)D_p f_\omega)(x, g'(x))| \\
&\leq Lip(f_{(p,\omega),1} - P(E^u(f_\omega p, \theta\omega))D_p f_\omega)|g(x) - g'(x)| \\
&< \mathcal{P}\epsilon'|g(x) - g'(x)|. \qquad (4.33)
\end{aligned}
$$

Then (4.33) and (4.32) imply that

$$
\begin{aligned}
&|f_{(p,\omega),2}(x, g(x)) - (\phi^*_{(p,\omega)} g')(f_{(p,\omega),1}(x, g(x)))| \\
&\leq |f_{(p,\omega),2}(x, g(x)) - f_{(p,\omega),2}(x, g'(x))| + |f_{(p,\omega),2}(x, g'(x)) - (\phi^*_{(p,\omega)} g')(f_{(p,\omega),1}(x, g(x)))| \\
&\leq \left(\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|\right) |g(x) - g'(x)| \\
&\qquad + |(\phi^*_{(p,\omega)} g')(f_{(p,\omega),1}(x, g'(x))) - (\phi^*_{(p,\omega)} g')(f_{(p,\omega),1}(x, g(x)))| \\
&\leq \left(\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|\right) |g(x) - g'(x)| + |f_{(p,\omega),1}(x, g(x)) - f_{(p,\omega),1}(x, g'(x))| \\
&< \left(2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|\right) |g(x) - g'(x)|. \qquad (4.34)
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
|f_{(p,\omega),1}(x,g(x))| &= |(f_{(p,\omega),1} - P(E^u(f_\omega p,\theta\omega))D_p f_\omega)(x,g(x)) + P(E^u(f_\omega p,\theta\omega))D_p f_\omega(x,g(x))| \\
&\geq (m(D_p f_\omega|_{E^u(p,\omega)}) - 2\mathcal{P}\epsilon')|x| \qquad (4.35)
\end{aligned}
$$

since $|x| \geq |g(x)|$. Hence (4.31) follows by (4.32) and (4.35). □

Remark 4.16. *It is easy to see that the above lemma also holds if we replace* $G_{(p,\omega)}$ *by* $G'_{(p,\omega)} := \{g \in G^*_{(p,\omega)} : \ Lip(g) \leq 1\}$.

The following lemma is a skew product version of the invariant section lemma.

Lemma 4.17 (Invariant Section Lemma For Skew Product)**.** *Suppose we have a bundle map* $F : E \to E$ *over* $\phi : M \times \Omega \to M \times \Omega$, *where* $E = \{(p,\omega,E_{(p,\omega)})|\ (p,\omega) \in M \times \Omega\}$, $E_{(p,\omega)}$ *is a bounded closed subset of a Banach space and* F *has the form* $F(p,\omega,y) = (f_\omega p, \theta\omega, F_{(p,\omega)}(y))$ *for* $y \in E_{(p,\omega)}$. *Denote* $d_{E_{(p,\omega)}}$ *to be the metric on* $E(p,\omega)$. *If for all* $(p,\omega) \in M \times \Omega$,

$$d_{E_{(f_\omega p,\theta\omega)}}(F_{(p,\omega)}(y), F_{(p,\omega)}(y')) \leq K_{(p,\omega)} d_{E_{(p,\omega)}}(y,y'),$$

and Moreover, $\sup_{(p,\omega)\in M\times\Omega} K_{(p,\omega)} = K < 1$, *then there exists a unique invariant section* $\sigma_* : M \times \Omega \to E$, $(p,\omega) \mapsto (p,\omega,\sigma_*(p,\omega))$, *in the sense that for all* $(p,\omega) \in M \times \Omega$,

$$\sigma_* \circ \phi(p,\omega) = F_{(p,\omega)} \circ \sigma_*(p,\omega).$$

Proof of Lemma 4.17. Let Σ' be the collection of all sections. Define the metric on Σ' by the sup metric, i.e.,

$$d_{\Sigma'}(\sigma,\sigma') = \sup\{d_{E_{(p,\omega)}}(\sigma(p,\omega), \sigma'(p,\omega)) : \ (p,\omega) \in M \times \Omega\}.$$

Σ' with $d_{\Sigma'}$ is a complete metric space since $E_{(p,\omega)}$ is closed. Define $F_* : \Sigma' \to \Sigma'$ by

$$(F_*\sigma)(x,\omega) = F_{((f_{\theta^{-1}\omega})^{-1}x,\theta^{-1}\omega)}\sigma((f_{\theta^{-1}\omega})^{-1}x,\theta^{-1}\omega).$$

Now, for all $\sigma, \sigma' \in \Sigma'$, we have

$$\begin{aligned} d_{\Sigma'}(F_*\sigma, F_*\sigma') &= \sup_{(p,\omega)\in M\times\Omega} \{|(F_*\sigma)(p,\omega) - (F_*\sigma')(p,\omega)|\} \\ &= \sup_{(p,\omega)\in M\times\Omega} \big\{|F_{((f_{\theta^{-1}\omega})^{-1}p,\theta^{-1}\omega)}\sigma((f_{\theta^{-1}\omega})^{-1}p,\theta^{-1}\omega) \\ &\qquad\qquad - F_{((f_{\theta^{-1}\omega})^{-1}p,\theta^{-1}\omega)}\sigma'((f_{\theta^{-1}\omega})^{-1}p,\theta^{-1}\omega)|\big\} \\ &\le \sup_{(p,\omega)\in M\times\Omega} \{K|\sigma((f_{\theta^{-1}\omega})^{-1}p,\theta^{-1}\omega) - \sigma'((f_{\theta^{-1}\omega})^{-1}p,\theta^{-1}\omega)|\} \\ &\le K d_{\Sigma'}(\sigma,\sigma'). \end{aligned}$$

Hence F_* is a contraction mapping. As a consequence, there exists a unique fixed point, named σ_*, i.e.,

$$\sigma_*(p,\omega) = F_{((f_{\theta^{-1}\omega})^{-1}p,\theta^{-1}\omega)}\sigma_*((f_{\theta^{-1}\omega})^{-1}p,\theta^{-1}\omega)$$

for all $(p,\omega) \in M \times \Omega$. By changing of variable,

$$\sigma_*(f_\omega p, \theta\omega) = F_{(p,\omega)}\sigma_*(p,\omega),$$

for all $(p,\omega) \in M \times \Omega$). □

We replace F and $E_{(p,\omega)}$ in Lemma 4.17 by ϕ^* and $G_{(p,\omega)}$, then we get a unique section $g^* : M\times\Omega \to G$ such that for each $(p,\omega) \subset M\times\Omega$, $g^*_{(p,\omega)}$ is a Lipschitz map from $E^u(p,\omega)(\epsilon)$ to $E^s(p,\omega)$, and g^* is invariant in the sense that

$$\phi^*_{(p,\omega)}g^*_{(p,\omega)} = g^*_{(\phi(x,\omega))}.$$

Note that one also can obtain the invariant section by iterating any section $g \in G$, i.e.,

$$g^*_{(p,\omega)} = \lim_{n\to\infty} ((\phi^*)^n g)_{(p,\omega)} = \lim_{n\to\infty} \phi^*_{(f_\omega^{-1}p,\theta^{-1}\omega)} \cdots \phi^*_{(f_\omega^{-n}p,\theta^{-n}\omega)} g_{(f_\omega^{-n}p,\theta^{-n}\omega)}, \tag{4.36}$$

uniformly for all $(p,\omega) \in M \times \Omega$.

By the stable and unstable manifolds theorem we know that the local unstable manifold passing p on M_ω is exactly $Exp_p(graph(g^*_{(p,\omega)}))$.

Next, we will show that the bundle map ϕ^* preserves the local Hölder property for an appropriate Hölder exponent.

Since $E^u(x,\omega)$ and $E^s(x,\omega)$ are uniformly continuous depending on $x \in M$, with the help of local coordinate charts, we may pick a sufficiently small $\delta_0 \in (0, \frac{\epsilon}{2P})$ such that whenever $d(p,q) < \delta_0$, $g_{(q,\omega)} : E^u(q,\omega)(\epsilon_0) \to E^s(q,\omega)$ with $Lip(g_{(q,\omega)}) < \frac{e^{-2\lambda}+1}{2}$ can be viewed as a Lipshitz function mapping from $E^u(p,\omega)(\delta_0)$ to $E^s(p,\omega)$, named $\tilde{g}_{(q,\omega)}$ with $Lip(\tilde{g}_{(q,\omega)}) < 1$. From now on, we fix this δ_0. We pick $N > 0$ depending on δ_0 such that $e^{-N\lambda} < \delta_0$. Define

$$G(\delta_0, \varrho, e^{-N\lambda}, K) := \Big\{ g \in G : \sup_{x \in E^u(p,\omega)(\delta_0)} |g_{(p,\omega)}(x) - \tilde{g}_{(q,\omega)}(x)| \le K d(p,q)^\varrho, \\ \text{whenever } e^{-N\lambda} < d(p,q) < \delta_0 \Big\}.$$

Lemma 4.18. *There exists a constant $C = C(\delta_0)$ such that $G \subset G(\delta_0, \varrho, e^{-N\lambda}, C(\delta_0)e^{N\lambda\varrho})$.*

Proof of Lemma 4.18. Notice that both the Lipschitz constant of $g_{(p,\omega)}$ and $\tilde{g}_{(q,\omega)}$ are less than 1, and $d(p,q) < \delta_0$, hence there exists a constant $C = C(\delta_0) > 0$ such that

$$\sup_{x \in E^u(p,\omega)(\delta_0)} |g_{(p,\omega)}(x) - \tilde{g}_{(q,\omega)}(x)| \le C.$$

Notice that $d(p,q) > e^{-N\lambda}$, so we have

$$\sup_{x \in E^u(p,\omega)(\delta_0)} |g_{(p,\omega)}(x) - \tilde{g}_{(q,\omega)}(x)| \le C(\delta_0) \le C(\delta_0)e^{N\lambda\varrho} d(p,q)^\varrho.$$

□

Lemma 4.19. *Let $g \in G$, if $d(p,q) < \delta_0$, $d(f_\omega p, f_\omega q) < \delta_0$, and $\sup_{x\in E^u(p,\omega)(\delta_0)} |g_{(p,\omega)}(x) - \tilde{g}_{(q,\omega)}(x)| \le Kd(p,q)^\varrho$, then*

$$\sup_{x\in E^u(f_\omega p,\theta\omega)(\delta_0)} \left|\left(\phi^*_{(p,\omega)} g_{(p,\omega)}\right)(x) - \left(\phi^*_{(p,\omega)} \tilde{g}_{(q,\omega)}\right)(x)\right| \le Kd(f_\omega p, f_\omega q)^\varrho \tag{4.37}$$

provided $\varrho \in (0,1)$ satisfying

$$\sup_{(p,\omega)\in M\times\Omega} \|D_p f_\omega|_{E^s(p,\omega)}\| t^{-\varrho}_{(p,\omega)} < 1, \tag{4.38}$$

where

$$t_{(p,\omega)} := \inf\left\{\frac{d(f_\omega p, f_\omega q)}{d(p,q)} : \ q \in M, \ d(p,q) < \epsilon\right\} > 0. \tag{4.39}$$

Proof of Lemma 4.19. We use the same notation as the proof of Lemma 4.15. Notice that by (4.38), we can pick a constant $\epsilon' > 0$ sufficiently small both satisfying the condition of Lemma 4.15 and

$$\sup_{(p,\omega)\in M\times\Omega} (2\mathcal{P}\epsilon' + \|D_p f_\omega\|_{E^s(p,\omega)}) t^{-\varrho}_{(p,\omega)} < 1. \tag{4.40}$$

Recall that $f_{(p,\omega)} : T_pM_\omega(\epsilon) \to T_{f_\omega p} M_{\theta\omega}$ is the local representation of f_ω with respect to Exp_p and $Exp_{f_\omega p}$. $\phi^*_{(p,\omega)}$ acts on $\tilde{g}_{(q,\omega)}$ by

$$graph(\phi^*_{(p,\omega)}\tilde{g}_{(q,\omega)}) = f_{(p,\omega)}(graph(\tilde{g}_{(q,\omega)})) \cap (E^u(f_\omega p, \theta\omega)(\delta_0) \oplus E^s(f_\omega p, \theta\omega)).$$

Notice that $graph(\phi^*_{(p,\omega)}\tilde{g}_{(q,\omega)}) = graph(\widetilde{\phi^*_{(q,\omega)} g_{(q,\omega)}})$, so by the choice of δ_0 and the invariance of bundle G, we have $Lip(\phi^*_{(p,\omega)}\tilde{g}_{(q,\omega)}) < 1$. Similar to the proof of (4.34),

$$
\begin{aligned}
&|f_{(p,\omega),2}(x, g_{(p,\omega)}(x)) - (\phi^*_{(p,\omega)}\tilde{g}_{(q,\omega)})(f_{(p,\omega),1}(x, g_{(p,\omega)}(x)))| \\
\le\ & (2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|)|g_{(p,\omega)}(x) - \tilde{g}_{(q,\omega)}(x)| \\
\le\ & (2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|) \sup_{x\in E^u(p,\omega)(\delta_0)} |g_{(p,\omega)} - \tilde{g}_{(q,\omega)}| \\
\le\ & (2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|)Kd(p,q)^\varrho \\
\le\ & (2\mathcal{P}\epsilon' + \|D_p f_\omega|_{E^s(p,\omega)}\|)Kt^{-\varrho}_{(p,\omega)}d(f_\omega p, f_\omega q)^\varrho \\
\le\ & Kd(f_\omega p, f_\omega q)^\varrho,
\end{aligned}
$$

provided (4.40). Notice that $h_{(p,\omega)}(\cdot) = f_{(p,\omega),1}(\cdot, g_{(p,\omega)}(\cdot))$ is a homeomorphism, hence we get (4.37). □

Now consider

$$R_n(\omega) := \{(f^n_{\theta^{-n}\omega}p, f^n_{\theta^{-n}\omega}q) \in M \times M|\ \max_{0\le k\le n-1} d(f^k_{\theta^{-n}\omega}p, f^k_{\theta^{-n}\omega}q) < \delta_0,\ e^{-N\lambda} < d(p,q) < \delta_0\},$$

and let $S_n(\omega) = \cup_{i=0}^n R_i(\omega)$. Applying Lemma 4.19 inductively and noticing that $\phi^*_{(p,\omega)}\tilde{g}_{(q,\omega)} = \widetilde{\phi^*_{(q,\omega)}g_{(q,\omega)}}$ for q in the δ_0-neighborhood of p, we see that for any $g \in G$, $(p,q) \in S_n(\omega)$,

$$\sup_{x\in E^u(p,\omega)(\delta_0)} |((\phi^*)^n g)_{(p,\omega)}(x) - (\widetilde{(\phi^*)^n g})_{(q,\omega)}(x)| \le C(\delta_0)e^{N\lambda\varrho}d(p,q)^\varrho.$$

By the stable and unstable manifolds theorem,

$$\{(p,q) \in M_\omega \times M_\omega|\ q \notin W^u_{\delta_0}(p,\omega),\ d(p,q) < \delta_0\} \subset \bigcup_{n=0}^\infty S_n(\omega).$$

Hence that fix point obtained by (4.36) has property that for any $p, q \in M$, $d(p,q) < \delta_0$,

$$\sup_{x\in E^u(p,\omega)(\delta_0)} \left|g^*_{(p,\omega)}(x) - \tilde{g^*}_{(q,\omega)}(x)\right| \le C(\delta_0)e^{N\lambda\varrho}d(p,q)^\varrho := H(\delta_0,\varrho)d(p,q)^\varrho. \quad (4.41)$$

Hence the local unstable foliation is Hölder continuous on the base point fiberwisely. A similar proof can be applied to the local stable foliation by reversing time.

Now let $y_0, y_1 \in graph(g^*_{(p,\omega)})$, let $q \in M$ and $d(p,q) < \delta_0$. Let

$$z_0 := (P(E^u(p,\omega)y_0, \tilde{g^*}_{(q,\omega)}(P(E^u(p,\omega)y_0)))$$

and

$$z_1 := (P(E^u(p,\omega)y_1, \tilde{g^*}_{(q,\omega)}(P(E^u(p,\omega)y_1))).$$

Then by (4.41),

$$|z_1 - y_1| \leq H(\delta_0, \varrho)|z_0 - y_0|^{\varrho}.$$

Denote

$$w_0 := \exp_p^{-1}(W^s_{loc}(y_0,\omega)) \cap graph(\tilde{g^*}_{(q,\omega)}),\ w_1 := \exp_p^{-1}(W^s_{loc}(y_1,\omega)) \cap graph(\tilde{g^*}_{(q,\omega)}).$$

Since that $\theta_0 := \inf_{(x,\omega)\in M\times\Omega} \Theta(E^s(x,\omega), E^u(x,\omega)) > 0$, hence when δ_0 sufficiently small, there exists a constant $C(\theta_0)$ independent of $(x,\omega) \in M \times \Omega$ such that

$$\frac{|y_1 - z_1|}{|y_1 - w_1|} \geq C(\theta_0)^{-1},\ \frac{|y_0 - z_0|}{|y_0 - w_0|} \leq C(\theta_0).$$

Hence, we get

$$|y_1 - w_1| \leq C(\theta_0)^{1+\varrho} H(\delta_0, \varrho)|y_0 - w_0|^{\varrho},$$

i.e., the fiber holonomy map between local unstable manifolds is uniformly $\varrho-$Hölder continuous at a small scale. A similar result holds for fiber holonomy map between local stable manifolds.

The proof of Proposition 4.13 is done. □

4.1.8 Properties of the Holonomy Map between Two Local Stable Leaves.

In this subsection, the properties of the Holonomy maps are further discussed.

For each $\omega \in \Omega$, $\tilde{\gamma}(\omega)$ and $\gamma(\omega)$ is said to be pair of nearby local stable leaves if the fiber holonomy map $\psi_\omega : \tilde{\gamma}(\omega) \to \gamma(\omega)$ by $\psi_\omega(x) = W^u_\epsilon(x,\omega) \cap \gamma(\omega)$ for $x \in \tilde{\gamma}(\omega)$ is a homeomorphism.

In the following, we restrict the size of local stable and unstable manifolds $W^s_\epsilon(x,\omega)$, $W^u_\epsilon(x,\omega)$ satisfying $\epsilon \le \min\{\epsilon_0, \delta_0\}$ to guarantee the Hölder continuity of the stable and unstable foliations, where δ_0 is the constant in Proposition 4.13.

Lemma 4.20. *There exists constants $a_0', \nu_0' > 0$ that only depend on system ϕ such that for any $\psi_\omega : \tilde{\gamma}(\omega) \to \gamma(\omega)$ fiber holonomy map of two nearby random local stable leaves, the followings hold:*

(i) ψ_ω *and* $\log|\det D\psi_\omega|$ *are* (a_0', ν_0')*-Hölder continuous;*

(ii) $\log|\det D_y\psi_\omega| \le a_0' d(y, \psi_\omega(y))^{\nu_0'}$ *for every* $y \in \tilde{\gamma}(\omega)$*;*

(iii) $d((f_{\theta^{-1}\omega})^{-1}x, (f_{\theta^{-1}\omega})^{-1}\psi_\omega(x)) \le e^{-\lambda} d(x, \psi_\omega(x))$.

Proof of Lemma 4.20. In Proposition 4.13, we already prove that ψ_ω is $(H, \varrho)-$Hölder continuous for all $\omega \in \Omega$.

Now we prove the Hölder continuity of $\log|\det D\psi_\omega|$. Pick any $x, y \in \tilde{\gamma}(\omega)$, we consider two cases: (case 1) $d(x, \psi_\omega(x)) \le d(x,y)$ and (case 2) $d(x, \psi_\omega(x)) > d(x,y)$.

In (case 1), by (4.22), we have

$$\frac{|\det D_x\psi_\omega|}{|\det D_y\psi_\omega|} \le \prod_{j=0}^{-\infty}(1 + C_{11}e^{\lambda j \nu_1} d(x, \psi_\omega(x))^{\nu_1}) \prod_{j=0}^{-\infty}(1 + C_{11}e^{\lambda j \nu_1} d(y, \psi_\omega(y))^{\nu_1}).$$

Then apply Proposition 4.13, we have

$$
\begin{aligned}
|\log|\det D_x\psi_\omega| - \log|\det D_y\psi_\omega|| &\le \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1}d(x,\psi_\omega(x))^{\nu_1} + \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1}d(y,\psi_\omega(y))^{\nu_1} \\
&\le \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1}d(x,\psi_\omega(x))^{\nu_1} + \sum_{j=0}^{-\infty} C_{11}H^{\nu_1}e^{\lambda j\nu_1}d(x,\psi_\omega(x))^{\nu_1\varrho} \\
&\le (\sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1} + \sum_{j=0}^{-\infty} C_{11}H^{\nu_1}e^{\lambda j\nu_1})d(x,y)^{\nu_1\varrho} \\
&:= S_1 d(x,y)^{\nu_1\varrho}. \qquad (4.42)
\end{aligned}
$$

In (case 2), since the expansion on stable manifolds and the contraction on unstable manifolds when reverse time, there exists an integer $m > 0$ such that

$$d(f_\omega^{-k}x, f_\omega^{-k}\psi_\omega(x)) > d(f_\omega^{-k}x, f_\omega^{-k}y) \text{ for } 0 \le k \le m-1;$$

and

$$d(f_\omega^{-m}x, f_\omega^{-m}\psi_\omega(x)) \le d(f_\omega^{-m}x, f_\omega^{-m}y).$$

Note that

$$\frac{|\det D_x\psi_\omega|}{|\det D_y\psi_\omega|} = \frac{|\det D_x f_\omega^{-m}|_{E^s(x,\omega)}|}{|\det D_y f_\omega^{-m}|_{E^s(y,\omega)}|} \cdot \frac{|\det D_{\psi_\omega(y)} f_\omega^{-m}|_{E^s(\psi_\omega(y),\omega)}|}{|\det D_{\psi_\omega(x)} f_\omega^{-m}|_{E^s(\psi_\omega(x),\omega)}|} \cdot \frac{|\det D_{f_\omega^{-m}x}(f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^{m})|}{|\det D_{f_\omega^{-m}y}(f_\omega^{-m}\psi_\omega f_{\theta^{-m}\omega}^{m})|}.$$

Denote $\beta := \sup\{\frac{d(f_\omega^{-1}x, f_\omega^{-1}y)}{d(x,y)} : d(x,y) \le \epsilon, \omega \in \Omega\} \in (1,\infty)$, and $\eta := \inf\{\frac{d(f_\omega^{-1}x, f_\omega^{-1}y)}{d(x,y)} : d(x,y) \le \epsilon, \omega \in \Omega\} \in (0,1)$, then by the choice of m, we have

$$\eta^m d(x,\psi_\omega(x)) \le \beta^m d(x,y).$$

As a consequence, $m \ge (\log \frac{d(x,\psi_\omega(x))}{d(x,y)})/\log(\beta/\eta)$. Hence

$$e^{-m} \le d(x,y)^{\frac{1}{\log(\beta/\eta)}} d(x,\psi_\omega(x))^{-\frac{1}{\log\beta/\eta}}.$$

By (4.21), we have

$$
\begin{aligned}
&|\log|\det D_x f_\omega^{-m}|_{E^s(x,\omega)}| - \log|\det D_y f_\omega^{-m}|_{E^s(y,\omega)}||\\
&\le \sum_{k=0}^{m-1} C_{11} d(f_\omega^{-k}x, f_\omega^{-k}y)^{\nu_1} \le \sum_{k=0}^{m-1} C_{11} e^{-\lambda(m-1-k)\nu_1} d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}y)^{\nu_1}\\
&\le (\sum_{k=0}^{m-1} C_{11} e^{-\lambda(m-1-k)\nu_1}) d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1}\\
&\le (\sum_{k=0}^{m-1} C_{11} e^{-\lambda(m-1-k)\nu_1}) e^{-\lambda(m-1)\nu_1} d(x, \psi_\omega(x))^{\nu_1}\\
&\le S_2 e^{-\lambda m \nu_1} d(x, \psi_\omega(x))^{\nu_1}\\
&\le S_2 d(x, \psi_\omega(x))^{\nu_1 - \frac{\lambda\nu_1}{\log(\beta/\eta)}} d(x,y)^{\frac{\lambda\nu_1}{\log(\beta/\eta)}}\\
&\le S_2 d(x,y)^{\frac{\lambda\nu_1}{\log(\beta/\eta)}},
\end{aligned}
$$

where $S_2 := (\sum_{k=0}^{m-1} C_{11} e^{-\lambda(m-1-k)\nu_1}) e^{\lambda\nu_1}$. Similar to above, we have

$$
\begin{aligned}
&|\log|\det D_{\psi_\omega(y)} f_\omega^{-m}|_{E^s(\psi_\omega(y),\omega)}| - \log|\det D_{\psi_\omega(x)} f_\omega^{-m}|_{E^s(\psi_\omega(x),\omega)}||\\
&\le \sum_{k=0}^{m-1} C_{11} d(f_\omega^{-k}\psi_\omega(x), f_\omega^{-k}\psi_\omega(y))^{\nu_1} \le \sum_{k=0}^{m-1} C_{11} e^{-\lambda(m-1-k)\nu_1} d(f_\omega^{-(m-1)}\psi_\omega(x), f_\omega^{-(m-1)}\psi_\omega(y))^{\nu_1}\\
&\le \sum_{k=0}^{m-1} C_{11} e^{-\lambda(m-1-k)\nu_1} H^{\nu_1} d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}y)^{\nu_1\varrho}\\
&\le \sum_{k=0}^{m-1} C_{11} e^{-\lambda(m-1-k)\nu_1} H^{\nu_1} d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1\varrho}\\
&\le S_3 e^{-\lambda m \nu_1 \varrho} d(x, \psi_\omega(x))^{\nu_1\varrho}\\
&\le S_3 d(x,y)^{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}},
\end{aligned}
$$

where $S_3 := \sum_{k=0}^{m-1} C_{11} e^{-\lambda(m-1-k)\nu_1} H^{\nu_1} e^{\lambda\nu_1\varrho}$. Note that $f_\omega^m \psi_\omega f_{\theta^{-m}\omega}^m$ is the holonomy map from

$f_\omega^{-m}\tilde{\gamma}(\omega)$ to $f_\omega^{-m}\gamma(\omega)$, hence similar to (4.42), we have

$$
\begin{aligned}
&|\log|\det D_{f_\omega^{-m}x}(f_\omega^m\psi_\omega f_{\theta^{-m}\omega}^m)| - \log|\det D_{f_\omega^{-m}y}(f_\omega^m\psi_\omega f_{\theta^{-m}\omega}^m)||\\
&\leq \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1} d(f_\omega^{-m}x, f_\omega^{-m}\psi_\omega(x))^{\nu_1} + \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1} d(f_\omega^{-m}y, f_\omega^{-m}\psi_\omega(y))^{\nu_1}\\
&\leq \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1}e^{-\lambda\nu_1} d(f_\omega^{-m}x, f_\omega^{-m}\psi_\omega(x))^{\nu_1} + \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1}e^{-\lambda\nu_1} d(f_\omega^{-(m-1)}y, f_\omega^{-(m-1)}\psi_\omega(y))^{\nu_1}\\
&\leq \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1}e^{-\lambda\nu_1}\left(d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1} + H^{\nu_1} d(f_\omega^{-(m-1)}x, f_\omega^{-(m-1)}\psi_\omega(x))^{\nu_1\varrho}\right)\\
&\leq S_4 d(x,y)^{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}},
\end{aligned}
$$

where $S_4 := (\sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1}e^{-\lambda\nu_1} + \sum_{j=0}^{-\infty} C_{11}e^{\lambda j\nu_1}e^{-\lambda\nu_1}H^{\nu_1})e^{\lambda\nu_1\varrho}$. Hence

$$
|\log|\det D_x\psi_\omega| - \log|\det D_y\psi_\omega|| \leq (S_2+S_3+S_4)d(x,y)^{\frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}}.
$$

For property (2), we use (4.22) to obtain

$$
\begin{aligned}
|\det D_y\psi_\omega| &\leq \prod_{j=0}^{-\infty}(1 + C_{11}e^{\lambda j\nu_1} d(y,\psi_\omega(y))^{\nu_1});\\
1/|\det D_y\psi_\omega| &\leq \prod_{j=0}^{-\infty}(1 + C_{11}e^{\lambda j\nu_1} d(y,\psi_\omega(y))^{\nu_1}).
\end{aligned}
$$

So

$$
|\log|\det D_y\psi_\omega|| \leq C_{11}\sum_{j=0}^{-\infty} e^{\lambda j\nu} d(y,\psi_\omega(y))^{\nu_1}.
$$

Now we define $a_0' := \max\{S_1, S_2+S_3+S_4\}$ and $\nu_0' := \frac{\lambda\nu_1\varrho}{\log(\beta/\eta)}$. Then property (1) and (2) are proved.

Property (3) follows the definition of holonomy map and contraction on local unstable manifolds when reverse time.

□

4.1.9 Fubini's Theorem on Rectangles.

We call $R(\omega) \subset M_\omega$ a rectangle if it is foliated by local stable manifolds and it has the local product structure. By the lemma 4.20, for any rectangle $R(\omega)$ in small scale, the holonomy map between stable manifolds lying in $R(\omega)$ is absolutely continuous and the Log of Jacobian of holonomy map is $(a_0', \nu_0')-$Hölder continuous, and the same results hold for holonomy map between the local unstable manifolds in $R(\omega)$. As a consequence, the Riemannian measure on $R(\omega)$ has disintegration on each local stable leaves in $R(\omega)$. Moreover, the density function on each local stable leaf has Hölder regularity. The proof of this statement is similar to the determinant case [13]. We supply a proof here for completeness.

Proposition 4.21. *There exist constants (a_0'', ν_0'') only depending on the system such that for each $\omega \in \Omega$ and any rectangle $R(\omega) = [W^u_\epsilon(x_0, \omega), W^s_\epsilon(x_0, \omega)]$ for some $x_0 \in M$, there exists a function $H(\omega) : R(\omega) \to \mathbb{R}$ satisfying*

$$|\log H(\omega)(x) - \log H(\omega)(y)| \leq a_0'' d(x, y)^{\nu_0''}, \ \forall x, y \in \gamma(\omega)$$

and for any bounded measurable function $\psi : M \to \mathbb{R}$, there is disintegration

$$\int_{R(\omega)} \psi(x) dm(x) = \int \int_{\gamma(\omega)} \psi(x) H_i(\omega)(x)|_{\gamma(\omega)} dm_{\gamma(\omega)}(x) d\tilde{m}_{R(\omega)}(\gamma(\omega)),$$

where $\gamma(\omega)$ denote the stable foliations in $R(\omega)$ and $\tilde{m}_{R(\omega)}$ the quotient measure induced by Riemannian volume measure in the space of local stable leaves in $R(\omega)$.

Proof of Proposition 4.21. With the help of a normal coordinate chart, we can view $R(\omega)$ as a subset in the vector space $T_{x_0}M$. We define a map $\Psi : (W^u_\epsilon(x_0, \omega) \times W^s_\epsilon(x_0, \omega), \mathcal{B}, m^u \times m^s) \to (R(\omega), \mathcal{B}, m)$ by

$$\Psi(\xi, \eta) = W^u_\epsilon(\eta, \omega) \cap W^s_\epsilon(\xi, \omega),$$

where m^u and m^s are the intrinsic Riemannian measures on $W^u_\epsilon(x_0, \omega)$ and $W^s_\epsilon(x_0, \omega)$ respectively and m is the Riemannian measure on $R(\omega)$.

For any $x \in R(\omega)$, we introduce the following sets:

$$\begin{aligned}
S_r(x,\omega) &= \{W^u_\epsilon(y,\omega) \cap W^u_\epsilon(z,\omega)|\ d(x,y) \le r,\ y \in W^s_\epsilon(x,\omega),\ d(x,z) \le r,\ z \in W^u_\epsilon(x,\omega)\} \\
&= [W^u_r(x,\omega), W^s_r(x,\omega)], \\
P_r(x,\omega) &= \{x+v+w|\ v \in E^s(x,\omega), \|v\| \le r,\ w \in E^u(x,\omega), \|w\| \le r\}, \\
P^s_r(x,\omega) &= \{x+v|\ v \in E^s(x,\omega)|,\ \|v\| \le r\}, \\
P^u_r(x,\omega) &= \{x+w|\ w \in E^u(x,\omega)|,\ \|w\| \le r\}.
\end{aligned}$$

Lemma 4.22. *There exists a constant K_4 independent of ω such that*

$$P_{\mathcal{P}r(1-K_4 r^{(\nu_0')^2})}(x,\omega) \subset S_r(x,\omega) \subset P_{\mathcal{P}r(1+K_4 r^{(\nu_0')^2})}(x,\omega). \tag{4.43}$$

Furthermore, there exists constant K_6 independent of ω and a function $\theta(x,\omega)$ satisfying that $\log\theta(x,\omega)$ is (K_6,ν_0') Hölder continuous on x and continuous on ω such that

$$m(S_r(x,\omega)) = \theta(x,\omega) m_s(P^s_{\mathcal{P}r}(x,\omega)) m_u(P^u_{\mathcal{P}r}(x,\omega))(1+\mathcal{O}(r^{(\nu_0')^2})), \tag{4.44}$$

where m_s is the induced measure on $E^s(x,\omega)$ and m_u is the induced measure on $E^u(x,\omega)$.

Proof of Lemma 4.22. For $z \in x + E^u(x,\omega)$, by the local stable manifolds theorem, there exists a map $\tilde{h}^s_{(z,\omega)} : E^s(x,\omega) \to E^u(x,\omega)$ such that the local stable manifold $W^s_\epsilon(z,\omega) := \{z + \xi + \tilde{h}^s_{(z,\omega)}(\xi)|\ \xi \in E^s(x,\omega)\}$. Recall that $\nu_0' < \min\{\nu_1, \varrho\}$ in Lemma 4.20. By Lemma 4.1 the Hölder continuity of stable subbundles, we have

$$\left\| D_\xi \tilde{h}^s_{(z,\omega)} \right\| \le C_1 \left\| z + \xi + \tilde{h}^s_{(z,\omega)}(\xi) \right\|^{\nu_0'}.$$

By Proposition 4.13, to represent points in $S_r(x,\omega)$, it is sufficient to consider $\|z\| \le \mathcal{P}r$ and $\|\xi\| \le \mathcal{P}H' r^{\nu_0'}$.

