2022-04-12

Stabilizing Controlled Systems in the Presence of Time-Delays

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A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

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ABSTRACT

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A dynamical system's state evolves over time, and when the system stays near a particular state this state is known as a stable state of the system. Through control methods, dynamical systems can be manipulated such that virtually any state can be made stable. Although most real systems evolve continuously in time the application of digital control methods to these systems is inherently discrete. States are sampled (with sensors) and fed back into the system in discrete-time to determine the input needed to control the continuous system. Additionally, dynamical systems often experience time delays. Some examples of time delays are delays due to transmission distances, processing software, sampling information, and many more. Such delays are often a cause of poor performance and, at times, instability in these systems. Recently a criterion referred to as intrinsic stability has been developed that ensures that a dynamic system cannot be destabilized by delays. The goal of this thesis is to broaden the definition of intrinsic stability to closed-loop systems, which are systems in which the control depends on the state of the system, and to determine control parameters that optimize this resilience to time delays. Here, we give criteria describing when a closed-loop system is intrinsically stable. This allows us to give examples in which systems controlled using Linear Quadratic Regulator (LQR) control can be made intrinsically stable.

Keywords: mathematical modeling, dynamics, control, time-delay, intrinsic stability

Acknowledgements

I wish to extend a special thanks to my advisor Ben Webb, who always encouraged me to take on projects that aligned with my interests, and who spent many hours helping me structure and write this thesis. I would also like to thank my committee members Jared Whitehead and Marc Killpack for taking time out of their busy schedules to answer my questions and concerns. I also wish to show my appreciation and deepest gratitude to my wife for always being supportive of my goals, encouraging me during difficult times, and for helping me time manage while working from home.

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Chapter 1. Introduction

A dynamical system is any system for which the state of the system evolves over time. Dynamical systems are all around us. Some examples of dynamical systems include mechanical, electrical, fluid, and thermal systems. In the study of dynamical systems mathematical models are used to describe and to investigate the evolution over time of real and theoretical systems. An important topic of study in dynamical systems is the study of stability, where stability can be thought of as the system's tendency to find its way to or relax back into a particular state after being perturbed. If so, the system is consider to be stable about that particular state. Otherwise, if the system does not relax to a particular state, then the system is classified as unstable [1]. As described in [2] a dynamical system can also be classified as intrinsically stable, meaning that no matter what types of time delays it experiences, the system will not be destabilized by these time delays.

In contrast, control theory can be used to change a system's dynamics to make it evolve in a desired way. For example, a method of control that implements this idea is called the "Linear Quadratic Regulator" or LQR. LQR is an optimization method that finds the best gains to control a system based on an optimization function. As a first step to understanding the interaction between control and intrinsic stability we develop a sufficient condition describing when a system can be controlled to be intrinsically stable. Specifically, we extend the criteria for intrinsic stability to closed-loop systems considering the case in which the system's dynamics and the control experience different time delays (see Theorem 5.6).

Some potential applications of this research could involve transmitting data to a system that is far away (for example planetary rovers), systems that are delayed due to high use (requests from users), etc. In terms of more general applications, control is often used to optimize the performance of a system relative to some specific task. However, it has not been used, as far as we know, to tune a system to be resilient to delays. The hope is that this work will allow engineers, or others, to consider a new measure of optimal performance for controlled dynamical systems that are destabilized by the presence of time-delays.

This thesis is structured as follows. In Chapter 2, we explore how dynamical systems with control are modeled. We also describe discretization methods and stability of linear systems. Following this, in Chapter 3 we describe the theory of intrinsic stability and its relation to time-delayed systems. In Chapter 4 we introduce LQR and discrete-LQR (dLQR), and we derive the new closed-loop system equations. In Chapter 5 we extend this theory to controlled systems, specifically closed loop systems and prove our main result. In Chapter 6 we give a few applications of this new theory to controlled systems.

Chapter 2. Background

2.1 Modeling Dynamical Systems and Control Theory

A dynamical system is a system in which the future state of the system depends on the current or past states of the system via some rule given by a function $f : \mathbb{R}^n \to \mathbb{R}^n$. In controlled systems, future states depend both on previous states and on inputs to the system, given by a function $f : (\mathbb{R}^n, \mathbb{R}^m) \to \mathbb{R}^n$. Dynamical systems can be time-variant or time-invariant often referred to a homogeneous and in-homogeneous, respectively. Time-variant means that time has a direct influence on the dynamics, and time-invariant means that the system only depends on states and input. Here we focus on time-invariant systems, since many real-world systems are time-invariant. For instance, all examples in this thesis are time-invariant (see Chapter 6).

The evolution of a continuous-time dynamical system with control is modeled using a system of differential equations given by,

$$\dot{\mathbf{x}} = f(\mathbf{x}, \mathbf{u}), \quad \mathbf{x}(t_0) = \mathbf{x_0}. \tag{2.1}$$

Here f defines the rate of change of the state based on the current state $\mathbf{x} \in \mathbb{R}^n$ and input $\mathbf{u}(t) \in \mathbb{R}^m$ applied to the system.

A discrete-time dynamical system with control is given by,

$$\mathbf{x}_{n+1} = F(\mathbf{x}_n, \mathbf{u}_n), \quad \mathbf{x}(0) = \mathbf{x}_\mathbf{0}. \tag{2.2}$$

Here F defines the next state $\mathbf{x}_{n+1} \in \mathbb{R}^n$ based on the current state $\mathbf{x}_n \in \mathbb{R}^n$ and current input $\mathbf{u}_n \in \mathbb{R}^m$. We note that this system can be a time-discretized version of Equation (2.1). There are several methods to discretize a continuous system (see Section 2.2). The methods vary in complexity and accuracy.

Additionally, we note that if the function F is linear with respect the the state and input separately, then it can be interpreted as the combination of two functions, $F(x, u) = G(x) + U(u)$ where $G : \mathbb{R}^n \to \mathbb{R}^n$ and $U : \mathbb{R}^m \to \mathbb{R}^n$. This is the case in a number of standard control schemes and is the case we consider in this thesis (see Equation 5.1).

In addition, non-linear models are often linearized to study the stability of the system about a local region (see Section 2.3). Linearization is done by taking the Jacobian of f or F with respect an *equilibrium* state $\mathbf{x}_e \in \mathbb{R}^n$ and equilibrium input $\mathbf{u}_e \in \mathbb{R}^m$. An equilibrium point for a continuous-time system is such that $f(\mathbf{x}_e, \mathbf{u}_e) = \mathbf{0}$ or $F(\mathbf{x}_{n_e}, \mathbf{u}_{n_e}) = \mathbf{x}_{n_e}$ for discrete-time systems. This results in a linear approximation of the system about the point $\tilde{\mathbf{x}} = \mathbf{x} - \mathbf{x}_e$ with input $\tilde{\mathbf{u}} = \mathbf{u} - \mathbf{u}_e$. The linearized system is known as the state-space model given by,

$$\dot{\tilde{\mathbf{x}}} = A\tilde{\mathbf{x}} + B\tilde{\mathbf{u}} \tag{2.3}$$

in continuous-time where the matrix $A \in \mathbb{R}^{n\times n}$ describes the evolution of the system based on its state. The matrix $B \in \mathbb{R}^{n\times m}$ describes how the inputs affect the evolution of the system's state.

Similarly, for discrete systems,

$$\tilde{\mathbf{x}}_{n+1} = A_d\tilde{\mathbf{x}}_n + B_d\tilde{\mathbf{u}}_n. \tag{2.4}$$

Here A_d and B_d denote the discrete-time system which, in most cases, is different from the continuous-time version of the system i.e. $A_d \neq A$ and $B_d \neq B$ (see Section 2.2).

The state-space equations are used in control methods to drive a system to a desired

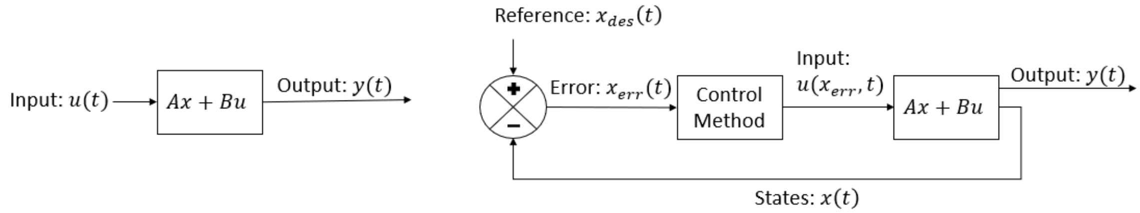

Figure 2.1: (a) The input $\mathbf{u} = \mathbf{u}(t)$ is set based on time and will not changed based on the states of the system. (b) The controller changes the input bases the difference between the desired state $\mathbf{x}_{des}$ and the current states $\mathbf{x}$.

state. Given a desired state $\mathbf{x}_{des} \in \mathbb{R}^n$, a system is *controllable* if it is possible choose $\mathbf{u} \in \mathbb{R}^m$ such that $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_{des}$. If $\mathbf{u} = \mathbf{u}(t)$ depends solely on time $t \in \mathbb{R}$ then the control is known as open-loop control (see Figure 2.1 a). The input for an open-loop system has to be calculated prior to starting the system in order to drive the system to $\mathbf{x}_{des}$, but ultimately, there is no guarantee the system will arrive at the desired state due to noise or unaccounted disturbances. If $\mathbf{u}$ depends on both time t and state $\mathbf{x}$ then the control is known as a closed-loop control (see Figure 2.1 b). Often closed-loop systems control based on the error between the current state and the desired state, that is $\mathbf{x}_{err} = \mathbf{x} - \mathbf{x}_{des}$. When we make this variable change into our linear system the desired state becomes our fixed point and the error will be driven to zero if the system is stable (see Section 2.3). The change of variable for a continuous-time system has the form

$$\begin{aligned} \dot{\mathbf{x}}_{err} &= A\mathbf{x}_{err} + B\mathbf{u}(\mathbf{x}_{err}, t) \\ \dot{\mathbf{x}} - \dot{\mathbf{x}}_{des} &= A\mathbf{x}_{err} + B\mathbf{u}(\mathbf{x}_{err}, t) \\ \dot{\mathbf{x}} &= A\mathbf{x}_{err} + B\mathbf{u}(\mathbf{x}_{err}, t) \end{aligned} \tag{2.5}$$

where $\dot{\mathbf{x}}_{des} = 0$ since $\mathbf{x}_{des}$ is constant. Similarly, controlling for error in a discrete-time system has the form

$$\begin{aligned} \mathbf{x}_{n+1} - \mathbf{x}_{des} &= A_d\mathbf{x}_{n_{err}} + B_d\mathbf{u}(\mathbf{x}_{n_{err}}, t) \\ \mathbf{x}_{n+1} &= A_d\mathbf{x}_{n_{err}} + B_d\mathbf{u}(\mathbf{x}_{n_{err}}, t) + \mathbf{x}_{des}. \end{aligned} \tag{2.6}$$

There are many methods for closed-loop control, and a few are listed for the interested reader: PID Control, Model Predictive Control, Linear-Quadratic Regulator (LQR), frequency response, etc. We will be focusing on LQR and discrete LQR (dLQR) (see Section 4.1).

