Weak Cayley Table Groups of Crystallographic Groups

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Weak Cayley Table Groups of Crystallographic Groups

Rebeca Ann Paulsen

A dissertation submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of
Doctor of Philosophy

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ABSTRACT

Weak Cayley Table Groups of Crystallographic Groups

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Let $G$ be a group. A weak Cayley table isomorphism $\varphi : G \to G$ is a bijection satisfying two conditions: (i) $\varphi$ sends conjugacy classes to conjugacy classes; and (ii) $\varphi(g_1)\varphi(g_2)$ is conjugate to $\varphi(g_1g_2)$ for all $g_1, g_2 \in G$. The set of all such mappings forms a group $\mathcal{W}(G)$ under composition. We study $\mathcal{W}(G)$ for fifty-six of the two hundred nineteen three-dimensional crystallographic groups $G$ as well as some other groups. These fifty-six groups are related to our previous work on wallpaper groups [HP].

Keywords: crystallographic groups, automorphisms, weak Cayley table isomorphisms
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Let $G$ and $H$ be groups. As defined in [JS, Hu], a \textit{weak Cayley table morphism} is a function $\varphi : G \rightarrow H$ satisfying two conditions: (i) for $g, g' \in G$, $\varphi(g)\varphi(g')$ is conjugate to $\varphi(gg')$; and (ii) $g$ is conjugate to $g'$ in $G$ if and only if $\varphi(g)$ is conjugate to $\varphi(g')$ in $H$. A \textit{weak Cayley table isomorphism} is a bijective weak Cayley table morphism. In the situation where $\varphi$ is bijective and $G = H$, $\varphi$ is a generalization of an automorphism, and we call these mappings \textit{weak Cayley table maps} or \textit{wcts}. The set of all such mappings $\varphi : G \rightarrow G$ forms the \textit{weak Cayley table group} $\mathcal{W}(G) = \mathcal{W}$ under composition and we note that the automorphism group $\text{Aut}(G)$ of $G$ is a subgroup of $\mathcal{W}(G)$. The inverse map $\iota : g \mapsto g^{-1}$ is also an element in $\mathcal{W}$. We define $\mathcal{W}_0 = \langle \text{Aut}(G), \iota \rangle$, and we call the wcts in this subgroup \textit{trivial}. If $\mathcal{W} = \mathcal{W}_0$ we say $\mathcal{W}(G)$ is \textit{trivial}.

Although this will not be relevant to this thesis, we now explain where this idea comes from. If $G = \{g_1 = 1, g_2, \ldots \}$, then the \textit{weak Cayley table} of $G$ is the $|G| \times |G|$ matrix whose $ij$ entry is the conjugacy class of $g_i g_j$. Two finite groups have the same weak Cayley table if and only if they have the same 1- and 2-characters, in the sense of Frobenius [JMS]. Here, for a complex character $\chi$, the corresponding 1-character is $\chi^{(1)} = \chi$, and the 2-character is $\chi^{(2)} : G^2 \rightarrow \mathbb{C}, \chi^{(2)}(x, y) = \chi(x)\chi(y) - \chi(xy)$. The group $\mathcal{W}(G)$ is the group of symmetries of the weak Cayley table of $G$ [JMS]. We note that the notion of a $k$-character only makes sense in the situation where $G$ is finite; however, the weak Cayley table is defined for any group, and $\mathcal{W}(G)$ makes sense for any group.

Given a group $G$ one would like to determine whether or not $\mathcal{W}(G)$ is trivial and to find the \textit{non-trivial} wcts in $\mathcal{W} \setminus \mathcal{W}_0$. Previous research has determined the weak Cayley table groups of various finite and infinite groups. For example, dihedral groups, symmetric groups, most finite irreducible Coxeter groups, most alternating groups, and $\text{PSL}(2, p^n)$ all have trivial weak Cayley table groups [Hu, HN, HN2]. In [HP] we studied the weak Cayley table groups of the seventeen two-dimensional crystallographic groups, also known as \textit{wallpaper}.
groups or plane groups. We found that thirteen wallpaper groups have trivial weak Cayley table groups while four have non-trivial weak Cayley table groups. In that paper we were able to generalize one of these results to show that if $G = A \ltimes \theta C_p$ (here $A$ is abelian, $p$ is an odd prime, and $\theta$ is not trivial) then $W(G)$ is non-trivial.