For $v \in E^s(x,\omega)$, $\|v\| = 1$, we define $h(t) = \|\tilde{h}^s_{(z,\omega)}(vt)\|$, then we have

$$\left|\frac{dh(t)}{dt}\right| = \left|\frac{(\tilde{h}^s_{(z,\omega)}(vt), D_{tv}\tilde{h}^s_{(z,\omega)}v)}{\|\tilde{h}^s_{(z,\omega)}(tv)\|}\right| \le \left\|D_{tv}\tilde{h}^s_{(z,\omega)}\right\| \le C_1 \left\|z + tv + \tilde{h}^s_{(z,\omega)}(tv)\right\|^{\nu_0'}.$$

So that if $\|z\| \le \mathcal{P}r$, $h(t)$ must satisfy the following inequality in the domain $\|t\| \le \mathcal{P}H'r^{\nu_0'}$

$$\begin{cases} \left|\frac{dh(t)}{dt}\right| \le C_1 \left(\mathcal{P}r + \mathcal{P}H'r^{\nu_0'} + h(t)\right)^{\nu_0'}, \\ h(0) = 0. \end{cases}$$

Solve the above equation we get

$$h(t) \le K_3 t r^{(\nu_0')^2} + \mathcal{O}\left(t^2 r^{-\nu_0' + 2(\nu_0')^2}\right),$$

where $K_3 = K_3(\mathcal{P}, H', \nu_0')$ is a constant. So if $\|z\| \le \mathcal{P}r$, and $\|v\| \le \mathcal{P}H'r^{\nu_0'}$, we have

$$\left\|\tilde{h}^s_{(z,\omega)}(v)\right\| \le K_4 \|v\| r^{(\nu_0')^2},$$

where constant $K_4 = K_4(\mathcal{P}, H', \nu_0')$ is independent of ω. The same estimates holds for unstable manifolds. These two estimate imply that

$$P_{\mathcal{P}r(1-K_4 r^{(\nu_0')^2})}(x,\omega) \subset S_r(x,\omega) \subset P_{\mathcal{P}r(1+K_4 r^{(\nu_0')^2})}(x,\omega).$$

As a consequence,

$$m\left(P_{\mathcal{P}r(1-K_4 r^{(\nu_0')^2})}(x,\omega)\right) \subset m\left(S_r(x,\omega)\right) \subset m\left(P_{\mathcal{P}r(1+K_4 r^{(\nu_0')^2})}(x,\omega)\right), \tag{4.45}$$

where m is the Riemannian measure on $R(\omega)$. Let $\{v_i\}_{i=1}^{\dim E^s}$ be an orthonormal basis for

$E^s(x,\omega)$ and $\{w_j\}_{j=1}^{\dim E^u}$ be an orthonormal basis for $E^u(x,\omega)$, we define

$$\theta(x,\omega) = |\det(v_1, ..., v_{\dim E^s}, w_1, ..., w_{\dim E^u})|.$$

Since $\Theta(E^s(x,\omega), E^u(x,\omega)) \geq \theta_0$, there exists a constant $K_5 = K_5(\theta_0)$ such that $\theta(x,\omega) \geq K_5$. By Lemma 4.1, $\theta(x,\omega)$ is $\nu_0'-$Hölder continuous on x and continuous on ω. Hence there exists a constant K_6 such that $\log\theta(x,\omega)$ is (K_6, ν_0') Hölder continuous. Combined with (4.45), we get

$$m(S_r(x,\omega)) = \theta(x,\omega) m_s\left(P^s_{\mathcal{P}r}(x,\omega)\right) m_u\left(P^u_{\mathcal{P}r}(x,\omega)\right)\left(1+\mathcal{O}\left(r^{(\nu_0')^2}\right)\right). \qquad \square$$

Note that by the local stable manifolds theorem, for each (x,ω), the local unstable manifold $W^s_\epsilon(x,\omega)$ is determined by a C^2 function $h^s_{(x,\omega)}: E^s(x,\omega) \to E^u(x,\omega)$. Moreover, $T_x W^s_\epsilon(x,\omega) = E^s(x,\omega)$ and $Dh^s_{(x,\omega)}$ is Lipschitz with Lipschitz constant L. Hence we have

$$m_s(P^s_{\mathcal{P}r}(x,\omega)) = m^s(W^s_r(x,\omega))\left(1+\mathcal{O}\left(r^{\dim E^s}\right)\right).$$

The same estimate holds for local unstable manifolds:

$$m_u(P^u_{\mathcal{P}r}(x,\omega)) = m^u(W^u_r(x,\omega))\left(1+\mathcal{O}\left(r^{\dim E^s}\right)\right).$$

The above two estimates and (4.44) imply that

$$m(S_r(x,\omega)) = \theta(x,\omega) m^s(W^s_r(x,\omega)) m^u(W^u_r(x,\omega))\left(1+\mathcal{O}\left(r^{(\nu_0')^2}\right)\right). \tag{4.46}$$

For $x \in R(\omega)$, we define $\psi^s_\omega : R(\omega) \mapsto W^u_\epsilon(x_0,\omega)$ by $\psi^s_\omega(x) := W^s_\epsilon(x,\omega) \cap W^u_\epsilon(x_0,\omega)$. It is easy to see that $\psi^s_\omega(\cdot)$ is constant on each stable foliation in $R(\omega)$. Similarly, we define $\psi^u_\omega : R(\omega) \mapsto W^s_\epsilon(x_0,\omega)$ by $\psi^u_\omega(x) = W^u_\epsilon(x,\omega) \cap W^s_\epsilon(x_0,\omega)$. We denote $W^\tau(x,\omega) \cap R(\omega)$ for

$x \in R(\omega)$ to be the connected part of $W^\tau(x,\omega)$ containing x in $R(\omega)$ for $\tau = s,u$. We define

$$\tilde{J}^s(x,\omega) := H_\omega(x, \psi^s_\omega(x), E^u(x,\omega), E^u(\psi^s_\omega(x),\omega)),$$

which is the Jacobian at x of the holonomy map from unstable manifolds $W^u(\psi^u_\omega(x),\omega) \cap R(\omega)$ to $W^u_\epsilon(x_0,\omega)$. Similarly, we define

$$\tilde{J}^u(x,\omega) := \lim_{n\to-\infty} \frac{|\det(D_x f^n_\omega|_{E^s(x,\omega)})|}{|\det(D_{\psi^u_\omega(x)} f^n_\omega|_{E^s(\psi^u_\omega(x),\omega)})|},$$

which is the Jacobian at x of the holonomy map between local stable manifolds $W^s(\psi^s_\omega(x),\omega) \cap R(\omega)$ and $W^s_\epsilon(x_0,\omega)$.

Lemma 4.23. *There exists a constant K_7 independent of ω such that both $\log \tilde{J}^s(x,\omega)$ and $\log \tilde{J}^u(x,\omega)$ are $(K_7,\nu_0')-$Hölder continuous on x.*

Proof. Let $x,y \in R(\omega)$, we denote $z := W^u_\epsilon(x,\omega) \cap W^s_\epsilon(y,\omega)$. Then we have $\psi^s_\omega(z) = \psi^s_\omega(y)$. Notice that z is the image of y under the holonomy map between $W^u(x,\omega) \cap R(\omega)$ and $W^u(y,\omega) \cap R(\omega)$, then by Lemma 4.20 (2), we have

$$\begin{aligned} &\frac{|\det D_z f^n_\omega|_{E^u(z,\omega)}|}{|\det D_{\psi^s_\omega(z)} f^n_\omega|_{E^u(\psi^s_\omega(z),\omega)}|} \Big/ \frac{|\det D_y f^n_\omega|_{E^u(y,\omega)}|}{|\det D_{\psi^s_\omega(y)} f^n_\omega|_{E^u(\psi^s_\omega(y),\omega)}|} \\ =& \frac{|\det D_z f^n_\omega|_{E^u(z,\omega)}|}{|\det D_y f^n_\omega|_{E^u(y,\omega)}|} \le e^{a_0' d(y,z)^{\nu_0'}}. \end{aligned}$$

Switch y and z and let n goes to infinity we have,

$$\left|\log \tilde{J}^s(z,\omega) - \log \tilde{J}^s(y,\omega)\right| \le a_0' d(y,z)^{\nu_0'}.$$

Notice that both x,z lie on the local unstable manifold $W^u_\epsilon(x,\omega)$, then by Lemma 4.20 (1), we have

$$\left|\log \tilde{J}^s(x,\omega) - \tilde{J}^s(z,\omega)\right| \le a_0' d(x,z)^{\nu_0'}.$$

Notice that $z = [y,x]_\omega$ and $\theta_0 = \inf \Theta(E^s(x,\omega), E^u(x,\omega)) > 0$, then there exists a constant $K_8 := K_8(\theta_0, \nu_0')$ such that

$$d(x,z)^{\nu_0'} + d(y,z)^{\nu_0'} \le K_8 d(x,y)^{\nu_0'}.$$

Hence

$$\left|\log \tilde{J}^s(x,\omega) - \log \tilde{J}^s(y,\omega)\right| \le a_0' K_8 d(x,y)^{\nu_0'} := K_7 d(x,y)^{\nu_0'}.$$

The similar proof can be applied to $\tilde{J}^u(x,\omega)$. □

Now consider set $C_r(\Psi^{-1}(x),\omega) := \Psi^{-1}(S_r(x,\omega))$. To obtain the Jacobian of the map Ψ, we need to compare $(m^u \times m^s)(C_r(\Psi^{-1}(x),\omega))$ and $m(S_r(x,\omega))$ and prove Ψ is absolutely continuous.

Lemma 4.24. *For $x \in R(\omega)$ and r sufficiently small, we have*

$$\frac{(m^u \times m^s)(C_r(\Psi^{-1}(x),\omega))}{m(S_r(x,\omega))} = \frac{\tilde{J}^u(x,\omega)\tilde{J}^s(x,\omega)}{\theta(x,\omega)} \left(1 + \mathcal{O}\left(r^{(\nu_0')^2}\right)\right). \tag{4.47}$$

Proof. By the definition of ψ^s_ω and ψ^u_ω, we have $\Psi^{-1}(x) = (\psi^s_\omega(x), \psi^u_\omega(x)) \in W^u_\epsilon(x_0,\omega) \times W^s_\epsilon(x_0,\omega)$. Since the set $S_r(x,\omega)$ has the local product structure, the set $C_r(\Psi^{-1}(x),\omega)$ is a product set

$$C_r(\Psi^{-1}(x),\omega) = C^u_r(\Psi^{-1}(x),\omega) \times C^s_r(\Psi^{-1}(x),\omega),$$

where

$$C^u_r(\Psi^{-1}(x),\omega) = \{\xi \in W^u_\epsilon(x_0,\omega) |\ \Psi(\xi, \psi^u_\omega(x)) \in S_r(x,\omega)\},$$
$$C^s_r(\Psi^{-1}(x),\omega) = \{\eta \in W^s_\epsilon(x_0,\omega) |\ \Psi(\psi^s_\omega(x), \eta) \in S_r(x,\omega)\}.$$

So $(m^u \times m^s)(C_r(\Psi^{-1}(x),\omega)) = m^u(C^u_r(\Psi^{-1}(x),\omega)) \times m^s(C^s_r(\Psi^{-1}(x),\omega))$. Note that the holonomy map between $W^u_\epsilon(x_0,\omega)$ and $W^u(x,\omega) \cap R(\omega)$ is absolutely continuous, hence we have

$$m^u(C_r^u(\Psi^{-1}(x),\omega)) = \int_{W_\epsilon^u(x,\omega)\cap S_r(x,\omega)} \tilde{J}^s(y,\omega)dm^u(y)$$
$$= \tilde{J}^s(x,\omega)m^u(W_r^u(x,\omega))\left(1+\mathcal{O}\left(r^{\nu_0'}\right)\right), \tag{4.48}$$

where the last equality follows the Hölder continuity of $\log \tilde{J}^s(\cdot,\omega)$. Similarly, we have

$$m^s(C_r^s(\Psi^{-1}(x),\omega)) = \tilde{J}^u(x,\omega)m^s(W_r^s(x,\omega))\left(1+\mathcal{O}\left(r^{\nu_0'}\right)\right). \tag{4.49}$$

Now (4.46), (4.48) and (4.49) imply that

$$\frac{(m^u\times m^s)(C_r(\Psi^{-1}(x),\omega))}{m(S_r(x,\omega))} = \frac{\tilde{J}^u(x,\omega)\tilde{J}^s(x,\omega)}{\theta(x,\omega)}\left(1+\mathcal{O}\left(r^{(\nu_0')^2}\right)\right). \qquad \square$$

Lemma 4.25. $\Psi : (W_\epsilon^u(x_0,\omega)\times W_\epsilon^s(x_0,\omega),\mathcal{B},m^u\times m^s)\to(R(\omega),\mathcal{B},m)$ *is absolutely continuous.*

Proof. By the local product structure, we know that Ψ is a homeomorphism. Let $\bar{m}=\Psi^*m$, i.e. for any $A\in\mathcal{B}(W_\epsilon^u(x_0,\omega)\times W_\epsilon^s(x_0,\omega))$, $\bar{m}(A)=m(\Psi(A))$. To prove the absolute continuity of Ψ, we just need to prove that $\bar{m}\ll m^u\times m^s$.

For any $A\subset W_\epsilon^u(x_0,\omega)\times W_\epsilon^s(x_0,\omega)$ Borel measurable set such that $\bar{m}(A)>0$, we want to prove that $m^u\times m^s(A)>0$. We prove it by way of contradiction. Suppose $m^u\times m^s(A)=0$, then for any $\zeta>0$, there exists an open set $U\supset A$ such that $m^u\times m^s(U)\le\zeta$. We pick a compact set $D\subset A$ such that $\bar{m}(D)\ge\frac{1}{2}\bar{m}(A)$. Since Ψ is a homeomorphism, $\Psi(D)$ is compact and $\Psi(U)$ is open.

We choose r_0 sufficiently small such that for any $r\in(0,r_0)$, the equation (4.44) implies

$$m(S_r(x,\omega))\ge\frac{1}{2}K_5 m_s(P_{\mathcal{P}r}^s(x,\omega))m_u(P_{\mathcal{P}r}^u(x,\omega)),$$

where recall K_5 satisfying $\inf \theta(x,\omega) \geq K_5$. The equation (4.47) implies

$$\frac{(m^u \times m^s)(C_r(\Psi^{-1}(x),\omega))}{m(S_r(x,\omega))} = \frac{1}{2} \inf_{x \in R(\omega)} \frac{\tilde{J}^u(x,\omega)\tilde{J}^s(x,\omega)}{\theta(x,\omega)} \geq \frac{1}{2} e^{-2a_0'}.$$

We can find a finite disjoint collection of cubes $\Gamma_n \subset \Psi(U)$ with diameter less than r_0 and

$$m(\cup_n \Gamma_n \cap \Psi(D)) \geq \frac{1}{2}\bar{m}(D).$$

By Lemma 4.22, we can find a set $S_n \subset \Gamma_n$ which has the same structure as $S_r(x,\omega)$ such that

$$m(S_n) \geq \frac{1}{2} \inf_{x \in R(\omega)} \theta_{x,\omega} m(\Gamma_n) \geq \frac{K_5}{2} m(\Gamma_n).$$

Now

$$\begin{aligned}
m^u \times m^s(U) &\geq m^u \times m^s(\cup_n \Psi^{-1}\Gamma_n) \geq m^u \times m^s(\cup_n \Psi^{-1}(S_n)) \\
&\geq \frac{1}{2} e^{-2a_0'} \sum_n m(S_n) \geq \frac{1}{2} e^{-2a_0'} \frac{K_5}{2} \sum_n m(\Gamma_n) \\
&\geq \frac{e^{-2a_0'} K_5}{4} \frac{1}{2} \bar{m}(D) \\
&\geq \frac{e^{-2a_0'} K_5}{16} \bar{m}(A).
\end{aligned}$$

This leads to a contradiction by choosing $\zeta < \frac{e^{-2a_0'} K_5}{16} \bar{m}(A)$. Therefore, Ψ is absolutely continuous. $\square$

Now by the Radon-Nikodym theorem, we have

$$Jac(\Psi)(\xi,\eta) = \lim_{r \to 0} \frac{m(S_r(\Psi(\xi,\eta),\omega))}{m^u \times m^s(C_r((\xi,\eta),\omega))} = \frac{\theta(\Psi(\xi,\eta),\omega)}{\tilde{J}^u(\Psi(\xi,\eta),\omega)\tilde{J}^s(\Psi(\xi,\eta),\omega)}.$$

Then for any $(\xi,\eta),(\xi',\eta')\in W^u_\epsilon(x_0,\omega)\times W^s_\epsilon(x_0,\omega)$, we have

$$\begin{aligned}
&|\log Jac(\Psi)(\xi,\eta)-\log Jac(\Psi)(\xi',\eta')| \\
&\leq \max\{K_6,K_7\}d(\Psi(\xi,\eta),\Psi(\xi',\eta'))^{\nu_0'} \\
&\leq \max\{K_6,K_7\}(d(\Psi(\xi,\eta),\Psi(\xi,\eta'))+d(\Psi(\xi,\eta'),\Psi(\xi',\eta')))^{\nu_0'} \\
&\leq \max\{K_6,K_7\}a_0'(d(\xi,\xi')+d(\eta,\eta'))^{\nu_0'}.
\end{aligned}$$

Hence $\log Jac(\Psi)$ is $(\max\{K_6,K_7\}a_0',\nu_0')-$Hölder continuous. By the Radon-Nikodym theorem, for any bounded measurable function $I: M\to\mathbb{R}$, we have

$$\int_{R(\omega)} I(x)dm(x)=\int_{W^u_\epsilon(x,0,\omega)}\int_{W^s_\epsilon(x_0,\omega)} I(\Psi(\xi,\eta))Jac\Psi(\xi,\eta)dm^s(\eta)dm^u(\xi).$$

For each stable foliation $\gamma\cap R(\omega)$, there exists a $\xi\in W^u_\epsilon(x_0,\omega)$ such that $\gamma\cap R(\omega)$ is the image of $\Psi_\xi:=\Psi|_{\{\xi\}\times W^s_\epsilon(x_0,\omega)}$ which is exactly the holonomy map between $W^s_\epsilon(x_0,\omega)$ and $\gamma\cap R(\omega)$. We denote this $\gamma\cap R(\omega)$ by γ_ξ. Denote the Jacobian of $\Psi|_{\{\xi\}\times W^s_\epsilon(x_0,\omega)}$ by $Jac_\xi(\Psi)$. Using Radon-Nikodym theorem again, we have

$$\int_{R(\omega)} I(x)dm(x)=\int_{W^u_\epsilon(x_0,\omega)}\int_{\gamma_\xi} I(x)\frac{Jac\Psi(\Psi_\xi^{-1}(x))}{Jac_\xi\Psi(\Psi_\xi^{-1}(x))}dm^s_{\gamma_\xi}(x)dm^u(\xi).$$

We define $H(\omega): R(\omega)\to R$ by

$$H(\omega)(x):=\frac{Jac\Psi(\Psi^{-1}_{\psi^s_\omega(x)}(x))}{Jac_{\psi^s_\omega(x)}\Psi(\Psi^{-1}_{\psi^s_\omega(x)}(x))}.$$

Note that for x,y in a same stable leaf in $R(\omega)$, $\psi^s_\omega(x)=\psi^s_\omega(y)\in W^u_\epsilon(x_0,\omega)$. Moreover, in Lemma 4.20, we have proved that the holonomy map $\Psi^{-1}_{\psi^s_\omega(x)}(x)$ is $(a_0',\nu_0')-$Hölder continuous on $\gamma_{\psi^s_\omega(x)}$ and $\log Jac_{\psi^s_\omega(x)}\Psi$ is $(a_0',\nu_0')-$Hölder continuous on $\{\psi^s_\omega(x)\}\times W^s_\epsilon(x_0,\omega)$. Hence for x, y in a same stable leaf in $R(\omega)$, we have

$$\begin{aligned}|\log H(\omega)(x) - \log H(\omega)(y)| &\le |\log Jac\Psi(\Psi^{-1}_{\psi^s_\omega(x)}(x)) - \log Jac\Psi(\Psi^{-1}_{\psi^s_\omega(x)}(y))| \\ &\quad + |\log Jac_{\psi^s_\omega(x)}\Psi(\Psi^{-1}_{\psi^s_\omega(x)}(x)) - \log Jac_{\psi^s_\omega(x)}\Psi(\Psi^{-1}_{\psi^s_\omega(x)}(y))| \\ &\le (\max\{K_6, K_7\}a'_0 + a'_0)d(\Psi^{-1}_{\psi^s_\omega(x)}(x), \Psi^{-1}_{\psi^s_\omega(x)}(y))^{\nu'_0} \\ &\le (\max\{K_6, K_7\}a'_0 + a'_0)a'_0 d(x, y)^{(\nu'_0)^2}.\end{aligned}$$

Hence $\log H(\omega)$ is $(a''_0, \nu''_0) := ((\max\{K_6, K_7\}a'_0 + a'_0)a'_0, (\nu'_0)^2)-$Hölder continuous on each local stable leaf in $R(\omega)$ The proof of Proposition 4.21 is done. □

4.2 For Random Partially Hyperbolic on Fibers Systems

In this section, we introduce several lemmas, including the strong unstable invariant manifolds theorem in Subsection 4.2.1 and a distortion lemma in Subsection 4.2.2, that will be used in the proof of existence of the random Gibbs $u-$states for random partially hyperbolic on fibers systems.

4.2.1 Strong Unstable Invariant Manifolds.

We first state the local strong unstable invariant manifolds theorem in our settings. Since $f : \Omega \to \text{Diff}^2(M)$ is a continuous mapping, the following condition holds naturally:

$$\int (\log^+ \|f_\omega\|_{C^2} + \log^+ \|f_\omega^{-1}\|_{C^2}) dP(\omega) < \infty.$$

The following lemma can be viewed as an adapted version of unstable invariant manifolds theorem with the help of Lusin's theorem in [37] by noticing that $E^{uu}(x, \omega)$ depends continuously on $(x, \omega) \in M \times \Omega$ and $f_\omega \in \text{Diff}^2(M)$ depends continuously on $\omega \in \Omega$. The proof of this lemma can be carried word for word from the arguments of Theorem III3.1 of [45].

Lemma 4.26 (Local Strong Unstable Invariant Manifolds Theorem)**.** *For random partially* *hyperbolic on fibers systems in our setting, the local strong unstable set is a* $C^{1,1}$ *embedded submanifold given by*

$$W^{uu}_\delta(x,\omega) = exp_x(Graph(h^u_{(x,\omega)})) \tag{4.50}$$

satisfying that:

(i) $h^u_{(x,\omega)} : E^{uu}_\delta(x,\omega) \to E^{cs}(x,\omega)$ *is a* $C^{1,1}-$*map with* $h^u_{(x,\omega)}(0) = 0$, $Dh^u_{(x,\omega)}(0) = 0$, $Liph^u_{(x,\omega)}(\cdot) \le \frac{1}{3}$, $LipD.h^u_{(x,\omega)} \le L$, *where* $E^{uu}_\delta(x,\omega) = \{\eta \in E^{uu}(x,\omega) : |\eta| < \delta\}$ *and* $L > 1$ *is a constant;*

(ii) $W^{uu}_\delta(\phi(x,\omega)) \subset f_\omega(W^{uu}_\delta(x,\omega))$, *and* $W^{uu}(x,\omega) = \bigcup_{n\ge 1} f^n_{\theta^{-n}\omega} W^{uu}_\delta(f^{-n}_\omega x, \theta^{-n}\omega)$ *where* $W^{uu}(x,\omega)$ *is given by* (3.1)*;*

(iii) $d^u(f^{-n}_\omega y, f^{-n}_\omega z) \le \gamma_0 e^{-n(\lambda_0-\epsilon_0)} d^u(y,z)$ *for any* $y, z \in W^{uu}_\delta(x,\omega)$ *where* d^u *denotes the distance along the strong unstable manifolds,* $\gamma_0 > 0$ *and* $0 < \epsilon_0 \ll \lambda_0$ *are constants;*

(iv) For any $\rho < \frac{1}{4}\delta$, *if* $W^u_\rho(x,\omega) := exp_x(Gragh(h^u_{(x,\omega)}|_{E^u_\rho(x,\omega)}))$ *intersects* $W^u_\rho(x',\omega)$, *then* $W^u_\rho(x,\omega) \subset W^{uu}_\delta(x',\omega)$*;*

(v) $W^{uu}_\delta(x,\omega)$ *depends continuously on* $(x,\omega) \in M \times \Omega$.

In the above lemma, we may shrink $\delta < \rho_0$ such that for any $(x,\omega) \in M_\omega$, $W^{uu}_\delta(x,\omega)$ lies in a normal neighborhood.

4.2.2 A Distortion Lemma.

We also need the following distortion lemma. Denote $J^u(x,\omega) := |det(D_x f_\omega|_{E^{uu}(x,\omega)})|$.

Lemma 4.27. *There exists a constant* $C > 0$ *independent of* $\omega \in \Omega$ *such that for any* $n \in \mathbb{N}$, *for any* (x,ω), *and* $y, z \in W^{uu}_\delta(x,\omega)$, *we have*

$$\frac{1}{C} \le \prod_{k=0}^{n-1} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} \le C. \tag{4.51}$$

Proof. We first prove the uniform Lipschitz variation of $J^u(x,\omega)$ along the local strong unstable manifolds, i.e., there exists a constant $K>0$ which is independent of x and ω such that for any $(x,\omega)\in M_\omega$, $y,z\in W^{uu}_\delta(x,\omega)$, one has

$$|J^u(y,\omega)-J^u(z,\omega)|\le Kd^u(y,z), \tag{4.52}$$

where $d^u(\cdot,\cdot)$ is the distance along $W^{uu}_\delta(x,\omega)$.

Denote $\pi^{uu}_{(x,\omega)},\pi^{cs}_{(x,\omega)}$ to be the projection from T_xM_ω to $E^{uu}(x,\omega)$ and $E^{cs}(x,\omega)$ respectively with respect to the splitting $T_xM_\omega=E^{uu}(x,\omega)\oplus E^{cs}(x,\omega)$, notice that $E^{uu}(x,\omega)$ and $E^{cs}_{(x,\omega)}$ are continuously depending on (x,ω), so $\|\pi^{uu}_{(x,\omega)}\|$ and $\|\pi^{cs}_{(x,\omega)}\|$ are uniformly bounded by the compactness of $M\times\Omega$. Notice that $f:\Omega\to \mathrm{Diff}^2(M)$ is continuous, $\|D_xf_\omega\|$ and $\|D^2_xf_\omega\|$ are uniformly bounded. Let $M\ge 1$ be a constant such that

$$\max\left\{\sup_{(x,\omega)\in M\times\Omega}|D_xf_\omega|,\sup_{(x,\omega)\in M\times\Omega}|D^2_xf_\omega|,\sup_{(x,\omega)\in M\times\Omega}LipDh^u_{(x,\omega)},\sup_{(x,\omega)}\{\|\pi^{cs}_{(x,\omega)}\|,\|\pi^{uu}_{(x,\omega)}\|\}\right\}\le M.$$

For sake of simplicity, we will identify $exp_x(\cdot)$ with $x+\cdot$ in the rest of the proof.

Notice that if $y,z\in W^{uu}_\delta(x,\omega)$, and $d^u(y,z)<\frac{\delta}{2M^3}$, then

$$\left|\pi^{uu}_{(y,\omega)}(z-y)\right|\le Md^u(y,z)<\frac{\delta}{2M^2};$$
$$\left|\pi^{uu}_{(f_\omega(y),\theta\omega)}(f_\omega(z)-f_\omega(y))\right|\le M|f_\omega(z)-f_\omega(y)|\le M^2|z-y|\le\frac{\delta}{2M}.$$

Therefore, $(z,\omega)\in W^{uu}_\delta(y,\omega)$ and $(f_\omega(z),\theta\omega)\in W^{uu}_\delta(f_\omega(y),\theta\omega)$. So it is sufficient to prove that there exists a constant $K>0$ independent of x and ω such that for any $y\in W^{uu}_{\frac{\delta}{2M^3}}(x,\omega)$,

$$|J^u(x,\omega)-J^u(y,\omega)|\le Kd^u(x,y). \tag{4.53}$$

With the help of the normal coordinate chart, and notice that $d(x,y)<\delta<\delta_0$ and $d(f_\omega x,f_\omega y)\le\delta<\delta_0$, we may view that x, y together with $W^s_{\frac{\delta}{2M^3}}(x,\omega)$ lie in a same Euclidean space and $f_\omega x$, $f_\omega y$ together with $W^s_\delta(f_\omega x, \theta\omega)$ lie in a same Euclidean space. By the local strong unstable manifolds theorem, there exist $\xi_y \in E^{uu}_{\frac{\delta}{2M^2}}(x,\omega)$ and $\xi_{f_\omega(y)} \in E^{uu}_\delta(f_\omega(x), \theta\omega)$ such that

$$y = x + \xi_y + h^u_{(x,\omega)}(\xi_y); \tag{4.54}$$

$$f_\omega(y) = f_\omega(x) + \xi_{f_\omega(y)} + h^u_{(f_\omega(x),\theta\omega)}(\xi_{f_\omega(y)}), \tag{4.55}$$

and $E^{uu}(y,\omega) = graph((Dh^u_{(x,\omega)})_{\xi_y})$, $E^{uu}(f_\omega(y), \theta\omega) = graph((Dh^u_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}})$. From (4.54) and (4.55), we have

$$\left(1 - \frac{1}{3}\right)|\xi_{f_\omega(y)}| \le |\xi_{f_\omega(y)} + h^u_{(f_\omega(x),\theta\omega)}(\xi_{f_\omega(y)})| = |f_\omega(y) - f_\omega(x)| \le M|y-x| \le M\left(1 + \frac{1}{3}\right)|\xi_y|,$$

so $|\xi_{f_\omega(y)}| \le 2M|\xi_y|$.

Now, we define the following linear maps $L_{(x,\omega)}, L_{(y,\omega)} : E^{uu}(x,\omega) \to E^{uu}(f_\omega(x), \theta\omega)$ by

$$L_{(x,\omega)} = D_x f_\omega|_{E^{uu}(x,\omega)};$$
$$\tilde{L}_{(y,\omega)} = \pi^{uu}_{(f_\omega(x),\theta\omega)} D_y f_\omega|_{E^{uu}(y,\omega)}(I + (Dh^u_{(x,\omega)})_{\xi_y}).$$

Then $\|L_{(x,\omega)}\|, \|\tilde{L}_{(y,\omega)}\| \le \frac{4}{3}M^2$. Now for any $v \in E^{uu}(x,\omega)$ with $\|v\| = 1$, we have

$$\begin{aligned}
&\sup_{\|v\|=1} \left\| D_x f_\omega v - \pi^u_{(f_\omega(x),\theta\omega)} D_y f_\omega (I + (Dh^u_{(x,\omega)})_{\xi_y}) v \right\| \\
&\le M\left(\|D_x f_\omega - D_y f_\omega\| + \|D_y f_\omega (Dh^u_{(x,\omega)})_{\xi_y}\|\right) \\
&\le M^2|y-x| + M^3|\xi_y| \\
&\le (M^2 + \frac{3}{2}M^3) d^u(x,y).
\end{aligned}$$

Hence, $\|L_{(x,\omega)} - \tilde{L}_{(y,\omega)}\| \le C_0(M^2 + \frac{3}{2}M^3)d^u(x,y)$, where C_0 only depends on the normal coordinate charts. Then by properties of determinant,

$$|det(L_{(x,\omega)}) - det(\tilde{L}_{(y,\omega)})| \leq C_1 d^u(x,y), \tag{4.56}$$

where C_1 is a polynomial of M and $\dim E^{uu}(x,\omega)$.

Notice that

$$\begin{aligned}
\|\pi^{uu}_{(f_\omega(x),\theta\omega)}|_{E^{uu}(f_\omega(y),\theta\omega)} - I\| &\leq \frac{\|(Dh^u_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|}{1 - \|(Dh^u_{(f_\omega(x),\theta\omega)})_{\xi_{f_\omega(y)}}\|} \\
&\leq \frac{M|\xi_{f_\omega(y)}|}{1 - M|\xi_{f_\omega(y)}|} \\
&\leq \frac{2M^2|\xi_y|}{1 - 2M^2|\xi_y|} \\
&\leq \frac{2M^2|\xi_y|}{1 - 2M^2\frac{\delta}{2M^2}} \\
&\leq 4M^2|\xi_y| \leq 6M^2 d^u(x,y).
\end{aligned}$$

So we have

$$|det(\pi^{uu}_{(f_\omega(x),\theta\omega)}|_{E^{uu}(f_\omega(y),\theta\omega)}) - 1| \leq C_2 d^u(x,y), \tag{4.57}$$

where C_2 is a polynomial of M and $\dim E^{uu}(x,\omega)$. Also

$$\|I + (Dh^u_{(x,\omega)})_{\xi_y} - I\| \leq M|\xi_y| \leq M d^u(x,y)$$

implies that there exists a constant C_3 such that

$$|det(I + (Dh^u_{(x,\omega)})_{\xi_y}) - 1| \leq C_3 d^u(x,y). \tag{4.58}$$

Combining (4.56), (4.57), and (4.58), we have

$$|J^u(x,\omega) - J^u(y,\omega)| \leq K d^u(x,y),$$

where K only depends on C_1, C_2, C_3.

Next, we prove (4.51). By (4.52), for any $y, z \in W_\delta^{uu}(x, \omega)$

$$|J^u(\phi^{-k}(y,\omega)) - J^u(\phi^{-k}(z,\omega))| \leq C_4 d^u(f_\omega^{-k}(y), f_\omega^{-k}(z)) \leq C_4\gamma_0 e^{-k(\lambda_0-\epsilon_0)} d^u(y,z) \tag{4.59}$$
$$\leq C_4\gamma_0 e^{-k(\lambda_0-\epsilon_0)}\delta.$$

Notice that $J^u(x,\omega) \geq e^{\lambda_0} > 1$ for all $(x,\omega) \in M \times \Omega$, hence we have

$$\left| \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} - 1 \right| \leq C_4\gamma_0\delta e^{-\lambda_0} e^{-k(\lambda_0-\epsilon_0)}. \tag{4.60}$$

As a consequence, there exists a constant C which is independent of x and ω such that

$$\frac{1}{C} \leq \prod_{k=0}^{n-1} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} \leq C$$

for any $n \in \mathbb{N}$. □

By (4.60), the function

$$D(x,y,\omega) := \lim_{n\to\infty} \prod_{k=0}^{n-1} \frac{J^u(\phi^{-k}(x,\omega))}{J^u(\phi^{-k}(y,\omega))} \tag{4.61}$$

is well defined if $x \in W_\delta^{uu}(y,\omega)$, and this limit converges uniformly on x.

Chapter 5. Random Specification

In this chapter, we prove Theorem 3.7, Theorem 3.8 and Theorem 3.9 in Section 5.1, Section 5.2 and Section 5.3 respectively.

5.1 Random Anosov and topological mixing on fibers systems has Random Specification

In this section, we prove Theorem 3.7. The proof is quite similar to the deterministic case. We start with the following lemma.

Lemma 5.1. *Let ϵ_0 be the size of local stable and unstable manifolds, $g \in L^\infty(\Omega, M)$ be a random periodic point of ϕ with period m, then for any $\epsilon \in (0, \epsilon_0]$ and any $\delta > 0$, there exists an integer $T = T(g, \epsilon, \delta)$ such that for all $\omega \in \Omega$*

$$\bigcup_{k=0}^{T} f^{km}_{\theta^{-km}\omega}(W^u_\epsilon(g(\theta^{-km}\omega), \theta^{-km}\omega))$$

is $\delta-$dense in M_ω.

Proof. Pick $\epsilon' \leq \frac{1}{4}\min\{\delta, \epsilon\}$, and let $\delta' \in (0, \epsilon')$ be the constant in the local product structure corresponding to ϵ', i.e., for all $\omega \in \Omega$, for any $x, y \in M$ with $d_M(x, y) < \delta'$, then $W^u_{\epsilon'}(x, \omega) \cap W^s_{\epsilon'}(y, \omega) \neq \emptyset$.

By the compactness of M, we can pick $\{x_j\}_{j=1}^n$ a $\delta/2-$dense subset of M. Define $U_j = B_{\delta'/8}(x_j)$ to be the ball centered at x_j with radius $\delta'/8$. By the definition of topological mixing on fibers, there exists an integer $T_{i,j}$ such that for any $n \geq T_{i,j}$, for any $\omega \in \Omega$, we have

$$\phi^n(B_{\delta'/8}(x_i) \times \{\theta^{-n}\omega\}) \cap (B_{\delta'/8}(x_j) \times \{\omega\}) \neq \emptyset.$$

Let $T_{\max} = \max_{1 \leq i,j \leq n}\{T_{i,j}\}$, and define T to be any integer such that

$$T \cdot m \geq T_{\max} \text{ and } e^{-Tm\lambda}\epsilon' < \delta'/8. \tag{5.1}$$

In the following, we are going to show that for all $\omega \in \Omega$

$$\bigcup_{k=1}^{T} f^{km}_{\theta^{-km}\omega}(W^u_\epsilon(g(\theta^{-km}\omega), \theta^{-km}\omega))$$

is $\delta-$dense in M_ω.

Now for any fixed $\omega \in \Omega$, $g(\theta^{-Tm}\omega) \in B_{\delta'/8}(x_i)$ for some x_i. Then by definition of T, for any x_j, we have

$$\phi^{Tm}(B_{\delta'/8}(x_i) \times \{\theta^{-Tm}\omega\}) \cap B_{\delta'/8}(x_j) \times \{\omega\} \neq \emptyset.$$

Pick a point $x(\omega) \in \phi^{Tm}(B_{\delta'/8}(x_i) \times \{\theta^{-Tm}\omega\}) \cap B_{\delta'/8}(x_j) \times \{\omega\}$. Then by the choice of δ', then

$$z(\omega) := W^u_{\epsilon'}(x(\omega), \omega) \cap W^s_{\epsilon'}(x_j, \omega)$$

is defined. By the local unstable manifolds theorem, we have

$$d(f_\omega^{-Tm} z(\omega), f_\omega^{-Tm} x(\omega)) \leq e^{-Tm\lambda}\epsilon' < \frac{\delta'}{8}.$$

As a consequence, we have

$$d(f_\omega^{-Tm} z(\omega), x_i) \leq d(f_\omega^{-Tm} z(\omega), f_\omega^{-Tm} x(\omega)) + d(f_\omega^{-Tm} x(\omega), x_i) \leq \frac{\delta'}{8} + \frac{\delta'}{8} = \frac{\delta'}{4}.$$

So we have

$$d(f_\omega^{-Tm} z(\omega), g(\theta^{-Tm}\omega)) \leq d(f_\omega^{-Tm} z(\omega), x_i) + d(x_i, g(\theta^{-Tm}\omega)) \leq \frac{\delta'}{4} + \frac{\delta'}{8} \leq \frac{\delta'}{2}.$$

Then by the choice of δ',

$$q(\theta^{-Tm}\omega) := W^u_{\epsilon'}(g(\theta^{-Tm}\omega), \theta^{-Tm}\omega) \cap W^s_{\epsilon'}(f_\omega^{-Tm} z(\omega), \theta^{-Tm}\omega)$$

is defined. Moreover, $f^{Tm}_{\theta^{-Tm}\omega} q(\theta^{-Tm}\omega) \in f^{Tm}_{\theta^{-Tm}\omega}(W^u_\epsilon(g(\theta^{-Tm}\omega), \theta^{-Tm}\omega))$ and

$$d(f^{Tm}_{\theta^{-Tm}\omega} q(\theta^{-Tm}\omega), x_j) \leq d(f^{Tm}_{\theta^{-Tm}\omega} q(\theta^{-Tm}\omega), z(\omega)) + d(z(\omega), x_j) \leq e^{-Tm\lambda}\epsilon' + \epsilon' \leq \frac{\delta}{2}.$$

Notice that in the above proof, x_j is arbitrarily chosen from a $\frac{\delta}{2}-$dense subset of M. Hence, the proof is done. $\square$

Lemma 5.2. *For any $\epsilon \in (0, \epsilon_0]$ there exists an integer N only depending on ϵ, such that for any $x, y \in M$, $\omega \in \Omega$, $n \geq N$*

$$f_\omega^n(W_\epsilon^u(x,\omega)) \cap W_\epsilon^s(y, \theta^n\omega) \neq \emptyset.$$

Proof. For any $\epsilon' > 0$, we denote $\delta(\epsilon') \in (0, \epsilon')$ to be the number in local product structure corresponding to ϵ'. Define $\eta = \min\{\epsilon/2, \delta(\epsilon/2)/4\}$. Pick a $\delta(\epsilon/2)/4$-dense set $\{x_k\}_{k=1}^l$ in M, then $\tilde{x_k} \equiv x_k$ is a continuous function from Ω to M for $k \in \{1, ..., l\}$. Apply Lemma 4.7, there exists a number $n \in \mathbb{N}$ (pick a least common multiple if necessary) and a set of random periodic point $\{\tilde{g}_k\}_{k=1}^l$ with period n and satisfying

$$d_{L^\infty(\Omega,M)}(\tilde{x}_k, \tilde{g}_k) \leq \delta(\epsilon/2)/4 \text{ for all } k \in \{1, ..., l\}.$$

Then for any $x \in M$, there exists a $\tilde{g} \in \{\tilde{g}_k\}$ such that

$$\sup_{\omega\in\Omega} d_M((x,\omega), (\tilde{g}(\omega), \omega)) \leq \delta(\epsilon/2)/2.$$

Use Lemma 5.1, there exists a $T_k = T_k(\tilde{g}_k, \eta, \delta(\eta))$ such that

$$\bigcup_{m=0}^{T_k} f_{\theta^{-nm}\omega}^{nm}(W_\eta^u(\tilde{g}_k(\theta^{-nm}\omega), \theta^{-nm}\omega))$$

is $\delta(\eta)-$dense in M_ω for all $\omega \in \Omega$.