2.2 Discretization

Continuous-time dynamical systems can be thought of as systems that allow for a smooth simulation and representation of real-world systems. However, discrete-time systems/methods are used to control the system because of the nature of calculations. Computers process information in discrete-time in that they sample the changing state of the system in discrete-time to determine the input in the case of closed-loop systems. There are several methods to create a time-discretized version of a continuous-time dynamical system. In this thesis we consider two of those methods. The first is Euler's method, which is the simplest but also the least accurate. The second is the matrix exponential (or zero-order hold), which is more involved and can be exact if the sampling rate is equal to the discrete time step τ. The sampling rate τ is the rate that continuous information is discretized. The time-step τ is a small value that we use to discretize the system (as τ goes to zero the system becomes continuous).

2.2.1 Euler Discretization. Using the following derivative approximation

$$\dot{\mathbf{x}} \approx \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\tau}$$

the Euler discretization of the continuous-time system $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u}$ is given by,

$$\begin{aligned} \frac{\mathbf{x}_{n+1} - \mathbf{x}_n}{\tau} &\approx A\mathbf{x}_n + B\mathbf{u} \\ \mathbf{x}_{n+1} &\approx I\mathbf{x}_n + \tau A\mathbf{x}_n + \tau B\mathbf{u} \\ \mathbf{x}_{n+1} &\approx (I + \tau A)\mathbf{x}_n + \tau B\mathbf{u}. \end{aligned} \tag{2.7}$$

Hence, the new discretized dynamics are,

$$\mathbf{x}_{n+1} = A_d\mathbf{x}_n + B_d\mathbf{u}_n$$

where

$$A_d = (I + \tau A_d), \quad B_d = \tau B. \tag{2.8}$$

2.2.2 Matrix Exponential.

Recall that the derivative of the matrix exponential

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A,$$

which can be used to derive the exponential method. Multiplying e^{-At} to the state-space model in (2.3) we have

$$\begin{aligned}
e^{-At}\dot{\mathbf{x}} &= e^{-At}A\mathbf{x} + e^{-At}B\mathbf{u} \\
\frac{d}{dt}(e^{-At}\mathbf{x}(t)) &= e^{-At}B\mathbf{u}(t) \\
e^{-At}\mathbf{x}(t) - e^{0}\mathbf{x}(0) &= \int_0^t e^{-At_1}B\mathbf{u}(t_1)\,dt_1 \\
\mathbf{x}(t) &= e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-t_1)}B\mathbf{u}(t_1)\,dt_1.
\end{aligned} \tag{2.9}$$

This formulation is known as the analytical solution to the continuous model. From here we can discretize the system by definiting $\mathbf{x}_k = \mathbf{x}(k\tau)$ where τ is the sampling time. Then, the new discretized equation is,

$$\mathbf{x}_k = e^{Ak\tau}\mathbf{x}_0 + \int_0^{k\tau} e^{A(k\tau - t_1)}B\mathbf{u}(t_1)\,dt_1. \tag{2.10}$$

For the $(k+1)^{st}$ time step we can use this equation to write this in terms of the previous state

$$\begin{aligned}
\mathbf{x}_{k+1} &= e^{A(k+1)\tau}\mathbf{x}_0 + \int_0^{(k+1)\tau} e^{A((k+1)\tau - t_1)}B\mathbf{u}(t_1)\,dt_1 \\
&= e^{A\tau}[e^{Ak\tau}\mathbf{x}_0 + \int_0^{k\tau} e^{A(k\tau - t_1)}B\mathbf{u}(t_1)\,dt_1] + \int_{k\tau}^{(k+1)\tau} e^{A(k\tau + \tau - t_1)}B\mathbf{u}(t_1)\,dt_1.
\end{aligned} \tag{2.11}$$

Subbing equation (2.10) into (2.11), doing the variable substitution $\nu(t_1) = k\tau + \tau - t_1$, and assuming $\mathbf{u}$ does not depend on t_1 we get

$$\begin{aligned}
\mathbf{x}_{k+1} &= e^{A\tau}\mathbf{x}_k - \left(\int_{\nu(k\tau)}^{\nu((k+1)\tau)} e^{A\nu}\, d\nu\right) B\mathbf{u}_k \\
&= e^{A\tau}\mathbf{x}_k - \left(\int_{\tau}^{0} e^{A\nu}\, d\nu\right) B\mathbf{u}_k \\
&= e^{A\tau}\mathbf{x}_k + \left(\int_{0}^{\tau} e^{A\nu}\, d\nu\right) B\mathbf{u}_k \\
&= e^{A\tau}\mathbf{x}_k + A^{-1}(e^{A\tau} - I)B\mathbf{u}_k.
\end{aligned} \tag{2.12}$$

This is the exact solution to the discretization problem, meaning it is the most accurate (as long as the sampling rate of the state is equal to the discretization). Though this method is the most accurate, it is also more computationally expensive to compute because of the matrix exponential operation. On top of that, if A is ill-conditioned, the inverse will prove to be difficult to compute and could introduce error to the system.

In Chapter 3 we consider how the Euler discretization can be used to define intrinsic stability for time-discretized versions of continuous-time systems, and consider the possibility to do the same for the matrix exponential discretization method.

2.3 Stability of Linear Systems

For a linear dynamical system it is relatively simple to determine the system's stability. The matrix A in the state-space Equation (2.3) (or A_d in Equation 2.4 for discrete-time) is the only matrix of interest when determining the stability of the system. This matrix effectively gives the *dynamics* of the system. Here a system is referred to as stable if the system will converge over time to a unique state irrespective of its initial state. This state is known as the system's globally attracting fixed point (see, for example, the paragraph following Definition 3.1). There is no local condition to this stability because the problem is linear. A continuous-time linear system is stable if the real part of the eigenvalues of the associated matrix A are less than zero. That is,

$$\mathbb{R}(\lambda_i) < 0 \quad \text{for} \quad \lambda_1, \ldots, \lambda_n \in \sigma(A) \tag{2.13}$$

where $\sigma(A)$ are the eigenvalues of A. Similarly, for a discrete-time linear system the modulus of the eigenvalues of the associated matrix A_d must be less than one. That is,

$$|\lambda_i| < 1 \quad \text{for} \quad \lambda_1, \ldots, \lambda_n \in \sigma(A_d). \tag{2.14}$$

In other words, the spectral radius of A_d must be less than 1.

As previously discussed, control is used to drive the system to a desired location. This can be thought of as changing the dynamics of the system to have a stable fixed point of our choosing. The control method has to be tuned to get the desired reaction from the system. This is important because control can potentially destabilize a system. Destabilization of a system is the main focus of the following chapter, however the destabilization considered is due to time-delays rather than control.

Chapter 3. Intrinsic Stability

Before describing the stability of a controlled system, e.g. systems given by either Equation (2.1) or Equation (2.2), we consider the stability of uncontrolled dynamical systems as this lends itself to the stability analysis of closed-loop systems. The specific systems we consider are discrete-time systems. The main reason is that the notion of intrinsic stability (see Definition 3.5) at this point has only been defined for discrete-time systems although here we make strides in extending this notion to continuous-time systems (see Theorem 3.10).

Definition 3.1. ***(Dynamical System)*** Let (X_i, d_i) be a complete metric space for $1 \leq i \leq n$. Let (X, d_{max}) be the complete metric space formed by endowing the product space $X = \bigoplus_{i=1}^{n} X_i$ with the metric

$$d_{max}(\mathbf{x}, \mathbf{y}) = \max_i d_i(x_i, y_i) \quad \text{where } \mathbf{x}, \mathbf{y} \in X \text{ and } \quad x_i, y_i \in X_i.$$

Let $F : X \to X$ be a continuous map, with i^{th} component function $F_i : X \to X_i$ given by

$$F_i = F_i(x_1, x_2, \ldots, x_n) \quad \text{in which} \quad x_j \in X_j \quad \text{for } j = 1, 2, \ldots, n$$

where it is understood that there may be no actual dependence of F_i on x_j. The dynamical system (F, X) generated by iterating the function F on X is called a discrete-time *dynamical system*. If an initial condition $\mathbf{x}^0 \in X$ is given, we define the k^{th} *iterate* of $\mathbf{x}^0$ as $\mathbf{x}^k = F^k(\mathbf{x}^0)$, with orbit $\{F^k(\mathbf{x}^0)\}_{k=0}^{\infty} = \{\mathbf{x}^0, \mathbf{x}^1, \mathbf{x}^2, \ldots\}$ in which $\mathbf{x}^k$ is the state of the system at time $k \geq 0$.