In this thesis we study the weak Cayley table groups of some of the three-dimensional crystallographic groups. Elements of an $n$-dimensional crystallographic group, also known as a space group, are isometries that act on Euclidean space $\mathbb{E}^n$. There are 219 three-dimensional space groups, up to isomorphism. As explained in [CFHT], 184 of these space groups consist of isometries that preserve one direction (up to orientation) and consequently, the space group can be given a fibration. (See §4.4 for a definition.) If a three-dimensional space group can be fibered, then the action of the space group on the set of fibers, a subspace isomorphic to $\mathbb{E}^2$, gives the action of some wallpaper group on $\mathbb{E}^2$. Thus there is a way to associate any of these 184 space groups to some wallpaper group.

In this thesis we chose to study the fifty-six space groups that correspond to one of the four wallpaper groups that have non-trivial weak Cayley table groups. We found that twenty have trivial weak Cayley table groups while thirty-six of them have non-trivial weak Cayley table groups.

Several of the groups examined here have cyclic quotients. Our study of these cases motivated us to study the general case where a group $G$ contains an abelian normal subgroup $A$ such that $G/A$ is cyclic. We were able to prove that if $2 < |G/A| < \infty$ and if the conjugation action of $G/A$ on $A$ is faithful, then $W(G)$ will be non-trivial (see Theorem 3.3).

### 1.1 Properties of weak Cayley table isomorphisms

The following properties of weak Cayley table isomorphisms in $W(G)$ will be helpful as we determine $W(G)$. In this section, $G, H$ and $K$ are groups. For $g, g' \in G$ we will write $g$ is conjugate to $g'$ in $G$ as $g \sim_G g'$, or $g \sim g'$ if the context is clear. We will write $g^G$ to denote the conjugacy class of $g \in G$. 

Recall that a wct $\varphi$ by definition satisfies $\varphi(g^g) = \varphi(g)^H$ for all $g \in G$. We will say that $\varphi$ preserves conjugacy classes when we refer to this property of wcts.

**Proposition 1.1.** Let $\varphi : G \to H$ be a weak Cayley table isomorphism. Then

(i) $\varphi$ maps the identity in $G$ to the identity in $H$: $\varphi(1_G) = 1_H$.

(ii) $\varphi$ respects inverses: $\varphi(g^{-1}) = \varphi(g)^{-1}$.

(iii) $\varphi^{-1} : H \to G$ is a weak Cayley table isomorphism.

(iv) For the centers of groups we have: $\varphi(Z(G)) = Z(H)$.

(v) $\varphi$ maps involutions to involutions: $g^2 = 1$ implies $\varphi(g)^2 = 1$.

(vi) $\varphi$ maps normal subgroups to normal subgroups: $N \trianglelefteq G$ implies $\varphi(N) \trianglelefteq H$.

(vii) If $N \trianglelefteq G$ and $\varphi(N) = M$, then $\varphi$ maps cosets of $N$ to cosets of $M$. In other words, $\varphi(Ng) = \varphi(N) \varphi(g)$ for all $g \in G$.

(viii) Let $N \trianglelefteq G$. Then $\varphi$ induces a map $\bar{\varphi} : G/N \to H/\varphi(N)$ which is also a weak Cayley table isomorphism.

**Proof.** Proofs of these same results in the case where $G$ and $H$ are finite can also be found in [JMS, p. 398].

(i) Let $\varphi(\alpha) = 1$. Then $\varphi(\alpha \cdot \alpha) \sim 1$. Then $\varphi(\alpha^2) = 1$. Since $\varphi$ is a bijection this implies $\alpha^2 = \alpha$; thus $\alpha = 1$.

(ii) Here $1 = \varphi(g \cdot g^{-1}) \sim \varphi(g)\varphi(g^{-1})$ implies $\varphi(g^{-1}) = \varphi(g)^{-1}$.