Now define $T = \prod_{k=1}^l T_k$ and $N = nT$, then for all $\omega \in \Omega$,

$$\bigcup_{m=0}^{T} f_{\theta^{-nm}\omega}^{nm}(W_\eta^u(\tilde{g}_k(\theta^{-nm}\omega), \theta^{-nm}\omega)) = \phi^N(W_\eta^u(\tilde{g}_k(\theta^{-N}\omega), \theta^{-N}\omega))$$

is $\delta(\eta)$-dense in M_ω for all k.

For any $x, y \in M$, pick $j \in \{1, 2, \ldots, l\}$ such that $\sup_{\omega\in\Omega} d((x,\omega), (\tilde{g}_j(\omega), \omega)) < \delta(\epsilon/2)/2$.

Let $z \in f^N_{\theta^{-N}\omega}(W^u_\eta(\tilde{g}_j(\theta^{-N}\omega), \theta^{-N}\omega))$ satisfying $d((y,\omega),(z,\omega)) \le \delta(\eta)$, and by the local product structure, there exists a point

$$v \in W^u_\eta(z,\omega) \cap W^s_\eta(y,\omega).$$

Then

$$\begin{aligned} f^{-N}_\omega v &\in W^u_\eta(\phi^{-N}(z,\omega)) \\ &\subset W^u_{2\eta}(\tilde{g}_j(\theta^{-N}\omega), \theta^{-N}\omega) \\ &\subset W^u_{\delta(\epsilon/2)/2}(\tilde{g}_j(\theta^{-N}\omega), \theta^{-N}\omega), \end{aligned}$$

so

$$\begin{aligned} & d\left(\phi^{-N}(v,\omega), (x, \theta^{-N}\omega)\right) \\ \le\ & d\left(\phi^{-N}(v,\omega), (\tilde{g}_j(\theta^{-N}\omega), \theta^{-N}\omega)\right) + d\left((\tilde{g}_j(\theta^{-N}\omega), \theta^{-N}\omega), (x, \theta^{-N}\omega)\right) \\ \le\ & \frac{\delta(\epsilon/2)}{2} + \frac{\delta(\epsilon/2)}{2} \\ =\ & \delta(\epsilon/2). \end{aligned}$$

Therefore, there exists a point

$$\rho \in W^s_{\epsilon/2}(\phi^{-N}(v,\omega)) \cap W^u_{\epsilon/2}(x, \theta^{-N}\omega).$$

Then We have

$$\begin{aligned} f^N_{\theta^{-N}\omega}\rho &\in W^s_{\epsilon/2}(v,\omega) \cap f^N_{\theta^{-N}\omega}(W^u_{\epsilon/2}(x, \theta^{-N}\omega)) \\ &\subset W^s_\eta(y,\omega) \cap f^N_{\theta^{-N}\omega}(W^u_\epsilon(x, \theta^{-N}\omega)). \end{aligned}$$

As a consequence, we have

$$W^s_\epsilon(y,\omega)\cap f^N_{\theta^{-N}\omega}(W^u_\epsilon(x,\theta^{-N}\omega))\neq\emptyset.$$

Now for any $n\geq N$

$$\begin{aligned}\emptyset &\neq W^s_\epsilon(y,\omega)\cap f^N_{\theta^{-N}\omega}(W^u_\epsilon(\phi^{n-N}(x,\theta^{-n}\omega)))\\ &\subset W^s_\epsilon(y,\omega)\cap f^n_{\theta^{-n}\omega}(W^u_\epsilon(x,\theta^{-n}\omega)).\end{aligned}$$

Since the above holds for arbitrary $\omega\in\Omega$, we can get the conclusion

$$f^n_\omega(W^u_\epsilon(x,\omega))\cap W^s_\epsilon(y,\theta^n\omega)\neq\emptyset$$

for $n\geq N$ and any $\omega\in\Omega$. □

Now we are ready to prove Theorem 3.7. For any fixed $\epsilon>0$, we first define $N=N(\epsilon)$ to be the desired space of the random specification.

Let $\beta\leq\min\{\epsilon/2,\epsilon_0,\delta(\epsilon_0),\alpha(\epsilon/2)\}$ be a positive number, where ϵ_0 is the size of local stable and unstable manifolds, $\delta(\epsilon_0)$ is the number in local product structure corresponding to ϵ_0 and $\alpha(\epsilon/2)$ is the number in shadowing lemma corresponding to $\epsilon/2$. Define $\gamma=\beta/8$, and let N be in Lemma 5.2 such that for any $x,y\in M$, $n\geq N$, we have

$$f^n_\omega(W^u_\gamma(x,\omega))\cap W^s_\gamma(y,\theta^n\omega)\neq\emptyset \tag{5.2}$$

for any $\omega\in\Omega$. Moreover, we pick N sufficiently large such that $e^{-N\lambda_0}\leq\frac{1}{2}$ and fix this N.

Now, let $S=(\tau,P)$ be any $N-$spaced random specification. For each fixed $\omega\in\Omega$, define $P_\omega(t):=P(t)(\theta^t\omega)$ for $t\in I\in\tau$, then $S_\omega=(\omega,\tau,P_\omega)$ is a $N-$spaced $\omega-$specification by Remark 3.4. We first prove that the N spaced ω-specification S_ω is shadowed by a point.

We define

$$(x_{a_1}, \theta^{a_1}\omega) = (P_\omega(a_1), \theta^{a_1}\omega)$$

and define $(x_{a_k}, \theta^{a_k}\omega)$ inductively: once $(x_{a_k}, \theta^{a_k}\omega)$ is defined, pick

$$x_{a_{k+1}} \in f_{\theta^{b_k}\omega}^{a_{k+1}-b_k}(W_\gamma^u(\phi^{b_k-a_k}(x_{a_k}, \theta^{a_k}\omega))) \cap W_\gamma^s(P_\omega(a_{k+1}), \theta^{a_{k+1}}\omega) \tag{5.3}$$

for $k \in \{1, 2, ..., m-1\}$, where the right hand set is not the emptyset since $a_{k+1} - b_k > N$ and (5.2). Define $x := \pi_M \phi^{-a_m}(x_{a_m}, \theta^{a_m}\omega)$, and we are going to show x is $(\omega, \beta/2)-$shadowing the $\omega-$specification S_ω, i.e.,

$$d(\phi^t(x, \omega), (P_\omega(t), \theta^t\omega)) < \beta/2 \;\text{ for } t \in \cup_{i=1}^m I_i. \tag{5.4}$$

For any fixed $t \in \cup_{i=1}^m I_i$, there exists a $j \in \{1, 2, ..., m\}$ such that $a_j \le t \le b_j$. Then

$$d(\phi^t(x, \omega), (P_\omega(t), \theta^t\omega)) \le d(\phi^t(x, \omega), \phi^{t-a_j}(x_{a_j}, \theta^{a_j}\omega)) + d(\phi^{t-a_j}(x_{a_j}, \theta^{a_j}\omega), (P_\omega(t), \theta^t\omega)),$$

and we have

$$d(\phi^{t-a_j}(x_{a_j}, \theta^{a_j}\omega), (P_\omega(t), \theta^t\omega)) = d(\phi^{t-a_j}(x_{a_j}, \theta^{a_j}\omega), \phi^{t-a_j}(P_\omega(a_j), \theta^{a_j}\omega)) \le \alpha \le \beta/4. \tag{5.5}$$

To estimate $d(\phi^t(x, \omega), \phi^{t-a_j}(x_{a_j}, \theta^{a_j}\omega))$, we are going to show that

$$f_\omega^{b_j}(x, \omega) \in W_{2\gamma}^u(\phi^{b_j-a_j}(x_{a_j}, \theta^{a_j}\omega)). \tag{5.6}$$

We prove (5.6) by induction, by construction of $x_{a_{k+1}}$, we know that

$$f_\omega^{b_j} \circ f_{\theta^{a_{j+1}}\omega}^{-a_{j+1}} x_{a_{j+1}} \in W_\gamma^u(\phi^{b_j-a_j}(x_{a_j}, \theta^{a_j}\omega)).$$

By

$$f_\omega^{b_{j+1}} \circ f_{\theta^{a_{j+2}}\omega}^{-a_{j+2}} x_{a_{j+2}} \in W_\gamma^u(\phi^{b_{j+1}-a_{j+1}}(x_{a_{j+1}}, \theta^{a_{j+1}}\omega)),$$

we can get

$$f_\omega^{b_j} \circ f_{\theta^{b_{j+1}}\omega}^{-b_{j+1}} \circ f_\omega^{b_{j+1}} \circ f_{\theta^{a_{j+2}}\omega}^{-a_{j+2}} x_{a_{j+2}} \in W_{\gamma \cdot e^{-\lambda M}}^u(\phi^{b_j-a_{j+1}}(x_{a_{j+1}}, \theta^{a_{j+1}}\omega)),$$

i.e.

$$\begin{aligned} f_\omega^{b_j} \circ f_{\theta^{a_{j+2}}\omega}^{-a_{j+2}} x_{a_{j+2}} &\in W_{\gamma \cdot e^{-\lambda M}}^u(\phi^{b_j-a_{j+1}}(x_{a_{j+1}}, \theta^{a_{j+1}}\omega)) \\ &\subset W_{\gamma+\gamma \cdot e^{-\lambda M}}^u(\phi^{b_j-a_j}(x_{a_j}, \theta^{a_j}\omega)). \end{aligned}$$

Inductively, we have

$$f_\omega^{b_j} \circ f_{\theta^{a_m}\omega}^{-a_m} x_{a_m} \in W_{2\gamma}^u(\phi^{b_j-a_j}(x_{a_j}, \theta^{a_j}\omega)),$$

then (5.6) is proved. As a consequence, we have

$$d(\phi^t(x,\omega), \phi^{t-a_j}(x_{a_j}, \theta^{a_j}\omega)) \le 2\gamma = \beta/4. \tag{5.7}$$

Combining (5.5) and (5.7), we conclude

$$d(\phi^t(x,\omega), (P_\omega(t), \theta^t\omega)) \le \beta/4 + \beta/4 < \beta/2.$$

Next, we are going to prove if $P_\omega(t)$ Borel measurably depends on $\omega \in \Omega$ for fixed t, then the shadowing point x also depends Borel measurably on ω.

We define a sequence of points

$$(y_t(\omega), \theta^t\omega) = \begin{cases} (P_\omega(t), \theta^t\omega), & \text{if } t \in \cup_{i=1}^m I_i \\ \phi^{t-a_1}(P_\omega(a_1), \theta^{a_1}\omega), & \text{if } t < a_1 \\ \phi^{i-b_m}(P_\omega(b_m), \theta^{b_m}\omega), & \text{if } t > b_m. \end{cases}$$

Define

$$\tilde{y}_t(\theta^t\omega) := y_t(\omega).$$

Since $P_\omega(t)$ is measurable for each fixed t, $\tilde{y}_t$ is a measurable function.

Notice that in (5.6) when $j = 1$, we have

$$f_\omega^{b_1} x \in W_{2\alpha}^u(\phi^{b_1-a_1}(x_{a_1}, \theta^{a_1}\omega)),$$

then

$$f_\omega^{a_1} x \in W_{2\alpha}^u(x_{a_1}, \theta^{a_1}\omega) = W_{2\alpha}^u(P_\omega(a_1), \theta^{a_1}\omega). \tag{5.8}$$

By the construction of x_{a_m}, we also have

$$f_\omega^{a_m} x \in W_\alpha^s(P_\omega(a_m), \theta^{a_m}\omega),$$

then

$$f_\omega^{b_m} x \in W_\alpha^s(P_\omega(b_m), \theta^{b_m}\omega). \tag{5.9}$$

By (5.8), (5.9) and (5.4), we know that (x, ω) is $(\omega, \beta/2)-$shadowing the sequence $\{(y_t(\omega), \theta^t\omega)\}_{t\in\tilde{\tau}}$ where $\tilde{\tau} = \cup_{i=1}^m I_i \cup \{t < a_1\} \cup \{t > b_m\}$.

Moreover, (x, ω) is the unique point $(\omega, \beta/2)-$shadowing the sequence $\{(y_t(\omega), \theta^t\omega)\}_{t\in\tilde{\tau}}$. In fact, if both (x, ω) and (x', ω) are $(\omega, \beta/2)-$shadowing the sequence, then

$$d((x, \omega), (x', \omega)) \leq \beta \leq \delta(\epsilon_0),$$

as a consequence, $(z,\omega) = [x, x']_\omega$ is defined. Pick $n > b_m$ big enough, then

$$\begin{aligned} d_M(z,x') &\le e^{-n\lambda} d_M(\pi_M\phi^n(z,\omega), \pi_M\phi^n(x',\omega)) \\ &\le e^{-n\lambda_0}(d_M(\pi_M\phi^n(x',\omega), \pi_M\phi^n(x,\omega)) + d_M(\pi_M\phi^n(x,\omega), \pi_M\phi^n(z,\omega))) \\ &\le e^{-n\lambda_0}(\beta + \epsilon_0), \end{aligned}$$

and

$$\begin{aligned} d_M(z,x) &\le e^{-n\lambda_0} d_M(\pi_M\phi^{-n}(z,\omega), \pi_M\phi^{-n}(x,\omega)) \\ &\le e^{-n\lambda_0}(d_M(\pi_M\phi^{-n}(z,\omega), \pi_M\phi^{-n}(x',\omega)) + d_M(\pi_M\phi^{-n}(x',\omega), \pi_M\phi^{-N}(x,\omega))) \\ &\le e^{-n\lambda_0}(\epsilon_0 + \beta). \end{aligned}$$

Hence

$$d_M(x,x') \le d_M(z,x') + d_M(z,x) \le 2e^{-n\lambda_0}(\epsilon_0 + \beta) \to 0 \tag{5.10}$$

as $n \to \infty$.

For $i > b_m$, let $\tilde{\tau}_i = [-i, a_i) \cup \cup_{i=1}^m I_i \cup (b_m, i]$, and define a multivalued function $\tilde{x}_i : \Omega \to 2^M$ for any $i \in \mathbb{N}$ by

$$\tilde{x}_i(\omega) := \bigcap_{t\in\tilde{\tau}_i} \pi_M\phi^{-t}\{(x, \theta^t\omega) |\ d((x,\theta^t\omega), (y_t(\omega), \theta^t\omega)) \le \beta/2\}$$

for all $\omega \in \Omega$. $\tilde{x}_i(\omega)$ is nonempty by the existence of the shadowing point, and $\tilde{x}_i(\omega)$ is closed by the continuity of $\phi(\cdot, \omega)$. Moreover, we also have that

$$graph(\tilde{x}_i) := \{(\tilde{x}_i(\omega), \omega) :\ \omega \in \Omega\} = \bigcap_{t\in\tilde{\tau}_i} \phi^{-t}(\overline{B_{\tilde{y}_t}(\beta/2)}),$$

where

$$\overline{B_{\tilde{y}_t}(\beta/2)} := \{(x, \theta^t\omega) |\ d((x,\theta^t\omega), (\tilde{y}_t(\theta^t\omega), \theta^t\omega)) \le \beta/2,\ \omega \in \Omega\}$$

is the $\beta/2$ neighborhood of $\tilde{y}_t$. Since $\tilde{y}_t$ is measurable, $\overline{B_{\tilde{y}_t}(\beta/2)}$ is a Borel measurable subset in $M \times \Omega$. Thus $\phi^{-t}(\overline{B_{\tilde{y}_t}(\beta/2)})$ is also a Borel subset of $M \times \Omega$. Then by the selection theorem (Proposition 2.17), $\tilde{x}_i$ can produce a selection $\tilde{x}_i' \in L^\infty(\Omega, M)$ with that $graph(\tilde{x}_i') \subset graph(\tilde{x}_i)$. Moreover, $\tilde{x}_i$ converges to a measurable function by (5.10). Hence the shadowing point x is Borel measurably depends on $\omega \in \Omega$.

Now for each fixed $t \in \cup_{i=1}^m I_i$, by (5.4), we have

$$d_M(P_\omega(t), f_\omega^t x(\omega)) < \beta/2 \text{ for all } \omega \in \Omega.$$

Then by the definition of $P_\omega(t)$, we have

$$d_M(P(t)(\theta^t\omega), f_\omega^t x(\omega)) < \beta/2 \text{ for all } \omega \in \Omega,$$

i.e.,

$$d_{L^\infty(\Omega,M)}(P(t), \tilde{\phi}^t(x)) < \beta/2. \tag{5.11}$$

Next, for the case $q \geq N + b_m - a_1$, let $\tau' := \tau \cup \{\{a_1 + q\}\}$ and define $P' : \tau' \to L^\infty(\Omega, M)$ by $P'|_\tau := P$ and $P'(a_1 + q) = P(a_1)$. This $S' = (\tau', P')$ is clearly $N-$spaced, there exists a $g \in L^\infty(\Omega, M)$ $\beta/2-$shadowing this new specification,i.e.,

$$d_{L^\infty(\Omega,M)}(\tilde{\phi}^t(g), P'(t)) < \beta/2$$

for $t \in \tau'$. Define $g' = \tilde{\phi}^{a_1}(g)$, then we have

$$\begin{aligned}
d_{L^\infty(\Omega,M)}(\tilde{\phi}^q(g'), g') &\leq d_{L^\infty(\Omega,M)}(\tilde{\phi}^q(g'), P'(q + a_1)) + d_{L^\infty(\Omega,M)}(P'(q + a_1), g')) \\
&\leq \beta/2 + d_{L^\infty}(P(a_1), g') \\
&\leq \beta/2 + \beta/2 \leq \beta.
\end{aligned}$$

Since $\beta < \alpha(\epsilon/2)$ and $\{\tilde{\phi}^t(g')\}_{t=0}^{q-1}$ is periodic $\beta-$pseudo orbit, then by Corollary 4.6, there exists a unique $z \in L^\infty(\Omega, M)$ such that $\tilde{\phi}^q(z) = z$ and z is $\epsilon/2-$shadowing this periodic pseudo orbit.

Now define $x := \tilde{\phi}^{-a_1} z$, then x is periodic under $\tilde{\phi}$ with period q, and for $t \in I \in \tau$, we have

$$d_{L^\infty(\Omega,M)}(\tilde{\phi}^t(x), P(t)) \le d_{L^\infty(\Omega,M)}(\tilde{\phi}^t(x), \tilde{\phi}^t(g)) + d_{L^\infty(\Omega,M)}(\tilde{\phi}^t(g), P(t)) \le \frac{\epsilon}{2} + \frac{\beta}{2} < \epsilon.$$

Thus, ϕ has the random specification property.

On the other hand, assume ϕ has the random specification property. For any nonempty open sets $U, V \subset M$, we can pick points $x \in U$ and $y \in V$ together with $\epsilon > 0$ such that $B_\epsilon(x) \subset U$, $B_\epsilon(y) \subset V$. Then there exists an $N = N_\epsilon$ corresponding to ϵ in the random specification property.

Let $a_1 = b_1 = 0$, $a_2 = b_2 = N$, and define $P(a_1) = P(b_1) \equiv x$, $P(a_2) = P(b_2) \equiv y$, then there exists a Borel measurable map $g \in L^\infty(\Omega, M)$ $\epsilon-$shadowing this random specification. As a consequence, we have

$$d_M(x, g(\omega)) < \epsilon, \text{ and } d_M(F_\omega^N g(\omega), y) < \epsilon, \ \forall \omega \in \Omega.$$

Hence we have

$$\phi^N(\{\omega\} \times U) \cap \{\theta^N \omega\} \times V \neq \emptyset, \ \forall \omega \in \Omega.$$

Now for any $n > N$, we define $a_1 = b_1 = 0$, $a_2 = N_\epsilon$, $b_2 = n$. Let $P(a_1) = P(b_1) \equiv x$, $P(b_2) \equiv y$ and $P(i) = \tilde{\phi}^{-(n-N)+(i-a_2)} P(b_2)$ for $a_2 \le i \le b_2$, then there exists a Borel measurable map $g' \in L^\infty(\Omega, M)$ $\epsilon-$shadowing this random specification, i.e.,

$$d_M(x, g(\omega)) < \epsilon, \text{ and } d_M(F_\omega^n g(\omega), y) < \epsilon, \ \forall \omega \in \Omega.$$

As a consequence,

$$\phi^n(\{\omega\} \times U) \cap \{\theta^n\omega\} \times V \neq \emptyset, \ \forall \omega \in \Omega, \ \forall n > N_\epsilon.$$

Hence ϕ is topological mixing on fibers. The proof of Theorem 3.7 is done.

5.2 Specification on the Space of Random Probability Measures

In this section, we prove Theorem 3.8.

First, we prove that $\phi^* : Pr_\Omega(M) \to Pr_\Omega(M)$ defines a homeomorphism with respect to the narrow topology. Pick any sequence $\mu^\alpha \to \mu$ in the narrow topology. For any random closed set C, define $C'(\omega) := (f_\omega)^{-1}C(\theta\omega)$. Using the selection theorem (Proposition 2.17), we can easily see that C' is a random closed set. Now

$$\begin{aligned}
\limsup_\alpha \phi^*\mu^\alpha(C) &= \limsup_\alpha \int_\Omega (\phi^*\mu)_\omega(C(\omega))dP(\omega) \\
&= \limsup_\alpha \int_\Omega (\phi^*\mu)_{\theta\omega}(C(\theta\omega))dP(\omega) \\
&= \limsup_\alpha \int_\Omega \mu^\alpha_\omega((f_\omega)^{-1}C(\theta\omega))dP(\omega) \\
&= \limsup_\alpha \int_\Omega \mu^\alpha_\omega(C'(\omega))dP(\omega) \\
&= \limsup_\alpha \mu^\alpha(C') \\
&\leq \mu(C') \\
&= \phi^*\mu(C).
\end{aligned}$$

Hence $\phi^*\mu^\alpha \to \phi^*\mu$ by the Portmanteau theorem (Proposition 2.18). Similarly $(\phi^*)^{-1} := (\phi^{-1})^*$ is also continuous with respect to the narrow topology.

For any $g \in L^\infty(\Omega, M)$, define δ_g by $\omega \mapsto (\delta_g)_\omega = \delta_{g(\omega)} \in Pr(M)$. Then for fixed ω, $(\delta_g)_\omega$

is a Borel probability measure on M and for each $B \in \mathcal{B}(M)$,

$$\omega \mapsto (\delta_g)_\omega(B) = \delta_{g(\omega)}(B) = \begin{cases} 0, & \text{if } g(\omega) \notin B \\ 1, & \text{if } g(\omega) \in B. \end{cases}$$

which is measurable. Hence $\delta_g \in Pr_\Omega(M)$ for each $g \in L^\infty(\Omega, M)$. We call δ_g by random Dirac measure, and it is also named random counting measure or point process [35]. We also have $\phi^*\delta_g = \delta_{\tilde{\phi}g}$.

For any $n \in \mathbb{N}$, let $Pr_{\Omega,n}(M)$ denote the collection of random probability measures of the form

$$\frac{1}{n}\sum_{i=1}^{n} \delta_{g_i},$$

where $g_i \in L^\infty(\Omega, M)$ and g_i are not necessarily distinct.

Lemma 5.3. $\cup_{n\in\mathbb{N}} Pr_{\Omega,n}(M)$ *is dense in* $Pr_\Omega(M)$ *with respect to the narrow topology.*

Proof. Pick any $\mu \in Pr_\Omega(M)$, and any $\eta > 0$ fix. Then $\omega \mapsto \mu_\omega$ is measurable with respect to the Borel $\sigma-$algebra of the narrow topology on $Pr(M)$. Notice that P on Ω is a Borel probability measure, hence regular. $Pr(M)$ is a compact metric space with respect to Prohorov metric, hence second countable. Then we can apply Lusin's theorem, we can choose a compact set $E \subset \Omega$ with $P(\Omega - E) < \frac{\eta}{4}$ such that $\omega \mapsto \mu_\omega$ restricted on E is continuous. Moreover, $\omega \mapsto \mu_\omega$ on E is uniformly continuous, i.e. there exists $\xi > 0$ for any $\omega_1, \omega_2 \in E$

$$d_\Omega(\omega_1, \omega_2) < \xi \text{ implies } d_p(\mu_{\omega_1}, \mu_{\omega_2}) < \eta/4. \tag{5.12}$$

Now let $\{\chi_i\}_{i=1}^k$ be a measurable partition of E, and size of each χ_i less than ξ. Pick any $\omega_i \in \chi_i$ and fix, then $\{\omega_i\}_{i=1}^k$ is a $\xi-$net in E. Consider $\{\mu_{\omega_i}\}_{i=1}^k$. Notice that $\cup_{n\in N} Pr_n(M)$ is dense in $Pr(M)$ by Proposition 2.14 in [27], where $Pr_n(M)$ is the set of measures on M of the form

$$\frac{1}{n}\sum_{i=1}^{n}\delta_{x_i},$$

where the x_i are (not necessarily distinct) elements of M. We can pick a sequence of points, $\{x_{i,j}\}_{i=1,\dots,k.j=1,\dots,n}$, in M for some n (we can pick a common n since we always can use the least common multiple) such that

$$d_p\left(\mu_{\omega_i}, \frac{1}{n}\sum_{j=1}^{n}\delta_{x_{i,j}}\right) < \eta/4 \text{ for all } i \in \{1,\dots,k\}.$$

Then for any $\omega \in \chi_i$, by using (5.12),

$$d_p\left(\mu_{\omega}, \frac{1}{n}\sum_{j=1}^{n}\delta_{x_{i,j}}\right) \le d_p(\mu_\omega, \mu_{\omega_i}) + d_p\left(\mu_{\omega_i}, \frac{1}{n}\sum_{j=1}^{n}\delta_{x_{i,j}}\right) < \eta/2.$$

Now define $\tilde{y}_j \in L^\infty(\Omega, M)$ for $j \in \{1, 2, ..., n\}$ by

$$\tilde{y}_j(\omega) = \begin{cases} x_{i,j}, & \text{if } \omega \in \chi_i \\ x_{1,j}, & \text{if } \omega \in \Omega - E. \end{cases} \tag{5.13}$$

Each $\tilde{y}_j$ is measurable map since $\tilde{y}_j$ is a simple function. Then for $\omega \in \chi_i$, we have

$$d_p\left(\mu_{\omega}, \frac{1}{n}\sum_{j=1}^{n}\delta_{\tilde{y}_j(\omega)}\right) = d_p\left(\mu_{\omega}, \frac{1}{n}\sum_{j=1}^{n}\delta_{x_{i,j}}\right) < \eta/2.$$

Now

$$
\begin{aligned}
& d_{rp}\left(\mu, \frac{1}{n}\sum_{j=1}^{n}\delta_{\tilde{y}_j}\right) \\
\le & \sum_{m\in\mathbb{N}} \frac{1}{2^m} \int_{G_m} d_p\left(\mu_\omega, \frac{1}{n}\sum_{j=1}^{n}\delta_{\tilde{y}_j(\omega)}\right) dP(\omega) \\
= & \sum_{m\in\mathbb{N}} \frac{1}{2^m} \left(\sum_{i=1}^{k} \int_{G_m\cap\chi_i} d_p(\mu_\omega, \frac{1}{n}\sum_{j=1}^{n}\delta_{\tilde{y}_j(\omega)}) dP(\omega) + \int_{G_m\cap(\Omega-E)} d_p\left(\mu_\omega, \frac{1}{n}\sum_{j=1}^{n}\delta_{\tilde{y}_j(\omega)}\right) dP(\omega)\right) \\
= & \sum_{m\in\mathbb{N}} \frac{1}{2^m} \left(\sum_{i=1}^{k} \int_{G_m\cap\chi_i} \frac{\eta}{2}\, dP(\omega) + \int_{G_m\cap(\Omega-E)} 2dP(\omega)\right) \\
\le & \sum_{m\in\mathbb{N}} \frac{1}{2^m} \left(\frac{\eta}{2} + 2\cdot\frac{\eta}{4}\right) \\
= & \eta.
\end{aligned}
$$

Hence $\cup_{n\in\mathbb{N}} Pr_{\Omega,n}(M)$ is dense in $Pr_\Omega(M)$. □

Now, we are ready to prove the topological dynamical system $(Pr_\Omega(M), \phi^*)$ has the specification property.

Let $\epsilon > 0$ and fix this ϵ, let $N := N(\epsilon/2)$ be the number in random specification property corresponding to $\epsilon/2$. Let $I_i = [a_i, b_i]$ for $i = 1, 2, \ldots, m$ with $a_{i+1} - b_i \ge N$ for $i \in \{1, 2 \ldots, m-1\}$. Given any random probability measures $\{\mu_i\}_{i=1}^m \subset Pr_\Omega(M)$, define $P : \cup_{i=1}^m I_i \to Pr_\Omega(M)$

$$P(j) = (\phi^*)^{j-a_i}(\mu_i) \text{ for } j \in [a_i, b_i].$$

and let $K \ge N(\epsilon/2) + b_m - a_1$,

Notice that $Pr_\Omega(M)$ is a compact metric space with respect to the random Prohorov metric, and $\phi^* : Pr_\Omega(M) \to Pr_\Omega(M)$ is continuous with respect to the narrow topology. Hence ϕ^* is uniformly continuous since $Pr_\Omega(M)$ is compact with respect to the narrow topology. Then there exists an $\eta > 0$ such that

$$d_{rp}(\mu, \nu) < \eta \text{ implies } d_{rp}((\phi^*)^j\mu, (\phi^*)^j\nu) < \epsilon/2 \text{ for } 1 \le j \le b_m.$$

Apply Lemma 5.3, there exists $\nu_i \in Pr_{\Omega,n}(M)$ for some integer n such that $d_p(\mu_i, \nu_i) < \eta$ for $1 \le i \le m$. Denote

$$\nu_i = \frac{1}{n}\sum_{l=1}^{n} \delta_{g_l^i} \text{ for } i \in \{1, 2..., m\}.$$

Since ϕ has the random specification property, then there exists a $z_l \in L^\infty(\Omega, M)$ with $\tilde{\phi}^K(z_l) = z_l$ and

$$d_{L^\infty(\Omega,M)}\left(\tilde{\phi}^j z_l, \tilde{\phi}^{j-a_i} g_l^i\right) < \epsilon/2$$

for $a_i \le j \le b_i$, $i = 1, 2 \ldots, m$ and $l = 1, 2, ..., n$.

Define $\rho := \frac{1}{n}\sum_{l=1}^{n} \delta_{z_l}$. Then

$$(\phi^*)^K(\rho) = \frac{1}{n}\sum_{l=1}^{n} \delta_{\tilde{\phi}^K z_l} = \frac{1}{n}\sum_{l=1}^{n} \delta_{z_l} = \rho.$$

For $a_i \le j \le b_i$ and $i \in \{1, ..., m\}$,

$$\begin{aligned}
d_{rp}((\phi^*)^j \rho, (\phi^*)^{j-a_i}\nu_i) &= d_{rp}\left(\frac{1}{n}\sum_{l=1}^{n} \delta_{\tilde{\phi}^j z_l}, \frac{1}{n}\sum_{l=1}^{n} \delta_{\tilde{\phi}^{j-a_i} g_l^i}\right) \\
&= \sum_{m\in\mathbb{N}} \frac{1}{2^m} \sup\Bigg\{ \int_{G_m} \frac{1}{n}\sum_{l=1}^{n} (\delta_{(\tilde{\phi}^j z_l)(\omega)}(g) - \delta_{(\tilde{\phi}^{j-a_i} g_l^i)(\omega)}(g)) dP(\omega), \\
&\qquad\qquad g \in BL(M), 0 \le g \le 1,\ [g]_L \le 1 \Bigg\} \\
&\le \sum_{m\in\mathbb{N}} \frac{1}{2^m}\frac{1}{n}\sum_{l=1}^{n} \int_{G_m} d_M((\tilde{\phi}^j z_l)(\omega), (\tilde{\phi}^{j-a_i} g_l^i)(\omega)) dP(\omega) \\
&\le \sum_{m\in\mathbb{N}} \frac{1}{2^m}\frac{1}{n}\sum_{l=1}^{n} d_{L^\infty(\Omega,M)}(\tilde{\phi}^j z_l, \tilde{\phi}^{j-a_i}(g_l^i)) \\
&\le \epsilon/2.
\end{aligned}$$

Then by the triangle inequality we have

$$\begin{aligned}
d_p((\phi^*)^j(\rho), (\phi^*)^{j-a_i}(\mu_i)) &\le d_p((\phi^*)^j(\rho), (\phi^*)^{j-a_i}(\nu_i)) + d_p((\phi^*)^{j-a_i}(\nu_i), (\phi^*)^{j-a_i}(\mu_i)) \\
&< \epsilon/2 + \epsilon/2 = \epsilon,
\end{aligned}$$

for $a_i \leq j \leq b_i$, $i = 1, ..., m$. Hence $(Pr_\Omega(M), \phi^*)$ has the specification property. The proof of Theorem 3.8 is done.

5.3 Positivity of Topological Fiber Entropy

In this section, we prove Theorem 3.9.

For any $\epsilon \in (0, \epsilon_0)$ a sufficiently small number and fix this ϵ. We pick any finite $3\epsilon-$seperated subset of M, named $\{x_i\}_{i=1}^k$, with respect to d_M. Define $\tilde{x}_i \in L^\infty(\Omega, M)$ by $\tilde{x}_i \equiv x_i$.

Let $N = N(\epsilon)$ be the spacing number in the random specification property corresponding to ϵ. For any $n-$tuple $(z_0, ..., z_{n-1})$ with $z_j \in \{x_i\}_{i=1}^k$ for $j = 0, 1, ..., n-1$, define $P(jN) = \tilde{z}_j$. Apply the random specification property to $I_j = \{jN\}$ and this P defined above, for any $\omega \in \Omega$, there exists a $z(\omega) \in M_\omega$ such that

$$d((z_j, \theta^{jN}\omega), \phi^{jN}(z(\omega), \omega)) < \epsilon.$$

If $(z_0, ...z_{n-1}) \neq (z'_0, ..., z'_{n-1})$, then there exists a j such that $z_j \neq z'_j$. Let $z(\omega)$ and $z'(\omega)$ be the corresponding shadowing point, then

$$\begin{aligned} d_{\omega,nN}(z(\omega), z'(\omega)) &\geq d(f_\omega^{jN} z(\omega), f_\omega^{jN} z'(\omega)) \\ &\geq d(z_j, z'_j) - d(f_\omega^{jN} z'(\omega), z'_j) - d(z_j, f_\omega^{jN} z(\omega)) \\ &> 3\epsilon - \epsilon - \epsilon = \epsilon. \end{aligned}$$

Hence there are at least k^n points in M_ω which are $(\omega, \epsilon, nN)-$seperated. Then

$$\begin{aligned} h_{top}(\phi|_{M_\omega}) &= \lim_{\eta\to 0^+}\limsup_{n\to\infty}\frac{1}{n}\log N(\omega,\eta,n)\\ &\geq \limsup_{n\to\infty}\frac{1}{n}\log N(\omega,\epsilon,n)\\ &\geq \limsup_{n\to\infty}\frac{1}{nN(\epsilon)}\log N(\omega,\epsilon,nN(\epsilon))\\ &\geq \limsup_{n\to\infty}\frac{1}{nN(\epsilon)}\log k^n\\ &= \frac{\log k}{N(\epsilon)} > 0. \end{aligned}$$

The proof of Theorem 3.9 is done.