The metric space we focus on for this thesis is $X = \mathbb{R}^n$ with the infinity norm $d_{max} = \|\mathbf{x}\|_\infty = \max_i |x_i|$. Also, we say that a dynamical system (F, X) is *globally stable* if there is an $\mathbf{x}^* \in \mathbb{R}^n$ that no matter the starting condition $\mathbf{x}_0 \in \mathbb{R}^n$ we get $F^k(\mathbf{x}_0) \to \mathbf{x}^*$ as $k \to \infty$. Here, $\mathbf{x}^*$ is refered to as a *globally attracting fixed point*.

3.1 The Lipschitz Matrix of a Dynamical System

To give a sufficient condition under which a system (F, X) is stable, we define a *Lipschitz matrix* (called a *stability matrix* in [4]).

Definition 3.2. ***(Lipschitz Matrix)*** For $F : X \to X$ suppose there are finite constants $a_{ij} \geq 0$ such that

$$d_i(F_i(\mathbf{x}), F_i(\mathbf{y})) \leq \sum_{j=1}^{n} a_{ij} d_j(x_j, y_j) \quad \text{for all } \mathbf{x}, \mathbf{y} \in X.$$

Then we call $A = [a_{ij}] \in \mathbb{R}^{n \times n}$ a *Lipschitz matrix* of the dynamical system (F, X).

It is worth noting that if A is a Lipschitz matrix of a dynamical system then any matrix B for which $A \preceq B$, where $\preceq$ denotes the element-wise inequality, is also a Lipschitz matrix of the system. However, if the function $F : X \to X$ is piecewise differentiable and each $X_i \subseteq \mathbb{R}$ then the matrix $A \in \mathbb{R}^{n \times n}$ given by

$$a_{ij} = \sup_{x \in X} \left| \frac{\partial F_i}{\partial x_j}(\mathbf{x}) \right| \tag{3.1}$$

is the Lipschitz matrix of minimal spectral radius of (F, X). That is, A has the smallest spectral radius of any Lipschitz matrix (see [4]). This will be the Lipschitz matrix we will use in what follows. From a computational point of view, the Lipschitz matrix $A = [a_{ij}]$ of (F, X) can be more straightforward to find by use of Equation (3.1) if the function $F : X \to X$ is differentiable, compared to the more general formulation in Definition 3.2. Furthermore, the following theorem from chapter 8 of [6] expounds on the relation between non-negative matrices and their spectral radii.

Theorem 3.3. *If matrices $A, B \in \mathbb{N}^{n\times n}$ such that $0_{n\times n} \preceq A \preceq B$ then $0 = \rho(0_{n\times n}) \leq \rho(A) \leq \rho(B)$*

It is straightforward to verify that a Lipschitz matrix exists for a dynamical system (F, X) if and only if the mapping F is Lipschitz continuous. Here, the idea is to use the Lipschitz matrix to simplify the stability analysis of nonlinear systems, using the following Theorem 2.3 found in [4]. By way of notation

$$\rho(A) = \max_{\lambda \in \sigma(A)} |\lambda|$$

denotes the spectral radius of a matrix A.

Theorem 3.4. (System Stability) *Let A be a Lipschitz matrix of a dynamical system (F, X). If $\rho(A) < 1$, then (F, X) is stable, i.e. has a globally attracting fixed point.*

This criteria for stability is stronger than the one stated previously in Equation (2.14). With this in place here we have the tools necessary to define *intrinsic stability*, and to extend the theory to controlled systems.

It is worth noting that if we use the Lipschitz matrix A of (F, X) to define the dynamical system $F(\mathbf{x}) = A\mathbf{x}$ then (F, X) is stable if and only if $\rho(A) < 1$. Thus, a Lipschitz matrix of a dynamical system (F, X) can be thought of as the worst-case linear approximation to F. If this approximation has a globally attracting fixed point, then the original dynamical system (F, X) must also be stable. Note, however, that the condition that $\rho(A) < 1$ is sufficient but not necessary for (F, X) to be stable. In fact, this stronger condition implies much more than system stability, so following the convention introduced in [4], we give it the name of *intrinsic stability*.

Definition 3.5. ***(Intrinsic Stability)*** Let $A \in \mathbb{R}^{n\times n}$ be a Lipschitz matrix of a dynamical system (F, X). If $\rho(A) < 1$, then we say (F, X) is *intrinsically stable.*

3.2 Time-Delayed Dynamical Systems

As previously discussed, most real-world dynamical systems are inherently *time-delayed.* This is due to processing time, transmission of information, sampling, etc. Following [2] we introduce delays into a dynamical system (F, X) using a *delay distribution matrix* $D = [d_{ij}]$ where each d_{ij} is a non-negative integer denoting the constant number of discrete time-steps by which the dependence of the component $F_j = F_j(x_1, \ldots, x_n)$ on x_i is delayed by d_{ij} time-steps. The following definition describes what happens to the dynamical system when a delay distribution matrix is introduced.

Definition 3.6. ***(Constant Time-Delayed Dynamical System)***
Let (F, X) be a dynamical system and $D = [d_{ij}] \in \mathbb{N}^{n\times n}$ a delay distribution matrix with $\max_{i,j} d_{ij} \leq L$, a bound on the delay length. Let X_L, the *extension of* X to delay-space, be defined as

$$X_L = \bigoplus_{\ell=0}^{L} \bigoplus_{i=1}^{n} X_{i,\ell} \quad \text{where} \quad X_{i,\ell} = X_i \quad \text{for} \quad 1 \leq i \leq n \quad \text{and} \quad 0 \leq \ell \leq L.$$

Component-wise, define $F_D : X_L \to X_L$ by

$$(F_D)_{i,\ell+1} : X_{i,\ell} \to X_{i,\ell+1} \quad \text{given by the identity map} \quad (F_D)_{i,\ell+1}(x_{i,\ell}) = x_{i,\ell} \tag{3.2}$$

for $0 \leq \ell \leq L - 1$ and

$$(F_D)_{i,0} : \bigoplus_{j=1}^{n} X_{j,d_{ij}} \to X_{i,0} \quad \text{given by} \quad (F_D)_{i,0} = F_i(x_{1,d_{i1}}, x_{2,d_{i2}}, \ldots, x_{n,d_{in}}) \tag{3.3}$$

where $F_i : X \to X_i$ is the i^{th} component function of F for $i = 1, 2, \ldots, n$. Then (F_D, X_L) is the *delayed version* of F corresponding to the fixed-delay distribution D with *delay bound* L.

We order the component spaces of X_L in the following way. If $\mathbf{x} \in X_L$ then

$$\mathbf{x} = [x_{1,0}, x_{2,0}, \ldots, x_{n,0}, x_{1,1}, x_{2,1}, \ldots, x_{n,L}]^T$$

where $x_{i,\ell} \in X_{i,\ell}$ for $i = 1, 2, \ldots, n$ and $\ell = 0, 1, \ldots, L$.

The formalization in Definition 3.6 captures the idea of adding a time-delay to the dynamics of length $d_{ij} \leq L$.

Each X_i is effectively copied L times, and past states of the i^{th} component are passed down this chain by the identity component functions $(F_D)_{i,\ell+1}$ for $0 \leq \ell \leq L-1$ in Equation (3.2). When a state of the i^{th} component has been passed through the chain d_{ij} times over d_{ij} time-steps it then influences the j^{th} component, as described by the entry-wise substitutions of $x_{j,d_{ij}}$ for x_j in $(F_D)_{i,0}$ in Equation (3.3).

In [4], the authors demonstrate that intrinsically stable systems are resilient to the addition of constant time-delays, as is stated in the following theorem.

Theorem 3.7. (Intrinsic Stability and Constant Delays) *Let (F, X) be a dynamical system and $D = [d_{ij}]$ a delay-distribution matrix. Let L satisfy $\max_{i,j} d_{ij} \leq L$. Then (F, X) is intrinsically stable if and only if (F_D, X_L) is intrinsically stable.*

Beyond maintaining stability, we note that any fixed point(s) of an undelayed system (F, X) will also be fixed point(s) of any delayed version (F_D, X_L). This is given in the following proposition, which is proven in the Appendix of [2]. Before stating this proposition, we require the following definition.

Definition 3.8. ***(Extension of a Point to Delay-Space)*** Let $E_L(\mathbf{x}) \in X_L$ be equal to $L + 1$ copies of $\mathbf{x} \in X$ stacked into a single vector, namely

$$E_L(\mathbf{x}) = \begin{bmatrix} \mathbf{x}_0 \\ \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_L \end{bmatrix} \quad \textit{where} \quad \mathbf{x}_\ell = \mathbf{x} \quad \textit{for} \quad 0 \leq \ell \leq L.$$

Proposition 3.9. (Fixed Points of Delayed System) *Let $\mathbf{x}^*$ be a fixed point of a dynamical system (F, X). Then for all delay distributions D with $\max_{ij} d_{ij} \leq L$, $E_L(\mathbf{x}^*)$ is a fixed point of (F_D, X_L).*

As an immediate consequence of Proposition 3.9 and Theorem 3.7, if an undelayed system (F, X) is intrinsically stable with a globally attracting fixed point $\mathbf{x}^* \in X$, then the delayed version (F_D, X_L) will have the "same" globally attracting fixed point $E_L(\mathbf{x}^*)$, in that $\mathbf{x}^*$ is the restriction of $\mathbf{y} = E_L(\mathbf{x}^*)$ to the first n component spaces of X_L. Thus, the asymptotic dynamics of an intrinsically stable system and any version of the system with constant time delays are essentially identical.

3.3 Euler Discrete and Intrinsic Stability

Here we give a criteria describing how intrinsic stability can, roughly speaking, be extended to continuous-time dynamical systems. What we show is that if a continuous-time system satisfies the described criteria then any Euler discretization of the system is intrinsically stable.