(iii) First we note that since $\varphi$ preserves conjugacy classes it is clear that $\varphi^{-1}$ also preserves conjugacy classes. It remains to show that $\varphi^{-1}$ respects the group operation, up to conjugacy. Let $f, g \in H$. Since $\varphi$ is bijective, there exists some $f', g' \in G$ such that $f = \varphi(f')$ and $g = \varphi(g')$. Using $\varphi(f')\varphi(g') \sim \varphi(f'g')$, we have

$$\varphi^{-1}(fg) = \varphi^{-1}(\varphi(f')\varphi(g')) \sim \varphi^{-1}(\varphi(f'g')) = f'g' = \varphi^{-1}(f)\varphi^{-1}(g).$$
(iv) For $g \in G$, we have $|g^G| = |\varphi(g^G)| = |\varphi(g)^H|$. The first equality follows from the bijectivity of $\varphi$ and the second is a consequence of $\varphi$ preserving conjugacy classes. It follows that $|g^G| = 1$ if and only if $|\varphi(g)^H| = 1$, i.e. $g \in Z(G)$ if and only if $\varphi(g) \in Z(H)$.

(v) Here $1 = g^2$ implies $\varphi(1) = \varphi(g \cdot g) \sim \varphi(g)\varphi(g)$. So $\varphi(g)^2 = 1$.

(vi) Since $\varphi$ preserves conjugacy classes, (i) implies that $\varphi$ will map a normal subgroup to a union of conjugacy classes that contains $1$. By the definition of a weak Cayley table map this union is closed under the group operation and by (ii) we have inverses, thus it is a normal subgroup.

(vii) Let $x, y \in G$ satisfy $Nx = Ny$ and so $xy^{-1} \in N$. Then $\varphi(xy^{-1}) \in M$ implies $\varphi(x)\varphi(y)^{-1} = \varphi(x)\varphi(y)^{-1} \in M$ and thus $M\varphi(x) = M\varphi(y)$.

(viii) Let $\varphi(N) = M$. Using (vii) we define $\bar{\varphi} : G/N \to H/M$ by $\bar{\varphi} : Ng \mapsto M\varphi(g)$. First we will show that $\bar{\varphi}$ maps conjugacy classes to conjugacy classes. Let $Ng, Nh \in G/N$ and suppose $Ng \sim_{G/N} Nh$. It follows that $g \sim_H hn$ for some $n \in N$. Since $\varphi$ preserves conjugacy classes, this gives $\varphi(g) \sim_H \varphi(n)\varphi(h)$. It follows that

$$\varphi(N)\varphi(g) \sim_{H/M} \varphi(N)\varphi(n)\varphi(h) = \varphi(N)\varphi(h).$$

In other words, $\bar{\varphi}(Ng) \sim_{H/M} \bar{\varphi}(Nh)$.

Now we show that that the converse also holds, i.e. that $\bar{\varphi}(Ng) \sim_{H/M} \bar{\varphi}(Nh)$ implies that $Ng \sim_{G/N} Nh$. We assume $\bar{\varphi}(Ng) \sim_{H/M} \bar{\varphi}(Nh)$, in other words, $\varphi(N)\varphi(g) \sim_{H/M} \varphi(N)\varphi(h)$. It follows that for some $n \in N$, $\varphi(g) \sim_H \varphi(n)\varphi(h) \sim_H \varphi(n)\varphi(h)$. Since $\varphi$ preserves conjugacy classes, this implies $g \sim_H nh$, thus $Ng \sim_{G/N} Nh = Nh$.

Lastly we show that for $Nx, Ny \in G/N$, $\varphi(Nx \cdot Ny) \sim_{H/M} \varphi(Nx) \cdot \varphi(Ny)$. We have $\varphi(xy) \sim_H \varphi(x)\varphi(y)$ which implies that $\varphi(N)\varphi(xy) \sim_{H/M} \varphi(N)\varphi(x)\varphi(y)$. Therefore,

$$\bar{\varphi}(Nx \cdot Ny) = \bar{\varphi}(Nxy) = \varphi(N)\varphi(xy) \sim_{H/M} \varphi(N)\varphi(x)\varphi(y) = \bar{\varphi}(Nx)\bar{\varphi}(Ny),$$

which shows that $\bar{\varphi}$ is a weak Cayley table map.