Chapter 6. Exponential Decay of Random Correlation

In this chapter, we prove Theorem 3.10. The proof is based on studying of the fiber transfer operator L_ω, which is defined by

$$L_\omega\varphi : M \to \mathbb{R},\ (L_\omega\varphi)(x) := \frac{\varphi((f_\omega)^{-1}x)}{|\det D_{(f_\omega)^{-1}(x)}f_\omega|} \tag{6.1}$$

for any measurable observable function $\varphi : M \to \mathbb{R}$. We denote

$$L^n_\omega := L_{\theta^{n-1}\omega}\circ\cdots\circ L_{\theta\omega}\circ L_\omega \text{ for } n\in\mathbb{N},\ \text{for all } \omega\in\Omega.$$

We first construct the suitable convex cone of observable functions C_ω on each fiber in Section 6.1. Then in Section 6.2, we prove that the transfer operator L_ω maps C_ω into $C_{\theta\omega}$ and the image of L^N_ω has finite diameter with respect to the Hilbert projective metric on the cone $C_{\theta^N\omega}$ uniformly for all $\omega\in\Omega$, where N comes from the topological mixing on fibers property. Birkhoff's inequality implies the contraction of L^N_ω for all $\omega\in\Omega$. In Section 6.3, we find the relationship between the unique random SRB measure and operator $L^n_{\theta^{-n}\omega}$. We show the exponential decay of past and future correlations in Section 6.4 and Section 6.5 respectively by using the contraction of $L^n_{\theta^{-n}\omega}$ and L^n_ω for $n \geq N$.

Before starting the proof, we recall some constants that will be used later on. Let K_1 be the constant in Lemma 4.9 such that

$$|\det(D_x f_\omega)|_{E^s(x,\omega)} - \det(D_x f_\omega)|_{E^s(x,\omega)}| \leq K_1 d(x,y) \text{ for any } x, y \in W^s_\epsilon(z,\omega),\ z \in M.$$

By the compactness of Ω and M and the continuity of f_ω on ω, there exists a constant $K_2 > 0$ independent of ω such that for any $x, y \in M$,

$$|\log|\det D_x f_\omega| - \log|\det D_y f_\omega|| \leq K_2 d(x,y). \tag{6.2}$$

Let $a_0 := \max\{a_0', a_0''\}$ and $\nu_0 := \min\{\nu_0', \nu_0''\}$. Then Lemma 4.20 and Proposition 4.21 hold for constants (a_0, ν_0). This ν_0 is the desired ν_0 in the statement of Theorem 3.10.

Now Let's pick any $\mu, \nu \in (0,1)$ satisfying $0 < \mu + \nu < \nu_0$ as in the statement of Theorem 3.10 and fix μ and ν. We also pick $\mu_1 \in (0,1)$ an auxiliary constant close to 1 and such that

$$0 < \mu + \nu < \mu_1 \nu_0. \tag{6.3}$$

Now we are going to prove Theorem 3.10 for fixed μ, ν.

6.1 Construction of Birkhoff Cone

In this section, we will first construct convex cones of density functions on each local stable leaf. With the help of these convex cones of density functions on each local stable leaf, we can define our desired convex cone of observable functions on each fiber. The definition of the convex cone, projective metric on the convex cone and Birkhoff's inequality are recalled in the Appendix.

In the following, we always consider the local stable leaf $\gamma(\omega)$ having size between $\epsilon/4$ and $\epsilon/2$, i.e. there exists a $x \in \gamma(\omega)$ such that $W^s_{\epsilon/4}(x,\omega) \subset \gamma(\omega) \subset W^s_{\epsilon/2}(x,\omega)$, without further clarifications.

The cone of Hölder continuous densities on a local stable leaf $\gamma(\omega)$ with constant (a,μ), $D(a,\mu,\gamma(\omega))$, is the collection of all function $\rho(\cdot,\omega) : \gamma(\omega) \to \mathbb{R}$ satisfying the following conditions:

(D1) $\rho(x,\omega) > 0$ for $x \in \gamma(\omega)$;

(D2) for all $\omega \in \Omega$ and any $x, y \in \gamma(\omega)$, $|\log \rho(x,\omega) - \log \rho(y,\omega)| \leq ad(x,y)^{\mu}$.

It is easy to check that $D(a,\mu,\gamma(\omega))$ is a convex cone (see Definition A.1 in the Appendix). Next, we will introduce the Hilbert projective metric $d_{\gamma(\omega)}(\cdot,\cdot)$ on $D(a,\mu,\gamma(\omega))$.

Now for any $\rho_1(\cdot,\omega), \rho_2(\cdot,\omega) \in D(a,\mu,\gamma(\omega))$, denote $\bar{\rho}_i(\omega)(\cdot) := \rho_i(\cdot,\omega) : \gamma(\omega) \to \mathbb{R}$ for $i = 1, 2$, define

$$\begin{aligned}
&\alpha(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) := \sup\{t > 0 : \ \bar{\rho}_2(\omega) - t\bar{\rho}_1(\omega) \in D(a,\mu,\gamma)(\omega)\};\\
&\beta(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = \inf\{s > 0 : \ s\bar{\rho}_1(\omega) - \bar{\rho}_2(\omega) \in D(a,\mu,\gamma)(\omega)\},
\end{aligned}$$

with the convention that $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$. By computation, we have

$$\alpha_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = \inf\left\{\frac{\rho_2(x,\omega)}{\rho_1(x,\omega)}, \frac{\exp(ad(x,y)^{\mu})\rho_2(x,\omega) - \rho_2(y,\omega)}{\exp(ad(x,y)^{\mu})\rho_1(x,\omega) - \rho_1(y,\omega)}, \ x, y \in \gamma(\omega)\right\}, \tag{6.4}$$

$$\beta_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = \sup\left\{\frac{\rho_2(x,\omega)}{\rho_1(x,\omega)}, \frac{\exp(ad(x,y)^{\mu})\rho_2(x,\omega) - \rho_2(y,\omega)}{\exp(ad(x,y)^{\mu})\rho_1(x,\omega) - \rho_1(y,\omega)}, \ x, y \in \gamma(\omega)\right\}. \tag{6.5}$$

Now define

$$d_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = \log \frac{\beta_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega))}{\alpha_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega))}, \tag{6.6}$$

with the convention that $d_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = \infty$ if $\alpha(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = 0$ or $\beta(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = \infty$.

By the property of projective metric, the followings hold:

(P1) $d_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = d_{\gamma(\omega)}(\bar{\rho}_2(\omega), \bar{\rho}_1(\omega))$;

(P2) $d_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) \leq d_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_3(\omega)) + d_{\gamma(\omega)}(\bar{\rho}_3(\omega), \bar{\rho}_2(\omega))$;

(P3) $d_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = 0$ for all $\omega \in \Omega$ if and only if there exists a constant $t \in \mathbb{R}^+$ such that $\rho_1(\cdot, \omega) = t\rho_2(\cdot, \omega)$.

Note that $(P2)$ and $(P3)$ implies that $d_{\gamma(\omega)}(\bar{\rho}_1(\omega), \bar{\rho}_2(\omega)) = d_{\gamma(\omega)}(t_1\bar{\rho}_1(\omega), t_2\bar{\rho}_2(\omega))$ for any $t_1, t_2 \in \mathbb{R}^+$.

Now for any $\omega \in \Omega$, and a local stable leaf $\gamma(\omega)$, we subdivide $f_\omega^{-1}\gamma(\omega)$ into connected local stable submanifolds of size between $\epsilon/4$ and $\epsilon/2$, named $\gamma_i(\theta^{-1}\omega)$ for i belonging to a finite index set. For every $\rho(\cdot, \omega) \in D(a, \alpha, \gamma(\omega))$, define

$$\rho_i(x, \theta^{-1}\omega) := \frac{|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}|}{|\det D_x f_{\theta^{-1}\omega}|}\rho(f_{\theta^{-1}\omega}x, \omega) \text{ for } x \in \gamma_i(\theta^{-1}\omega). \tag{6.7}$$

For $\rho(\cdot, \omega) \in D(a, \alpha, \gamma(\omega))$ and for any bounded and measurable function $\varphi : M \to \mathbb{R}$, we have

$$\begin{aligned}
&\int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y, \omega)dm_{\gamma(\omega)}(y) \\
&= \sum_i \int_{f_{\theta^{-1}\omega}\gamma_i(\theta^{-1}\omega)} \frac{\varphi((f_{\theta^{-1}\omega})^{-1}y)}{|\det D_{(f_{\theta^{-1}\omega})^{-1}y} f_{\theta^{-1}\omega}|} \cdot \rho(y, \omega)dm_{\gamma(\omega)}(y) \\
&= \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \frac{\varphi(x)}{|\det D_x f_{\theta^{-1}\omega}|} \cdot \rho(f_{\theta^{-1}\omega}x, \omega) \cdot |\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}| dm_{\gamma_i(\theta^{-1}\omega)}(x) && (6.8) \\
&= \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x, \theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x). && (6.9)
\end{aligned}$$

Given pair of local stable leaves $\gamma(\omega)$ and $\tilde{\gamma}(\omega)$. With the help of holonomy map $\psi_\omega : \tilde{\gamma}(\omega) \to \gamma(\omega)$, we can define the distance between $\tilde{\gamma}(\omega)$ and $\gamma(\omega)$ by

$$d(\tilde{\gamma}(\omega), \gamma(\omega)) := \sup\{d(x, \psi_\omega(x)),\ x \in \tilde{\gamma}(\omega)\}.$$

For every $\rho(\cdot,\omega) \in D(a_1,\mu_1,\gamma(\omega))$, we associate the random density $\tilde{\rho}(\cdot,\omega)$ on $\tilde{\gamma}(\omega)$ by

$$\tilde{\rho}(x,\omega) = \rho(\psi_\omega(x),\omega) \cdot |\det D\psi_\omega(x)|. \tag{6.10}$$

By changing of variable, we have

$$\int_{\tilde{\gamma}(\omega)} \tilde{\rho}(x,\omega) dm_{\tilde{\gamma}(\omega)}(x) = \int_{\gamma(\omega)} \rho(y,\omega) dm_{\gamma(\omega)}(y).$$

Lemma 6.1. *Let a_1 be any number depending on μ such that*

$$\frac{K_1 + K_2}{1 - e^{-\lambda\mu}} < a_1. \tag{6.11}$$

Let a, depending on μ and a_1, be any number such that

$$a_1 a_0^{\mu_1} + a_0 < \frac{a}{2}. \tag{6.12}$$

Then there are $\lambda_1 = \lambda_1(a_1,\mu) > 0$ and $\Lambda_1 = \Lambda_1(\lambda_1, a) < 1$ such that

(i) if $\rho(\cdot,\omega) \in D(a_1,\mu,\gamma(\omega))$, then $\rho_i(\cdot,\theta^{-1}\omega) \in D(e^{-\lambda_1}a_1,\mu,\gamma_i(\theta^{-1}\omega)) \subset D(a_1,\mu,\gamma_i(\theta^{-1}\omega))$;

(ii) if $\rho(\cdot,\omega) \in D(\frac{a}{2},\mu,\gamma(\omega))$, then $\rho_i(\cdot,\theta^{-1}\omega) \in D(e^{-\lambda_1}\frac{a}{2},\mu,\gamma_i(\theta^{-1}\omega)) \subset D(\frac{a}{2},\mu,\gamma_i(\theta^{-1}\omega))$;

(iii) if $\rho(\cdot,\omega) \in D(a,\mu,\gamma(\omega))$, then $\rho_i(\cdot,\theta^{-1}\omega) \in D(e^{-\lambda_1}a,\mu,\gamma_i(\theta^{-1}\omega)) \subset D(a,\mu,\gamma_i(\theta^{-1}\omega))$;

(iv) let $\rho'(\cdot,\omega), \rho''(\cdot,\omega) \in D(a,\mu,\gamma(\omega))$, then

$$d_{\gamma_i(\theta^{-1}\omega)}(\bar{\rho}_i'(\theta^{-1}\omega), \bar{\rho}_i''(\theta^{-1}\omega)) \le \Lambda_1 d_{\gamma(\omega)}(\bar{\rho}'(\omega), \bar{\rho}''(\omega)),$$

where $\bar{\rho}_i'(\theta^{-1}\omega)(\cdot) := \rho_i'(\cdot,\theta^{-1}\omega)$, $\bar{\rho}_i''(\theta^{-1}\omega)(\cdot) := \rho_i''(\cdot,\theta^{-1}\omega)$, $\bar{\rho}'(\theta^{-1}\omega)(\cdot) := \rho'(\cdot,\theta^{-1}\omega)$ and $\bar{\rho}''(\theta^{-1}\omega)(\cdot) := \rho''(\cdot,\theta^{-1}\omega)$, $d_{\gamma(\omega)}$ and $d_{\gamma_i(\theta^{-1}\omega)}$ are the Hilbert projective metric on $D(a,\mu,\gamma(\omega))$ and $D(a,\mu,\gamma_i(\theta^{-1}\omega))$ respectively, *and*

$$\textit{if } \rho(\cdot,\omega) \in D(a_1,\mu_1,\gamma(\omega)), \textit{then } \tilde{\rho}(\cdot,\omega) \in D(a/2,\mu_1\nu_0,\tilde{\gamma}(\omega)) \subset D(\frac{a}{2},\mu,\tilde{\gamma}(\omega)). \tag{6.13}$$

Proof of lemma 6.1. We first prove (1). Let $\rho(\cdot,\omega) \in D(a_1,\mu,\gamma(\omega))$, clearly, $\rho_i(x,\theta^{-1}\omega) > 0$ for all $x \in \gamma_i(\theta^{-1}\omega)$. Since $a_1 > \frac{K_1+K_2}{1-\exp(-\lambda\mu)}$, we pick $\lambda_1 > 0$ so that $a_1 > \frac{K_1+K_2}{e^{-\lambda_1}-\exp(-\lambda\mu)} > 0$. Then for any $x, y \in \gamma_i(\theta^{-1}\omega)$

$$\begin{aligned}
&|\log\rho_i(x,\theta^{-1}\omega) - \log\rho_i(y,\theta^{-1}\omega)| \\
&\le |\log\rho(f_{\theta^{-1}\omega}x,\omega) - \log\rho(f_{\theta^{-1}\omega}y,\omega)| + |\log|\det D_x f_{\theta^{-1}\omega}| - \log|\det D_y f_{\theta^{-1}\omega}|| \\
&\qquad + |\log|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}| - \log|\det D_y f_{\theta^{-1}\omega}|_{E^s(y,\theta^{-1}\omega)}|| \\
&\le a_1 d(f_{\theta^{-1}\omega}x, f_{\theta^{-1}\omega}y)^\mu + K_1 d(x,y) + K_2 d(x,y) \\
&\le a_1 e^{-\lambda\mu} d(x,y)^\mu + (K_1+K_2) d(x,y) \\
&\le a_1 e^{-\lambda_1} d(x,y)^\mu,
\end{aligned}$$

This proves part (i). Similar proof can be applied to (ii) and (iii) by the choice of a.

Next, we prove (iv). Now we have a linear operator that maps from cone $D(a,\mu,\gamma(\omega))$ to cone $D(e^{-\lambda_1}a,\mu,\gamma_i(\theta^{-1}\omega)) \subset D(a,\mu,\gamma_i(\theta^{-1}\omega))$ defined by (6.7). By Birkhoff's inequality (Proposition A.4), if

$$R(\theta^{-1}\omega) := \sup\{d_{\gamma_i(\theta^{-1}\omega)}(\bar{\rho}'_i(\theta^{-1}\omega), \bar{\rho}''_i(\theta^{-1}\omega)) : \ \rho', \rho'' \in D(a,\mu,\gamma)\} < \infty,$$

then

$$d_{\gamma_i(\theta^{-1}\omega)}(\bar{\rho}'_i(\theta^{-1}\omega), \bar{\rho}''_i(\theta^{-1}\omega)) \le (1 - e^{-R(\theta^{-1}\omega)}) d_{\gamma(\omega)}(\bar{\rho}'(\omega), \bar{\rho}''(\omega)), \tag{6.14}$$

where $d_{\gamma(\omega)}$ and $d_{\gamma_i(\theta^{-1}\omega)}$ are the Hilbert projective metric on $D(a,\mu,\gamma(\omega))$ and $D(a,\mu,\gamma_i(\theta^{-1}\omega))$ respectively. To estimate $R(\theta^{-1}\omega)$. It suffices to estimate the diameter of $D(e^{-\lambda_1}a,\mu,\gamma_i(\theta^{-1}\omega))$ in $D(a,\mu,\gamma_i(\theta^{-1}\omega))$.

Denote $D_+(\gamma_i(\theta^{-1}\omega))$ by the collection of all measurable functions $\bar\rho(\theta^{-1}\omega):\gamma_i(\theta^{-1}\omega)\to\mathbb{R}$ such that $\bar\rho(\theta^{-1}\omega)(x)>0$ for $x\in\gamma_i(\theta^{-1}\omega)$. $D_+(\gamma_i(\theta^{-1}\omega))$ is a convex cone obviously.

$$\alpha_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))=\sup\{t>0:\ \bar\rho_2(\theta^{-1}\omega)-t\bar\rho_1(\theta^{-1}\omega)\in D_+(\gamma_i(\theta^{-1}\omega))\}; \tag{6.15}$$

$$\beta_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))=\inf\{s>0:\ s\bar\rho_1(\theta^{-1}\omega)-\bar\rho_2(\theta^{-1}\omega)\in D_+(\gamma_i(\theta^{-1}\omega))\}, \tag{6.16}$$

with the convention that $\sup\emptyset=0$ and $\inf\emptyset=+\infty$. We define the Hilbert projective metric on $D_+(\gamma_i(\theta^{-1}\omega))$ for $\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)\in D_+(\gamma_i(\theta^{-1}\omega))$ by

$$d_{+,\gamma_i(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)):=\log\frac{\beta_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))}{\alpha_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))}, \tag{6.17}$$

with the convention that $d_{+,\gamma_i(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))=\infty$ if $\alpha_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))=0$ or $\beta_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))=\infty$. By computation, we have

$$\alpha_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))=\inf\left\{\frac{\bar\rho_2(\theta^{-1}\omega)(x)}{\bar\rho_1(\theta^{-1}\omega)(x)},\ x\in\gamma_i(\theta^{-1}\omega)\right\}, \tag{6.18}$$

$$\beta_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))=\sup\left\{\frac{\bar\rho_2(\theta^{-1}\omega)(x)}{\bar\rho_1(\theta^{-1}\omega)(x)},\ x\in\gamma_i(\theta^{-1}\omega)\right\}. \tag{6.19}$$

Given $\bar\rho_1(\theta^{-1}\omega)$, $\bar\rho_2(\theta^{-1}\omega)\in D(e^{-\lambda_1}a,\mu,\gamma_i(\theta^{-1}\omega))$, $\bar\rho_1(\theta^{-1}\omega)$ and $\bar\rho_2(\theta^{-1}\omega)$ belong to $D_+(\gamma_i(\theta^{-1}\omega))$ automatically. For any $x,y\in\gamma_i(\theta^{-1}\omega)$,

$$\begin{aligned}\frac{\exp(ad(x,y)^\mu)-\bar\rho_2(\theta^{-1}\omega)(y)/\bar\rho_2(\theta^{-1}\omega)(x)}{\exp(ad(x,y)^\mu)-\bar\rho_1(\theta^{-1}\omega)(y)/\bar\rho_1(\theta^{-1}\omega)(x)}&\geq\frac{\exp(ad(x,y)^\mu)-\exp(e^{-\lambda_1}ad(x,y)^\mu)}{\exp(ad(x,y)^\mu)-\exp(-e^{-\lambda_1}ad(x,y)^\mu)}\\&\geq\tau_1,\end{aligned}$$

where

$$\tau_1=\inf\left\{\frac{z-z^{\exp(-\lambda_1)}}{z-z^{-\exp(-\lambda_1)}}:\ z>1\right\}=\frac{1-\exp(-\lambda_1)}{1+\exp(-\lambda_1)}\in(0,1).$$

Therefore, comparing (6.4) and (6.18), we have

$$\alpha_{\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)) \geq \tau_1\alpha_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)).$$

Similarly, let

$$\tau_2 = \sup\left\{\frac{z - z^{-\exp(-\lambda_1)}}{z - z^{\exp(-\lambda_1)}} :\ z > 1\right\} = \frac{1+\exp(-\lambda_1)}{1-\exp(-\lambda_1)} \in (1,\infty),$$

we have

$$\beta_{\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)) \leq \tau_2\beta_{+,\gamma(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)).$$

Thus, we conclude

$$d_{\gamma_i(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)) \leq d_{+,\gamma_i(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)) + \log(\tau_2/\tau_1). \tag{6.20}$$

Next, we estimate $d_{+,\gamma_i(\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega))$ for $\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega) \in D(e^{-\lambda_1}a,\alpha,\gamma_i(\theta^{-1}\omega))$. By property $(P3)$, we can normalize $\bar\rho_1(\theta^{-1}\omega)$ and $\bar\rho_2(\theta^{-1}\omega)$ by

$$\int_{\gamma_i(\theta^{-1}\omega)} \bar\rho_1(\theta^{-1}\omega)(x)dm_{\gamma_i(\theta^{-1}\omega)}(x) = \int_{\gamma_i(\theta^{-1}\omega)} \bar\rho_2(\theta^{-1}\omega)(x)dm_{\gamma_i(\theta^{-1}\omega)}(x) = 1.$$

Then condition $(D2)$ in $D(e^{-\lambda_1}a,\mu,\gamma_i(\theta^{-1}\omega))$ implies for all $x \in \gamma_i(\theta^{-1}\omega)$

$$\frac{\bar\rho_2(\theta^{-1}\omega)(x)}{\bar\rho_1(\theta^{-1}\omega)(x)} \geq \frac{\exp(-e^{-\lambda_1}a(diam\gamma_i(\theta^{-1}\omega))^{\mu})}{\exp(e^{-\lambda_1}a(diam\gamma_i(\theta^{-1}\omega))^{\mu})} \geq e^{-2\exp(-\lambda_1)a} \geq e^{-2a}.$$

It follows that $\alpha_{+,\gamma_i(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)) \geq e^{-2a}$. Similarly, $\beta_{+,\gamma_i(\theta^{-1}\omega)}(\theta^{\bar{-1}}\rho_1(\omega),\bar\rho_2(\theta^{-1}\omega)) \leq e^{2a}$. So we have

$$\begin{aligned} d_{\gamma_i(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)) &\leq d_{+,\gamma_i(\theta^{-1}\omega)}(\bar\rho_1(\theta^{-1}\omega),\bar\rho_2(\theta^{-1}\omega)) + \log(\tau_2/\tau_1) \\ &\leq 4a + \log(\tau_2/\tau_1). \end{aligned} \tag{6.21}$$

As a consequence, $R(\theta^{-1}\omega) \leq 4a + \log(\tau_2/\tau_1)$. By (6.14), let $\Lambda_1 = 1 - e^{-(4a+\log(\tau_2/\tau_1))}$. The proof of (4) is done.

Now, let's prove (6.13). For all $\omega \in \Omega$, $\tilde{\rho}(x,\omega) > 0$ is obvious. Moreover, for any $x, y \in \tilde{\gamma}(\omega)$, we have

$$
\begin{aligned}
& |\log \tilde{\rho}(x,\omega) - \log \tilde{\rho}(y,\omega)| \\
\leq\; & |\log \rho(\psi_\omega(x),\omega) - \log \rho(\psi_\omega(y),\omega)| + |\log|\det D\psi_\omega(x)| - \log|\det D\psi_\omega(y)|| \\
\leq\; & a_1 d(\psi_\omega(x),\psi_\omega(y))^{\mu_1} + a_0 d(x,y)^{\nu_0} \\
\leq\; & a_1 a_0^{\mu_1} d(x,y)^{\mu_1\nu_0} + a_0 d(x,y)^{\nu_0} \\
\leq\; & (a_1 a_0^{\mu_1} + a_0) d(x,y)^{\mu_1\nu_0} \\
\leq\; & a/2 d(x,y)^{\mu_1\nu_0},
\end{aligned}
$$

provided assumption (6.12). So if $\rho(\cdot,\omega) \in D(a_1,\mu_1,\gamma(\omega))$, then $\tilde{\rho}(\cdot,\omega) \in D(a/2,\mu_1\nu_0,\tilde{\gamma}(\omega))$. □

Now, we use convex cone $D(a_1,\mu,\gamma(\omega))$, $D(\frac{a}{2},\mu,\gamma(\omega))$ and $D(a,\mu,\gamma(\omega))$ to define the convex cone of observable functions on each fiber. Let $b, c > 0$ be parameters to be determined later. For any $\omega \in \Omega$, define $C_\omega(b,c,\nu)$ be the collection of all bounded measurable functions $\varphi : M \to \mathbb{R}$ satisfying:

(C1) $\int_{\gamma(\omega)} \varphi(x)\rho(x,\omega) dm_{\gamma(\omega)}(x) > 0$ for every local stable submanifold $\gamma(\omega) \subset M_\omega$ having size between $\epsilon/4$ and $\epsilon/2$, and every $\rho(\cdot,\omega) \in D(a/2,\mu,\gamma(\omega))$ satisfying that $\int_{\gamma(\omega)} \rho(x,\omega) dm_{\gamma(\omega)}(x) = 1$;

(C2) $|\log \int_{\gamma(\omega)} \varphi(x)\rho'(x,\omega) dm_{\gamma(\omega)}(x) - \log \int_{\gamma(\omega)} \varphi(x)\rho''(x,\omega) dm_{\gamma(\omega)}(x)| \leq b d_{\gamma(\omega)}(\vec{\rho}'(\omega), \vec{\rho}''(\omega))$ for every local stable submanifold $\gamma(\omega) \subset M_\omega$ having size between $\epsilon/4$ and $\epsilon/2$, and $\rho'(\cdot,\omega)$, $\rho''(\cdot,\omega) \in D(a/2,\mu,\gamma(\omega)) \subset D(a,\mu,\gamma(\omega))$ satisfying $\int_{\gamma(\omega)} \rho'(x,\omega) dm_{\gamma(\omega)}(x) =$

$\int_{\gamma(\omega)} \rho''(x,\omega) dm_{\gamma(\omega)}(x) = 1$ and $d_{\gamma(\omega)}(\cdot,\cdot)$ is the Hilbert projective metric defined on $D(a,\mu,\gamma(\omega))$.

(C3) $|\log \int_{\gamma(\omega)} \varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) - \log \int_{\tilde{\gamma}(\omega)} \varphi(x)\tilde{\rho}(x,\omega)dm_{\tilde{\gamma}(\omega)}(x)| \leq cd(\gamma(\omega),\tilde{\gamma}(\omega))^\nu$ for every pair of local stable leaves $\gamma(\omega), \tilde{\gamma}(\omega) \subset M_\omega$ having size between $\epsilon/4$ and $\epsilon/2$, and $\gamma(\omega)$ is the holonomy image of $\tilde{\gamma}(\omega)$, $\rho(\cdot,\omega) \in D(a_1,\mu_1,\gamma(\omega))$ and $\tilde{\rho}(\cdot,\omega)$ corresponding to $\rho(\cdot,\omega)$ defined as (6.10).

Remark 6.2. *The choice of parameters a, a_1, μ_1, b and c is used for proving the contraction of the transfer operator on the convex cone of observable functions. We just need to guarantee that all auxiliary parameters only depend on μ and ν.*

Remark 6.3. *Note that $(C2)$ is automatically fulfilled if φ is nonnegative. In fact, notice that*

$$\int_{\gamma(\omega)} \rho''(x,\omega)dm_{\gamma(\omega)}(x) = \int_{\gamma(\omega)} \rho'(x,\omega)dm_{\gamma(\omega)}(x) = 1,$$

so we have

$$\begin{aligned}\frac{\rho'(x,\omega)}{\rho''(x,\omega)} &\leq \sup_{x\in\gamma(\omega)}\{\frac{\rho'(x,\omega)}{\rho''(x,\omega)}\}/\inf_{y\in\gamma(\omega)}\{\frac{\rho'(x,\omega)}{\rho''(x,\omega)}\} = \exp(d_{+,\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega)))\\ &\leq \exp(d_{\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega))).\end{aligned}$$

Switch ρ' and ρ'', we get $\frac{\rho''(x,\omega)}{\rho'(x,\omega)} \leq \exp(d_{\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega)))$. $(C2)$ is a consequence of these two inequalities as long as $b > 1$.

Also note that positive constant functions belong to $\cap_{\omega\in\Omega} C_\omega(b,c,\nu)$ obviously.

Lemma 6.4. *For each $\omega \in \Omega$, $C_\omega(b,c,\nu)$ is a convex cone (see Definition A.1 in the Appendix).*

Proof of Lemma 6.4. For any $\varphi \in C_\omega(b,c,\nu)$, and $t > 0$, $t\varphi \in C_\omega(b,c,\nu)$ obviously.

Now, we prove the convexity, i.e., $\varphi_1, \varphi_2 \in C_\omega(b, c, \nu)$ and $t_1, t_2 > 0$, we are going to prove $t_1\varphi + t_2\varphi_2 \in C_\omega(b, c, \nu)$. (C1) is automatically fulfilled. For condition (C2),

$$e^{-bd_{\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega))} \leq \frac{\int_{\gamma(\omega)} \varphi_i(x)\rho'(x,\omega)dm_{\gamma(\omega)}(x)}{\int_{\gamma(\omega)} \varphi_i(x)\rho''(x,\omega)dm_{\gamma(\omega)}(x)} \leq e^{bd_{\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega))}$$

for $i = 1, 2$. The above implies that

$$e^{-bd_{\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega))} \leq \frac{\int_{\gamma(\omega)} (t_1\varphi_1(x) + t_2\varphi_2(x))\,\rho'(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)} (t_1\varphi_1(x) + t_2\varphi_2(x))\,\rho''(x,\omega)dm_{\gamma(\omega)}} \leq e^{bd_{\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega))}.$$

So $(C2)$ is verified. Similarly, $(C3)$ is also verified. Therefore, $t_1\varphi + t_2\varphi_2 \in C(b, c, \nu)$.

Now, we prove that $-\overline{C_\omega(b, c, \nu)} \cap \overline{C_\omega(b, c, v)} = \{0\}$. Suppose $\varphi \in -\overline{C_\omega(b, c, \nu)} \cap \overline{C_\omega(b, c, v)}$, then there exists $\varphi_1, \varphi_2 \in C_\omega(b, c, \nu)$ and $t_n^1, t_n^2 \downarrow 0$ such that $\varphi + t_n^1(\varphi_1) \in C_\omega(b, c, \nu)$ and $-\varphi + t_n^2(\varphi_2) \in C_\omega(b, c, \nu)$. Hence, for any local stable leaf $\gamma(\omega)$ and $\rho(\cdot, \omega) \in D(a/2, \mu, \gamma(\omega))$, we have

$$\int_{\gamma(\omega)} (\varphi + t_n^1(\varphi_1))(x)\rho(x,\omega)dm_{\gamma(\omega)} > 0;$$
$$\int_{\gamma(\omega)} (-\varphi + t_n^2(\varphi_2))(x)\rho(x,\omega)dm_{\gamma(\omega)} > 0.$$

Let $n \to \infty$, we have

$$\int_{\gamma(\omega)} \varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) = 0 \tag{6.22}$$

for any $\rho(\cdot, \omega) \in D(a/2, \mu, \gamma(\omega))$ and any local stable leaf $\gamma(\omega)$. Now pick $\tilde{g} \in C^{0,\mu}(M)$ any $\mu-$Hölder continuous function, and define $\bar{g}(x, \omega) \equiv \tilde{g}(x)$. Choose $B = \frac{2|\tilde{g}|_\mu}{a}$, then

$$\log(\bar{g}^+(x,\omega) + B), \text{ and } \log(\bar{g}^-(x,\omega) + B)$$

are $(a/2, \mu)-$Hölder continuous, where

$$|\tilde{g}|_\mu := \sup_{x\neq y\in M} \frac{|g(x) - g(y)|}{d(x,y)^\mu}, \ \bar{g}^+ := \frac{1}{2}(|\bar{g}| + \bar{g}), \ \bar{g}^- := \frac{1}{2}(|\bar{g}| - \bar{g}).$$

Then $(\bar{g}^+(\cdot,\omega)+B)|_{\gamma(\omega)}$, $(\bar{g}^-(\cdot,\omega)+B)|_{\gamma(\omega)}$ are in $D(a/2,\mu,\gamma(\omega))$. By (6.22) and linearity of integration, we have

$$\int_{\gamma(\omega)} \varphi(x)\tilde{g}(x)dm_{\gamma(\omega)}(x) = 0.$$

For any fixed ω, we can pick $\tilde{g} \in C^{0,\mu}(M)$ L^1- approximating $\varphi(\cdot)$, hence we have $\int_{\gamma(\omega)} \varphi^2(x)dm_{\gamma(\omega)}(x) = 0$. So $\varphi(x) = 0$ for $x \in \gamma(\omega)$. Since $\gamma(\omega) \subset M_\omega$ is arbitrary, $\varphi(\cdot) \equiv 0$. □

Now $C_\omega(b,c,\nu)$ is a convex cone, so we can define the Hilbert projective metric on $C_\omega(b,c,\nu)$. For any $\varphi_1, \varphi_2 \in C_\omega(b,c,\nu)$, define

$$\alpha_\omega(\varphi_1,\varphi_2) := \sup\{t > 0 : \ \varphi_2 - t\varphi_1 \in C_\omega(b,c,\nu)\},$$

$$\beta_\omega(\varphi_1,\varphi_2) := \sup\{s > 0 : \ s\varphi_1 - \varphi_2 \in C_\omega(b,c,\nu)\},$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$, and let

$$d_\omega(\varphi_1,\varphi_2) := \log\frac{\beta_\omega(\varphi_1,\varphi_2)}{\alpha_\omega(\varphi_1,\varphi_2)},$$

with the convention that $d_\omega(\varphi_1,\varphi_2) = \infty$ if $\alpha_\omega(\varphi_1,\varphi_2) = 0$ or $\beta_\omega(\varphi_1,\varphi_2) = \infty$. Without ambiguity, we write $\int_{\gamma(\omega)} \varphi\rho dm_{\gamma(\omega)}$ instead of $\int_{\gamma(\omega)} \varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x)$. By computation, we have that

$$\begin{aligned}\alpha_\omega(\varphi_1,\varphi_2) = \inf\Bigg\{&\frac{\int_{\gamma(\omega)}\varphi_2\rho' dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho' dm_{\gamma(\omega)}}, \frac{\int_{\gamma(\omega)}\varphi_2\rho' dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho' dm_{\gamma(\omega)}}\xi_\omega(\rho',\rho'',\varphi_1,\varphi_2) \\ &\frac{\int_{\gamma(\omega)}\varphi_2\rho dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho dm_{\gamma(\omega)}}\eta_\omega(\rho,\tilde{\rho},\varphi_1,\varphi_2), \frac{\int_{\tilde{\gamma}(\omega)}\varphi_2\tilde{\rho} dm_{\tilde{\gamma}(\omega)}}{\int_{\tilde{\gamma}(\omega)}\varphi_1\tilde{\rho} dm_{\tilde{\gamma}(\omega)}}\eta_\omega(\tilde{\rho},\rho,\varphi_1,\varphi_2)\Bigg\},\end{aligned} \tag{6.23}$$

where

$$\xi_\omega(\rho',\rho'',\varphi_1,\varphi_2) := \frac{\exp(bd_{\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega))) - \int_{\gamma(\omega)}\varphi_2\rho'' dm_{\gamma(\omega)}/\int_{\gamma(\omega)}\varphi_2\rho' d_{m(\gamma(\omega))}}{\exp(bd_{\gamma(\omega)}(\bar{\rho}'(\omega),\bar{\rho}''(\omega))) - \int_{\gamma(\omega)}\varphi_1\rho'' dm_{\gamma(\omega)}/\int_{\gamma(\omega)}\varphi_1\rho' d_{m(\gamma(\omega))}}, \tag{6.24}$$

$$\eta_\omega(\rho,\tilde\rho,\varphi_1,\varphi_2) := \frac{\exp(cd(\gamma(\omega),\tilde\gamma(\omega))^\nu) - \int_{\tilde\gamma}\varphi_2\tilde\rho dm_{\tilde\gamma(\omega)}/\int_{\gamma(\omega)}\varphi_2\rho dm_{\gamma(\omega)}}{\exp(cd(\gamma(\omega),\tilde\gamma(\omega))^\nu) - \int_{\tilde\gamma}\varphi_1\tilde\rho dm_{\tilde\gamma(\omega)}/\int_{\gamma(\omega)}\varphi_1\rho dm_{\gamma(\omega)}}, \tag{6.25}$$

and the infimum runs over all $\rho'(\cdot,\omega),\rho''(\cdot,\omega)\in D(a/2,\mu,\gamma(\omega))$ with $\int_{\gamma(\omega)}\rho^\tau(x,\omega)dm_{\gamma(\omega)}(x) = 1$ for $\tau = {\prime},{\prime\prime}$, every pair of local stable leaves $\gamma(\omega)$ and $\tilde\gamma(\omega)$, $\rho(\cdot,\omega)\in D(a_1,\mu_1,\gamma(\omega))$ and corresponding $\tilde\rho(\cdot,\omega)\in D(a/2,\mu,\tilde\gamma(\omega))$. Similarly,

$$\beta_\omega(\varphi_1,\varphi_2) = \sup\left\{\frac{\int_{\gamma(\omega)}\varphi_2\rho' dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho' dm_{\gamma(\omega)}}, \frac{\int_{\gamma(\omega)}\varphi_2\rho' dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho' dm_{\gamma(\omega)}}\xi_\omega(\rho',\rho'',\varphi_1,\varphi_2)\right. \tag{6.26}$$
$$\left.\frac{\int_{\gamma(\omega)}\varphi_2\rho dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi_1\rho dm_{\gamma(\omega)}}\eta_\omega(\rho,\tilde\rho,\varphi_1,\varphi_2), \frac{\int_{\tilde\gamma(\omega)}\varphi_2\tilde\rho dm_{\tilde\gamma(\omega)}}{\int_{\tilde\gamma(\omega)}\varphi_1\tilde\rho dm_{\tilde\gamma(\omega)}}\eta_\omega(\tilde\rho,\rho,\varphi_1,\varphi_2)\right\},$$

where the supreme runs over all $\rho'(\cdot,\omega),\rho''(\cdot,\omega)\in D(a/2,\mu,\gamma(\omega))$, $\int_{\gamma(\omega)}\rho^\tau(x,\omega)dm_{\gamma(\omega)}(x) = 1$ for $\tau = {\prime},{\prime\prime}$, every pair of local stable leaves $\gamma(\omega)$ and $\tilde\gamma(\omega)$, $\rho(\cdot,\omega)\in D(a_1,\mu_1,\gamma(\omega))$ and corresponding $\tilde\rho(\cdot,\omega)\in D(a/2,\mu,\tilde\gamma(\omega))$.