Theorem 3.10. ***(B. and Webb)*** *For the continuous-time dynamical system* $\dot{\mathbf{x}} = f(\mathbf{x})$ *with* f *differentiable on* $\mathbb{R}^n$ *let* $\tilde{A} = [\tilde{a}_{ij}] \in \mathbb{R}^{n\times n}$ *be given by*

$$\tilde{a}_{ij} = \begin{cases} \sup_{x\in X} \ \frac{\partial f_i}{\partial x_j}(\mathbf{x}) & \text{if} \quad i = j \\ \sup_{x\in X} \left|\frac{\partial f_i}{\partial x_j}(\mathbf{x})\right| & \text{if} \quad i \neq j. \end{cases} \tag{3.4}$$

If the real part of all eigenvalues of $\tilde{A}$ *are strictly negative then the Euler discretization of* $\dot{\mathbf{x}} = f(\mathbf{x})$ *is intrinsically stable for small enough* τ.

Proof. The Euler discretization of $\dot{\mathbf{x}} = f(\mathbf{x})$ is given by $\mathbf{x}_{n+1} = \mathbf{x}_n + \tau f(\mathbf{x}_n)$ via the derivative approximation in Equation (2.7). Using Equation (3.1), a Lipschitz matrix of this discrete-time dynamical system is then the matrix $M = |(I_n + \tau\tilde{A})|$, since for sufficiently small $\tau > 0$ we have $|1 + \tau a_{ii}| = 1 + \tau a_{ii}$ in the diagonal entries of M. Thus, for small enough τ the matrix $I_n + \tau\tilde{A} \succeq 0$ and we have $M = I_n + \tau\tilde{A}$.

Note, $\sigma(M) = \{1 + \tau\lambda : \lambda \in \sigma(\tilde{A})\}$. The claim is that all eigenvalues of M lie within the unit circle for small enough τ. To see this consider, $\lambda_j = 1 + \tau(a + bi)$ for arbitrary eigenvalue of M. From our assumption on $\tilde{A}$ we have $a < 0$ so,

$$|\lambda_j|^2 = |1 + \tau(a + bi)|^2 = 1 + 2\tau a + \tau^2 a^2 + \tau^2 b^2.$$

For τ small enough we have $\tau^2 << \tau$. Hence, $|\lambda_j| < 1$ since $a < 0$ for small enough τ in which case $\rho(M) < 1$ which means that the Euler discretization of $\dot{\mathbf{x}} = f(\mathbf{x})$ is intrinsically stable. $\square$

Taking Theorem 3.10 together with Proposition 3.9 and Theorem 3.4, the time-delayed continuous-time dynamical system $\dot{\mathbf{x}} = f_D(\mathbf{x})$ with delay distribution $D \in \mathbb{R}^{n\times n}$, defined analogous to (F_D, X_L) in Definition 3.2, will have a globally attracting fixed point for any D if it is discretized using the Euler method so long as $\mathbb{R}(\sigma(\tilde{A})) < 0$ and τ is small enough.

Future work will be to identify under what conditions the Exponential discretization of a continuous-time dynamical system is intrinsically stable analogous to the result in Theorem 3.10.

3.4 Switched Systems

Beyond the constant time-delays introduced by Definition (3.6) a system may also experience time-varying time-delays. In order to define a system with time-varying time-delays, we first define the more general concept of a *switched system*.

Definition 3.11. ***(Switched System)*** Let M be a set of Lipschitz continuous mappings on X, such that for every $F \in M$, (F, X) is a dynamical system. Then we call (M, X) a *switched system* on X. Given some sequence $\{F^{(k)}\}_{k=1}^{\infty} \subset M$, we say that $(\{F^{(k)}\}_{k=1}^{\infty}, X)$ is an instance of (M, X), with orbits determined at time k by the function

$$F^k(\mathbf{x}) = F^{(k)} \circ \ldots \circ F^{(2)} \circ F^{(1)}(\mathbf{x}) \quad \text{for} \quad \mathbf{x} \in X.$$

For the switched system (M, X) we construct a *Lipschitz set* S consisting of a set of $n \times n$ matrices as follows: For each $F \in M$, identify exactly one Lipschitz matrix A of (F, X) to the set S.

In Definition (3.11), (M, X) is an ensemble of dynamical systems formed by taking all possible sequences of mappings $\{F^{(k)}\}_{k=1}^{\infty} \subset M$. For a switched system (M, X), the set S serves an analogous purpose to the Lipschitz matrix A of a dynamical system (F, X).

Since each system in a switches system has its own Lipschitz matrix that determines its respective stability the following extension of the notion of spectral radius is necessary to determine the stability of a switched system.

Definition 3.12. ***(Joint Spectral Radius)*** Given some $\mathbf{z}^0 \in \mathbb{R}^n$ and some set of matrices $S \subset \mathbb{R}^{n\times n}$, let $\mathbf{z}^k = A_k \dots A_2 A_1 \mathbf{z}^0$ for some sequence $\{A_i\}_{i=1}^{\infty} \subset S$. The *joint spectral radius* $\overline{\rho}(S)$ of the set of matrices S is the smallest value $\overline{\rho} \geq 0$ such that for every $\mathbf{z}^0 \in \mathbb{R}^n$ there is some constant $C > 0$ for which

$$||\mathbf{z}^k|| \leq C(\overline{\rho})^k.$$

The following is the main result of [2].

Theorem 3.13. (Intrinsic Stability and Time-Varying Time-Delayed Systems) *Suppose (F, X) is intrinsically stable with $\rho(A) < 1$, where $A \in \mathbb{R}^{n\times n}$ is a Lipschitz matrix of F and $\mathbf{x}^*$ is the system's globally attracting fixed point. Let $L > 0$ and*

$$M_d = \{F_D | D \in \mathbb{N}^{n\times n} \text{ with } \max_{ij} d_{ij} \leq L\}$$

and let S_d be the Lipschitz set of M_d. Then $E_L(\mathbf{x}^*)$ is a globally attracting fixed point of every instance $(\{F_{D^{(k)}}\}_{k=1}^{\infty}, X_L)$ of (M_d, X_L). Furthermore, $\overline{\rho}(S_d) = \rho(A_L) < 1$, where

$$A_L = \begin{bmatrix} \mathbf{0}_{n\times nL} & A \\ \mathbf{I}_{nL\times nL} & \mathbf{0}_{nL\times n} \end{bmatrix}.$$

In the following Chapter our goal is to extend Theorem 3.13 to the case of closed-loop systems under some mild conditions.

Chapter 4. Controlled Systems

In this chapter we first consider the motivating case of control using LQR and DLQR. We then show that it is possible to give sufficient conditions under which a closed-loop system with delays in both its dynamics and control is intrinsically stable. This is our main result (Theorem 5.6).

4.1 LQR and dLQR

The Linear-Quadratic Regulator (LQR) is an optimal control method. This means that the controller is determined by minimizing a *cost function* which depends on both the states and the control. The goal of an LQR system is to bring the system to the origin. There are two variations to LQR based on the length of the horizon. In this thesis we focus on infinite-horizon LQR for both continuous and discrete-LQR (dLQR). The reason being that finite-horizon requires an update to the control solution, which would complicate identifying whether the controlled system is intrinsically stable for all time. We expect that our theory can expand to LQR with finite horizon, but will require further analysis.

The cost functional for a continuous-time system takes the form,

$$J = \int_0^\infty \left(\mathbf{x}^T Q \mathbf{x} + \mathbf{u}^T R \mathbf{u}\right) dt \tag{4.1}$$

and for a discrete-time system,

$$J = \sum_{k=0}^{\infty} \left(\mathbf{x}_k^T Q \mathbf{x}_k + \mathbf{u}_k^T R \mathbf{u}_k\right) \tag{4.2}$$

where Q and R are weight factors chosen by the engineer tuning the control. Both matrices must be positive semi-definite in order for a solution to exist.

The solution to the optimization problem has a closed form solution. First we take a look at the LQR continuous infinite-horizon solution. The input $\mathbf{u}$ is,

$$\mathbf{u} = -K\mathbf{x} \tag{4.3}$$

where $K \in \mathbb{R}^{m \times n}$ is defined as

$$K = R^{-1}(B^T P)$$

and P is the solution to the continuous algebraic Ricatti equation

$$A^T P + PA - PBR^{-1}B^T P + Q = 0. \tag{4.4}$$

The dLQR infinite-horizon solution closed form derivation is slightly different. Starting at the input

$$\mathbf{u}_n = -T\mathbf{x}_n \tag{4.5}$$

where $T \in \mathbb{R}^{m \times n}$ is defined as

$$T = (R + B_d^T P B_d)^{-1}(B_d^T P A_d)$$

and P is the solution to the discrete time algebraic Ricatti equation

$$A_d^T P A_d + (A_d^T P B_d)(R + B_d^T P B_d)^{-1}(B_d^T P A_d) + Q = 0. \tag{4.6}$$

Because of the nature of this control method we can analyze the closed-loop dynamics as a brand new system without inputs. In continuous time we have

$$\begin{aligned} \dot{\mathbf{x}} &= A\mathbf{x} + B\mathbf{u} \\ &= A\mathbf{x} - BK\mathbf{x} \\ &= (A - BK)\mathbf{x} \end{aligned} \tag{4.7}$$

where BK is an outer product. And the new discrete system,

$$\begin{aligned} \mathbf{x}_{n+1} &= A_d\mathbf{x}_n + B_d\mathbf{u}_n \\ &= A_d\mathbf{x}_n - B_d T\mathbf{x}_n \\ &= (A_d - B_d T)\mathbf{x}_n \end{aligned} \tag{4.8}$$

where $B_d T$ is an outer product.

These closed-loop systems will drive state to the **0** state. Recall from Section 2.1, Equations (2.5) and (2.6), that we control for the error for closed-loop systems. LQR and dLQR are used in Chapter 6 to control real systems in a way that focuses on intrinsic stability.

Chapter 5. Main Result

The closed-loop systems we consider have an additive relation between dynamics and control. Hence, when we consider a closed-loop system (F, X) with *additive control* we can write

$$F(\mathbf{x}) = G(\mathbf{x}) + U(\mathbf{x}). \tag{5.1}$$

Note that for LQR, $G = A$ and $U = BK$, and for dLQR, $G = A_d$ and $U = B_d T$ (see the second equations in both (4.7) and (4.8)).