Lemma 1.2. Let $\varphi_1 : G \to H$, $\varphi_2 : H \to K$ be weak Cayley table morphisms. Then $\varphi_2 \circ \varphi_1 : G \to K$ is also a weak Cayley table morphism.
Proof. Since \( \varphi_1 \) and \( \varphi_2 \) preserve conjugacy classes, it is clear that \( \varphi_1 \circ \varphi_2 \) also preserves conjugacy classes. It remains to show that \( \varphi_1 \circ \varphi_2 \) respects the group operation, up to conjugacy. Let \( g, h \in G \). Since \( \varphi_2(g \cdot h) \sim \varphi_2(g)\varphi_2(h) \) and \( \varphi_1 \) preserves conjugacy classes, we have

\[
(\varphi_1 \circ \varphi_2)(g \cdot h) \sim \varphi_1(\varphi_2(g)\varphi_2(h)).
\]

The above is conjugate to \( (\varphi_1 \circ \varphi_2)(g) \cdot (\varphi_1 \circ \varphi_2)(h) \) because \( \varphi_1 \) respects the group operation up to conjugacy. \( \square \)

This result, together with Proposition 1.1 \((iii)\), shows that \( W(G) \) is a group.

Lemma 1.3. The inverse map \( \iota : g \mapsto g^{-1} \) is in the center of \( W(G) \).

Proof. Let \( g, h \in G \). It is clear that \( g \sim h \) if and only if \( g^{-1} \sim h^{-1} \), thus \( \iota \) preserves conjugacy classes. We also have

\[
\iota(gh) = h^{-1}g^{-1} \sim g^{-1}h^{-1} = \iota(g)\iota(h).
\]

We conclude that \( \iota \in W(G) \).

The fact that \( \iota \) commutes with any \( \varphi \in W \) follows from Proposition 1.1 \((ii)\): \( (\varphi \circ \iota)(g) = \varphi(g^{-1}) = \varphi(g)^{-1} = (\iota \circ \varphi)(g) \). \( \square \)

An anti-automorphism is a bijective map \( \alpha : G \rightarrow G \) that satisfies \( \alpha(gh) = \alpha(h)\alpha(g) \) for all \( g, h \in G \). Let \( \alpha \) be an anti-automorphism of \( G \) and let \( \psi = \iota \circ \alpha \). Then for any \( g, h \in G \),

\[
\psi(gh) = \alpha(gh)^{-1} = (\alpha(h)\alpha(g))^{-1} = \alpha(g)^{-1}\alpha(h)^{-1} = \psi(g)\psi(h).
\]

Thus \( \psi \) is an automorphism. It follows that any anti-automorphism is the composition of an automorphism with the inverse map. By Lemma 1.2 an anti-automorphism is a wct. This shows that \( W_0(G) \) is the group of all automorphisms and anti-automorphisms of \( G \).
Chapter 2. Preliminary results

2.1 Commutators and commutator subgroups

For \( a, b \in G \) we will write \( a^b = b^{-1}ab \). Also, we define the commutator \((a, b) = a^{-1}b^{-1}ab = a^{-1}a^b \). For subgroups \( H, K \leq G \), we define \([H, K] = \langle (h, k) : h \in H, k \in K \rangle \).

Throughout this thesis, whenever we have a group \( G \) that contains a normal abelian subgroup \( A \) then we will let \( F \) denote a set of coset representatives for \( G/A \). We will also assume that \( A \) is countably generated with generators \( x_1, x_2, \cdots \).

In what follows we will frequently use the Witt-Hall identities, which can be found in [MKS, p. 290]. These are

\[
(a, b) = (b, a)^{-1};
\]

\[
(a, b \cdot c) = (a, c) \cdot (a, b) \cdot ((a, b), c)
\]

\[
= (a, c) \cdot (a, b)^c;
\]

\[
(a \cdot b, c) = (a, c) \cdot ((a, c), b) \cdot (b, c)
\]

\[
= (a, c)^b \cdot (b, c).
\]

Definitions For \( G \) a group with an abelian normal subgroup \( A = \langle x_1, x_2, \cdots \rangle \) we define

\[
K = \langle (x_i, h) : i \in \{1, 2, \cdots \}, h \in F \rangle.
\]

We will show later that \( K \) does not depend on \( F \). For \( g \in G \), we define

\[
K_g = \langle (x_i, g) : i \in \{1, 2, \cdots \} \rangle.
\]

We will see that an equivalent definition of \( K \) is \( K = \{(a, h) : a \in A, h \in F \} \), and also that an equivalent definition of \( K_g \) is \( K_g = \{(a, g) : a \in A \} \). It will frequently be helpful to think of \( K_g \) this way. To show that the definitions are equivalent, we will first prove a lemma.
Lemma 2.1. Let $G$ be a group with an abelian subgroup $A \trianglelefteq G$. For $a, b \in A, g \in G$ we have

$$(ab, g) = (a, g) \cdot (b, g).$$

It follows that for $k \in \mathbb{Z}, (a, g)^k = (a^k, g)$.