6.2 Contraction of the Fiber Transfer Operator

In this section, we will prove that the fiber transfer operator L_ω maps $C_\omega(b,c,\nu)$ into $C_{\theta\omega}(b,c,\nu)$ for all $\omega\in\Omega$. Moreover, the diameter of $L_\omega^N C_\omega(b,c,\nu)$ with respect to the Hilbert projective metric on $C_{\theta^N\omega}(b,c,\nu)$ is finite uniformly for all $\omega\in\Omega$, where the number N comes from the topological mixing on fibers property. Birkhoff's inequality (Proposition A.4 in the Appendix) implies the contraction of the fiber transfer operator L_ω^N.

Lemma 6.5. *Let $\lambda_2\in(\max\{\Lambda_1,e^{-\lambda\nu}\},1)$, then there exist constants b_0 and c_0 such that for any $b > b_0 = b_0(\lambda_2,\Lambda_1)$, $c > c_0 = c_0(\nu)$ and for all $\omega\in\Omega$, we have $L_{\theta^{-1}\omega}(C_{\theta^{-1}\omega}(b,c,\nu))\subset C_\omega(\lambda_2 b,\lambda_2 c,\nu)\subset C_\omega(b,c,\nu)$. Recall that the fiber transfer operator $L_{\theta^{-1}\omega}$ is defined by*

$$(L_{\theta^{-1}\omega}\varphi)(x) = \frac{\varphi((f_{\theta^{-1}\omega})^{-1}x)}{|\det D_{(f_{\theta^{-1}\omega})^{-1}(x)}f_{\theta^{-1}\omega}|}.$$

Proof of Lemma 6.5. Pick any $\omega\in\Omega$ and fix it. For any $\varphi: M\to\mathbb{R}$ bounded and measurable, it is easy to see that $L_{\theta^{-1}\omega}\varphi: M\to\mathbb{R}$ is bounded and measurable.

Let $\gamma(\omega)$ be a local stable leaf having size between $\epsilon/4$ and $\epsilon/2$ and every $\rho(\cdot,\omega) \in D(a/2,\mu,\gamma(\omega))$. we subdivide $f_\omega^{-1}\gamma(\omega)$ into connected local stable submanifolds of size between $\epsilon/4$ and $\epsilon/2$, named $\gamma_i(\theta^{-1}\omega)$ for i belonging to a finite index set. let $\rho_i(\cdot,\theta^{-1}\omega)$ be defined as (6.7). By Lemma 6.1,

$$\rho_i(\cdot,\theta^{-1}\omega) \in D(e^{-\lambda_1}a/2,\mu,\gamma_i(\theta^{-1}\omega)) \subset D(a/2,\mu,\gamma_i(\theta^{-1}\omega)).$$

Hence by (6.9), for any $\varphi \in C_{\theta^{-1}\omega}(b,c,\nu)$, we have

$$\omega \mapsto \int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega)dm_{\gamma(\omega)}(y) = \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x) > 0.$$

So $(C1)$ is verified.

Now for any $\rho'(\cdot,\omega),\rho''(\cdot,\omega) \in D(a/2,\mu,\gamma(\omega))$ such that

$$\int_{\gamma(\omega)} \rho'(x,\omega)dm_{\gamma(\omega)}(x) = 1 \text{ and } \int_{\gamma(\omega)} \rho''(x,\omega)dm_{\gamma(\omega)}(x) = 1,$$

we denote $\rho_i' := (\rho')_i$ and $\rho_i'' := (\rho'')_i$ which are defined as (6.7) on $\gamma_i(\theta^{-1}\omega)$. Define

$$\rho_i^-(x,\theta^{-1}\omega) = \rho_i'(x,\theta^{-1}\omega)/\int_{\gamma_i(\theta^{-1}\omega)} \rho_i'(y,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(y) \text{ for } x \in \gamma_i(\theta^{-1}\omega),$$
$$\rho_i^=(x,\theta^{-1}\omega) = \rho_i''(x,\theta^{-1}\omega)/\int_{\gamma_i(\theta^{-1}\omega)} \rho_i''(y,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(y) \text{ for } x \in \gamma_i(\theta^{-1}\omega).$$

We have $\rho_i^-(\cdot,\theta^{-1}\omega), \rho_i^=(\cdot,\theta^{-1}\omega) \in D(e^{-\lambda_1}\frac{a}{2},\mu,\gamma_i(\theta^{-1}\omega)) \subset D(\frac{a}{2},\mu,\gamma_i(\theta^{-1}\omega))$. Then

$$\begin{aligned}
&\int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho''(y,\omega)dm_{\gamma(\omega)}(y)\\
&= \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i''(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x)\\
&= \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \rho_i''(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x) \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i^=(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x)\\
&\le \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \rho_i''dm_{\gamma_i(\theta^{-1}\omega)} \cdot \exp(bd_{\gamma_i(\theta^{-1}\omega)}(\bar\rho_i^-(\theta^{-1}\omega),\bar\rho_i^=(\theta^{-1}\omega))) \cdot \int_{\gamma_i(\theta^{-1}\omega)} \varphi\rho_i^- dm_{\gamma_i(\theta^{-1}\omega)}
\end{aligned}$$

$$\begin{aligned}
&\le \sum_i \frac{\int_{\gamma_i(\theta^{-1}\omega)} \rho_i''dm_{\gamma_i(\theta^{-1}\omega)}}{\int_{\gamma_i(\theta^{-1}\omega)} \rho_i'dm_{\gamma_i(\theta^{-1}\omega)}} \cdot \exp(bd_{\gamma_i(\theta^{-1}\omega)}(\bar\rho_i'(\theta^{-1}\omega),\bar\rho_i''(\theta^{-1}\omega))) \cdot \int_{\gamma_i(\theta^{-1}\omega)} \varphi\rho_i' dm_{\gamma_i(\theta^{-1}\omega)}\\
&= \sum_i \frac{\int_{\gamma_i(\theta^{-1}\omega)} \rho_i''dm_{\gamma_i(\theta^{-1}\omega)}}{\int_{\gamma_i(\theta^{-1}\omega)} \rho_i'dm_{\gamma_i(\theta^{-1}\omega)}} \cdot \exp(b\Lambda_1 d_{\gamma(\omega)}(\bar\rho'(\omega),\bar\rho''(\omega))) \cdot \int_{\gamma_i(\theta^{-1}\omega)} \varphi\rho_i' dm_{\gamma_i(\theta^{-1}\omega)},
\end{aligned}$$

where $d_{\gamma_i(\theta^{-1}\omega)}$ and $d_{\gamma(\omega)}$ are the Hilbert projective metric on the cone $D(a,\mu,\gamma_i(\theta^{-1}\omega))$ and $D(a,\mu,\gamma(\omega))$ respectively. Note that

$$\frac{\rho_i''(x,\theta^{-1}\omega)}{\rho_i'(x,\theta^{-1}\omega)} = \frac{\rho''(f_{\theta^{-1}\omega}x,\omega)}{\rho'(f_{\theta^{-1}\omega}x,\omega)} \le \exp(d_{\gamma(\omega)}(\bar\rho'(\omega),\bar\rho''(\omega))) \text{ for } x \in \gamma_i(\theta^{-1}\omega).$$

Hence we have

$$\begin{aligned}
&\int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho''(y,\omega)dm_{\gamma(\omega)}(y)\\
&\le e^{b\Lambda_1 d_{\gamma(\omega)}(\bar\rho'(\omega),\bar\rho''(\omega))} e^{d_{\gamma(\omega)}(\bar\rho'(\omega),\bar\rho''(\omega))} \sum_{i=1}^{\infty} \int_{\gamma_i(\theta^{-1}\omega)} \varphi\rho_i' dm_{\gamma_i(\theta^{-1}\omega)}\\
&\le e^{b\lambda_2 d_{\gamma(\omega)}(\bar\rho'(\omega),\bar\rho''(\omega))} \int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho'(y,\omega)dm_{\gamma(\omega)}(y),
\end{aligned}$$

provided $\lambda_2 \in (\Lambda_1, 1)$ and $b > \frac{1}{\lambda_2 - \Lambda_1} := b_0$. Switch ρ' and ρ'', we get

$$\int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho'(y,\omega)dm_{\gamma(\omega)} \leq e^{b\lambda_2 d_{\gamma(\omega)}(\bar{\rho}''(\omega),\bar{\rho}''(\omega))} \int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho''(y,\omega)dm_{\gamma(\omega)}(y).$$

So condition $(C2)$ is verified.

Next, we verify the condition $(C3)$. Given any pair of local stable leaves $\gamma(\omega)$ and $\tilde{\gamma}(\omega)$ such that $\tilde{\gamma}(\omega)$ is the holonomy image of $\gamma(\omega)$. For each fixed $\omega \in \Omega$, let $\gamma_i(\theta^{-1}\omega)$ be defined as before such that $\gamma(\omega) = \cup f_{\theta^{-1}\omega}\gamma_i(\theta^{-1}\omega)$, and let $\tilde{\gamma}_i(\theta^{-1}\omega)$ be the holonomy image of $\gamma_i(\theta^{-1}\omega)$ inside of $(f_{\theta^{-1}\omega})^{-1}\tilde{\gamma}(\omega)$. Naturally, we have $\tilde{\gamma}(\omega) = \cup f_{\theta^{-1}\omega}\tilde{\gamma}_i(\theta^{-1}\omega)$. For any $\rho(\cdot,\omega) \in D(a_1, \mu_1, \gamma(\omega))$, let $\tilde{\rho}(\cdot,\omega)$ be defined as (6.10), and we already see that $\tilde{\rho}(\cdot,\omega) \in D(\frac{a}{2}, \mu_1\nu_0, \tilde{\gamma}(\omega)) \subset C(a, \mu, \tilde{\gamma}(\omega))$. Let $\rho_i(\cdot, \theta^{-1}\omega)$ and $(\tilde{\rho})_i(\cdot, \theta^{-1}\omega)$ be defined as (6.7) corresponding to ρ and $\tilde{\rho}$ respectively, then we have

$$\int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega)dm_{\gamma(\omega)}(y) = \sum_i \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)}(x),$$
$$\int_{\tilde{\gamma}(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\tilde{\rho}(y,\omega)dm_{\tilde{\gamma}(\omega)}(y) = \sum_i \int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)(\tilde{\rho})_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x).$$

By Lemma 6.1, $\rho_i(\cdot, \theta^{-1}\omega) \in D(e^{-\lambda_1}a_1, \mu_1, \gamma_i(\theta^{-1}\omega)) \subset D(a_1, \mu_1, \gamma_i(\theta^{-1}\omega))$. Since $\varphi \in C_{\theta^{-1}\omega}(b, c, \nu)$, we conclude for each i,

$$\begin{aligned} &\left|\log \int_{\gamma_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x,\theta^{-1}\omega)dm_{\gamma_i(\theta^{-1}\omega)} - \log \int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)\tilde{\rho}_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}\right| \quad (6.27)\\ &\leq cd(\gamma_i(\theta^{-1}\omega), \tilde{\gamma}_i(\theta^{-1}\omega))^\nu \\ &\leq c \cdot e^{-\lambda\nu} d(\gamma(\omega), \tilde{\gamma}(\omega))^\nu, \end{aligned}$$

where

$$\tilde{\rho}_i(x, \theta^{-1}\omega) = \rho_i(\psi^i_{\theta^{-1}\omega}(x), \theta^{-1}\omega) \cdot |\det D\psi^i_{\theta^{-1}\omega}(x)| \text{ for } x \in \tilde{\gamma}_i(\theta^{-1}\omega),$$

and $\psi^i_{\theta^{-1}\omega}$ is the holonomy map between $\tilde{\gamma}_i(\theta^{-1}\omega)$ and $\gamma_i(\theta^{-1}\omega)$. To prove the condition $(C3)$, we need the following Sublemma:

Sublemma 6.2.1. *There exists a $K_0 > 0$ only depending on μ such that for each i, the following inequality holds*

$$\left|\log \int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)\tilde{\rho}_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)} - \log \int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)(\tilde{\rho})_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}\right| \tag{6.28}$$

$$\leq K_0 d(\gamma(\omega),\tilde{\gamma}(\omega))^\nu.$$

Once Sublemma 6.2.1 is proved, we combine (6.27) and (6.28) to obtain

$$\left|\log \int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)\rho_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)} - \log \int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \varphi(x)(\tilde{\rho})_i(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}\right|$$
$$\leq (c\cdot e^{-\lambda\nu} + K_0)d(\gamma(\omega),\tilde{\gamma}(\omega))^\nu.$$

As a consequence,

$$\begin{aligned}&\left|\log \int_{\gamma(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\rho(y,\omega)dm_{\gamma(\omega)}(y) - \log \int_{\tilde{\gamma}(\omega)} (L_{\theta^{-1}\omega}\varphi)(y)\tilde{\rho}(y,\omega)dm_{\tilde{\gamma}(\omega)}(y)\right| \\ &\leq (c\cdot e^{-\lambda\nu} + K_0)d(\gamma(\omega),\tilde{\gamma}(\omega))^\nu \\ &\leq \lambda_2 c d(\gamma(\omega),\tilde{\gamma}(\omega))^\nu,\end{aligned}$$

provided $\lambda_2 \in (e^{-\lambda\nu}, 1)$, and $c \geq \frac{K_0}{\lambda_2 - \exp(-\lambda\nu)} := c_0$. The proof of Lemma 6.5 is done. □

Proof of Sublemma 6.2.1. Applying Lemma 6.1 and (6.13) to $\rho(\cdot,\omega) \in D(a_1,\mu_1,\gamma(\omega))$, we see that $(\tilde{\rho})_i(\cdot,\theta^{-1}\omega)$, $\tilde{\rho}_i(\cdot,\theta^{-1}\omega)$ both belong to $D(a/2,\mu_1\nu_0,\tilde{\gamma}_i(\theta^{-1}\omega)) \subset D(a/2,\mu,\tilde{\gamma}_i(\theta^{-1}\omega))$. Without ambiguity, we denote $\rho' := (\tilde{\rho})_i$ and $\rho'' := \tilde{\rho}_i$ for short.

We normalize the random density ρ' and ρ'' by

$$\frac{\rho'(x,\theta^{-1}\omega)}{\int_{\tilde{\gamma}_i(\theta^{-1}\omega)}\rho'(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)},\ \frac{\rho''(x,\theta^{-1}\omega)}{\int_{\tilde{\gamma}_i(\theta^{-1}\omega)}\rho''(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)}.$$

Then by condition $(C2)$, we have

$$\begin{aligned}
&\left|\log\int_{\tilde{\gamma}_i(\theta^{-1}\omega)}\varphi(x)\rho'(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta)^{-1}\omega}-\log\int_{\tilde{\gamma}_i(\theta^{-1}\omega)}\varphi(x)\rho''(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta)^{-1}\omega}\right|\\
&\le bd_{\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega),\bar{\rho}''(\theta^{-1}\omega))\\
&\qquad+\left|\log\int_{\tilde{\gamma}_i(\theta^{-1}\omega)}\rho'(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)-\log\int_{\tilde{\gamma}_i(\theta^{-1}\omega)}\rho''(x,\theta^{-1}\omega)dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)\right|.
\end{aligned}$$

Next, we are going to estimate the terms of the right hand of the above inequality. By definition, we have expressions

$$\begin{aligned}
\rho'(x,\theta^{-1}\omega)&=\frac{|\det D_xf_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}|}{|\det D_xf_{\theta^{-1}\omega}|}\cdot\rho(\psi_\omega f_{\theta^{-1}\omega}x,\omega)|\det D_{f_{\theta^{-1}\omega}x}\psi_\omega|;\\
\rho''(x,\theta^{-1}\omega)&=\frac{|\det D_{\psi^i_{\theta^{-1}\omega}(x)}f_{\theta^{-1}\omega}|_{E^s(\psi^i_{\theta^{-1}\omega}(x),\theta^{-1}\omega)}|}{|\det D_{\psi^i_{\theta^{-1}\omega}(x)}f_{\theta^{-1}\omega}|}\cdot\rho(f_{\theta^{-1}\omega}\psi^i_{\theta^{-1}\omega}(x),\omega)\cdot|\det D_x\psi^i_{\theta^{-1}\omega}|.
\end{aligned}$$

By definition of holonomy map, we have

$$\rho(\psi_\omega f_{\theta^{-1}\omega}x,\omega)=\rho(f_{\theta^{-1}\omega}\psi^i_{\theta^{-1}\omega}(x),\omega)\text{ for }x\in\tilde{\gamma}_i(\theta^{-1}\omega).\tag{6.29}$$

By Lemma 4.20, we have

$$\begin{aligned}
|\log|\det D_{f_{\theta^{-1}\omega}x}\psi_\omega|-\log|\det D_x\psi^i_{\theta^{-1}\omega}||&\le a_0d(f_{\theta^{-1}\omega}(x),\psi_\omega f_{\theta^{-1}\omega}(x))^{\nu_0}+a_0d(x,\psi^i_{\theta^{-1}\omega}(x))^{\nu_0}\\
&\le a_0(1+e^{-\lambda\nu_0})d(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}.
\end{aligned}\tag{6.30}$$

Combing Lemma 4.1 and Lemma 4.8, for all $x, y \in M$, $\omega \in \Omega$, we have

$$||\det D_x f_\omega|_{E^s(x,\omega)}| - |\det D_y f_\omega|_{E^s(y,\omega)}|| \leq 2C_2 d(x,y)^{\nu_0}. \tag{6.31}$$

Then, by (6.31) and uniformly boundness of $|\det D_x f_\omega|_{E^s(x,\omega)}|$, there exists a constant R independent of x and ω such that

$$\begin{aligned}
&\left|\log|\det D_x f_{\theta^{-1}\omega}|_{E^s(x,\theta^{-1}\omega)}| - \log|\det D_{\psi^i_{\theta^{-1}\omega}(x)} f_{\theta^{-1}\omega}|_{E^s(\psi^i_{\theta^{-1}\omega}(x),\theta^{-1}\omega)}|\right| \\
&\leq R d(x, \psi^i_{\theta^{-1}\omega}(x))^{\nu_0} \\
&\leq R d(\gamma_i(\theta^{-1}\omega), \tilde{\gamma}_i(\theta^{-1}\omega))^{\nu_0} \\
&\leq R e^{-\lambda\nu_0} d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0}.
\end{aligned} \tag{6.32}$$

Applying (6.2), we have

$$\left|\log|\det D_{\psi^i_{\theta^{-1}\omega}x} f_{\theta^{-1}\omega}| - \log|\det D_x f_{\theta^{-1}\omega}|\right| \leq K_2 d(x, \psi^i_{\theta^{-1}\omega}x) \leq K_2 e^{-\lambda} d(\gamma(\omega), \tilde{\gamma}(\omega)). \tag{6.33}$$

Then (6.29), (6.30), (6.32) and (6.33) imply

$$\begin{aligned}
&\left|\log\int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \rho'(x,\theta^{-1}\omega) dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x) - \log\int_{\tilde{\gamma}_i(\theta^{-1}\omega)} \rho''(x,\theta^{-1}\omega) dm_{\tilde{\gamma}_i(\theta^{-1}\omega)}(x)\right| \\
&\leq (a_0(1+e^{-\lambda\nu_0}) + Re^{-\lambda\nu_0} + K_2 e^{-\lambda}) d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0} \\
&:= K_3 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0},
\end{aligned} \tag{6.34}$$

and

$$e^{-K_3 d(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}} \leq \inf_{x\in\tilde{\gamma}_i(\theta^{-1}\omega)} \frac{\rho'(x,\theta^{-1}\omega)}{\rho''(x,\theta^{-1}\omega)} \leq \sup_{x\in\tilde{\gamma}_i(\theta^{-1}\omega)} \frac{\rho'(x,\theta^{-1}\omega)}{\rho''(x,\theta^{-1}\omega)} \leq e^{K_3 d(\gamma(\omega),\tilde{\gamma}(\omega))^{\nu_0}}.$$

The above inequality implies that

$$d_{+,\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega), \bar{\rho}''(\theta^{-1}\omega)) \leq 2K_3 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0}.$$

Similar to (6.20), we have an estimate

$$d_{\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega), \bar{\rho}''(\theta^{-1}\omega)) \leq d_{+,\tilde{\gamma}_i(\theta^{-1}\omega)}(\bar{\rho}'(\theta^{-1}\omega), \bar{\rho}''(\theta^{-1}\omega)) + \log\left(\hat{\tau}_2(\theta^{-1}\omega)/\hat{\tau}_1(\theta^{-1}\omega)\right), \tag{6.35}$$

where

$$\hat{\tau}_1(\theta^{-1}\omega) = \inf_{x \neq y \in \gamma_i(\theta^{-1}\omega)} \frac{\exp(ad(x,y)^\mu) - \rho''(y,\theta^{-1}\omega)/\rho''(x,\theta^{-1}\omega)}{\exp(ad(x,y)^\mu) - \rho'(y,\theta^{-1}\omega)/\rho'(x,\theta^{-1}\omega)},$$

$$\hat{\tau}_2(\theta^{-1}\omega) = \sup_{x \neq y \in \gamma_i(\theta^{-1}\omega)} \frac{\exp(ad(x,y)^\mu) - \rho''(y,\theta^{-1}\omega)/\rho''(x,\theta^{-1}\omega)}{\exp(ad(x,y)^\mu) - \rho'(y,\theta^{-1}\omega)/\rho'(x,\theta^{-1}\omega)}.$$

Denote

$$B_1(x,y,\theta^{-1}\omega) := \frac{\rho'(y,\theta^{-1}\omega)}{\rho'(x,\theta^{-1}\omega)} \cdot \exp(-ad(x,y)^\nu),$$

$$B_2(x,y,\theta^{-1}\omega) := \frac{\rho''(y,\theta^{-1}\omega)}{\rho''(x,\theta^{-1}\omega)} \cdot \exp(-ad(x,y)^\nu).$$

Since $\rho'(\cdot,\theta^{-1}\omega), \rho''(\cdot,\theta^{-1}\omega) \in D(a/2, \mu, \tilde{\gamma}_i(\theta^{-1}\omega))$, we have

$$\begin{aligned} \log B_1(x,y,\theta^{-1}\omega) &= \log \rho'(y,\theta^{-1}\omega) - \log \rho'(x,\theta^{-1}\omega) - ad(x,y)^\mu \\ &\leq \frac{a}{2} d(x,y)^\mu - ad(x,y)^\mu \\ &\leq -\frac{a}{2} d(x,y)^\mu < 0. \end{aligned}$$

As a consequence $B_1(x,y,\theta^{-1}\omega) < 1$. Similarly, $B_2(x,y,\theta^{-1}\omega) < 1$. Hence, on one hand

$$\begin{aligned}
&|B_1(x,y,\theta^{-1}\omega) - B_2(x,y,\theta^{-1}\omega)| \\
&\le \max\{B_1, B_2\}|\log B_1(x,y,\theta^{-1}\omega) - \log B_2(x,y,\theta^{-1}\omega)| \\
&\le |\log \rho'(x,\theta^{-1}\omega) - \log \rho''(x,\theta^{-1}\omega)| + |\log \rho'(y,\theta^{-1}\omega) - \log \rho''(y,\theta^{-1}\omega)| \\
&\le 2K_3 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0} \\
&\le 2K_3 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\mu+\nu}. \qquad (6.36)
\end{aligned}$$

On the other hand,

$$\begin{aligned}
|B_1(x,y,\theta^{-1}\omega) - B_2(x,y,\theta^{-1}\omega)| &\le |\log \rho'(x,\theta^{-1}\omega) - \log \rho'(y,\theta^{-1}\omega)| + |\log \rho''(x,\theta^{-1}\omega) - \log \rho''(y,\theta^{-1}\omega)| \\
&\le 2 \cdot \frac{a}{2} d(x,y)^{\mu_1\nu_0} \\
&\le a d(x,y)^{\mu+\nu}.
\end{aligned}$$

(6.36) and (6.37) imply that

$$\begin{aligned}
|B_1(x,y,\theta^{-1}\omega) - B_2(x,y,\theta^{-1}\omega)| &\le \max\{a, 2K_3\} d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu} \cdot d(x,y)^{\mu} \\
&:= K_4 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu} \cdot d(x,y)^{\mu}. \qquad (6.37)
\end{aligned}$$

Then

$$\begin{aligned}
\left|\log \frac{1 - B_2(x,y,\theta^{-1}\omega)}{1 - B_1(x,y,\theta^{-1}\omega)}\right| &\le \frac{|B_1(x,y,\theta^{-1}\omega) - B_2(x,y,\theta^{-1}\omega)|}{1 - \max\{B_1(x,y,\theta^{-1}\omega), B_2(x,y,\theta^{-1}\omega)\}} \\
&\le \frac{K_4 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu} \cdot d(x,y)^{\mu}}{1 - \exp(-\frac{a}{2} d(x,y)^{\mu})} \\
&\le K_5 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu}, \qquad (6.38)
\end{aligned}$$

where $K_5 := K_4 \cdot \sup_{z\in(0,1)} \frac{z^\mu}{1-\exp(-\frac{a}{2}z^\mu)} < \infty$. Hence we have

$$|\log \hat{\tau}_2(\theta^{-1}\omega)/\hat{\tau}_1(\theta^{-1}\omega)| \le 2K_5 d(\gamma(\omega), \tilde{\gamma}(\omega))^\nu. \tag{6.39}$$

Let $K_0 = 3K_3 + 2K_5$, then by (6.34), (6.35) and (6.39), Sublemma 6.2.1 is proved. □

Let $\delta \in (0, \epsilon/8)$ be the constant in Lemma 4.3 corresponding to $\epsilon/8$, i.e., for any $x, y \in M$, $d(x, y) < \delta$, for all $\omega \in \Omega$, we have

$$W^s_{\epsilon/8}(x, \omega) \cap W^u_{\epsilon/8}(y, \omega) \ne \emptyset.$$

Now let $\{B_{\delta/4}(x)\}_{x\in M}$ be an open cover of M. Pick a subcover $\{B_{\delta/4}(x_i)\}_{i=1}^l$ by the compactness of M. Now by the definition of topological mixing on fibers, there exists a $N \in \mathbb{N}$ such that for any $n \ge N$,

$$\phi^n(\{\omega\} \times B_{\delta/4}(x_i)) \cap (\{\theta^n\omega\} \times B_{\delta/4}(x_j)) \ne \emptyset \text{ for any } 1 \le i, j \le l. \tag{6.40}$$

Moreover, we pick N large enough such that

$$e^{\lambda N} \ge 24, \tag{6.41}$$

and

$$e^{-\lambda N}\epsilon < 2\delta. \tag{6.42}$$

From now on, we fix this constant N. Next, we are going to show that the diameter of image of $L^N_{\theta^{-N}\omega} : C_{\theta^{-N}\omega}(b, c, \nu) \to C_\omega(b, c, \nu)$ with respect to the Hilbert projective metric on $C_\omega(b, c, \nu)$ is finite uniformly for all $\omega \in \Omega$.

Lemma 6.6. *For $b > b_0$, $c > c_0$, there exists a constant $D_2 = D_2(\lambda_2, a, b, c, N)$ such that for* *any* $\omega \in \Omega$,

$$\sup\{d_\omega(L^N_{\theta^{-N}\omega}\varphi_1, L^N_{\theta^{-N}\omega}\varphi_2) : \ \varphi_1, \varphi_2 \in C_{\theta^{-N}\omega}(b, c, \nu)\} \le D_2 < \infty, \tag{6.43}$$

where d_ω *is the Hilbert projective metric on* $C_\omega(b, c, \nu)$.

Proof of Lemma 6.6. By Lemma 6.5, $L_{\theta^{-1}\omega}(C_{\theta^{-1}\omega}(b, c, \nu)) \subset C_\omega(\lambda_2 b, \lambda_2 c, \nu) \subset C_\omega(b, c, \nu)$, for all $\omega \in \Omega$. Pick any $\varphi_1, \varphi_2 \in L^N_{\theta^{-N}\omega} C_{\theta^{-N}\omega}(b, c, \nu) \subset C_\omega(\lambda_2 b, \lambda_2 c, \nu)$, by (6.24) and condition $(C2)$, we have

$$\begin{aligned}
\xi_\omega(\rho', \rho'', \varphi_1, \varphi_2) &= \frac{\exp(bd_{\gamma(\omega)}(\bar\rho'(\omega), \bar\rho''(\omega))) - \int_{\gamma(\omega)} \varphi_2 \rho'' dm_{\gamma(\omega)} / \int_{\gamma(\omega)} \varphi_2 \rho' d_{m(\gamma(\omega))}}{\exp(bd_{\gamma(\omega)}(\bar\rho'(\omega), \bar\rho''(\omega))) - \int_{\gamma(\omega)} \varphi_1 \rho'' dm_{\gamma(\omega)} / \int_{\gamma(\omega)} \varphi_1 \rho' d_{m(\gamma(\omega))}} \\
&\ge \frac{\exp(bd_{\gamma(\omega)}(\bar\rho'(\omega), \bar\rho''(\omega))) - \exp(b\lambda_2 d_{\gamma(\omega)}(\bar\rho'(\omega), \bar\rho''(\omega)))}{\exp(bd_{\gamma(\omega)}(\bar\rho'(\omega), \bar\rho''(\omega))) - \exp(-b\lambda_2 d_{\gamma(\omega)}(\bar\rho'(\omega), \bar\rho''(\omega)))} \\
&\ge \tau_3,
\end{aligned}$$

where $\tau_3 := \inf\{\frac{z - z^{\lambda_2}}{z - z^{-\lambda_2}} : \ z > 1\} = \frac{1-\lambda_2}{1+\lambda_2} \in (0, 1)$. Similarly, we have

$$\xi_\omega(\rho', \rho'', \varphi_1, \varphi_2) \le \tau_4$$

for $\tau_4 = \sup\{\frac{z - z^{-\lambda_2}}{z - z^{\lambda_2}} : \ z > 1\} = \frac{1+\lambda_2}{1-\lambda_2} \in (1, \infty)$. Likewise, $\eta_\omega(\rho, \tilde\rho, \varphi_1, \varphi_2) \in [\tau_3, \tau_4]$.

Let $C_{+,\omega}$ be the collection of all bounded measurable functions $\varphi : M \to \mathbb{R}$ only satisfying condition $(C1)$, which is a convex cone obviously. Next, we introduce the Hilbert projective metric on $C_{+,\omega}$. We define

$$\alpha_{+,\omega}(\varphi_1, \varphi_2) := \sup\{t > 0 : \varphi_2 - t\varphi_1 \in C_{+,\omega}\};$$

$$\beta_{+,\omega}(\varphi_1, \varphi_2) := \inf\{s > 0 : \ s\varphi_1 - \varphi_2 \in C_{+,\omega}\};$$

with the convention that $\sup\emptyset = 0$ and $\inf\emptyset = +\infty$, and let

$$d_{+,\omega}(\varphi_1, \varphi_2) := \log \frac{\beta_{+,\omega}(\varphi_1, \varphi_2)}{\alpha_{+,\omega}(\varphi_1, \varphi_2)}$$

with the convention that $d_{+,\omega}(\varphi_1, \varphi_2) = \infty$ if $\alpha_{+,\omega}(\varphi_1, \varphi_2) = 0$ or $\beta_{+,\omega}(\varphi_1, \varphi_2) = \infty$. By computation, we have

$$\alpha_{+,\omega}(\varphi_1, \varphi_2) = \inf \left\{ \frac{\int_{\gamma(\omega)} \varphi_2 \rho dm_{\gamma(\omega)}}{\int_{\gamma(\omega)} \varphi_1 \rho dm_{\gamma(\omega)}} \right\}, \tag{6.44}$$

where the infimum runs over all $\rho(\cdot, \omega) \in D(\frac{a}{2}, \mu, \gamma(\omega))$, $\gamma(\omega)$ any local stable leaf having size between $\epsilon/4$ and $\epsilon/2$.

$$\beta_{+,\omega}(\varphi_1, \varphi_2) = \sup \left\{ \frac{\int_{\gamma(\omega)} \varphi_2 \rho dm_{\gamma(\omega)}}{\int_{\gamma(\omega)} \varphi_1 \rho dm_{\gamma(\omega)}} \right\}, \tag{6.45}$$

where the supremum runs over all $\rho(\cdot, \omega) \in D(\frac{a}{2}, \mu, \gamma(\omega))$, $\gamma(\omega)$ any local stable leaf having size between $\epsilon/4$ and $\epsilon/2$.