In what follows we analyse closed-loop systems with additive control that experience *independently delayed dynamics*. That is, delays of differing lengths that impact the system's dynamics, given by $G(\mathbf{x})$, and the system's control given by $U(\mathbf{x})$, respectively. Consequently, the delay distribution matrix structure previously considered in [2] needs to be extended. We let the delay distribution for closed-loop systems with additive control be given by $\vec{D} = (D, C)$ where $D, C \in \mathbb{N}^{n\times n}$ and $\max_{i,j}\{d_{ij}, c_{ij}\} \leq L$. Here D delays the system's dynamics and C delays the system's control. The result is the *delayed closed-loop system*

$$F_{\vec{D}}(\mathbf{x}) = G_D(\mathbf{x}) + U_C(\mathbf{x}) \tag{5.2}$$

with constant time-delays.

In the following section our goal is to extend the results of Chapter 3 to delayed closed-loop systems.

5.1 Splitting Dynamics from Control Via Delays

Given a controlled system (F, X) where $F(\mathbf{x}) = G(\mathbf{x}) + U(\mathbf{x})$, where G describes the system's dynamics, and U to the system's control, our objective is to give a criteria describing when the controlled system (F, X) is resilient to delays in both its dynamics and control.

Definition 5.1. We say the delay distribution $\vec{D}^{(k)} = (D^k, C^k)$ maintains the interactions

$I = \{(i,j) : 1 \leq i,j \leq n\}$ if $D^k_{ij} = C^k_{ij}$ for all $(i,j) \in I$ and $k \geq 1$.

The term *maintains* refers to the case when dynamics and control are equally delayed. In this case, the dynamics and the control are coupled and there is no need to split these terms when checking if the system is intrinsically stable.

To prove a result similar to that of Main Result 2 in [2] (and Theorem (3.13) in this thesis), for closed-loop system of the form given by Equation (5.2) we need a Lipschitz matrix A of (F,X), and a Lipschitz matrix $A_{\vec{D}}$ for $(F_{\vec{D}}, X_L)$.

Lemma 5.2. ***(B. and Webb)*** *For the closed-loop system $F(\mathbf{x}) = G(\mathbf{x}) + U(\mathbf{x})$ suppose $\vec{D}^{(k)} = (D^k, C^k)$ maintains the interactions in set I. If we define the entries of the matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$ by*

$$a_{ij} = \begin{cases} \sup_{x\in X} \left|\frac{\partial F_i}{\partial x_j}(\mathbf{x})\right| & for\ (i,j) \in I \\ \sup_{x\in X} \left|\frac{\partial G_i}{\partial x_j}(\mathbf{x})\right| + \sup_{x\in X} \left|\frac{\partial U_i}{\partial x_j}(\mathbf{x})\right| & for\ (i,j) \notin I \end{cases} \tag{5.3}$$

then A is a Lipschitz matrix of F.

Proof. The matrix $\tilde{A}$ defined element-wise by,

$$\tilde{a}_{ij} = \sup_{x\in X} \left|\frac{\partial F_i}{\partial x_j}(\mathbf{x})\right|$$

is a Lipschitz matrix with the smallest spectral radius. Thus, any matrix A is a Lipschitz matrix of F if $\tilde{A} \preceq A$. Note, by the triangle inequality that $\tilde{a}_{ij} \leq a_{ij}$, with equality only when $(i,j) \in I$. Thus, we have,

$$\begin{aligned} d_i(F_i(\mathbf{x}), F_i(\mathbf{y})) &\leq \sum_{j=1}^n \tilde{a}_{ij} d_j(x_j, y_j) \\ &\leq \sum_{j=1}^n a_{ij} d_j(x_j, y_j) \quad \text{for all } \mathbf{x}, \mathbf{y} \in X. \end{aligned}$$

Hence, A is a Lipschitz matrix of (F, X). □

Lemma 5.3. ***(B. and Webb)*** *Given the delay distribution $\vec{D} = (D, C)$, let $F_{\vec{D}}(\mathbf{x}) =$*

$G_D(\mathbf{x}) + U_C(\mathbf{x})$. *Define* $A_\ell \in \mathbb{R}^{n\times n}$ *for* $\ell \in \{0, 1, \ldots, L\}$ *by*

$$(A_\ell)_{ij} = \begin{cases} \sup_{x\in X} \left|\frac{\partial F_i}{\partial x_j}(\mathbf{x})\right| & \text{if } D_{ij} = C_{ij} = \ell \\ \sup_{x\in X} \left|\frac{\partial G_i}{\partial x_j}(\mathbf{x})\right| & \text{if } D_{ij} = \ell,\ C_{ij} \neq \ell \\ \sup_{x\in X} \left|\frac{\partial U_i}{\partial x_j}(\mathbf{x})\right| & \text{if } D_{ij} \neq \ell,\ C_{ij} = \ell \\ 0 & \text{if } D_{ij} \neq \ell,\ C_{ij} \neq \ell \end{cases}. \tag{5.4}$$

Let $A_{\vec{D}} \in \mathbb{R}^{(L+1)n\times(L+1)n}$ *be the matrix*

$$A_{\vec{D}} = \begin{bmatrix} A_0 & A_1 & \ldots & & A_L \\ I_n & 0 & \ldots & & 0 \\ 0 & \ddots & \ddots & \ldots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & \ddots & 0 & I_n & 0 \end{bmatrix}. \tag{5.5}$$

Then $A_{\vec{D}}$ *is a Lipschitz matrix for* $(F_{\vec{D}}, X_L)$.

Proof. Recall that for $\mathbf{x} \in X_L$, we order the components $x_{i,\ell}$ of $\mathbf{x}$ as

$$\mathbf{x} = [x_{1,0}, x_{2,0}, \ldots, x_{n,0}, x_{1,1}, x_{2,1}, \ldots, x_{n,L}]^T$$

where $x_{i,\ell} \in X_{i,\ell}$ for $i = 1, 2, \ldots, n$ and $\ell = 0, 1, \ldots, L$. Let $\mathbf{x}, \mathbf{y} \in X$ be given, and let $m_{ij} = \max\{d_{ij}, c_{ij}\} \leq L$. Then,

$$\begin{aligned} d_{i,0}\left((F_{\vec{D}})_{i,0})(\mathbf{x}), (F_{\vec{D}})_{i,0})(\mathbf{y})\right) &= d_i\big[G_i(x_{1,d_{i1}}, x_{2,d_{i2}}, \ldots, x_{n,d_{in}}) + U_i(x_{1,c_{i1}}, x_{2,c_{i2}}, \ldots, x_{n,c_{in}}), \\ &\qquad G_i(y_{1,d_{i1}}, y_{2,d_{i2}}, \ldots, y_{n,d_{in}}) + U_i(y_{1,c_{i1}}, y_{2,c_{i2}}, \ldots, y_{n,c_{in}}))\big] \\ &\leq \sum_{j=1}^n a_{ij} d_j(x_{j,m_{ij}}, y_{j,m_{ij}}) = \sum_{\ell=0}^L \sum_{j=1}^n a_{ij} 1_{m_{ij}=\ell} d_{j,\ell}(x_{j,\ell}, y_{j,\ell}) \end{aligned}$$

where

$$1_{d_{ij}=\ell} = \begin{cases} 1 & \text{if } d_{ij} = \ell \\ 0 & \text{otherwise.} \end{cases} \quad \text{for } 0 \leq \ell \leq L.$$

This matches the first n rows of $A_{\vec{D}}$ i.e. the first block row in $A_{\vec{D}}$. For $\ell \geq 1$,

$$d_{i,\ell}\left((F_{\vec{D}})_{i,\ell})(\mathbf{x}), (F_{\vec{D}})_{i,\ell})(\mathbf{y})\right) = d_{i,\ell-1}(x_{i,\ell-1}, y_{i,\ell-1}),$$

which yields the identity matrices I_n in A_D. □

With this in place we can prove the following result analogous to Theorem 3.7 for delayed closed-loop systems.

Theorem 5.4. ***(B. and Webb: Intrinsic Stability and Closed-Loop Systems with Constant Time-Delays)*** *Let (F, X) be a closed-loop system with $F(\mathbf{x}) = G(\mathbf{x}) + U(\mathbf{x})$ differentiable on $X \subseteq \mathbb{R}^n$. For the delay distribution $\vec{D} = (C, D)$ let $A \in \mathbb{R}^{n\times n}$ be defined by Equation (5.3). Then $\rho(A) < 1$ if and only if $\rho(A_{\vec{D}}) < 1$.*

Proof. From Lemma 3 in [7] it follows that $\rho(A_{\vec{D}}) < 1$ if and only if $\rho\left(\sum_{i=0}^{L} A_i\right) < 1$. Since $\sum_{i=0}^{L} A_i = A$ then the result follows. □

Now we extend Proposition 3.9 to delayed closed-loop systems.

Proposition 5.5. ***(B. and Webb: Fixed Points of Delayed Closed-Loop Systems)*** *Let $\mathbf{x}^*$ be a fixed point of the closed loop system (F, X) where $F(\mathbf{x}) = G(\mathbf{x}) + U(\mathbf{x})$. Then for any delay distribution $\vec{D} = (D, C)$ with $\max_{i,j}(d_{ij}, c_{ij}) \leq L$, $E_L(\mathbf{x}^*)$ is a fixed point of $(F_{\vec{D}}, X_L)$.*

Proof. Let $\mathbf{x}^* = [x_1^*, x_2^*, \ldots, x_n^*]^T$ be a fixed point of a dynamical network (F, X). Then by definition

$$F_i(x_1^*, x_2^*, \ldots, x_n^*) = G_i(x_1^*, x_2^*, \ldots, x_n^*) + U_i(x_1^*, x_2^*, \ldots, x_n^*) = x_i^* \quad \textit{for all} \quad 1 \leq i \leq n.$$

Let $\vec{D}$ be a delay distribution with $\max_{ij}\{d_{ij}, c_{ij}\} \leq L$ for some finite $L > 0$. With the usual ordering of the component spaces of $\mathbf{x} \in X_L$, we have

$$F_{\vec{D}}(E_L(\mathbf{x}^*)) = \begin{bmatrix} (F_{\vec{D}})_{1,0}(x_1^*, x_2^*, \ldots, x_n^*) \\ (F_{\vec{D}})_{2,0}(x_1^*, x_2^*, \ldots, x_n^*) \\ \vdots \\ (F_{\vec{D}})_{n,0}(x_1^*, x_2^*, \ldots, x_n^*) \\ (F_{\vec{D}})_{1,1}(x_1^*) \\ (F_{\vec{D}})_{2,1}(x_2^*) \\ \vdots \\ (F_{\vec{D}})_{n,L}(x_n^*) \end{bmatrix}$$

$$= \begin{bmatrix} G_1(x_1^*, x_2^*, \ldots, x_n^*) + U_1(x_1^*, x_2^*, \ldots, x_n^*) \\ G_2(x_1^*, x_2^*, \ldots, x_n^*) + U_2(x_1^*, x_2^*, \ldots, x_n^*) \\ \vdots \\ G_n(x_1^*, x_2^*, \ldots, x_n^*) + U_n(x_1^*, x_2^*, \ldots, x_n^*) \\ x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix}$$

$$= \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \\ x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix} = E_L(\mathbf{x}^*).$$

Hence, the extended fixed point $E_L(x^*)$ is a fixed point of $(F_{\vec{D}}, X_L)$. □

Thus, we have that extended fixed points $E_L(\mathbf{x}^*) \in \mathbb{R}^{(L+1)n}$ of closed-loop systems are fixed points of time-delayed versions of these systems. Now we proceed by extending the theory of intrinsic stability with time-varying time-delays to closed-loop systems.

Theorem 5.6. ***(B. and Webb: Intrinsic Stability for Closed-Loop Systems with Time-Varying Time-Delays)*** *For the closed-loop system* (F, X) *where* $F(\mathbf{x}) = G(\mathbf{x}) + U(\mathbf{x})$ *is differentiable on* $X \subseteq \mathbb{R}^n$ *suppose* $\vec{D}^{(k)} = (D^{(k)}, C^{(k)})$ *maintains the set* I*. Let the Lipschitz matrix* $A \in \mathbb{R}^{n \times n}$ *be the Lipschitz matrix given by Equation (5.3) and assume that* $\rho(A) < 1$ *so that* (F, X) *has a globally attracting point* $\mathbf{x}^* \in \mathbb{R}^n$*. Let* $L > 0$ *and*

$$M_d = \{F_{\vec{D}} | \vec{D} = (D, C) \quad \textit{with} \quad \max_{i,j}\{d_{ij}, c_{ij}\} \leq L\}, \tag{5.6}$$

and let S_d *be the Lipschitz set of* M_d*. Then,* $E_L(\mathbf{x}^*)$ *is a globally attracting fixed point of every instance* $(\{F_{\vec{D}^{(k)}}\}_{k=1}^{\infty}, X_L)$ *where* $F_{\vec{D}^{(k)}}(\mathbf{x}) = G_{D^{(k)}}(\mathbf{x}) + U_{C^{(k)}}(\mathbf{x})$*.*
Additionally, $\rho(\bar{S}_d) = \rho(A_{\vec{L}}) < 1$*, where*

$$A_{\vec{L}} = \begin{bmatrix} \mathbf{0}_{n \times nL} & A \\ \mathbf{I}_{nL \times nL} & \mathbf{0}_{nL \times n} \end{bmatrix} \tag{5.7}$$

Proof. Consider the matrix $A_{\vec{D}}$ (5.5) in lemma (5.3) for some $\vec{D}$. Recall, by the result inLlemma 5.3, that this is a Lipschitz matrix of F and therefore $A_{\vec{D}} \in S_d$. Note also that $A = \sum_{\ell=0}^{L} A_\ell$ for any $\vec{D}$. Since we assume that $\rho(A) < 1$ then Lemma 3.3 of [3] implies $\rho(A) \leq \rho(A_{\vec{D}}) < 1$. Let $A_{\vec{L}}$ be given by Equation (5.7). Similarly, if $\rho(A_{\vec{D}}) < 1$ then Lemma 3.3 of [3] implies $\rho(A_{\vec{D}}) < \rho(A_{\vec{L}}) < 1$. Lastly, by Proposition (3.9) we have that $E_L(\mathbf{x}^*)$ is a globally attracting fixed point of every instance $(\{F_{\vec{D}^{(k)}}\}_{k=1}^{\infty}, X_L)$. Theorem 5.6 then follows by a direct modification of the proof of Main Result in [2]. □

Chapter 6. Examples

In this Chapter we consider a number of applications of the theory we developed in Chapter 5. Specifically, we focus on two controlled systems. First, we consider the control of a mass-spring-damper system. In our second example we consider the control of a circuit. These systems are simple cases of mechanical and electrical systems, respectively, but we note that our theory can be scaled to more complex systems. Other general systems we hope to consider in the future is the control of fluid and thermal systems.

6.1 Mass-Spring-Damper-System

A mass-spring-damper is a simple two-dimensional mechanical system. The system's equations of motion are derived using the Newtonian physics equation $F = ma$. There are three forces acting on the mass of the system. The spring constant k generates a force as the spring is lengthened or contracted. The damper b slows the system based on the speed $\dot{z}$ of the mass, and an input force F drives the movement of the system. The dynamics of the system is given by

$$\frac{d\dot{z}}{dt} = \frac{F - \dot{z}b - zk}{m}$$

$$\frac{dz}{dt} = \dot{z}$$

where z is the distance from the equilibrium of the spring. We can formulate the system as a state-space model by taking the Jacobian which results in the system

$$\dot{\mathbf{x}} = \begin{bmatrix} -b/m & -k/m \\ 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1/m \\ 0 \end{bmatrix} F$$

where,

$$\mathbf{x} = \begin{bmatrix} \dot{z} \\ z \end{bmatrix}.$$

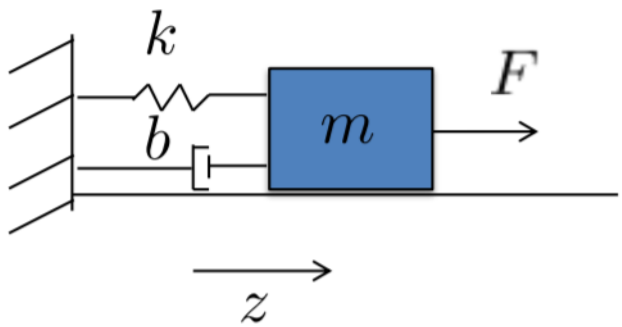

Figure 6.1: Forces acting on a mass-spring-damper system. The forces are produced by the spring with spring constant k, the damper with damper coefficient b, and input force F. The displacement z measures the distance of the mass from spring at rest.

Using the values $m = 5\,kg$, $k = 3\,N/m$, $b = 0.5\,N\frac{s}{m}$ we have the system:

$$A = \begin{bmatrix} -0.1 & -0.6 \\ 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}.$$

The system without control has eigenvalues $\{-0.05 + 0.77j,\ -0.05 - 0.77j\}$. Recall that this implies that the system is stable by the condition in Equation (2.13). Using the results from Theorem 3.10, we can check if this system is intrinsically stable before discretization by finding the eigenvalues of the matrix $\tilde{A}$ as defined in Equation (3.4). In this particular case $\tilde{A}$ has eigenvalues $\sigma(\tilde{A}) = \{-0.826,\ 0.726\}$. Hence, we cannot conclude that the Euler discretized version of this system is not intrinsically stable.

We try to intrinsically stabilize the system by implementing LQR control. The control matrix BK (from Equation (4.7)) is

$$BK = \begin{bmatrix} 0.2k_1 & 0.2k_2 \\ 0 & 0 \end{bmatrix}$$

and the closed-loop dynamics plus control is given by

$$A - BK = \begin{bmatrix} -0.1 - 0.2k_1 & -0.6 - 0.2k_2 \\ 1 & 0 \end{bmatrix}.$$

Theorem 3.10 does not have a clear result here because k_1 and k_2 depend on LQR tuning.

Using Equation (2.7) we write the discretized system using the Euler discretization which results in the discrete-time system

$$\mathbf{x}_{n+1} = \left(I_n + \tau \begin{bmatrix} -0.1 - 0.2k_1 & -0.6 - 0.2k_2 \\ 1 & 0 \end{bmatrix} \right) \mathbf{x}_n$$

Unfortunately, this matrix is not intrinsically stable for any values of k_1 and k_2. We show this by considering the matrix

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

with spectral radius equal to 1, and which is element-wise less than or equal to $|I_n + \tau(A - BK)|$ for all K and τ. Hence, by Theorem 3.3 we know $\rho(|I_n + \tau(A - BK)|) \geq 1$ and therefore not intrinsically stable. This result that the Euler discretized mass-spring-damper system cannot be intrinsically stable is a special case of the following theorem.

Theorem 6.1. ***(B. and Webb)*** *Suppose the continuous-time closed-loop system has the form $\dot{\mathbf{x}} = g(\mathbf{x}) + BK\mathbf{x}$ where $g(\mathbf{x})$ is differentiable on $X \subset \mathbb{R}^n$ where $B = [b_i] \in \mathbb{R}^n$. If*

$$\sup_{\mathbf{x} \in X} \frac{\partial g_i}{\partial x_i}(\mathbf{x}) \geq 0 \quad \text{for some } i \in \{1, \ldots, n\}$$

and $b_i = \mathbf{0}$ then it is not possible to find a matrix K such that the Euler discretization of this system is intrinsically stable.