Compare (6.23) with (6.44) and (6.26) with (6.45), and notice that $D(a_1, \mu_1, \gamma) \subset D(\frac{a}{2}, \mu, \gamma)$, we have for $\varphi_1, \varphi_2 \in L^N_{\theta^{-N}\omega} C_{\theta^{-N}\omega}(b, c, \nu) \subset C_\omega(\lambda_2 b, \lambda_2 c, \nu)$,

$$\alpha_\omega(\varphi_1, \varphi_2) \geq \tau_3 \alpha_{+,\omega}(\varphi_1, \varphi_2)$$
$$\beta_\omega(\varphi_1, \varphi_2) \leq \tau_4 \beta_{+,\omega}(\varphi_1, \varphi_2).$$

As a consequence, we have

$$d_\omega(\varphi_1, \varphi_2) \leq d_{+,\omega}(\varphi_1, \varphi_2) + \log \frac{\tau_4}{\tau_3}.$$

To estimate $d_{+,\omega}(\varphi_1, \varphi_2)$ for $\varphi_1, \varphi_2 \in L^N_{\theta^{-N}\omega} C_{\theta^{-N}\omega}(b, c, \nu)$, it is sufficient to estimate the upper bound of

$$\frac{\int_{\gamma''(\omega)}\varphi_2\rho'' dm_{\gamma''(\omega)}/\int_{\gamma''(\omega)}\varphi_1\rho'' dm_{\gamma''(\omega)}}{\int_{\gamma'(\omega)}\varphi_2\rho' dm_{\gamma'(\omega)}/\int_{\gamma'(\omega)}\varphi_1\rho' dm_{\gamma'(\omega)}}$$

$$=\frac{\int_{\gamma''(\omega)}\varphi_2\rho'' dm_{\gamma''(\omega)}}{\int_{\gamma'(\omega)}\varphi_2\rho' dm_{\gamma'(\omega)}}\cdot\frac{\int_{\gamma'(\omega)}\varphi_1\rho' dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)}\varphi_1\rho'' dm_{\gamma''(\omega)}}$$

$$=\frac{\int_{\gamma''(\omega)}\varphi_2\rho'' dm_{\gamma''(\omega)}/\int_{\gamma''(\omega)}\rho'' dm_{\gamma''(\omega)}}{\int_{\gamma'(\omega)}\varphi_2\rho' dm_{\gamma'(\omega)}/\int_{\gamma'(\omega)}\rho' dm_{\gamma'(\omega)}}\cdot\frac{\int_{\gamma'(\omega)}\varphi_1\rho' dm_{\gamma'(\omega)}/\int_{\gamma'(\omega)}\rho' dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)}\varphi_1\rho'' dm_{\gamma''(\omega)}/\int_{\gamma''(\omega)}\rho'' dm_{\gamma''(\omega)}}$$

for any random local stable leaves $\gamma'(\omega)$, $\gamma''(\omega)$, $\rho'(\cdot,\omega)\in D(\frac{a}{2},\mu,\gamma'(\omega))$ and $\rho''(\cdot,\omega)\in D(\frac{a}{2},\mu,\gamma''(\omega))$. Next, we are going to estimate

$$\frac{\int_{\gamma''(\omega)}\varphi(x)\rho''(x,\omega)dm_{\gamma''(\omega)}(x)}{\int_{\gamma'(\omega)}\varphi(x)\rho'(x,\omega)dm_{\gamma'(\omega)}(x)} \tag{6.46}$$

for $\varphi\in L^N_{\theta^{-N}\omega}C_{\theta^{-N}\omega}(b,c,\nu)$, $\rho'(\cdot,\omega)\in D(a/2,\mu,\gamma'(\omega))$, and $\rho''(\cdot,\omega)\in D(a/2,\mu,\gamma''(\omega))$ with $\int_{\gamma''(\omega)}\rho'' dm_{\gamma''(\omega)}=1=\int_{\gamma'(\omega)}\rho' dm_{\gamma'(\omega)}$. Let

$$\bar{k}_1(\omega)=\left(\int_{\gamma'(\omega)}\varphi(x)dm_{\gamma'(\omega)}(x)\right)^{-1},$$
$$\bar{k}_2(\omega)=\left(\int_{\gamma''(\omega)}\varphi(x)dm_{\gamma''(\omega)}(x)\right)^{-1}.$$

Then we define

$$k_1(x,\omega):=\bar{k}_1(\omega)/\int_{\gamma'(\omega)}\bar{k}_1(\omega)dm_{\gamma'(\omega)}\text{ for }x\in\gamma'(\omega),$$
$$k_2(x,\omega):=\bar{k}_2(\omega)/\int_{\gamma''(\omega)}\bar{k}_2(\omega)dm_{\gamma''(\omega)}\text{ for }x\in\gamma''(\omega).$$

By construction, we have $k_1(\cdot,\omega)\in D(a/2,\mu,\gamma'(\omega))$ and $k_2(\cdot,\omega)\in D(a/2,\mu,\gamma''(\omega))$. Now by $(C2)$

$$\frac{\int_{\gamma''(\omega)} \varphi(x)\rho''(x,\omega)dm_{\gamma''(\omega)}(x)}{\int_{\gamma'(\omega)} \varphi(x)\rho'(x,\omega)dm_{\gamma'(\omega)}(x)}$$

$$= \frac{\int_{\gamma''(\omega)} \varphi(x)\rho''(x,\omega)dm_{\gamma''(\omega)}(x)}{\int_{\gamma''(\omega)} \varphi(x)k_2(x,\omega)dm_{\gamma''(\omega)}(x)} \cdot \frac{\int_{\gamma'(\omega)} \varphi(x)k_1(x,\omega)dm_{\gamma'(\omega)}(x)}{\int_{\gamma'(\omega)} \varphi(x)\rho'(x,\omega)dm_{\gamma'(\omega)}(x)} \cdot \frac{\int_{\gamma''(\omega)} \bar{k}_2(\omega)dm_{\gamma''(\omega)}}{\int_{\gamma'(\omega)} \bar{k}_1(\omega)dm_{\gamma'(\omega)}}$$

$$\leq e^{\lambda_2 b d_{\gamma''(\omega)}(\rho''(\cdot,\omega),k_2(\cdot,\omega))} \cdot e^{\lambda_2 b d_{\gamma'(\omega)}(\rho'(\cdot,\omega),k_1(\cdot,\omega))} \cdot \frac{\int_{\gamma''(\omega)} \bar{k}_2(\omega)dm_{\gamma''(\omega)}}{\int_{\gamma'(\omega)} \bar{k}_1(\omega)dm_{\gamma'(\omega)}}.$$

Sublemma 6.2.2. *There exists a constant $D_1 = D_1(a,b,c,N) < \infty$ such that*

$$\frac{\int_{\gamma'(\omega)} \bar{k}_1(\omega)dm_{\gamma'(\omega)}}{\int_{\gamma''(\omega)} \bar{k}_2(\omega)dm_{\gamma''(\omega)}} \leq D_1. \tag{6.47}$$

Now we let $\tau_5 = \sup\{\frac{z-z^{-1/2}}{z-z^{1/2}} : z > 1\}$ and $\tau_6 = \inf\{\frac{z-z^{1/2}}{z-z^{-1/2}} : z > 1\}$, similar to (6.21), the diameter of $D(a/2,\mu,\gamma(\omega))$ with respect to the Hilbert projective metric on $D(a,\mu,\gamma(\omega))$ is finite, i.e.,

$$d_{\gamma''(\omega)}(\bar{\rho}''(\omega),\bar{k}_2(\omega)) \leq 4a + \log \tau_5/\tau_6, \tag{6.48}$$

$$d_{\gamma'(\omega)}(\bar{\rho}'(\omega),\bar{k}_1(\omega)) \leq 4a + \log \tau_5/\tau_6. \tag{6.49}$$

Hence (6.46) $\leq e^{2\lambda_2 b(4a+\log\tau_5/\tau_6)}D_1$. As a consequence, we have $d_{+,\omega}(\varphi_1,\varphi_2) \leq e^{4\lambda_2 b(4a+\log\tau_5/\tau_6)}D_1^2$, and

$$d_\omega(\varphi_1,\varphi_2) \leq e^{4\lambda_2 b(4a+\log\tau_5/\tau_6)}D_1^2 + \log\tau_4/\tau_3 := D_2. \tag{6.50}$$

Then the proof of Lemma 6.6 is done. □

Proof of Sublemma 6.2.2. For $\varphi \in C_\omega(b,c,\nu)$, we define

$$\|\varphi\|_{\omega,+} = \sup \frac{\int_{\gamma(\omega)} \varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma(\omega)} \rho(x,\omega)dm_{\gamma(\omega)}(x)},$$

where the supremum runs over all local stable leaf $\gamma(\omega) \subset M_\omega$ having size between $\epsilon/4$ and

$\epsilon/2$ and $\rho(\cdot, \omega) \in D(a/2, \mu, \gamma(\omega))$. Similarly, we define

$$\|\varphi\|_{\omega,-} = \inf \frac{\int_{\gamma(\omega)} \varphi(x)\rho(x,\omega) dm_{\gamma(\omega)}}{\int_{\gamma(\omega)} \rho(x,\omega) dm_{\gamma(\omega)}(x)},$$

where the infimum runs over all local stable leaf $\gamma(\omega) \subset M_\omega$ having size between $\epsilon/4$ and $\epsilon/2$ and $\rho(\cdot, \omega) \in D(a/2, \mu, \gamma(\omega))$. Then (6.47) is a consequence if there exists $D_1 = D_1(a, b, c, N)$ such that:

$$\sup_{\omega\in\Omega} \frac{\|L_\omega^N \varphi\|_{\theta^N\omega,+}}{\|L_\omega^N \varphi\|_{\theta^N\omega,-}} < D_1 \text{ for any } \varphi \in C_\omega(b, c, \nu). \tag{6.51}$$

We need some preliminary inequalities before we start proof. By the continuity of f_ω in ω, there exists a constant K_6 such that

$$K_6^{-1} \leq |\det D_x f_\omega| \leq K_6 \text{ for all } (x, \omega) \in M \times \Omega. \tag{6.52}$$

For any $n \geq 1$, for any $\gamma(\theta^n\omega) \subset M_{\theta^n\omega}$ local stable leaf having size between $\epsilon/4$ and $\epsilon/2$, we break $f_{\theta^n\omega}^{-n}\gamma(\theta^n\omega)$ into finite pieces of connected local stable leaves having size between $\epsilon/4$ and $\epsilon/2$, named $\gamma_i(\omega)$. Let $\rho(\cdot, \theta^n\omega) \in D(a/2, \mu, \gamma(\theta^n\omega))$. Define

$$\rho_i(x, \omega) = \frac{|\det D_x f_\omega^n|_{E^s(x,\omega)}|}{|\det D_x f_\omega^n|} \rho(f_\omega^n x, \theta^n\omega) \text{ for } x \in \gamma_i(\omega).$$

Likewise (6.9), we have

$$\int_{\gamma(\theta^n\omega)} (L_\omega^n \varphi)(x)\rho(x, \theta^n\omega) dm_{\gamma(\theta^n\omega)} = \sum_i \int_{\gamma_i(\omega)} \varphi(x)\rho_i(x, \omega) dm_{\gamma_i(\omega)}.$$

Now we have

$$\frac{\int_{\gamma(\theta^n\omega)}(L^n\varphi)(x)\rho(x,\theta^n\omega)dm_{\gamma(\theta^n\omega)}}{\int_{\gamma(\theta^n\omega)}\rho(x,\theta^n\omega)dm_{\gamma(\theta^n\omega)}} = \frac{\sum_i\int_{\gamma_i(\omega)}\varphi(x)\rho_i(x,\omega)dm_{\gamma_i(\omega)}}{\int_{\gamma(\theta^n\omega)}\rho(x,\theta^n\omega)dm_{\gamma(\theta^n\omega)}}$$

$$\leq \frac{\sum_i\int_{\gamma_i(\omega)}\rho_i(x,\omega)dm_{\gamma_i(\omega)}\|\varphi\|_{\omega,+}}{\int_{\gamma(\theta^n\omega)}\rho(x,\theta^n\omega)dm_{\gamma^n\omega}} = \frac{\int_{\gamma(\theta^n\omega)}(L^n_\omega 1)(x)\rho(x,\theta^n\omega)dm_{\gamma(\theta^n\omega)}}{\int_{\gamma(\theta^n\omega)}\rho(x,\theta^n\omega)dm_{\gamma^n\omega}}\cdot\|\varphi\|_{\omega,+}$$

$$\leq (K_6)^n\cdot\|\varphi\|_{\omega,+}.$$

Since $\gamma(\theta^n\omega)$ and $\rho(\cdot,\theta^n\omega)$ are arbitrary, we get

$$\|L^n_\omega\varphi\|_{\theta^n\omega,+}\leq (K_6)^n\|\varphi\|_{\omega,+}. \tag{6.53}$$

For any $\omega\in\Omega$, for any $\gamma(\omega)\subset M_\omega$ local stable leaf having size between $\epsilon/4$ and $\epsilon/2$, $\rho_1(\cdot,\omega),\rho_2(\cdot,\omega)\in D(a/2,\mu,\gamma(\omega))$, any $\varphi\in C_\omega(b,c,\nu)$, we have

$$\sup_{z\in\gamma(\omega)}\frac{\rho_2(z,\omega)\int_{\gamma(\omega)}\varphi(x)\rho_1(x,\omega)dm_{\gamma(\omega)}}{\rho_1(z,\omega)\int_{\gamma(\omega)}\varphi(x)\rho_2(x,\omega)dm_{\gamma(\omega)}} := D_3<\infty. \tag{6.54}$$

In fact, for any $z\in\gamma(\omega)$, then by condition (C2) and finite diameter of $D(a/2,\mu,\gamma(\omega))$ in $D(a,\mu,\gamma(\omega))$ with respect to the Hilbert projective metric on $D(a,\mu,\gamma(\omega))$, we have

$$\frac{\rho_2(z,\omega)}{\rho_1(z,\omega)}\cdot\frac{\int_{\gamma(\omega)}\varphi(x)\rho_1(x,\omega)dm_{\gamma(\omega)}/\int_{\gamma(\omega)}\rho_1 dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\varphi(x)\rho_2(x,\omega)dm_{\gamma(\omega)}/\int_{\gamma(\omega)}\rho_2 dm_{\gamma(\omega)}}\cdot\frac{\int_{\gamma(\omega)}\rho_1 dm_{\gamma(\omega)}}{\int_{\gamma(\omega)}\rho_2 dm_{\gamma(\omega)}}$$

$$\leq \frac{\rho_2(z,\omega)}{\int_{\gamma(\omega)}\rho_2(x,\omega)dm_{\gamma(\omega)}}\frac{\int_{\gamma(\omega)}\rho_1(x,\omega)dm_{\gamma(\omega)}}{\rho_1(z,\omega)}e^{bd_{\gamma(\omega)}(\bar\rho_1(\omega),\bar\rho_2(\omega))}$$

$$\leq \frac{\rho_2(z,\omega)}{\int_{\gamma(\omega)}\rho_2(x,\omega)dm_{\gamma(\omega)}}\frac{\int_{\gamma(\omega)}\rho_1(x,\omega)dm_{\gamma(\omega)}}{\rho_1(z,\omega)}e^{(4a+\log\tau_5/\tau_6)b}$$

$$\leq e^{a/2(diam(\gamma(\omega)))^\mu}\cdot e^{a/2(diam(\gamma(\omega)))^\mu}\cdot e^{(4a+\log\tau_5/\tau_6)b}$$

$$= e^{a+b(4a+\log\tau_5/\tau_6)} := D_3.$$

Now we are in the position to prove (6.51). For any $\omega\in\Omega$, any $\varphi\in C_\omega(b,c,\nu)$, we choose

$\gamma_*(\omega)$ and $\rho_*(\cdot,\omega) \in D(a/2,\mu,\gamma_*(\omega))$ such that

$$\frac{\int_{\gamma_*(\omega)} \varphi(x)\rho_*(x,\omega)dm_{\gamma(\omega)}}{\int_{\gamma_*(\omega)} \rho_*(x,\omega)dm_{\gamma_*(\omega)}} \geq \frac{1}{2}\|\varphi\|_{\omega,+}.$$

Recall we pick N satisfying (6.40), (6.41) and (6.42) and fix it. Pick any $\gamma(\theta^N\omega) \subset M_{\theta^N\omega}$ local stable leaf having size between $\epsilon/4$ and $\epsilon/2$. We pick $x_*(\omega) \in \gamma_*(\omega)$ such that

$$W^s_{\epsilon/4}(x_*(\omega),\omega) \subset \gamma_*(\omega) \subset W^s_{\epsilon/2}(x_*(\omega),\omega),$$

and $x(\theta^N\omega) \in \gamma(\theta^N\omega)$ such that

$$W^s_{\epsilon/4}(x(\theta^N\omega),\theta^N\omega) \subset \gamma(\theta^N\omega) \subset W^s_{\epsilon/2}(x(\theta^N\omega),\theta^N\omega).$$

Then there exists i and j such that $x_*(\omega) \in B_{\delta/4}(x_i)$ and $x(\theta^N\omega) \in B_{\delta/4}(x_j)$. Then by the choice of N, $\phi^N(B_{\delta/4}(x_i) \times \{\omega\}) \cap (B_{\delta/4}(x_j) \times \{\theta^N\omega\}) \neq \emptyset$. Pick $y(\theta^N\omega) \in f^N_\omega B_{\delta/4}(x_i) \cap B_{\delta/4}(x_j)$, then

$$d(y(\theta^N\omega),x(\theta^N\omega)) \leq d(y(\theta^N\omega),x_j) + d(x_j,x(\theta^N\omega)) \leq \delta/4 + \delta/4 < \delta.$$

Then

$$y_1(\theta^N\omega) := W^u_{\epsilon/8}(y(\theta^N\omega),\theta^N\omega) \cap W^s_{\epsilon/8}(x(\theta^N\omega),\theta^N\omega) \subset W^u_{\epsilon/8}(y(\theta^N\omega),\theta^N\omega) \cap \gamma(\theta^N\omega)$$

exists. Note that by (6.42), we have

$$d(f^{-N}_{\theta^N\omega}y(\theta^N\omega), f^{-N}_{\theta^N\omega}y_1(\theta^N\omega)) \leq e^{-\lambda N}\epsilon/8 \leq \delta/4.$$

So

$$\begin{aligned}d(x_*(\omega), f^{-N}_{\theta^N\omega}y_1(\theta^N\omega)) &\leq d(x_*(\omega), x_i) + d(x_i, f^{-N}_{\theta^N\omega}y(\theta^N\omega)) + d(f^{-N}_{\theta^N\omega}y(\theta^N\omega), f^{-N}_{\theta^N\omega}y_1(\theta^N\omega))\\ &\leq \delta/4 + \delta/4 + \delta/4 < \delta.\end{aligned}$$

As a consequence,

$$W^s_{\epsilon/8}(x_*(\omega), \omega) \cap W^u_{\epsilon/8}(f^{-N}_{\theta^N\omega}y_1(\theta^N\omega), \omega) \neq \emptyset.$$

Notice that $W^u_{\epsilon/8}(f^{-N}_{\theta^N\omega}y_1(\theta^N\omega), \omega)$ breaks $f^{-N}_{\theta^N\omega}\gamma(\theta^N\omega)$ into two parts, and each part has size at least $e^{\lambda N}\frac{\epsilon}{8} \geq 3\epsilon$ by (6.41). Hence $f^{-N}_{\theta^N\omega}\gamma(\theta^N\omega)$ contains a holonomy image of $\gamma_*(\omega)$, named $\gamma_1(\omega)$.

Now let $\rho(\cdot, \theta^N\omega) \in D(a/2, \mu, \gamma(\theta^N\omega))$, define

$$\rho_1(x, \omega) = \frac{|\det D_x f^N_\omega|_{E^s(x,\omega)}|}{|\det D_x f^N_\omega|}\rho(f^N_\omega x, \theta^N\omega) \text{ for } x \in \gamma_1(\omega).$$

Let $\tilde{\rho}_1(\cdot, \omega)$ be the density function on $\gamma_*(\omega)$ defined by (6.10) corresponding to ρ_1. Then

$$\begin{aligned}&\frac{\int_{\gamma(\theta^N_\omega)}(L^N_\omega\varphi)(x)\rho(x, \theta^N\omega)dm_{\gamma(\theta^N\omega)}}{\int_{\gamma(\theta^N\omega)}\rho(x, \theta^N\omega)dm_{\gamma(\theta^N\omega)}}\\ &\geq \frac{\int_{\gamma_1(\omega)}\varphi(x)\rho_1(x, \omega)dm_{\gamma_1(\omega)}}{\int_{\gamma(\theta^N\omega)}\rho(x, \theta^N\omega)dm_{\gamma(\theta^N\omega)}}\\ &\geq \frac{\int_{\gamma_*(\omega)}\varphi(x)\tilde{\rho}_1(x, \omega)dm_{\gamma_*(\omega)}}{\int_{\gamma(\theta^N\omega)}\rho(x, \theta^N\omega)dm_{\gamma(\theta^N\omega)}} \cdot e^{-cd(\gamma_1(\omega), \gamma_*(\omega))^\nu},\end{aligned}$$

pick any $z \in \gamma_*(\omega)$, by (6.54),

$$\text{the above} \geq \frac{\int_{\gamma_*(\omega)} \varphi(x)\rho_*(x,\omega)dm_{\gamma_*(\omega)}}{\int_{\gamma(\theta^N\omega)} \rho(x,\theta^N\omega)dm_{\gamma(\theta^N\omega)}} \cdot e^{-cd(\gamma_1(\omega),\gamma_*(\omega))^\nu} \cdot D_3^{-1} \cdot \frac{\tilde{\rho}_1(z,\omega)}{\rho_*(z,\omega)}$$

$$\geq \frac{\frac{1}{2}\|\varphi\|_{\omega,+} \int_{\gamma_*(\omega)} \rho_*(x,\omega)dm_{\gamma_*(\omega)}}{\int_{\gamma(\theta^N\omega)} \rho(x,\theta^N\omega)dm_{\gamma(\theta^N\omega)}} \cdot e^{-cd(\gamma_1(\omega),\gamma_*(\omega))^\nu} \cdot D_3^{-1} \cdot \frac{\tilde{\rho}_1(z,\omega)}{\rho_*(z,\omega)}$$

$$\geq \frac{\|\varphi\|_{\omega,+} \int_{\gamma_*(\omega)} \tilde{\rho}_1(x,\omega)dm_{\gamma_*(\omega)}}{2\int_{\gamma(\theta^N\omega)} \rho(x,\theta^N\omega)dm_{\gamma(\theta^N\omega)}} \cdot e^{-cd(\gamma_1(\omega),\gamma_*(\omega))^\nu} \cdot D_3^{-2}$$

$$= \frac{1}{2}e^{-cd(\gamma_1(\omega),\gamma_*(\omega))^\nu} \cdot D_3^{-2} \cdot \|\varphi\|_{\omega,+} \cdot \frac{\int_{\gamma_1(\omega)} \rho_1(x,\omega)dm_{\gamma_1(\omega)}}{\int_{\gamma(\theta^N\omega)} \rho(x,\theta^N\omega)dm_{\gamma(\theta^N\omega)}}$$

$$= \frac{1}{2}e^{-cd(\gamma_1(\omega),\gamma_*(\omega))^\nu} \cdot D_3^{-2} \cdot \|\varphi\|_{\omega,+} \cdot \frac{\int_{f_\omega^N\gamma_1(\omega)} (L_\omega^N 1)(x)\rho(x,\theta^N\omega)dm_{\gamma(\theta^N\omega)}}{\int_{\gamma(\theta^N\omega)} \rho(x,\theta^N\omega)dm_{\gamma(\theta^N\omega)}}$$

$$\geq \frac{1}{2}e^{-cd(\gamma_1(\omega),\gamma_*(\omega))^\nu} \cdot D_3^{-2} \cdot \|\varphi\|_{\omega,+} \cdot (K_6)^{-N} \cdot \frac{\int_{f_\omega^N\gamma_1(\omega)} \rho(x,\theta^N\omega)dm_{\gamma(\theta^N\omega)}}{\int_{\gamma(\theta^N\omega)} \rho(x,\theta^N\omega)dm_{\gamma(\theta^N\omega)}},$$

pick some $y \in f_\omega^N\gamma_1(\omega) \subset \gamma(\theta^N\omega)$, then

$$\text{the above} \geq \frac{1}{2}e^{-cd(\gamma_1(\omega),\gamma_*(\omega))^\nu} \cdot D_3^{-2} \cdot \|\varphi\|_{\omega,+} \cdot (K_6)^{-N} \cdot \frac{\int_{f_\omega^N\gamma_1(\omega)} \rho(x,\theta^N\omega)/\rho(y,\theta^N\omega)dm_{\gamma(\theta^N\omega)}}{\int_{\gamma(\theta^N\omega)} \rho(x,\theta^N\omega)/\rho(y,\theta^N\omega)dm_{\gamma(\theta^N\omega)}}$$

$$\geq \frac{1}{2}e^{-cd(\gamma_1(\omega),\gamma_*(\omega))^\nu} \cdot D_3^{-2} \cdot \|\varphi\|_{\omega,+} \cdot (K_6)^{-N} \cdot e^{-a/2(diam(\gamma(\theta^N\omega)))^\mu\cdot 2} \cdot \frac{\int_{f_\omega^N\gamma_1(\omega)} dm_{\gamma(\theta^N\omega)}}{\int_{\gamma(\theta^N\omega)} dm_{\gamma(\theta^N\omega)}}$$

$$\geq \frac{1}{2}e^{-c\epsilon^\nu - a\epsilon^\mu} \cdot D_3^{-2} \cdot \|\varphi\|_{\omega,+} \cdot (K_6)^{-N} \cdot \frac{\inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega|_{E^s(p,\omega)})^N}{2^N}$$

$$\geq \frac{1}{2^{N+1}}e^{-c\epsilon^\nu - a\epsilon^\mu} \cdot D_3^{-2} \cdot (K_6)^{-2N} \cdot \inf_{(x,\omega)\in M\times\Omega} m(D_x f_\omega|_{E^s(p,\omega)})^N \cdot \|L_\omega^N\varphi\|_{\theta^N\omega,+}$$

$$:= (D_1)^{-1}\|L_\omega^N\varphi\|_{\theta^N\omega,+}.$$

Since $\gamma(\theta^N\omega)$ and $\rho(\cdot,\theta^N\omega) \in D(a/2,\mu,\gamma(\theta^N\omega))$ are arbitrary, we have

$$\|L_\omega^N\varphi\|_{\theta^N\omega,-} \geq (D_1)^{-1}\|L_\omega^N\varphi\|_{\theta^N\omega,+}.$$

Hence (6.51) is proved. $\square$

Remark 6.7. *Note that the Lemma 6.6 is proved for all $\omega \in \Omega$, so we also have*

$$\sup\{d_{\theta^N\omega}(L_\omega^N\varphi_1, L_\omega^N\varphi_2) : \varphi_1, \varphi_2 \in C_\omega(b,c,\nu)\} \le D_2, \text{ for all } \omega \in \Omega, \tag{6.55}$$

where $d_{\theta^N\omega}$ is the Hilbert projective metric on $C_{\theta^N\omega}(b,c,\nu)$.

Lemma 6.8. *There exist a number D_4 and a number $\Lambda \in (0,1)$ both depending on D_2 and N such that for all $n \ge N$, for all $\omega \in \Omega$,*

$$d_\omega(L^n_{\theta^{-n}\omega}\varphi^1_{\theta^{-n}\omega}, L^n_{\theta^{-n}\omega}\varphi^2_{\theta^{-n}\omega}) \le D_4\Lambda^n \text{ for any } \varphi^1_{\theta^{-n}\omega}, \varphi^2_{\theta^{-n}\omega} \in C_{\theta^{-n}\omega}(b,c,\nu); \tag{6.56}$$

$$d_{\theta^n\omega}(L^n_\omega\varphi^1_\omega, L^n_\omega\varphi^2_\omega) \le D_4\Lambda^n \text{ for any } \varphi^1_\omega, \varphi^2_\omega \in C_\omega(b,c,\nu). \tag{6.57}$$

Proof. Let's define the Bowen metric on $C_\omega(b,c,\nu)$ by

$$d_{\omega,B}(\varphi_1,\varphi_2) = \max_{0\le i\le N-1} d_{\theta^i\omega}(L^i_\omega\varphi_1, L^i_\omega\varphi_2) \text{ for } \varphi_1,\varphi_2 \in C_\omega(b,c,\nu).$$

Now we have a linear operator $L^N_{\theta^{-N}\omega}$ maps cone $C_{\theta^{-N}\omega}(b,c,\nu)$ into cone $C_\omega(b,c,\nu)$ with finite diameter of $L^N_{\theta^{-N}\omega}(C_{\theta^{-N}\omega}(b,c,\nu))$ in $C_\omega(b,c,\nu)$, then we apply Birkhoff's inequality (Proposition A.4) to obtain that for all $\omega \in \Omega$,

$$d_\omega(L^N_{\theta^{-N}\omega}\varphi_1, L^N_{\theta^{-N}\omega}\varphi_2) \le \Lambda' d_{\theta^{-N}\omega}(\varphi_1,\varphi_2), \text{ for all } \varphi_1,\varphi_2 \in C_{\theta^{-N}\omega}(b,c,\nu), \tag{6.58}$$

where $\Lambda' = 1 - e^{-D_2}$. Note that $L^N_\omega(C_\omega(b,c,\nu))$ has finite diameter in $C_{\theta^N\omega}(b,c,\nu)$ for all $\omega \in \Omega$, so

$$d_{\theta^N\omega,B}(L^N_\omega\varphi_1, L^N_\omega\varphi_2) = \max_{0\le i\le N-1} d_{\theta^{i+N}\omega}(L^N_{\theta^i\omega}L^i_\omega\varphi_1, L^N_{\theta^i\omega}L^N_\omega\varphi_2) \le D_2.$$

Then (6.58) implies that for all $\omega \in \Omega$,

$$\begin{aligned} d_\omega(L^n_{\theta^{-n}\omega}\varphi^1_{\theta^{-n}\omega}, L^n_{\theta^{-n}\omega}\varphi^2_{\theta^{-n}\omega}) &\le (\Lambda')^{[\frac{n-N}{N}]} d_{\theta^{N-n}\omega,B}(L^N_{\theta^{-n}\omega}\varphi^1_{\theta^{-n}\omega}, L^N_{\theta^{-n}\omega}\varphi^2_{\theta^{-n}\omega}) \\ &\le \Lambda^{n-2N} D_2 = \frac{D_2}{\Lambda^{2N}}\Lambda^n := D_4\Lambda^n \end{aligned}$$

for all $n \ge N$ and $\varphi^1_{\theta^{-n}\omega}, \varphi^2_{\theta^{-n}\omega} \in C_{\theta^{-n}\omega}(b, c, \nu)$, where $\Lambda = (\Lambda')^{\frac{1}{N}} < 1$.

Similarly, for all $\omega \in \Omega$

$$\begin{aligned} d_{\theta^n\omega}(L^n_\omega\varphi^1_\omega, L^n_\omega\varphi^2_\omega) &\le (\Lambda')^{[\frac{n-N}{N}]} d_{\theta^N\omega,B}(L^N_\omega\varphi^1_\omega, L^N_\omega\varphi^2_\omega) \\ &\le D_4\Lambda^n \end{aligned}$$

for all $n \ge N$ and $\varphi^1_\omega, \varphi^2_\omega \in C_\omega(b, c, \nu)$. □

6.3 Construction of the Random SRB measure

In this section, we will prove that the sequence $(f^n_{\theta^{-n}\omega})_*m$ converges with respect to the narrow topology on $Pr(M)$ by using the contraction of $L^n_{\theta^{-n}\omega}$ when $n \ge N$. Moreover, we will prove that the random probability measure μ_ω defined by the weak$*$ limit of $(f^n_{\theta^{-n}\omega})_*m$ is $\phi-$invariant.

Before we introduce the next lemma, we need some preparation. Since the local stable leaves form a partition in a neighborhood of a point on each M_ω, we can divide M_ω into some rectangles foliated by local stable leaves having size between $\epsilon/4$ and $\epsilon/2$. We can realize this partition by first filling M_ω by disjoints rectangle $[W^u_{\epsilon/4}(x,\omega), W^s_{\epsilon'}(x,\omega)]$ as far as possible for any $\epsilon' < \epsilon/4$, then we attach 'crevices' to the rectangle, which 'crevices' belongs to the same local stable leaves. After attaching, we divide the rectangle together with the attachment into several parts (at most three) according to the 'edge of local stable leaves' of 'crevices'. By this method, we can partition M_ω into finite rectangles, and the local stable leaves lying in each rectangle have size between $\epsilon/4$ and $\epsilon/2$. We name this partition by

$$\mathcal{R}(\omega) = \{R_1(\omega), ..., R_i(\omega), ..., R_{k(\omega)}(\omega)\},\ k(\omega) < \infty.$$

By Proposition 4.21, for any $R_i(\omega)$, there exists a function $H_i(\omega) : R_i(\omega) \to \mathbb{R}$ with $\log H_i(\omega)$ $(a_0, \nu_0)-$Hölder continuous on each local stable leaf and for all bounded measurable functions $\psi : M \to \mathbb{R}$, we have disintegration

$$\int_{R_i(\omega)} \psi(x) dm(x) = \int \int_{\gamma^i(\omega)} \psi(x) H_i(\omega)(x)|_{\gamma^i(\omega)} dm_{\gamma^i(\omega)}(x) d\tilde{m}_{R_i(\omega)}(\gamma^i(\omega)), \tag{6.59}$$

where $\gamma^i(\omega)$ denotes the stable leaves in $R_i(\omega)$ and $\tilde{m}_{R_i(\omega)}$ the quotient measure induced by Riemannian volume measure in the space of local stable leaves in $R_i(\omega)$.

Lemma 6.9. *For any fixed $\omega \in \Omega$, given any sequence $\{\varphi_n\}_{n\in\mathbb{N}} \subset C_\omega(b, c, \nu)$ satisfying*

$$\int_M \varphi_n(x) dm(x) = 1 \text{ for all } n \in \mathbb{N},$$

and

$$d_{+,\omega}(\varphi_n, \varphi_m) \to 0 \text{ exponentially as } n, m \to \infty.$$

Then for any continuous function $\psi : M \to \mathbb{R}$, the sequence

$$\left\{ \int_M \varphi_n(x) \psi(x) dm(x) \right\}_{n\in\mathbb{N}}$$

is a Cauchy sequence.

Proof. First, we consider the case for positive continuous function $\psi : M \to \mathbb{R}$, satisfying

$$|\log \psi(\cdot)|_\mu = \sup_{x,y\in M, x\neq y} \frac{|\log \psi(x) - \log \psi(y)|}{d(x, y)^\mu} < \frac{a}{4}.$$

Now for any $\omega \in \Omega$. Let $R_i(\omega)$ and $H_i(\omega)$ be defined as above for $i \in \{1, ..., k(\omega)\}$. Note that

$\psi(\cdot)H_i(\omega)(\cdot)|_{\gamma^i(\omega)}$ is strictly positive on $\gamma^i(\omega)$. Moreover, $\log(\psi(\cdot)H_i(\omega)(\cdot))$ is $(a/2,\mu)-$Hölder continuous on $\gamma^i(\omega)$ by assumption (6.12). Therefore, by the representation of $\beta_{+,\omega}(\varphi_k,\varphi_l)$ and $\alpha_{+,\omega}(\varphi_k,\varphi_l)$ as in (6.45) and (6.44), we have

$$\frac{\int_{\gamma^i(\omega)}\varphi_k(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)}{\int_{\gamma^i(\omega)}\varphi_l(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)}\leq\beta_{+,\omega}(\varphi_k,\varphi_l), \tag{6.60}$$

$$\frac{\int_{\gamma^i(\omega)}\varphi_k(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)}{\int_{\gamma^i(\omega)}\varphi_l(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)}\geq\alpha_{+,\omega}(\varphi_k,\varphi_l) \tag{6.61}$$

for all $i\in\{1,...,k(\omega)\}$, any $\gamma^i(\omega)\subset R_i(\omega)$ and $k,l\in\mathbb{N}$. On the other hand, notice that

$$\int_M\varphi_k(x)dm(x)=\int_M\varphi_l(x)dm(x)=1,$$

so by (6.59), there exists a $\hat{i}$ and $\gamma^{\hat{i}}(\omega)\subset R_{\hat{i}}(\omega)$ such that

$$\int_{\gamma^{\hat{i}}(\omega)}\varphi_k(x)H_{\hat{i}}(\omega)(x)|_{\gamma^{\hat{i}}(\omega)}dm_{\gamma^i(\omega)}(x)\leq\int_{\gamma^{\hat{i}}(\omega)}\varphi_l(x)H_{\hat{i}}(\omega)(x)|_{\gamma^{\hat{i}}(\omega)}dm_{\gamma^i(\omega)}(x).$$

Now for any i and $\gamma^i(\omega)\subset R_i(\omega)$ stable leaf, we have

$$\begin{aligned}&\frac{\int_{\gamma^i(\omega)}\varphi_k(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)}{\int_{\gamma^i(\omega)}\varphi_l(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)}\\
&\leq\frac{\beta_{+,\omega}(\varphi_k,\varphi_l)}{\alpha_{+,\omega}(\varphi_k,\varphi_l)}\cdot\alpha_{+,\omega}(\varphi_k,\varphi_l)\\
&\leq\frac{\beta_{+,\omega}(\varphi_k,\varphi_l)}{\alpha_{+,\omega}(\varphi_k,\varphi_l)}\cdot\frac{\int_{\gamma^{\hat{i}}(\omega)}\varphi_k(x)H_{\hat{i}}(\omega)(x)|_{\gamma^{\hat{i}}(\omega)}dm_{\gamma^i(\omega)}(x)}{\int_{\gamma^{\hat{i}}(\omega)}\varphi_l(x)H_{\hat{i}}(\omega)(x)|_{\gamma^{\hat{i}}(\omega)}dm_{\gamma^i(\omega)}(x)}\\
&\leq\frac{\beta_{+,\omega}(\varphi_k,\varphi_l)}{\alpha_{+,\omega}(\varphi_k,\varphi_l)}\cdot 1\\
&=\exp(d_{+,\omega}(\varphi_k,\varphi_l)),\ \text{for all } k,l\geq 1.\end{aligned}$$

Now fix $N' > 0$ such that for any $k, l > N'$, $d_{+,\omega}(\varphi_k, \varphi_l) < \frac{1}{2}$. Then, we have

$$\begin{aligned}
&\left|\int_M \varphi_k(x)\psi(x)dm(x) - \int_M \varphi_l(x)\psi(x)dm(x)\right| \\
&= \left|\int_M \varphi_l(x)\psi(x)dm(x)\right| \cdot \left|\frac{\int_M \varphi_k(x)\psi(x)dm(x)}{\int_M \varphi_l(x)\psi(x)dm(x)} - 1\right| \\
&\leq \sup_{x\in M} |\psi(x)| \cdot \left|\frac{\sum_{i=1}^{k(\omega)} \int_{R_i(\omega)} \varphi_k(x)\psi(x)dm(x)}{\sum_{i=1}^{k(\omega)} \int_{R_i(\omega)} \varphi_l(x)\psi(x)dm(x)} - 1\right| \\
&= \|\psi\|_{C^0(M)} \cdot \left|\frac{\sum_{i=1}^{k(\omega)} \int\int_{\gamma^i(\omega)} \varphi_k(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)d\tilde{m}_{R^i(\omega)}}{\sum_{i=1}^{k(\omega)} \int\int_{\gamma^i(\omega)} \varphi_l(x)\psi(x)H_i(\omega)(x)|_{\gamma^i(\omega)}dm_{\gamma^i(\omega)}(x)d\tilde{m}_{R^i(\omega)}} - 1\right| \\
&\leq \|\psi\|_{C^0(M)} \cdot (e^{d_{+,\omega}(\varphi_k,\varphi_l)} - 1) \\
&\leq 2\|\psi\|_{C^0(M)} \cdot d_{+,\omega}(\varphi_k, \varphi_l). \qquad (6.62)
\end{aligned}$$

Hence $\{\int_M \varphi_n(x)\psi(x)dm(x)\}_{n\in\mathbb{N}}$ is Cauchy sequence in this case.