Proof. Consider the Euler discretized system $\mathbf{x}_{n+1} = \mathbf{x}_n + \tau(g(\mathbf{x}_n) + BK\mathbf{x_n})$ where $\tau > 0$ and let $A \in \mathbb{R}^{n \times n}$ be the Lipscitz matrix of this system. Now, assume $\sup_{\mathbf{x} \in X} \frac{\partial g_i}{\partial x_i}(\mathbf{x}) = v \geq 0$ and $b_i = \mathbf{0}$ for some $i \in \{1, \ldots, n\}$. Then, $b_i k_i = 0$ where k_i is the i^{th} column of K. Therefore, the $(i,i)^{th}$ entry of A is equal to $1 + \tau v \geq 1$. Now consider the sparse matrix $L \in \mathbb{R}^{n \times n}$ with 0's everywhere except for the $(i,i)^{th}$ entry which equals 1. Note that $L \preceq A$ and by Theorem 3.3, we have that $1 = \rho(L) \leq \rho(A)$, and hence, the Euler discretized version of this system is not intrinsically stable for any values of τ or K. $\square$

We now consider the matrix exponential discretization (see Equation (2.12)) of the orig-

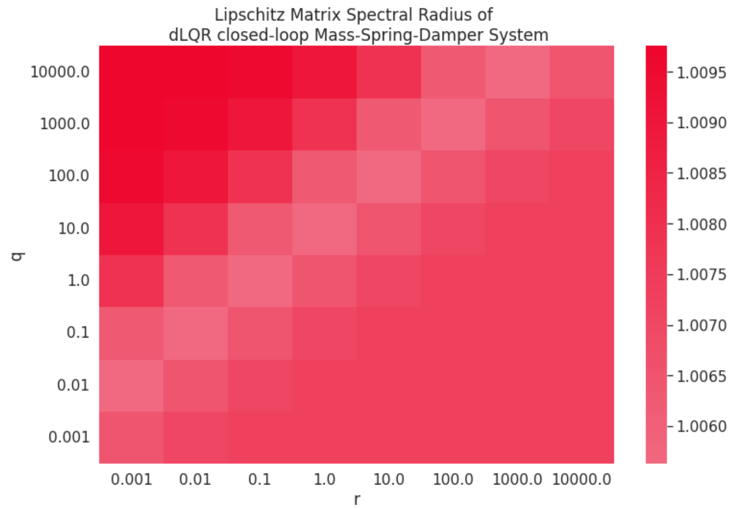

Figure 6.2: Parameter search for values of Q and R to minimize spectral radius of Lipschitz matrix of mass-spring-damper system where $Q = I_n q$ and $R = I_m r$. Note that there is no spectral radius less than 1 and therefore the closed-loop system is not intrinsically stable with dLQR control.

inal mass-spring-damper system without control. For computational purposes we will fix $\tau = 0.01$, which results in values

$$A_d = \begin{bmatrix} 0.99897052 & -0.00599694 \\ 0.0099949 & 0.99997001 \end{bmatrix}, \quad B_d = \begin{bmatrix} 1.99898034e-03 \\ 9.99661752e-06 \end{bmatrix}.$$

For this system we have $\rho(|A|) = 1.0072284$ which means the exponential matrix discretization results in a discrete-time system that is also not intrinsically stable. Curiously, the matrix B_d has values in both entries, so it might be possible to intrinsically stabilize through dLQR. We do a parameter search over values of Q and R in Equations (4.1) and (4.2) to minimize the spectral radius of the Lipschitz matrix of the closed-loop system. As can be seen in Figure 6.2 varying the values of Q and R does not produce an intrinsically stable mass-spring-damper system. We can choose Q and R to minimize the spectral radius of the Lipschitz matrix of the system which would make the system more resilient to time-delays, but would not guarantee the system won't become unstable due to time-delays.

Something to consider for future work is a control method that allows the user to choose the eigenvalues of the system because, as it turns out, as long as $500 < k_1 < 1000$ and $k_2 > 0$

then the system will be intrinsically stable. For example, $\rho(|A - BK|) = 0.99999998 < 1$ when $k_1 = 600$ and $k_2 = 1$, and therefore intrinsically stable. That is, this system's behavior needs more analysing to draw more informative conclusions.

We conclude that the mass-spring-damper system and closed-loop systems that have the properties described in Theorem 6.1 cannot be intrinsically stable.

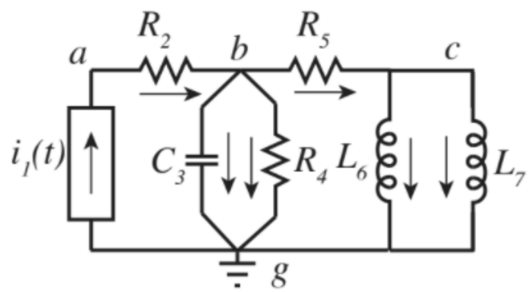

Figure 6.3: A simple circuit system. The input is current $i(t)$, which flows through resistors R, capacitors C, and inductors L. The letters a, b, and c are the places where the system's state (current i and voltage e) take on different values, and g is the ground which means current and voltage are zero.

6.2 Electrical System

Circuits are, for us, an interesting topic in control because states are not solely defined in terms of another state like position and velocity in the case of mass-spring systems. The states for simple circuits are voltage e and current i. The following case study is Example 3 from [5]. The circuit in this example uses three basic elements of passive electrical systems. These elements are resistors R, capacitors C, and inductors L. The relation of these constants to the state of the system are

$$e_1 - e_2 = Ri \tag{6.1}$$

$$i = Cs(e_1 - e_2) \tag{6.2}$$

$$si = \frac{1}{L}(e_1 - e_2) \tag{6.3}$$

where s represents the derivative with respect to time of the variable adjacent to it (i.e. $si = \frac{di}{dt}$).

The following differential equations are the least number of equations needed to model the circuit system as derived in [5],

$$se_3 = \frac{1}{C_3}\left(i_1(t) - \frac{e_3}{R_4} - i_{eq}\right)$$

$$si_{eq} = \frac{1}{L_{eq}} \left(e_3 - i_{eq} R_5 \right).$$

This can be written in state-space form:

$$\dot{\mathbf{x}} = \begin{bmatrix} \frac{-1}{R_4 C_3} & \frac{-1}{C_3} \\ \frac{1}{L_{eq}} & \frac{-R_5}{L_{eq}} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \frac{1}{C_3} \\ 0.0 \end{bmatrix} i_1(t)$$

where

$$\mathbf{x} = \begin{bmatrix} e_3 \\ i_{eq} \end{bmatrix}$$

For this example we will consider two cases. One where the system is intrinsically stable without control, and another where it is only stable and it is made intrinsically stable through control.

6.2.1 Case 1: An Intrinsically Stable System.

We use the following parameter values: $L_{eq} = .8,\ C_3 = 2,\ R_4 = 2,\ R_5 = 3$. The dynamics of this system are

$$A = \begin{bmatrix} -0.25 & -0.5 \\ 1.25 & -3.75 \end{bmatrix}$$

with Lipschitz matrix $\tilde{A}$, as defined in Equation (3.4). Note that $\sigma(\tilde{A}) = \{-0.078, -3.9203\}$ which implies the Euler discretized version of this system is intrinsically stable by Theorem 3.10. So, we take the closed-loop system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bi_1(t), \quad B = \begin{bmatrix} 0.5 \\ 0.0 \end{bmatrix}$$

and proceed by using Euler discretization to get the discrete-time system in Equation (2.7). Here we get

$$A_d = \begin{bmatrix} 1 - 0.25\tau & -0.5\tau \\ 1.25\tau & 1 - 3.75\tau \end{bmatrix}, \quad B_d = \tau \begin{bmatrix} 0.5 \\ 0.0 \end{bmatrix}.$$

Now the goal is to control this system while keeping the dynamics intrinsically stable.

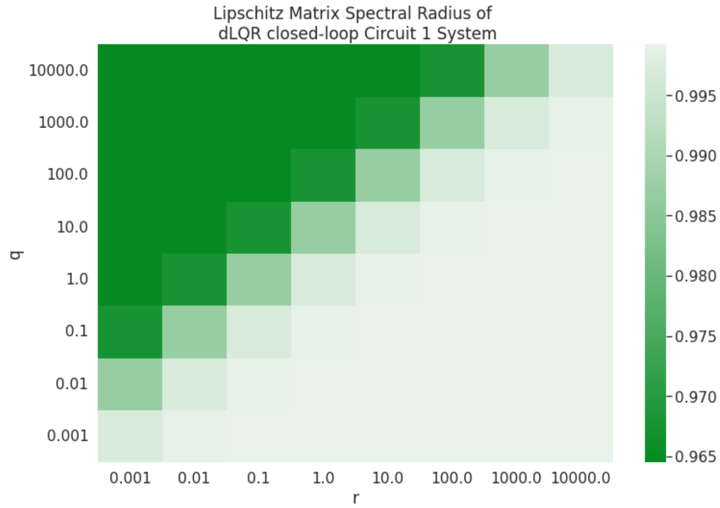

Figure 6.4: Parameter search for values of Q and R to minimize spectral radius of Lipschitz matrix of circuit (A) system where $Q = I_n q$ and $R = I_m r$. Note that all values of q and r produce a closed-loop system that is intrinsically stable with dLQR control.

We use dLQR so the input becomes $u_k = -Tx_k$ where $T = [t_1, t_2]$. This results in the closed loop system

$$A_d - B_d T = \begin{bmatrix} 1-(0.25+0.5t_1)\tau & (-0.5-0.5t_2)\tau \\ 1.25\tau & 1-3.75\tau \end{bmatrix}.$$

At this point we know that for small enough τ the open-loop system is intrinsically stable, so we have to check if dLQR would cause the system to no longer be intrinsically stable. We let $\tau = 0.01$ and do a parameter search over values of Q and R in Equation (4.2). We see in Figure (6.4) that there are parameters such that the closed-loop dLQR system remains intrinsic stable. We choose $Q = 10I_n$ and $R = 0.1I_m$ to continue our example. The closed-loop system will referred to as A_c where $A_c = A_d - B_d T$ which is equal to

$$A_c = \begin{bmatrix} 0.95067034 & -0.00665859 \\ 0.01196397 & 0.96315311 \end{bmatrix}.$$

From Theorem 3.13 we know A_c is stable in the presence of time-varying time-delays.

Now we will work up to our main result to show that this system is intrinsically stable in the presence of delay distribution of the form $\vec{D} = (D, C)$ where dynamics and control can

I	$\rho(\lvert A \rvert)$
$\{(1,1),(1,2)\}$	0.9674
$\{(1,1)\}$	0.9674
$\{(1,2)\}$	1.0454
$\{\}$	1.0454

Table 6.1: Given $\vec{D}^{(k)} = (D^{(k)}, C^{(k)})$ and we define the set $I = \{(i,j) | d_{ij}^{(k)} = c_{ij}^{(k)} \text{ for } k \geq 0\}$. Note that for Case 1 the second row of $B_d T$ is all 0's and therefore it does not matter if $(2,1),(2,2) \in I$, which is why they are not considered in this table. The table shows the spectral radii for different sets of indices. The spectral radii $\rho(|A|) < 1$ are the systems that are intrinsically stable and therefore stable regardless of time-varying time-delays.

experience delays that are not coupled. The dynamics and control matrix of our system are

$$A_d = \begin{bmatrix} 0.9975 & -0.005 \\ 0.0125 & 0.9625 \end{bmatrix}, \quad B_d T = \begin{bmatrix} 0.0469094645631977 & 0.00160645169761599 \\ 0 & 0 \end{bmatrix}.$$

Whether the dynamics and the control experience different times delays comes from field knowledge, but for our examples we will check all cases. Recall that if $d_{ij} = c_{ij}$ then $(i,j) \in I$ and that the Lipschitz matrix A as defined in Equation (5.3) depends on the set I. Note that the second row of $B_d T$ is $\mathbf{0}$ and hence does not affect the spectral radius of A. Therefore, the only (i,j) entries of interest are $(1,1),(1,2)$. As can be see in Table (6.2) if the dynamics and the control do not maintain the $(1,1)$ interaction then the closed-loop system will no longer be intrinsically stable by Theorem (5.4). Furthermore, given the delay distribution $\vec{D}^{(k)} = (D^{(k)}, C^{(k)})$ if $(1,1) \in I$ for all k, then the closed-loop system is intrinsically stable for all time-varying time-delays by Thorem (5.6).

6.2.2 Case 2: Intrinsically Stable through Control.

We use the following parameter values: $L_{eq} = .8,\ C_3 = 2,\ R_4 = 2,\ R_5 = 1$. The dynamics of this system are

$$A = \begin{bmatrix} -0.25 & -0.5 \\ 1.25 & -1.25 \end{bmatrix}$$

with Lipschitz matrix $\tilde{A}$, as defined in Equation (3.4). Note that $\sigma(\tilde{A}) = \{0.1854, -1.685\}$ which does not implies the Euler discretized version of this system is intrinsically stable by

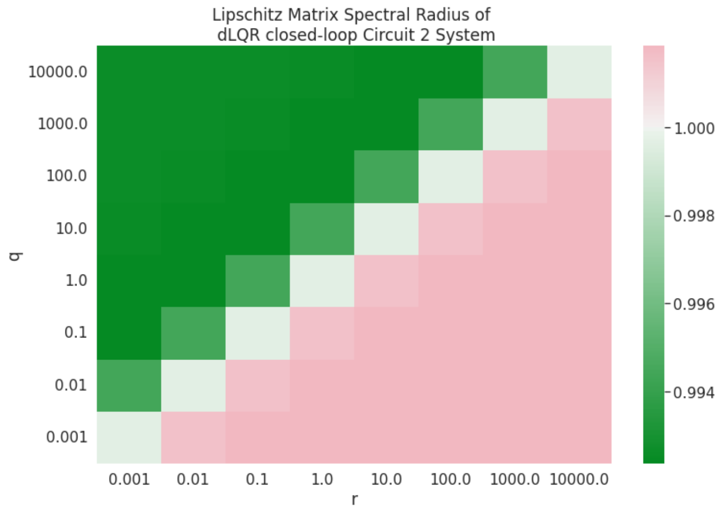

Figure 6.5: Parameter search for values of Q and R to minimize the spectral radius of Lipschitz matrix of Case 2 system where $Q = I_n q$ and $R = I_m r$. Note that depending on the values of Q and R the closed-loop system is intrinsically stable with dLQR control (see green area).

Theorem 3.10. So, we take the closed-loop system

$$\dot{\mathbf{x}} = A\mathbf{x} + Bi_1(t), \quad B = \begin{bmatrix} 0.5 \\ 0.0 \end{bmatrix}$$

and proceed by using Euler discretization to get the discrete-time system in Equation (2.7). We get

$$A_d = \begin{bmatrix} 1 - 0.25\tau & -0.5\tau \\ 1.25\tau & 1 - 1.25\tau \end{bmatrix}, \quad B_d = \tau \begin{bmatrix} 0.5 \\ 0.0 \end{bmatrix}.$$

We let $\tau = 0.01$, the same value given in Case 1, to check if this system is intrinsically stable. Note, $\rho(|A_d|) = 1.002$ hence this system is not intrinsically stable. We attempt to use dLQR to make the system intrinsically stable. We do a parameter search over values of Q and R in Equation (4.2). We see in Figure (6.5) dLQR can produce a closed-loop system that is intrinsically stable. We chose $Q = 10I_n$ and $R = 0.1I_m$ to continue our example. The closed-loop system is given by the matrix A_c where $A_c = A_d - B_d T$ and

$$A_c = \begin{bmatrix} 0.947925999046769 & -0.0172496249884508 \\ 0.0125 & 0.9875 \end{bmatrix}.$$

I	$\rho(\|A\|)$
$\{(1,1),(1,2)\}$	0.9924
$\{(1,1)\}$	0.9924
$\{(1,2)\}$	1.0505
$\{\}$	1.0505

Table 6.2: Given $\vec{D}^{(k)} = (D^{(k)}, C^{(k)})$ and we define the set $I = \{(i,j) | d_{ij}^{(k)} = c_{ij}^{(k)} \text{ for } k \geq 0\}$. Note that for Case 2 the second row of $B_d T$ is all 0's and therefore it does not matter if $(2,1),(2,2) \in I$, which is why they are not considered in this table. The table shows the spectral radii for different sets of indices. The spectral radii $\rho(|A|) < 1$ are the systems that are intrinsically stable and therefore stable regardless of time-varying time-delays.

From Theorem 3.13 we know A_c is stable in the presence of time-varying time-delays.

Now we will work up to our main result to show that this system is intrinsically stable in the presence of delay distribution of the form $\vec{D} = (D, C)$ where dynamics and control can experience delays that are not coupled. The dynamics and control matrix of our system are

$$A_d = \begin{bmatrix} 0.9975 & -0.005 \\ 0.0125 & 0.9875 \end{bmatrix}, \quad B_d T = \begin{bmatrix} 0.0495740009532309 & 0.0122496249884508 \\ 0 & 0 \end{bmatrix}.$$

Again, whether the dynamics and the control experience different times delays comes from field knowledge, but for our examples we will check all cases. Recall that if $d_{ij} = c_{ij}$ then $(i,j) \in I$ and that the Lipschitz matrix A as defined in Equation (5.3) depends on the set I. Note that the second row of $B_d T$ is $\mathbf{0}$ and hence does not affect the spectral radius of A. Hence the only (i,j) entries of interest are $(1,1),(1,2)$. As can be seen in Table (6.1) if the dynamics and the control do not maintain the $(1,1)$ interaction then the closed-loop system would no longer be intrinsically stable by Theorem (5.4). Furthermore, for any delay distribution $\vec{D}^{(k)} = (D^{(k)}, C^{(k)})$, if $(1,1) \in I$ for all k, then the closed-loop system is intrinsically stable for all time-varying time-delays by Thorem (5.6).

Chapter 7. Conclusion

In this thesis we have explored criteria for LQR and dLQR closed-loop dynamical systems to be intrinsically stable. We have shown some Euler discretized physical systems cannot be controlled to be intrinsically stable (e.g. mass spring damper system). Alternatively, these same physical systems can be made intrinsically stable using a different discretization method and other control methods (i.e. pole-placement control). The results of the latter are not explored here because the control method is not LQR or dLQR.

We have shown that other physical systems can be made or remain intrinsically stable using dLQR (e.g. a circuit system). For these systems, we have applied the main result of this thesis, that is, a criteria for determining if the closed-loop system is stable in the presence of decoupled time-delays between the dynamics and the control. Our results show that for each decoupled time-delay the spectral radius of the new Lipschitz matrix is equal or larger than the original Lipschitz matrix. Therefore, the system is less likely to be intrinsically stable for each decoupling between dynamics and control. Furthermore, some decoupling have a larger impact on spectral radius, where even one split in the time-delay the dynamics and control are experiencing can cause the system to no longer be intrinsically stable.

7.1 Future Work

This work primarily focuses on systems that are linear or linearized around a fixed point in the specific case where the fixed point is the global attracting fixed point. Future work to be done on the subject of intrinsic stability for closed-loop systems is to look at systems that need new linearization as the state evolves. Examples of these type of systems include fluid and thermal systems, and even some mechanical systems, like robot arms. The idea of repetitive linearization can be thought of as a series of switched systems as discussed previously in this thesis.

Similarly, the finite-horizon version of LQR and dLQR are control methods that change with respect to time and state. This requires further analysis to determine if the system is intrinsically stable over time under this type of control.

Lastly, it would be interesting to build the analogous version of intrinsic stability for continuous-time systems. We showed a sufficient condition for Euler discretization in Theorem (3.10), which we believe can be done for the Exponential discretization method. Hopefully, a proof that shows an if and only if condition would construct the analogous criteria we alluded to in this thesis.

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