Next for any $\psi \in C^{0,\mu}(M)$, let

$$B = \frac{5|\psi(\cdot)|_\mu}{a}.$$

We define

$$\psi_B^+(\cdot) := \frac{1}{2}(|\psi(\cdot)| + \psi(\cdot)) + B, \ \psi_B^-(\cdot) := \frac{1}{2}(|\psi(\cdot)| - \psi(\cdot)) + B.$$

It's easy to see that

$$|\log\psi_B^\pm(\cdot)|_\mu = \sup_{x,y\in M, x\neq y} \frac{|\log\psi_B^\pm(x) - \log\psi_B^\pm(y)|}{d(x,y)^\nu} < \frac{a}{4}.$$

Then we apply (6.62) and linearity of integration to get

$$\left|\int_M \varphi_k(x)\psi(x)dm(x) - \int_M \varphi_l(x)\psi(x)dm(x)\right| \leq \max\left\{4, \frac{20}{a}\right\} \|\psi\|_{C^{0,\mu}(M)} \cdot d_{+,\omega}(\varphi_k, \varphi_l).$$

Finally, for any continuous function $\psi : M \to \mathbb{R}$, for any $\epsilon > 0$, we can pick a function

$\tilde{\psi} \in C^{0,\mu}(M)$ such that

$$\sup_{x \in M} |\psi(x) - \tilde{\psi}(x)| < \epsilon/4.$$

Now, pick $N' > 0$ depending on $\tilde{\psi}$ and ϵ such that for all $k, l \geq N'$

$$\max\left\{4, \frac{20}{a}\right\} \cdot \|\tilde{\psi}\|_{C^{0,\mu}(M)} \cdot d_{+,\omega}(\varphi_k, \varphi_l) < \epsilon/2.$$

Then

$$\begin{aligned}
&\left|\int_M \varphi_k(x)\psi(x)dm(x) - \int_M \varphi_l(x)\psi(x)dm(x)\right| \\
\leq &\left|\int_M \varphi_k(x)\tilde{\psi}(x)dm(x) - \int_M \varphi_l(x)\tilde{\psi}(x)dm(x)\right| + \epsilon/4 + \epsilon/4 \\
\leq &\max\left\{4, \frac{20}{a}\right\} \cdot \|\tilde{\psi}(\cdot)\|_{C^{0,\mu}(M)} \cdot d_{+,\omega}(\varphi_k, \varphi_l) + \epsilon/2 \\
\leq &\,\epsilon.
\end{aligned}$$

Hence, for any continuous function $\psi : M \to \mathbb{R}$, the sequence $\{\int_M \varphi_n(x)\psi(x)dm(x)\}_{n\in\mathbb{N}}$ is Cauchy sequence. □

For any measurable function $\varphi : M \to \mathbb{R}$, we define the fiber Koopman operator

$$U_\omega\varphi : M \to \mathbb{R},\ (U_\omega\varphi)(x) := \varphi(f_{\theta^{-1}\omega}x). \tag{6.63}$$

We denote

$$U^n_\omega := U_{\theta^{-(n-1)}\omega} \circ \cdots \circ U_{\theta^{-1}\omega} \circ U_\omega \text{ for all } n \in \mathbb{N} \text{ and } \omega \in \Omega.$$

For each fixed $\omega \in \Omega$, by changing variable, for measurable functions φ_1, φ_2, we have

$$\int_M (L_{\theta^{-1}\omega}\varphi_1)(y)\varphi_2(y)dm(y) = \int_M \frac{\varphi_1((f_{\theta^{-1}\omega})^{-1}y)}{|\det D_{(f_{\theta^{-1}\omega})^{-1}(y)} f_{\theta^{-1}\omega}|}\varphi_2(y)dm(y)$$
$$= \int_M \frac{\varphi_1(x)}{|\det D_x f_{\theta^{-1}\omega}|}\varphi_2(f_{\theta^{-1}\omega}x)|\det D_x f_{\theta^{-1}\omega}|dm(x)$$
$$= \int_M \varphi_1(x)(U_\omega\varphi_2)(x)dm(x). \tag{6.64}$$

Let 1 be the constant function $1(x) \equiv 1$, then $1 \in \cap_\omega C_\omega(b, c, \nu)$ by the Remark 6.3. Now consider $\varphi_n(x) = (L^n_{\theta^{-n}\omega}1)(x)$ for $n \geq N$ and notice that for all $\omega \in \Omega$,

$$\int_M (L^n_{\theta^{-n}\omega}1)(x)dm(x) = \int_M 1(x)(U^n_\omega 1)(x)dm(x) = \int_M 1dm(x) = 1.$$

Moreover, by (6.56), we have

$$d_{+,\omega}(L^n_{\theta^{-n}\omega}1, L^{n+k}_{\theta^{-(n+k)}\omega}1) \leq d(L^n_{\theta^{-n}\omega}1, L^n_{\theta^{-n}\omega}(L^k_{\theta^{-(n+k)}\omega}1)) \leq \Lambda^n \cdot D_4 \text{ for } n \geq N. \tag{6.65}$$

Hence the sequence $\varphi_n = L^n_{\theta^{-n}\omega}1 \subset C_\omega(b, c, \nu)$ satisfies the condition of Lemma 6.9. So for any $g \in C^0(M)$, $\{\int_M (L^n_{\theta^{-n}\omega}1)(x)g(x)dm\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Now define $\mathcal{F}_\omega : C(M) \to \mathbb{R}$ by $\mathcal{F}_\omega(g) = \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega}1)(x)g(x)dm(x)$. Then $\mathcal{F}_\omega$ is obviously a positive linear functional on $C(M)$. By Riesz representation theorem, there exists a regular Borel measure μ_ω such that

$$\int_M g(x)d\mu_\omega(x) = \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega}1)(x)g(x)dm(x). \tag{6.66}$$

Moreover, μ_ω is a probability measure since $\int_M (L^n 1)(x, \omega)dm(x) = 1$.

Note that for each $g \in C(M)$, $\omega \mapsto \int_M g(x)d\mu_\omega(x)$ is measurable because of the measurability of $\omega \mapsto \int_M (L^n_{\theta^{-n}\omega}1)(x)g(x)dm(x)$. For any closed set $B \subset M$, let $g_k(x) := 1 - \min\{kd(x, B), 1\}$ for $k \in \mathbb{N}$ where $d(x, B) := \inf\{d(x, y) : y \in B\}$, then $g_k(x) \in C^0(M)$

and $g_k(x) \searrow 1_B(x)$. Then by Monotone convergence theorem, we have

$$\mu_\omega(B) = \lim_{k\to\infty} \int_M g_k(x) d\mu_\omega(x) = \lim_{k\to\infty} \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega} 1)(x,\omega) g_k(x) dm(x).$$

Hence $\omega \mapsto \mu_\omega(B)$ is measurable for all closed sets $B \subset M$. By the Definition in Section 2.3, $\omega \mapsto \mu_\omega$ defines a random probability measure.

Notice that for any $g \in C^0(M)$, we have

$$\int_M g(x) d\mu_\omega(x) = \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega} 1)(x) g(x) dm(x) = \lim_{n\to\infty} \int_M g(f^n_{\theta^{-n}\omega} x) dm(x)$$
$$= \lim_{n\to\infty} \int_M g(y) d(f^n_{\theta^{-n}\omega})_* m(y).$$

So μ_ω is actually the weak$^*-$limit of $(f^n_{\theta^{-n}\omega})_* m$.

Now for any continuous $g : M \to \mathbb{R}$, by (6.64),

$$\int_M g(f_\omega x) d\mu_\omega = \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega} 1)(x) g(f_\omega x) dm(x) = \lim_{n\to\infty} \int_M g(f_\omega f^n_{\theta^{-n}\omega} x) dm(x)$$
$$= \lim_{n\to\infty} \int_M g(f^{n+1}_{\theta^{-(n+1)}\theta\omega} x) dm(x) = \lim_{n\to\infty} \int_M (L^{n+1}_{\theta^{-(n+1)}\theta\omega} 1)(x) g(x) dm(x)$$
$$= \int_M g(x) d\mu_{\theta\omega}.$$

Thus the random probability measure μ_ω is $\phi-$invariant.

Remark 6.10. *Note that for each fixed $\omega \in \Omega$, for any $\varphi_{\theta^{-k}\omega} \in C_{\theta^{-k}\omega}(b, c, \nu)$ such that $\int_M \varphi_{\theta^{-k}\omega}(x) dm = 1$ for all $k \in \mathbb{N}$, then we have*

$$\lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega} \varphi_{\theta^{-n}\omega})(x) g(x) dm(x) = \int_M g(x) d\mu_\omega(x), \tag{6.67}$$

for any $g \in C^0(M)$. In fact, we define the sequence $\hat{\varphi}_n$ by $\hat{\varphi}_{2k} = L^k_{\theta^{-k}\omega} 1$, $\hat{\varphi}_{2k+1} = L^k_{\theta^{-k}\omega} \varphi_{\theta^{-k}\omega}$

for all $k \geq N$, then by noticing

$$d_{+,\omega}(\hat{\varphi}_{2k}, \hat{\varphi}_{2k+1}) \leq d(L^k_{\theta^{-k}\omega}1, L^k_{\theta^{-k}\omega}\varphi_{\theta^{-k}\omega}) \leq \Lambda^k D_4$$

and by (6.64)

$$\int_M (L^k_{\theta^{-k}\omega}\varphi_{\theta^{-k}\omega})(x)dm(x) = \int_M \varphi_{\theta^{-k}\omega}(x)(U^k_\omega 1)(x)dm(x) = 1.$$

So $\hat{\varphi}_n \subset C_\omega(b, c, \nu)$ satisfying the condition of Lemma 6.9. Thus the sequence $\{\int_M \hat{\varphi}_n g dm\}$ is Cauchy sequence for all $g \in C^0(M)$. As a consequence, we have

$$\begin{aligned}\int_M g(x)d\mu_\omega(x) &= \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega}1)(x)g(x)dm(x) \\ &= \lim_{n\to\infty} \int_M (L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)g(x)dm(x). \qquad (6.68)\end{aligned}$$

6.4 Proof of The Exponential Decay of the Past Random Correlations

In this section, we prove the exponential decay of the past random correlations.

Lemma 6.11. *Let $\psi : M \to \mathbb{R}$ be a positive function such that $\log \psi$ is $(\frac{a}{4}, \mu)$ Hölder continuous. Then for each fixed $\omega \in \Omega$, let $\varphi_{\theta^{-k}\omega} \in C_{\theta^{-k}\omega}(b, c, \nu)$ for $k \in \mathbb{N}$, for any $n \geq N$, the following holds:*

$$\begin{aligned}&\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi_{\theta^{-n}\omega}(x)dm(x) - \int_M \psi(x)d\mu_\omega(x)\int_M \varphi_{\theta^{-n}\omega}(x)dm(x)\right| \\ &\leq K(D_4)\sup_{x\in M}|\psi(x)|\int_M \varphi_{\theta^{-n}\omega}(x)dm(x)\Lambda^n, \qquad (6.69)\end{aligned}$$

where $K(D_4)$ is a constant only depending on D_4. Recall that D_4 from (6.56).

Proof. We first prove the case that $\varphi_{\theta^{-k}\omega} \in C_{\theta^{-k}\omega}(b, c, \nu)$ and $\int_M \varphi_{\theta^{-k}\omega}(x)dm(x) = 1$ for all

$k \in \mathbb{N}$. Similarly as (6.62), for any $n \geq N,\ k \geq 0$, we have

$$
\begin{aligned}
&\left|\int_M \psi(x)(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)dm(x) - \int_M \psi(x)(L^{n+k}_{\theta^{-(n+k)}\omega}\varphi_{\theta^{-(n+k)}\omega})(x)dm(x)\right| \\
&\leq \|\psi\|_{C^0(M)}(e^{d_{+,\omega}(L^n_{\theta^{-n}\omega}\varphi, L^{n+k}_{\theta^{-(n+k)}\omega}\varphi_{\theta^{-(n+k)}\omega})} - 1) \\
&\leq \|\psi\|_{C^0(M)}(e^{d_{+,\omega}(L^n_{\theta^{-n}\omega}\varphi, L^n_{\theta^{-n}\omega}L^k_{\theta^{-(n+k)}\omega}\varphi_{\theta^{-(n+k)}\omega})} - 1) \\
&\leq \|\psi\|_{C^0(M)}(e^{D_4\Lambda^n} - 1) \\
&\leq K(D_4)\|\psi\|_{C^0(M)}\Lambda^n,
\end{aligned}
$$

where $K(D_4)$ is a constant only depending on D_4. Let $k \to \infty$, by Remark 6.10, we have

$$
\left|\int_M \psi(x)(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)dm(x) - \int_M \psi(x)d\mu_\omega(x)\right| \leq K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n. \tag{6.70}
$$

Note that by (6.64), we have

$$
\int_M \psi(x)(L^n_{\theta^{-n}\omega}\varphi_{\theta^{-n}\omega})(x)dm(x) = \int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi_{\theta^{-n}\omega}(x)dm(x).
$$

Hence (6.70) becomes

$$
\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi_{\theta^{-n}\omega}(x)dm(x) - \int_M \psi(x)d\mu_\omega(x)\right| \leq K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n. \tag{6.71}
$$

Now for any $\varphi_{\theta^{-n}\omega} \in C_{\theta^{-n}\omega}(b,c,\nu)$, let $\tilde{\varphi}_{\theta^{-n}\omega}(x) := \varphi_{\theta^{-n}\omega}(x)/\int_M \varphi_{\theta^{-n}\omega}(x)dm(x)$, then (6.69) is proved by replacing $\varphi_{\theta^{-n}\omega}$ by $\tilde{\varphi}_{\theta^{-n}\omega}$ in (6.71). □

We still need the following lemma:

Lemma 6.12. *Pick the number c in the definition of $C_\omega(b,c,\nu)$ satisfying*

$$
2(2a_0 + \frac{K_2}{1-e^{-\lambda}} + \frac{2C_2}{1-e^{-\lambda\nu_0}}) < c \tag{6.72}
$$

where K_2 *is defined in* (6.2), C_2 *is defined in Lemma 4.8. Let* c_1 *be a constant such that*

$$1 < c_1 < 2a_0 + \frac{K_2}{1-e^{-\lambda}} + \frac{2C_2}{1-e^{-\lambda\nu_0}}. \tag{6.73}$$

Given any positive continuous function $\varphi : M \to \mathbb{R}^+$ *with*

$$\sup_{x,y\in M,\ x\neq y} \frac{|\log\varphi(x) - \log\varphi(y)|}{d(x,y)^\nu} < c_1,$$

then $\varphi(L^l_{\theta^{-l}\omega}1) \in C_\omega(b,c,\nu)$ *for every* $l \geq 1$ *and all* $\omega \in \Omega$.

Proof. We prove this lemma for each fixed $\omega \in \Omega$. Let $l \geq 1$ be fixed. $\varphi\cdot(L^l_{\theta^{-l}\omega}1)$ is obviously bounded and measurable function.

For every random local stable manifold $\gamma(\omega)$ and $\rho(\cdot,\omega) \in D(a/2,\mu,\gamma(\omega))$, we have

$$\int_{\gamma(\omega)} \varphi(x)(L^l_{\theta^{-l}\omega}1)(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) \geq \inf\varphi \cdot \int_{\gamma(\omega)} (L^l_{\theta^{-l}\omega}1)(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) > 0$$

since $L^l_{\theta^{-l}\omega}1 \in C_\omega(b,c,\nu)$ and φ is positive.

By Remark 6.3, $\varphi \cdot L^l_{\theta^{-l}\omega}1$ fulfills $(C2)$ since $\varphi \cdot L^l_{\theta^{-l}\omega}1$ is nonnegative. So it is left to verify $(C3)$.

Let $\gamma(\omega), \tilde{\gamma}(\omega)$ be any pair of local stable manifolds such that $\tilde{\gamma}(\omega)$ is the holonomy image of $\gamma(\omega)$. Let $\rho(\cdot,\omega) \in D(a_1,\mu_1,\gamma(\omega))$ and $\tilde{\rho}(\cdot,\omega) \in D(a/2,\mu,\tilde{\gamma}(\omega))$ which is defined as (6.10) corresponds to $\rho(\cdot,\omega)$. We subdivide $f^{-l}_\omega\gamma(\omega)$ into $\gamma_i(\theta^{-l}\omega)$ such that $\gamma_i(\theta^{-l}\omega)$ are local stable manifolds having size between $\epsilon/4$ and $\epsilon/2$. Let $\tilde{\gamma}_i(\theta^{-l}\omega)$ be the holonomy image of $\gamma_i(\theta^{-l}\omega)$ which lies in $f^{-l}_\omega\tilde{\gamma}(\omega)$. Denote $\psi_\omega : \tilde{\gamma}(\omega) \to \gamma(\omega)$ to be the holonomy map between $\tilde{\gamma}(\omega)$ and $\gamma(\omega)$, and $\psi^i_{\theta^{-l}\omega} : \tilde{\gamma}_i(\theta^{-l}\omega) \to \gamma_i(\theta^{-l}\omega)$ the holonomy map between $\tilde{\gamma}_i(\theta^{-l}\omega)$ and $\gamma_i(\theta^{-l}\omega)$.

By using the definition of L_ω, we have

$$\begin{aligned}
&\int_{\gamma(\omega)} (L^l_{\theta^{-l}\omega}1)(x)\varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) \\
&= \sum_i \int_{\gamma_i(\theta^{-l}\omega)} \frac{|\det D_x f^l_{\theta^{-l}\omega}|_{E^s(x,\theta^{-l}\omega)}|}{|\det D_x f^l_{\theta^{-l}\omega}|}\rho(f^l_{\theta^{-l}\omega}x,\omega)\varphi(f^l_{\theta^{-l}\omega}x)dm_{\gamma_i(\theta^{-l}\omega)}(x) \\
&= \sum_i \int_{\tilde{\gamma}_i(\theta^{-l}\omega)} \frac{|\det D_{\psi^i_{\theta^{-l}\omega}(x)} f^l_{\theta^{-l}\omega}|_{E^s(\psi^i_{\theta^{-l}\omega}(x),\theta^{-l}\omega)}|}{|\det D_{\psi^i_{\theta^{-l}\omega}(x)} f^l_{\theta^{-l}\omega}|} \cdot \rho(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x),\omega)\cdot\varphi(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x)) \\
&\qquad \cdot |\det D\psi^i_{\theta^{-l}\omega}(x)|dm_{\tilde{\gamma}_i(\theta^{-l}\omega)}(x).
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
&\int_{\tilde{\gamma}(\omega)} (L^l_{\theta^{-l}\omega}1)(x)\varphi(x)\tilde{\rho}(x,\omega)dm_{\tilde{\gamma}(\omega)}(x) \\
&= \sum_i \int_{\tilde{\gamma}_i(\theta^{-l}\omega)} \frac{|\det D_x f^l_{\theta^{-l}\omega}|_{E^s(x,\theta^{-l}\omega)}|}{|\det D_x f^l_{\theta^{-l}\omega}|}\tilde{\rho}(f^l_{\theta^{-l}\omega}x,\omega)\varphi(f^l_{\theta^{-l}\omega}x)dm_{\tilde{\gamma}_i(\theta^{-l}\omega)}(x) \\
&= \sum_i \int_{\tilde{\gamma}_i(\theta^{-l}\omega)} \frac{|\det D_x f^l_{\theta^{-l}\omega}|_{E^s(x,\theta^{-l}\omega)}|}{|\det D_x f^l_{\theta^{-l}\omega}|}\rho(\psi_\omega(f^l_{\theta^{-l}\omega}x),\omega)\varphi(f^l_{\theta^{-l}\omega}x)|\det D\psi_\omega(f^l_{\theta^{-l}\omega}x)|dm_{\tilde{\gamma}_i(\theta^{-l}\omega)}.
\end{aligned}$$

Note that $f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x) = \psi_\omega(f^l_{\theta^{-l}\omega}\omega)$ by the invariance of stable and unstable manifolds, so

$$\rho(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x),\omega) = \rho(\psi_\omega(f^l_{\theta^{-l}\omega}\omega),\omega). \tag{6.74}$$

Since $\log\varphi(\cdot)$ is $(c_1,\nu)-$Hölder continuous on local unstable manifolds,

$$\begin{aligned}
\left|\log\varphi(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x)) - \log\varphi(f^l_{\theta^{-l}\omega}x)\right| &\le c_1 d(f^l_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x), f^l_{\theta^{-l}\omega}(x))^\nu \\
&\le c_1 d(\gamma(\omega),\tilde{\gamma}(\omega))^\nu. \qquad (6.75)
\end{aligned}$$

By Lemma 4.20,

$$
\begin{aligned}
&\Big|\log|\det D\psi_\omega(f^l_{\theta^{-l}\omega}x)| - \log|\det D\psi^i_{\theta^{-l}\omega}(x)|\Big| \\
&\le a_0 d(f^l_{\theta^{-l}\omega}x, \psi_\omega f^l_{\theta^{-l}\omega}x)^{\nu_0} + a_0 d(x, \psi^i_{\theta^{-l}\omega}x)^{\nu_0} \\
&\le a_0 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0} + a_0 d(\gamma_i(\theta^{-l}\omega), \tilde{\gamma}_i(\theta^{-l}\omega))^{\nu_0} \\
&< 2a_0 d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0}. \qquad (6.76)
\end{aligned}
$$

By (6.2), for $x \in \tilde{\gamma}_i(\theta^{-l}\omega)$, we deduce that

$$
\begin{aligned}
&|\log|\det D_{\psi^i_{\theta^{-l}\omega}(x)} f^l_{\theta^{-l}\omega}| - \log|\det D_x f^l_{\theta^{-l}\omega}|| \\
&\le K_2 d(x, \psi^i_{\theta^{-l}\omega}(x)) + K_2 d(f_{\theta^{-l}\omega}x, f_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x)) + \cdots + K_2 d(f^{l-1}_{\theta^{-l}\omega}x, f^{l-1}_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x)) \\
&\le K_2 e^{-l\lambda} d(\gamma(\omega), \tilde{\gamma}(\omega)) + K_2 e^{-(l-1)\lambda} d(\gamma(\omega), \tilde{\gamma}(\omega)) + \cdots K_2 d(\gamma(\omega), \tilde{\gamma}(\omega)) \\
&\le K_2/(1-e^{-\lambda}) \cdot d(\gamma(\omega), \tilde{\gamma}(\omega)). \qquad (6.77)
\end{aligned}
$$

By applying (6.31),

$$
\begin{aligned}
&\Big|\log|\det D_{\psi^i_{\theta^{-l}\omega}(x)} f^l_{\theta^{-l}\omega}|_{E^s(\psi^i_{\theta^{-l}\omega}(x), \theta^{-l}\omega)}| - \log|\det D_x f^l_{\theta^{-l}\omega}|_{E^s(x,\theta^{-l}\omega)}|\Big| \\
&\le 2C_2 d(\psi^i_{\theta^{-l}\omega}(x), x)^{\nu_0} + \cdots + 2C_2 d(f^{l-1}_{\theta^{-l}\omega}\psi^i_{\theta^{-l}\omega}(x), f^{l-1}_{\theta^{-l}\omega}x)^{\nu_0} \\
&\le 2C_2/(1-e^{-\lambda\nu_0}) d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu_0}. \qquad (6.78)
\end{aligned}
$$

Combing (6.74), (6.75), (6.76), (6.77) and (6.78), we conclude

$$
\begin{aligned}
&\left|\log\int_{\gamma(\omega)} (L^l_{\theta^{-l}\omega}1)(x)\varphi(x)\rho(x,\omega)dm_{\gamma(\omega)}(x) - \log\int_{\tilde{\gamma}(\omega)} (L^l_{\theta^{-l}\omega}1)(x)\varphi(x)\tilde{\rho}(x,\omega)dm_{\tilde{\gamma}(\omega)}(x)\right| \\
&\le (c_1 + 2a_0 + \frac{K_2}{1-e^{-\lambda}} + \frac{2C_2}{1-e^{-\lambda\nu_0}}) d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu} \\
&\le c d(\gamma(\omega), \tilde{\gamma}(\omega))^{\nu}.
\end{aligned}
$$

Hence $(C3)$ is verified. □

Now we assume continuous function $\psi \in C^0(M)$ such that $\psi > 0$, $\log \psi(\cdot)$ is $(a/4, \mu)-$Hölder continuous. Let positive continuous function $\varphi : M \to \mathbb{R}$ with $\log \varphi$ is $(c_1, \nu)-$Hölder continuous. Then by the Lemma 6.12, for each $n \in \mathbb{N}$

$$\varphi \cdot L^l_{\theta^{-(l+n)}\omega} 1 = \varphi \cdot L^l_{\theta^{-l}\theta^{-n}\omega} 1 \in C_{\theta^{-n}\omega}(b, c, \nu) \text{ for all } l \in \mathbb{N} \text{ all } \omega \in \Omega.$$

Now, we apply Lemma 6.11 to obtain that for all $\omega \in \Omega$,

$$\begin{aligned}
&\left| \int_M \psi(f^n_{\theta^{-n}\omega} x)(\varphi \cdot L^l_{\theta^{-l}\theta^{-n}\omega} 1)(x) dm(x) - \int_M \psi(x) d\mu_\omega \int_M (\varphi \cdot L^l_{\theta^{-l}\theta^{-n}\omega} 1)(x) dm(x) \right| \\
&\le K(D_4) D_4 \sup_{x \in M} |\psi(x)| \int_M (\varphi \cdot (L^l_{\theta^{-l}\theta^{-n}\omega} 1))(x) dm \cdot \Lambda^n, \text{ for all } l \in \mathbb{N}. \qquad (6.79)
\end{aligned}$$

Let $l \to \infty$, by (6.66), for all $\omega \in \Omega$,

$$\begin{aligned}
&\left| \int_M \psi(f^n_{\theta^{-n}\omega} x) \varphi(x) d\mu_{\theta^{-n}\omega} - \int_M \psi(x) d\mu_\omega(x) \int_M \varphi(x) d\mu_{\theta^{-n}\omega} \right| \\
&\le K(D_4) \|\psi\|_{C^0(M)} \int_M \varphi(x) d\mu_{\theta^{-n}\omega} \cdot \Lambda^n \\
&\le K(D_4) \|\psi\|_{C^0(M)} \cdot \|\varphi\|_{C^0(M)} \cdot \Lambda^n. \qquad (6.80)
\end{aligned}$$

Finally, given $\psi \in C^{0,\mu}(M)$ and $\varphi \in C^{0,\nu}(M)$, let

$$B_\psi = \frac{5|\psi(\cdot)|_\mu}{a}, \; B_\varphi = \frac{2|\varphi(\cdot)|_\nu}{c_1},$$

and define

$$\begin{aligned}
\psi^+_{B_\psi} \left(\cdot = \frac{1}{2} (|\psi(\cdot)| + \psi(\cdot) \right) + B_\psi, \; \psi^-_{B_\psi}(\cdot) = \frac{1}{2} (|\psi(\cdot)| - \psi(\cdot)) + B_\psi, \\
\varphi^+_{B_\varphi}(\cdot) = \frac{1}{2} (|\varphi(\cdot)| + \varphi(\cdot)) + B_\varphi, \; \varphi^-_{B_\varphi}(\cdot) = \frac{1}{2} (|\varphi(\cdot)| - \varphi(\cdot)) + B_\varphi.
\end{aligned}$$

Then $\log \psi^\pm_{B_\psi}(\cdot)$ are $(a/4, \mu)-$Hölder continuous, and $\log \varphi^\pm_{B_\varphi}(\cdot)$ are $(c_1, \nu)-$Hölder continuous. By (6.80) and the linearity of integration, we conclude

$$\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega}(x) - \int_M \psi(x)d\mu_\omega(x)\int_M \varphi(x)d\mu_{\theta^{-n}\omega}(x)\right|$$
$$\leq 4K(D_4)\cdot \max\left\{1,\frac{5}{a}\right\}\cdot \max\left\{1,\frac{2}{c_1}\right\}\cdot \|\psi\|_{C^{0,\mu}(M)}\cdot \|\varphi\|_{C^{0,\nu}(M)}\cdot \Lambda^n.$$

Note that the above is true for all $n \geq N$. Next, we let

$$K := \max\left\{4K(D_4)\cdot \max\left\{1,\frac{5}{a}\right\}\cdot \max\left\{1,\frac{2}{c_1}\right\}, 2\Lambda^{-N}\right\}, \tag{6.81}$$

then

$$\left|\int_M \psi(f^n_{\theta^{-n}\omega}x)\varphi(x)d\mu_{\theta^{-n}\omega} - \int_M \psi(x)d\mu_\omega(x)\int_M \varphi(x)d\mu_{\theta^{-n}\omega}\right| \leq K\|\psi\|_{C^{0,\mu}(M)}\cdot \|\varphi\|_{C^{0,\nu}(M)}\cdot \Lambda^n$$

for all $n \geq 0$.

This finishes the proof for the past random correlations.

6.5 Proof of The Exponential Decay of the Future Random Correlation

In this subsection, we prove the exponential decay of the future random correlations.

Lemma 6.13. *Let positive continuous function $\psi : M \to \mathbb{R}$ satisfy that $\log\psi$ is $(\frac{a}{4},\mu)-$Hölder continuous. Then for each fixed $\omega \in \Omega$, $\varphi_\omega \in C_\omega(b,c,\nu)$, for any $n \geq N$, the following holds:*

$$\left|\int_M \psi(f^n_\omega x)\varphi_\omega(x)dm - \int_M \psi(x)d\mu_{\theta^n\omega}\int_M \varphi_\omega(x)dm\right| \leq K(D_4)\cdot \|\psi\|_{C^0(M)}\int_M \varphi_\omega(x)dm\cdot \Lambda^n. \tag{6.82}$$

Recall that $K(D_4)$ is defined in Lemma 6.11.

Proof. We first prove the case that $\int_M \varphi_\omega(x)dm(x) = 1$. Note that for any $n \geq N$, $k \geq 0$, $L^n_\omega\varphi_\omega$, $L^{n+k}_{\theta^{-k}\omega}1 \in C_{\theta^n\omega}(b,c,\nu)$. Similar proof as (6.62) can be applied on the fiber $\{\theta^n\omega\}$ to show that

$$\begin{aligned}
&\left|\int_M \psi(x)(L_\omega^n\varphi_\omega)(x)dm(x) - \int_M \psi(x)(L_{\theta^{-k}\omega}^{n+k}1)(x)dm(x)\right| \\
&\le \|\psi\|_{C^0(M)}(e^{d_{+,\theta^n\omega}(L_\omega^n\varphi_\omega, L_{\theta^{-k}\omega}^{n+k}1)} - 1) \\
&\le \|\psi\|_{C^0(M)}(e^{d_{+,\theta^n\omega}(L_\omega^n\varphi_\omega, L_\omega^n L_{\theta^{-k}\omega}^{k}1)} - 1) \\
&\le \|\psi\|_{C^0(M)}(e^{D_4\Lambda^n} - 1) \\
&\le K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n,
\end{aligned}$$

where we apply (6.57) in the last inequality. Notice that $L_{\theta^{-k}\omega}^{n+k}1 = L_{\theta^{-k}\theta^n\omega}^{k}L_{\theta^{-k}\omega}^{n}1$, $L_{\theta^{-k}\omega}^{n}1 \in C_{\theta^{n-k}\omega}(b,c,\nu) = C_{\theta^{-k}\theta^n\omega}(b,c,\nu)$ and $\int_M L_{\theta^{-k}\omega}^{n}1dm = 1$ for all $k\in\mathbb{N}$. Let $k\to\infty$, by Remark 6.10, we have

$$\left|\int_M \psi(x)(L_\omega^n\varphi_\omega)(x)dm(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\right| \le K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n. \tag{6.83}$$

Note that by (6.64),

$$\int_M \psi(x)(L_\omega^n\varphi_\omega)(x)dm(x) = \int_M \psi(f_\omega^n x)\varphi_\omega(x)dm(x).$$

Hence (6.70) becomes

$$\left|\int_M \psi(f_\omega^n x)\varphi_\omega(x)dm(x) - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\right| \le K(D_4)\cdot\|\psi\|_{C^0(M)}\Lambda^n. \tag{6.84}$$

Now for any $\varphi_\omega\in C_\omega(b,c,\nu)$, let $\tilde\varphi_\omega(x) := \varphi_\omega(x)/\int_M\varphi_\omega(x)dm(x)$, then (6.82) is proved by replacing φ_ω by $\tilde\varphi_\omega$ in (6.84). □

Now assume function $\psi: M\to\mathbb{R}$ such that $\psi>0$, $\log\psi(\cdot)$ is $(a/4,\mu)-$Hölder continuous. Let $\varphi: M\to\mathbb{R}$ with $\log\varphi$ is $(c_1,\nu)-$Hölder continuous. Then by the Lemma 6.12, we have

$$\varphi\cdot L_{\theta^{-l}\omega}^{l}1 \in C_\omega(b,c,\nu) \text{ for all } l\in\mathbb{N} \text{ all } \omega\in\Omega.$$

Now, we apply Lemma 6.13 to obtain that for all $\omega \in \Omega$,

$$\left|\int_M \psi(f_\omega^n x)(\varphi \cdot L_{\theta^{-l}\omega}^l 1)(x)dm(x) - \int_M \psi(x)d\mu_{\theta^n\omega}\int_M (\varphi \cdot L_{\theta^{-l}\omega}^l 1)(x)dm(x)\right|$$
$$\leq K(D_4)\|\psi\|_{C^0(M)}\int_M (\varphi \cdot (L_{\theta^{-l}\omega}^l 1))(x)dm \cdot \Lambda^n, \text{ for all } l \in \mathbb{N}.$$

Let $l \to \infty$, by (6.66), for all $\omega \in \Omega$,

$$\begin{aligned}&\left|\int_M \psi(f_\omega^n x)\varphi(x)d\mu_\omega - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\int_M \varphi(x)d\mu_\omega\right|\\ &\leq K(D_4)\|\psi\|_{C^0(M)}\int_M \varphi(x)d\mu_\omega \cdot \Lambda^n\\ &\leq K(D_4)\|\psi\|_{C^0(M)} \cdot \|\varphi\|_{C^0(M)} \cdot \Lambda^n. \qquad (6.85)\end{aligned}$$

Finally, given $\psi \in C^{0,\mu}(M)$ and $\varphi \in C^{0,\nu}(M)$, let

$$B_\psi = \frac{5|\psi(\cdot)|_\mu}{a},\ B_\varphi = \frac{2|\varphi(\cdot)|_\nu}{c_1},$$

and define

$$\psi_{B_\psi}^+\left(\cdot = \frac{1}{2}\left(|\psi(\cdot)| + \psi(\cdot)\right)\right) + B_\psi,\ \psi_{B_\psi}^-(\cdot) = \frac{1}{2}\left(|\psi(\cdot)| - \psi(\cdot)\right) + B_\psi,$$
$$\varphi_{B_\varphi}^+(\cdot) = \frac{1}{2}\left(|\varphi(\cdot)| + \varphi(\cdot)\right) + B_\varphi,\ \varphi_{B_\varphi}^-(\cdot) = \frac{1}{2}\left(|\varphi(\cdot)| - \varphi(\cdot)\right) + B_\varphi.$$

Then $\log \psi_{B_\psi}^\pm(\cdot)$ are $(a/4, \mu)-$Hölder continuous, and $\log \varphi_{B_\varphi}^\pm(\cdot)$ are (c_1, ν) Hölder continuous. By (6.85) and the linearity of integration, we conclude

$$\left|\int_M \psi(f_\omega^n x)\varphi(x)d\mu_\omega - \int_M \psi(x)d\mu_{\theta^n\omega}(x)\int_M \varphi(x)d\mu_\omega\right| \leq K\|\psi\|_{C^{0,\mu}(M)} \cdot \|\varphi\|_{C^{0,\nu}(M)} \cdot \Lambda^n.$$

for all $n \geq 0$. Recall that K is defined in (6.81).

This finishes the proof for the future random correlations. The proof of Theorem 3.10 is done.

Chapter 7. Existence of the random Gibbs $u-$state

In this chapter, we prove the Theorem 3.12 by using the geometric 'push-forward' approach, appeared in [48], [42] and [43], on each fiber. We refer to a survey about this approach in deterministic dynamical systems [19]. We proceed with the proof by a reference measure λ_x which is a random probability measure and its disintegration coincides with the normalized intrinsic Riemmanian measure on a local strong unstable manifold. Then we consider the Krylov-Bogolyubov type sequence $\frac{1}{n}\sum_{k=0}^{n-1}(\phi^*)^k\lambda_x$. We keep track of the density function when the system is stretched along the unstable direction, and finally, we prove that any weak* limit point in the Krylov-Bogolyubov sequence satisfies the definition of the random Gibbs $u-$state.

Pick any $x \in M$ and fix, define $L_x : \Omega \to 2^M$ by

$$L_x(\omega) := W_\delta^{uu}(x, \omega).$$

We define $(\lambda_x)_\omega \in Pr(M)$ by the normalized intrinsic Riemannian volume measure on $L_x(\omega)$ as a submanifold. Then the disintegration $\{(\lambda_x)_\omega\}_{\omega\in\Omega}$ defines a random probability measure, named λ_x. In fact, for any $g : M \to \mathbb{R}$ bounded and Lipschitz continuous function, we have

$$\begin{aligned}&|(\lambda_x)_\omega(g) - (\lambda_x)_{\omega'}(g)|\\ =&\left|\frac{1}{\lambda^u_{(x,\omega)}(W^{uu}_\delta(x,\omega))}\int_{W^{uu}_\delta(x,\omega)} g(y)d\lambda^u_{(x,\omega)}(y) - \frac{1}{\lambda^u_{(x,\omega')}(W^{uu}_\delta(x,\omega'))}\int_{W^{uu}_\delta(x,\omega')} g(y)d\lambda^u_{(x,\omega')}(y)\right|,\end{aligned}$$

where $\lambda^u_{(x,\omega)}$ and $\lambda^u_{(x,\omega')}$ are the intrinsic Riemannian volume measure on $W^{uu}_\delta(x, \omega)$ and $W^{uu}_\delta(x, \omega')$ induced by the inherited Riemannian structure respectively. So $\omega \mapsto (\lambda_x)_\omega(g)$ is continuous due to the continuity of $W^{uu}_\delta(x, \omega)$ on ω for fixed x. Now for any closed set

$B \subset M$, define $g_n(x) := 1 - \min\{nd(x,B), 1\}$ which is bounded and Lipshitz. By Monotone convergence theorem, we have $(\lambda_x)_\omega(g_n) \to (\lambda_x)_\omega(B)$. Hence $\omega \mapsto (\lambda_x)_\omega(B)$ is measurable. By Remark 2.13, λ_x defined by disintegration $\{(\lambda_x)_\omega\}_{\omega\in\Omega}$ is a random probability measure by definition.

Consider the sequence of random probability measures $\{\frac{1}{n}\sum_{k=0}^{n-1}(\phi^*)^k\lambda_x\}_{n=1}^{\infty} \subset Pr_\Omega(M)$. Since $Pr_\Omega(M)$ is compact with respect to the narrow topology, then there exists a subsequence $\{\frac{1}{n_i}\sum_{k=0}^{n_i-1}(\phi^*)^k\lambda_x\}_{i=1}^{\infty}$ converging in the narrow topology. We denote μ by the limit of this subsequence in narrow topology. μ is obviously a $\phi-$invariant random probability measure.

Let $\{x_i\}_{i=1}^{\infty} \subset M$ be a countable dense subset. For any $\omega \in \Omega$, we define

$$V_{x_i,\epsilon}(\omega) = \bigcup_{y\in\Sigma_{x_i,\epsilon}(\omega)} W_\rho^{uu}(y,\omega) - \partial(\bigcup_{y\in\Sigma_{x_i,\epsilon}(\omega)} W_\rho^{uu}(y,\omega))$$

where $\Sigma_{x_i,\epsilon}(\omega) := exp_{x_i}(E_\epsilon^{cs}(x_i,\omega))$ and ∂ denotes the boundary. Here, by property (4) in Lemma 4.26, we pick ϵ sufficiently small such that $W_\rho^u(y,\omega) \cap W_\rho^u(z,\omega) = \emptyset$ for any $y, z \in \Sigma_{x_i,\epsilon}(\omega)$. This $\{V_{x_i,\epsilon}(\omega)\}_{i=1}^{\infty}$ forms an open cover of M_ω. We choose a finite cover of it, named $\{V_{x_i,\epsilon}(\omega)\}_{i=1}^{m(\omega)}$. By continuity of $W_\delta^{uu}(x,\omega)$ and $E^{cs}(x,\omega)$ on ω, there exists a $\gamma(\omega) > 0$ sufficiently small such that whenever $d(\omega,\omega') < \gamma(\omega)$, $\{V_{x_i,\epsilon}(\omega')\}_{i=1}^{m(\omega)}$ is still an open cover of $M_{\omega'}$. Keep doing this process and use the compactness of Ω, we can find a finite measurable partition $\{F_i\}_{i=1}^{n}$ of Ω in small scale and a sequence of numbers $\{m_i\}_{i=1}^{n}$ such that whenever $\omega \in F_i$, $\{V_{x_j,\epsilon}(\omega)\}_{j=1}^{m_i}$ is an open cover of M_ω.

Now define $g_{j_1,j_2,\ldots,j_n} \in L^\infty(\Omega, M)$ for $j_i \in \{1,\ldots,m_i\}$ by $g_{j_1,\ldots,j_n}(\omega) = x_{j_i}$ when $\omega \in F_i$. Now for any $g \in \{g_{j_1,\ldots,j_n}\}$, define $V_{g,\epsilon} : \Omega \to 2^M$ by

$$V_{g,\epsilon}(\omega) := V_{g(\omega),\epsilon}(\omega).$$

Notice that $\omega \mapsto \overline{V_{g,\epsilon}(\omega)}$ is obviously a random closed set by the continuity of $W_\rho^{uu}(y,\omega)$ on ω and the measurability of $\Sigma_{g(\omega),\epsilon}(\omega)$ on ω. As a consequence, $V_{g,\epsilon}(\omega) = V_{g,\epsilon}^o(\omega) := int(\overline{V_{g,\epsilon}(\omega)})$

is an random open set by Proposition 2.16. We have for any $\omega \in \Omega$,

$$\bigcup_{g \in \{g_{j_1,\dots,j_n}\}} V_{g,\epsilon}(\omega) = M_\omega.$$

By shrinking ϵ and ρ if necessary, without losing any generality, one can assume that $\mu(\partial V_{g,\epsilon}) = \int_\Omega \mu_\omega(\partial V_{g,\epsilon}(\omega))dP(\omega) = 0$. For each $\omega \in \Omega$, we divide $V_{g,\epsilon}(\omega)$ into pieces $\{W^{uu}_\rho(y,\omega)\}_{y \in \Sigma_{g(\omega),\epsilon}(\omega)}$, which produces a measurable partition of $V(\omega)$. Let $(\mu_\omega|_{V_{g,\epsilon}(\omega)})_y$ be the conditional probability measure of $\mu_\omega|_{V_{g,\epsilon}(\omega)}$ on $W^{uu}_\rho(y,\omega)$ for $y \in \Sigma_{g(\omega),\epsilon}(\omega)$. Then μ is a random Gibbs u-state if for $P-$a.s. $\omega \in \Omega$, and neglecting a $\mu_\omega|_{V_{g,\epsilon}(\omega)}-$null set, the following holds

$$(\mu_\omega|_{V_{g,\epsilon}(\omega)})_y \ll \lambda^u_{(y,\omega)} \text{ on every piece } W^{uu}_\rho(y,\omega),\ y \in \Sigma_{g(\omega),\epsilon}(\omega), \tag{7.1}$$

where $\lambda^u_{(y,\omega)}$is the intrinsic Riemannian volume measure on $W^{uu}_\rho(y,\omega)$.

For each $n \geq 0$, for all $\omega \in \Omega$, let

$$L_n(\omega) := \Big\{ z \in L_x(\omega) :\ f^n_\omega(z) \in W^{uu}_\rho(y,\theta^n\omega) \text{ for some } y \in \Sigma_{g,\epsilon}(\theta^n\omega) \\ \text{but } W^{uu}_\rho(y,\theta^n\omega) \not\subset f^n_\omega L_x(\omega) \Big\}.$$

By the local strong unstable invariant manifolds theorem, we have that for any $z \in L_n(\omega)$, $d^u(z, \partial L_x(\omega)) \leq \delta\gamma_0 e^{-n(\lambda_0-\epsilon_0)}$. Otherwise, since for any $z' \in W^{uu}_\rho(y,\theta^n\omega)$, $d^u(f^n_\omega(z), z') < \rho$, then

$$d^u(z, f^{-n}_\omega z') \leq \gamma_0 e^{-n(\lambda_0-\epsilon_0)} \cdot \rho \leq \gamma_0 e^{-n(\lambda_0-\epsilon_0)}\delta < d^u(z, \partial L_x(\omega)),$$

which implies that $z' \in f^n_\omega L_x(\omega)$, contradiction. Therefore, by Lemma 4.26 (1), there exists a constant C independent of $\omega \in \Omega$ and $x \in M$ such that

$$\lambda_{L_x(\omega)}(L_n(\omega)) \leq C(\delta\gamma_0 e^{-n(\lambda_0-\epsilon_0)})^{dim(E^{uu}(x,\omega))} \to 0 \text{ as } n \to \infty$$

uniformly for all $\omega \in \Omega$. Thus we have

$$\lim_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k (\lambda_x|_{L_x-L_n}) = \mu. \tag{7.2}$$

where $\lambda_x|_{L_x-L_n}$ is the random measure with disintegration $\omega \mapsto (\lambda_x)_\omega|_{L_x(\omega)-L_n(\omega)}$.

For each random box $V_{g,\epsilon}$, $\int_\Omega \mu_\omega(\partial V_{g,\epsilon}(\omega))dP(\omega) = 0$, we claim that

$$\lim_{i\to\infty} \left(\frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k (\lambda_x|_{L_x-L_n}) \right) (V_{g,\epsilon}) = \mu(V_{g,\epsilon}). \tag{7.3}$$

In fact, by the portmanteau theorem (Proposition 2.18) and notice that $\omega \mapsto \overline{V_{g,\epsilon}}(\omega)$ is random closed set, and $\omega \mapsto V_{g,\epsilon}(\omega)$ is a random open set, then on one hand

$$\begin{aligned} \mu(V_{g,\epsilon}) &= \int_\Omega \mu_\omega(V_{g,\epsilon}(\omega))dP(\omega) \\ &\le \liminf_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k (\lambda_x|_{L_x-L_n})(V_{g,\epsilon}), \end{aligned}$$

and on the other hand

$$\begin{aligned} \mu(V_{g,\epsilon}) &= \int_\Omega \mu_\omega(V_{g,\epsilon}(\omega))dP(\omega) \\ &= \int_\Omega \mu_\omega(\overline{V_{g,\epsilon}(\omega)})dP(\omega) \\ &\ge \limsup_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k (\lambda_x|_{L_x-L_n})(\overline{V_{g,\epsilon}}) \\ &\ge \limsup_{i\to\infty} \frac{1}{n_i} \sum_{k=0}^{n_i-1} (\phi^*)^k (\lambda_x|_{L_x-L_n})(V_{g,\epsilon}). \end{aligned}$$

Hence the equality (7.3) holds.

Recall that $J^u(x,\omega)$ is defined in Subsection 4.2.2. Now for any $n \in \mathbb{N}$, and any $z \in$

$V_{g,\epsilon}(\omega)$, define

$$h_n(z,\omega) = \frac{\prod_{k=0}^{n} \frac{1}{J^u(\phi^{-k}(z,\omega))}}{\int_{W^u_\rho(y,\omega)} \prod_{k=0}^{n} \frac{1}{J^u(\phi^{-k}(z,\omega))} d\lambda^u_{(y,\omega)}(z)}$$
$$= \frac{\prod_{k=0}^{n} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))}}{\int_{W^u_\rho(y,\omega)} \prod_{k=0}^{n} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} d\lambda^u_{(y,\omega)}(z)},$$

where $y \in \Sigma_{g(\omega),\epsilon}(\omega)$ is the point such that $z \in W^{uu}_\rho(y,\omega)$.

Suppose that $W^{uu}_\rho(y,\omega) \subset f^n_{\theta^{-n}\omega}(L_x(\theta^{-n}\omega) - L_n(\theta^{-n}\omega))$ for some $y \in \Sigma_{g(\omega),\epsilon}(\omega)$, and let $m^n_{(y,\omega)}$ be the conditional measure of $(f^n_{\theta^{-n}\omega})^*((\lambda_x)_{\theta^{-n}\omega}|_{L_x(\theta^{-n}\omega)-L_n(\theta^{-n}\omega)})$ on $W^{uu}_\rho(y,\omega)$, then by definition we have

$$h_n(\cdot,\omega)|_{W^{uu}_\rho(y,\omega)} = \frac{dm^n_{(y,\omega)}}{d\lambda^u_{(y,\omega)}}. \tag{7.4}$$

Notice that $h_n : graph(V_{g,\epsilon}) = \{(x,\omega)|\ x \in V_{g,\epsilon}(\omega)\} \subset M \times \Omega \to \mathbb{R}$ is obviously measurable with respect to the $\sigma-$algebra $graph(V_{g,\epsilon}) \cap \mathcal{B}(M) \otimes \mathcal{B}(\Omega)$. By (4.61), h_n converges uniformly to a measurable function $h : graph(V_{g,\epsilon}) \to (0,+\infty)$ defined by

$$h(z,\omega) := \frac{\prod_{k=1}^{\infty} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))}}{\int_{W^{uu}_\rho(y,\omega)} \prod_{k=1}^{\infty} \frac{J^u(\phi^{-k}(y,\omega))}{J^u(\phi^{-k}(z,\omega))} d\lambda^u_{(y,\omega)}(z)} \tag{7.5}$$

for any $(z,\omega) \in W^{uu}_\rho(y,\omega)$ and $y \in \Sigma_{g(\omega),\epsilon}(\omega)$. Moreover, for each $\omega \in \Omega$, note that $J^u(\cdot,\omega)$ is a continuous function and $\phi^{-1}(\cdot,\omega)$ is continuous function, so $h_n(\cdot,\omega)$ is a continuous function on each local strong unstable leaf. Thus, $h(\cdot,\omega)$ is continuous on each local strong unstable leaf since $h_n(\cdot,\omega) \to h(\cdot,\omega)$ uniformly on each local strong unstable leaf as $n \to \infty$.

Now define a random measure ν on $graph(V_{g,\epsilon})$ by

$$\nu(A) = \int_\Omega \int_{\Sigma_{g(\omega),\epsilon}(\omega)} \int_{W^{uu}_\rho(y,\omega)\cap A(\omega)} h(z,\omega) d\lambda^u_{(y,\omega)}(z) d(\widetilde{\mu_\omega|_{V_{g,\epsilon}(\omega)}})(y) dP(\omega) \tag{7.6}$$

for any measurable set $A \subset M \times \Omega$ with $A \subset graph(V_{g,\epsilon})$ and $A(\omega) := \{x \in M :\ (x,\omega) \in A\}$, where $d(\widetilde{\mu_\omega|_{V_{g,\epsilon}(\omega)}})(y)$ is the projection measure on $\Sigma_{g(\omega),\epsilon}(\omega)$ with respect to the measurable partition $\{W^{uu}_\rho(y,\omega)\}_{y\in\Sigma_{g(\omega),\epsilon}(\omega)}$.

For any random continuous function $c(x,\omega)$, and $z\in W^{uu}_\rho(y,\omega)$ with $y\in\Sigma_{g(\omega),\epsilon}(\omega)$, define

$$\bar c(z,\omega)=\int_{W^{uu}_\rho(y,\omega)} c(z',\omega)h(z',\omega)d\lambda^u_{(y,\omega)}(z').$$

Then we have that $\bar c(z,\omega)$ is a bounded measurable function defined on $graph(V_{g,\epsilon})$ and for each fixed ω, $\bar c(z,\omega)$ is constant on each $W^u_\rho(y,\omega)$ for $y\in\Sigma_{g(\omega),\epsilon}(\omega)$.

Denote $\Lambda_k(\omega):=f^k_{\theta^{-k}\omega}(L_x(\theta^{-k}\omega)-L_k(\theta^{-k}\omega))\cap\Sigma_{g(\omega),\epsilon}(\omega)$, and denote $(\phi^*)^k(\lambda_x|_{L_x-L_k})$ by μ_k, and $\widetilde{(\mu_k)_\omega}$ the projection of $(\mu_k)_\omega$ on $\Lambda_k(\omega)$ with respect to the measurable partition $\{W^{uu}_\rho(y,\omega)\}_{y\in\Lambda_k(\omega)}$. Then for any random continuous function $c(z,\omega)$, by (7.2), and the uniform convergence of h_k, we have

$$\begin{aligned}
&\mu|_{V_{g,\epsilon}}(c)\\
&=\lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}(\phi^*)^k(\lambda_x|_{L_x-L_k})|_{V_{g,\epsilon}}(c)\\
&=\lim_{i\to\infty}\int_\Omega\int_{V_{g,\epsilon}(\omega)}c(z,\omega)d(\frac{1}{n_i}\sum_{k=0}^{n_i-1}(\phi^*)^k(\lambda_x|_{L_x-L_k})|_{V_{g,\epsilon}})_\omega dP(\omega)\\
&=\lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}\int_\Omega\int_{V_{g,\epsilon}(\omega)}c(z,\omega)d((\phi^*)^k(\lambda_x|_{L_x-L_k})|_{V_{g,\epsilon}})_\omega dP(\omega)\\
&=\lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}\int_\Omega\int_{\Lambda_k(\omega)}((\phi^*)^k(\lambda_x|_{L_x\setminus L_k}))_\omega(W^{uu}_\rho(y,\omega))\int_{W^{uu}_\rho(y,\omega)}c(z,\omega)h_k(z,\omega)d\lambda^u_{(y,\omega)}(z)d\widetilde{(\mu_k)_\omega}(y)dP\\
&=\lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}\int_\Omega\int_{\Lambda_k(\omega)}((\phi^*)^k(\lambda_x|_{L_x\setminus L_k}))_\omega(W^{uu}_\rho(y,\omega))\int_{W^{uu}_\rho(y,\omega)}c(z,\omega)h(z,\omega)d\lambda^u_{(y,\omega)}(z)d\widetilde{(\mu_k)_\omega}(y)dP\\
&=\lim_{i\to\infty}\frac{1}{n_i}\sum_{k=0}^{n_i-1}\int_\Omega\int_{\Lambda_k(\omega)}((\phi^*)^k(\lambda_x|_{L_x\setminus L_k}))_\omega(W^{uu}_\rho(y,\omega))\int_{W^{uu}_\rho(y,\omega)}\bar c(z,\omega)h(z,\omega)d\lambda^u_{(y,\omega)}(z)d\widetilde{(\mu_k)_\omega}(y)dP\\
&=\mu|_{V_{g,\epsilon}}(\bar c)\\
&=\int_\Omega\int_{V_{g,\epsilon}(\omega)}\bar c(z,\omega)d(\mu|_{V_{g,\epsilon}})_\omega(y)dP(\omega)\\
&=\int_\Omega\int_{\Sigma_{g(\omega),\epsilon}(\omega)}\int_{W^{uu}_\rho(y,\omega)}c(z,\omega)h(z,\omega)d\lambda^u_{(y,\omega)}(z)d(\widetilde{\mu|_{V_{g,\epsilon}})_\omega}(y)dP(\omega).
\end{aligned}$$

Hence, we have $\mu|_{V_{g,\epsilon}} = \nu$, and therefore (7.1) holds. This proves that μ is a random Gibbs $u-$state.

The proof of Theorem 3.12 is done.

Appendix A. Convex Cone, Projective Metric and Birkhoff's inequality

In this appendix, we review the notion of Hilbert projective metric associated to a convex cone in a topological vector space. The following notions are borrowed from [62].

Definition A.1. *Let E be a topological vector space. A subset $C \subset E$ is said to be a convex cone if*

(i) $tv \in C$ for $v \in C$ and $t \in \mathbb{R}^+$;

(ii) for any $t_1, t_2 \in \mathbb{R}^+$, $v_1, v_2 \in C$, then $t_1 v_1 + t_2 v_2 \in C$;

(iii) $\bar{C} \cap -\bar{C} = \{0\}$, where $\bar{C}$ the closure of C is defined by: $w \in \bar{C}$ if and only if there are $v \in C$ and $t_n \searrow 0$ such that $w + t_n v \in C$ for all $n \geq 1$.

Definition A.2. *For a convex cone $C \subset E$, given any $v_1, v_2 \in C$, we define*

$$\alpha(v_1, v_2) := \sup\{t > 0 : \; v_2 - tv_1 \in C\};$$

$$\beta(v_1, v_2) := \inf\{s > 0 : \; sv_1 - v_2 \in C\},$$

with the convention that $\sup \emptyset = 0$ and $\inf \emptyset = +\infty$. The Hilbert projective metric between $v_1, v_2 \in C$ is defined by

$$d_C(v_1, v_2) = \log \frac{\beta(v_1, v_2)}{\alpha(v_1, v_2)}$$

with the convention that $d_C(v_1, v_2) = \infty$ if $\alpha(v_1, v_2) = 0$ or $\beta(v_1, v_2) = \infty$.

Proposition A.3. *d_C is a metric in the projective quotient of C, i.e.,* *(i)* $d_C(v_1, v_2) = d_C(v_2, v_1)$ *for all* $v_1, v_2 \in C$*;*

(ii) $d_C(v_1, v_3) \le d_C(v_1, v_2) + d_C(v_2, v_3)$ *for all* $v_1, v_2, v_3 \in C$*;*

(iii) $d_C(v_1, v_2) = 0$ *if and only if there exists* $t \in \mathbb{R}^+$ *such that* $v_1 = tv_2$*.*

An important property of the Hilbert projective metric is Birkhoff's inequality. We use this theorem to prove that the iterations of fiber transfer operator is a contraction on a suitable fiber observations cone in Section 6.2.

Proposition A.4 (Birkhoff's inequality)**.** *Let* E_1*,* E_2 *be two topological vector spaces, and* $C_i \subset E_i$*, for* $i = 1, 2$ *be convex cones. Let* $L : E_1 \to E_2$ *be a linear operator and assume that* $L(C_1) \subset C_2$*. Let* $D = \sup\{d_{C_2}(L(v_1), L(v_2)) :\ v_1, v_2 \in C_1\}$*. If* $D < \infty$*, then*

$$d_{C_2}(L(v_1), L(v_2)) \le (1 - e^{-D}) d_{C_1}(v_1, v_2) \text{ for all } v_1, v_2 \in C_1.$$

Appendix B. The random SRB measure for random hyperbolic Systems

Let $F : \mathbb{Z} \times \Omega \times M \to M$ be a continuous random dynamical system over an invertible ergodic metric dynamical systems $(\Omega, \mathcal{B}, P, \theta)$. A random variable $\gamma : \Omega \mapsto \mathbb{R}^+$ will be called tempered, if it satisfies $\lim_{n\to\pm\infty} \frac{1}{n} \log \gamma(\theta^n \omega) = 0$ $P-$a.s.

Definition B.1. *A random compact nonempty set* $\omega \mapsto \Lambda(\omega)$ *is called invariant under* F *if* $F(\omega)\Lambda(\omega) = \Lambda(\theta\omega)$ *for* $P-$*a.s.* $\omega \in \Omega$*. Such a* Λ *is called a random hyperbolic set for* F *if there exists an open set* V *with a compact closure* $\bar{V}$*, tempered random variables* $\lambda > 0$*,* $\alpha > 0$*,* $C > 0$*, and subbundles* $\Gamma^u(\omega)$ *and* $\Gamma^s(\omega)$ *of the tangent bundle* $T\Lambda(\omega)$*, depending measurably on* ω *such that*

(i) for $P-$*a.s.* $\omega \in \Omega$*, there exist a measurable in* ω *family of open sets* $U(\omega)$ *such that* $\{x : d(x, \Lambda(\omega)) < \alpha(\omega)\} \subset U(\omega) \subset V$*,* $F(\omega)U(\omega) \subset V$*, and* $F(\omega)$ *restricted to* $U(\omega)$

is a diffeomorphism and both $\log^+ \sup_{x\in U(\omega)} \|D_x F(\omega)\|$ *and* $\log^+ \sup_{x\in U(\omega)} \|D_x F(\omega^{-1})\|$ *belong to* $L^1(\Omega, P)$*;*

(ii) $T\Lambda(\omega) = \Gamma^u(\omega) \oplus \Gamma^s(\omega)$*,* $DF(\omega)\Gamma^\tau(\omega) = \Gamma^\tau(\theta\omega)$ *for* $\tau = u, s$*, and* $\angle(\Gamma^u(\omega), \Gamma^s(\omega)) \geq \alpha(\omega)$ $P-$*a.s.*

(iii) for $n \in \mathbb{N}$ *and* $\lambda(n,\omega) = \lambda(\omega)\cdots\lambda(\theta^{n-1}\omega)$ *and* $P-$*a.s.* ω

$$\|DF(n,\omega)\xi\| \leq C(\omega)\lambda(n,\omega)\|\xi\| \text{ for } \xi \in \Gamma^s(\omega);$$
$$\|DF(-n,\omega)\eta\| \leq C(\omega)\lambda(n,\theta^{-n}\omega)\|\eta\| \text{ for } \eta \in \Gamma^u(\omega);$$

(iv) $\int \log \lambda dP < 0$*;*

(v) $\log \alpha \in L^1(\Omega, P)$

If in addition, $F(\omega)U(\omega) \subset U(\theta\omega)$ $P-$*a.s. and* $\cap_{n\in\mathbb{N}} F(n, \theta^{-n}\omega)U(\theta^{-n}\omega) = \Lambda(\omega)$*, then we call* Λ *a random hyperbolic attractor of* F*. If* M *is compact and all* $\Lambda(\omega) = M$ *and satisfy assumptions above, then we call* F *a random Anosov system.*

It is obviously that the random Anosov on fibers systems defined in Section 2.1 is random Anosov systems.

Definition B.2. *F is called random topological transitive if for any given open random sets U and V with $U(\omega), V(\omega) \neq \emptyset$ for all $\omega \in \Omega$, there exists a random variable n taking values in $\mathbb{Z}$ such that the intersection $F(n(\omega), \theta^{-n(\omega)})U(\theta^{-n}\omega) \cap V(\omega) \neq \emptyset$ $P-$a.s..*

The following lemma is the Lemma A.1 in [34].

Lemma B.3. *If F is topological mixing on fibers, then F is random topological transitive.*

The following theorem is Theorem 4.3 in [32], which is the main result of the SRB measure for random hyperbolic systems.

Theorem B.4. *Let F be a $C^{1+\alpha}$ RDS with a random topological transitive hyperbolic attractor $\Lambda \subset M \times \Omega$. Then there exists a unique $F-$invariant measure (SRB-measure) ν supported by Λ and characterized by each of the following:*

(i) $h_\nu(F) = \int \sum \lambda_i^+ d\nu$, *where λ_i are the Lyapunov exponents corresponding to ν;*

(ii) $P-$*a.s. the conditional measure of ν_ω on the unstable manifolds are absolutely continuous with respect to the Riemannian volume on these submanifolds;*

(iii) $h_\nu(F)+\int f d\nu = \sup_{F-invariant\ \nu}\{h_\nu(F)+\int f d\nu\}$ *and the later is the topological pressure $\pi_F(f)$ of f which satisfies $\pi_F(f) = 0$;*

(iv) $\nu = \psi\tilde{\mu}$, *where ψ is the conjugation between F on Λ and two-sided shift σ on Σ_A, and $\tilde{\mu}$ is the equilibrium state for the σ and function $f \circ \psi$. The measure $\tilde{\mu}$ can be obtained as a natural extension of the probability measure μ which is invariant with respect to the one-sided shift on Σ_A^+ and such that $L_\eta^* \mu_{\theta\omega} = \mu_\omega$ $P-$a.s. where $\eta - f \circ \psi = h - h \circ (\theta \times \sigma)$ for some random Hölder continuous function h;*

(v) ν *can be obtained as a weak limit $\nu_\omega = \lim_{n\to} F(n, \theta^{-n}\omega) m_{\theta^{-n}\omega}$ $P-$a.s. for any measure m_ω absolutely continuous with respect to the Riemannian volume such that supp$\mu_\omega \subset U(\omega)$.*

Bibliography

[1] Alves, J. F., Luzzatto, S., and Pinheiro, V. Markov structures and decay of correlations for non-uniformly expanding dynamical systems. *Ann. Inst. H. Poincaré Anal. Non Linéaire 22*, 6 (2005), 817–839.

[2] Anosov, D. V. *Geodesic flows on closed Riemann manifolds with negative curvature.* Proceedings of the Steklov Institute of Mathematics, No. 90 (1967). Translated from the Russian by S. Feder. American Mathematical Society, Providence, R.I., 1969.

[3] Armando, A., and Júnior, C. Backward inducing and exponential decay of correlations for partially hyperbolic attractors. *Israel Journal of Mathematics 130*, 1 (2002), 29–75.

[4] Arnold, L. *Random dynamical systems.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998.

[5] Bahnmüller, J., and Liu, P.-D. Characterization of measures satisfying the pesin entropy formula for random dynamical systems. *Journal of Dynamics and Differential Equations 10*, 3 (1998), 425–448.

[6] Baladi, V. *Positive transfer operators and decay of correlations*, vol. 16. World scientific, 2000.

[7] Baladi, V., Kondah, A., and Schmitt, B. Random correlations for small perturbations of expanding maps. *Random and Computational Dynamics 4*, 2/3 (1996), 179–204.

[8] Bauer, W., and Sigmund, K. Topological dynamics of transformations induced on the space of probability measures. *Monatshefte für Mathematik 79*, 2 (1975), 81–92.

[9] Bonatti, C., Díaz, L. J., and Viana, M. *Dynamics beyond uniform hyperbolicity: A global geometric and probabilistic perspective*, vol. 102. Springer Science & Business Media, 2006.

[10] Bowen, R. Periodic points and measures for axiom a diffeomorphisms. *Transactions of the American Mathematical Society 154* (1971), 377–397.

[11] Bowen, R. Some systems with unique equilibrium states. *Mathematical systems theory 8*, 3 (1974), 193–202.

[12] Bowen, R. Ergodic theory of axiom a diffeomorphisms. In *Equilibrium States and the Ergodic Theory of Anosov Diffeomorphisms*. Springer, 1975, pp. 90–107.

[13] Bressaud, X., and Liverani, C. Anosov diffeomorphisms and coupling. *Ergodic Theory and Dynamical Systems 22*, 1 (2002), 129–152.

[14] Brin, M. I., and Pesin, J. B. Partially hyperbolic dynamical systems. *Izv. Akad. Nauk SSSR Ser. Mat. 38* (1974), 170–212.

[15] BUZZI, J. Exponential decay of correlations for random lasota–yorke maps. *Communications in mathematical physics 208*, 1 (1999), 25–54.

[16] CARMO, M. P. D. *Riemannian geometry.* Birkhäuser, 1992.

[17] CASTRO, A., AND NASCIMENTO, T. Statistical properties of the maximal entropy measure for partially hyperbolic attractors. *Ergodic Theory and Dynamical Systems 37*, 4 (2017), 1060.

[18] CHERNOV, N. Decay of correlations and dispersing billiards. *Journal of Statistical Physics 94*, 3-4 (1999), 513–556.

[19] CLIMENHAGA, V., LUZZATTO, S., AND PESIN, Y. The geometric approach for constructing Sinai-Ruelle-Bowen measures. *J. Stat. Phys. 166*, 3-4 (2017), 467–493.

[20] CLIMENHAGA, V., AND PAVLOV, R. One-sided almost specification and intrinsic ergodicity. *Ergodic Theory and Dynamical Systems 39*, 9 (2019), 2456–2480.

[21] CRAUEL, H. *Random probability measures on Polish spaces*, vol. 11. CRC press, 2002.

[22] CROVISIER, S., AND POTRIE, R. Introduction to partially hyperbolic dynamics. *School on Dynamical Systems, ICTP, Trieste 3* (2015), 1.

[23] DE GUZMÁN, M. *Differentiation of integrals in R^n*. Lecture Notes in Mathematics, Vol. 481. Springer-Verlag, Berlin-New York, 1975. With appendices by Antonio Córdoba, and Robert Fefferman, and two by Roberto Moriyón.

[24] DE SIMOI, J., AND LIVERANI, C. Statistical properties of mostly contracting fast-slow partially hyperbolic systems. *Inventiones mathematicae 206*, 1.

[25] DEMERS, M., AND ZHANG, H.-K. Spectral analysis of the transfer operator for the lorentz gas. *Journal of Modern Dynamics 5*, 4 (2011).

[26] DEMERS, M. F., AND ZHANG, H.-K. Spectral analysis of hyperbolic systems with singularities. *Nonlinearity 27*, 3 (2014), 379.

[27] DENKER, M., GRILLENBERGER, C., AND SIGMUND, K. *Ergodic theory on compact spaces*, vol. 527. Springer, 2006.

[28] DOLGOPYAT, D. On dynamics of mostly contracting diffeomorphisms. *Communications in Mathematical Physics 213*, 1 (2000), 181–201.

[29] DOLGOPYAT, D. Lectures on u-gibbs states. In *Lecture Notes, Conference on Partially Hyperbolic Systems (Northwestern University, 2001)* (2001).

[30] DOLGOPYAT, D. On mixing properties of compact group extensions of hyperbolic systems. *Israel journal of mathematics 130*, 1 (2002), 157–205.

[31] GUNDLACH, V., AND KIFER, Y. Expansiveness, specification, and equilibrium states for random bundle transformations. *Discrete & Continuous Dynamical Systems-A 6*, 1 (2000), 89.

[32] Gundlach, V. M., and Kifer, Y. Random hyperbolic systems. In *Stochastic dynamics*. Springer, 1999, pp. 117–145.

[33] Hirsch, M. W. *Differential topology*, vol. 33. Springer Science & Business Media, 2012.

[34] Huang, W., Lian, Z., and Lu, K. Ergodic theory of random anosov systems mixing on fibers. *arXiv preprint arXiv:1612.08394* (2019).

[35] Kallenberg, O. *Random measures*, fourth ed. Akademie-Verlag, Berlin; Academic Press, Inc., London, 1986.

[36] Kifer, Y. Thermodynamic formalism for random transformations revisited. *Stochastics and Dynamics 8*, 01 (2008), 77–102.

[37] Kifer, Y., and Liu, P.-D. Random dynamics. *Handbook of dynamical systems 1* (2006), 379–499.

[38] Klünger, M. Periodicity and Sharkovsky's theorem for random dynamical systems. *Stoch. Dyn. 1*, 3 (2001), 299–338.

[39] Korepanov, A., Kosloff, Z., and Melbourne, I. Explicit coupling argument for non-uniformly hyperbolic transformations. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics 149*, 1 (2019), 101–130.

[40] Kwietniak, D., Lacka, M., and Oprocha, P. A panorama of specification-like properties and their consequences. *Contemporary Mathematics 669* (2016), 155–186.

[41] Li, X., and Vilarinho, H. Almost sure mixing rates for non-uniformly expanding maps. *Stochastics and Dynamics 18*, 04 (2018), 1850027.

[42] Lian, Z., Liu, P., and Lu, K. Srb measures for a class of partially hyperbolic attractors in hilbert spaces. *Journal of Differential Equations 261*, 2 (2016), 1532–1603.

[43] Lian, Z., Liu, P., and Lu, K. Existence of srb measures for a class of partially hyperbolic attractors in banach spaces. *Discrete & Continuous Dynamical Systems-A 37*, 7 (2017), 3905.

[44] Liu, P., and Lu, K. A note on partially hyperbolic attractors: entropy conjecture and SRB measures. *Discrete Contin. Dyn. Syst. 35*, 1 (2015), 341–352.

[45] Liu, P.-D., and Qian, M. *Smooth ergodic theory of random dynamical systems*. Springer, 2006.

[46] Liverani, C. Decay of correlations. *Annals of Mathematics 142*, 2 (1995), 239–301.

[47] Pesin, Y., Senti, S., and Zhang, K. Thermodynamics of the katok map. *Ergodic Theory and Dynamical Systems 39*, 3 (2019), 764–794.

[48] Pesin, Y. B., and Sinai, Y. G. Gibbs measures for partially hyperbolic attractors. *Ergodic Theory and Dynamical Systems 2*, 3-4 (1982), 417–438.

[49] Pfister, C.-E., and Sullivan, W. G. On the topological entropy of saturated sets. *Ergodic Theory and Dynamical Systems 27*, ARTICLE (2007), 929–956.

[50] Poincaré, H. Sur le problème des trois corps et les équations de la dynamique. *Acta mathematica 13*, 1 (1890), A3–A270.

[51] Pugh, C., Shub, M., and Wilkinson, A. Hölder foliations. *Duke Math. J. 86*, 3 (1997), 517–546.

[52] Ruelle, D. *Thermodynamic formalism, volume 5 of Encyclopedia of Mathematics and its Applications*. Addison-Wesley Publishing Co., Reading, Mass, 1978.

[53] Ruziboev, M. Almost sure rates of mixing for random intermittent maps. In *Differential equations and dynamical systems*, vol. 268 of *Springer Proc. Math. Stat.* Springer, Cham, 2018, pp. 141–152.

[54] Salman, M., and Das, R. Specification properties for non-autonomous discrete systems. *Topological Methods in Nonlinear Analysis* (2020), 1–17.

[55] Sarkooh, J. N., and Ghane, F. Specification and thermodynamic properties of topological time-dependent dynamical systems. *Qualitative Theory of Dynamical Systems 18*, 3 (2019), 1161–1190.

[56] Shah, S., Das, R., and Das, T. Specification property for topological spaces. *Journal of Dynamical and Control Systems 22*, 4 (2016), 615–622.

[57] Shub, M. *Global stability of dynamical systems.* Springer Science & Business Media, 2013.

[58] Sigmund, K. Generic properties of invariant measures for axioma-diffeomorphisms. *Inventiones mathematicae 11*, 2 (1970), 99–109.

[59] Sinai, Y. G. Gibbs measures in ergodic theory. *Russian Mathematical Surveys 27*, 4 (1972), 21.

[60] Smale, S. Differentiable dynamical systems. *Bull. Amer. Math. Soc. 73* (1967), 747–817.

[61] Stadlbauer, M., Suzuki, S., and Varandas, P. Thermodynamic formalism for random non-uniformly expanding maps. *arXiv preprint arXiv:2006.03749* (2020).

[62] Viana, M. *Stochastic dynamics of deterministic systems*, vol. 21. IMPA Rio de Janeiro, 1997.

[63] Wang, Q., and Young, L.-S. Strange attractors with one direction of instability. *Comm. Math. Phys. 218*, 1 (2001), 1–97.

[64] Wang, X., Wu, W., and Zhu, Y. Unstable entropy and unstable pressure for random partially hyperbolic dynamical systems. *Stochastics and Dynamics* (2020), 2150021.

[65] Yamamoto, K. On the one-way specification property and large deviations for systems with non-dense ergodic measures. *Kyushu Journal of Mathematics 72*, 1 (2018), 71–94.

[66] Young, L.-S. Ergodic theory of differentiable dynamical systems. In *Real and complex dynamical systems (Hillerød, 1993)*, vol. 464 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.* Kluwer Acad. Publ., Dordrecht, 1995, pp. 293–336.

[67] Young, L.-S. Statistical properties of dynamical systems with some hyperbolicity. *Annals of Mathematics 147*, 3 (1998), 585–650.

[68] Young, L.-S. What are SRB measures, and which dynamical systems have them? vol. 108. 2002, pp. 733–754. Dedicated to David Ruelle and Yasha Sinai on the occasion of their 65th birthdays.

[69] Zhao, H., and Zheng, Z.-H. Random periodic solutions of random dynamical systems. *J. Differential Equations 246*, 5 (2009), 2020–2038.