Proof. First we note that since $A$ is normal, $(a, g) = a^{-1}a^g \in A$. Since $A$ is abelian, 

$$(a, g) = (b, g) = 1.$$  

Then by Eq. (2.4) we have

$$(a \cdot b, g) = (a, g) \cdot ((a, g), b) \cdot (b, g) = (a, g) \cdot (b, g).$$

The second statement can be proven for $k \in \mathbb{N}$ by letting $b = a$ and applying an inductive argument. Putting $b = a^{-1}$ we see that $(a, g)^{-1} = (a^{-1}, g)$ and thus for $k \in \mathbb{Z}$ we have 

$$(a, g)^k = (a^k, g).$$

We are now ready to prove the equivalence of the two definitions of $K_g$ and $K$.

Lemma 2.2. An equivalent definition of $K_g$ is $K_g = \{(a, g) : a \in A\}$. An equivalent definition of $K$ is $K = \{(a, h) : a \in A, h \in F\}$.

Proof. Since $K_g$ is generated by commutators of the form $(x_i, g)$ where $\langle x_1, x_2, \cdots \rangle = A$, an arbitrary element of $K_g$ is a word in $\{(x_i, g) : i \in \mathbb{N}\}$. By Lemma 2.1, this word can be written as the commutator $(a, g)$ where $a$ is a word in $\{x_i^{\pm 1} : i \in \mathbb{N}\}$. It follows that the two definitions of $K_g$ are equivalent. The same argument shows that the two definitions of $K$ are equivalent.

Clarification about notation: In Lemma 2.3 below we show that $K_f = K_{af}$ for any $a \in A$. It follows that each $K_f$ subgroup corresponds to a coset $Af \in G/A$. Thus, whenever possible, we may denote each $K_g$ subgroup by $K_f$ where $f \in F$ and $Af = Ag$. However, in general, when we write $K_g$ that does not necessarily imply that $g \in F$.

Lemma 2.3. Let $G$ be a group with normal abelian subgroup $A$. If $f_1$ and $f_2$ are in the same coset of $G/A$ (i.e. $Af_1 = Af_2$) then for $a \in A$ we have $(a, f_1) = (a, f_2)$. This implies that $K_{f_1} = K_{f_2}$. In other words, $K_f$ does not depend on which coset representative is used. This also shows that $K = [A, G]$. 

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Proof. Assume \( A f_1 = A f_2 \). Then \( f_2 = b f_1 \) for some \( b \in A \). Then by Eq. (2.2), for \( a \in A \), (since \( A \) is abelian),

\[
(a, f_2) = (a, b f_1) = (a, f_1)(a, b)((a, b), f_1) = (a, f_1),
\]

and the result follows by Lemma 2.2.

In Lemmas 2.4 through 2.11 we prove results about commutators and \( K_f \) subgroups that will be helpful in various upcoming proofs.

**Lemma 2.4.** Let \( G \) be a group with normal abelian subgroup \( A \). If \( G/A \) is abelian, then for \( f, g \in G \) and \( a \in A \), \( (a, f)^g = (a^g, f) \).

**Proof.** Since \( G/A \) is abelian, \( f^g = b f \) for some \( b \in A \). Thus

\[
(a, f)^g = (a^g, f^g) = (a^g, b f) = (a^g, f).
\]

The last equality follows from Lemma 2.3.

**Lemma 2.5.** Let \( G \) be a group with normal abelian subgroup \( A \). For \( f \in F \) we have \( K_f \leq K \leq A \). Additionally, \( K \trianglelefteq G \). If \( G/A \) is abelian, then \( K_f \trianglelefteq G \).

**Proof.** It is clear from the definition of \( K_f \) and \( K \) that \( K_f \leq K \). By Lemma 2.2 any element of \( K \) can be written as \( (a, f) \) for some \( a \in A, f \in F \). Since \( A \trianglelefteq G \) the commutator \( (a, f) = a^{-1} a^f \in A \), thus \( K \leq A \).

Now we show that \( K \trianglelefteq G \). An arbitrary element of \( K \) is \( (b, h) \) for some \( b \in A \), and \( h \in G \). For \( g \in G \), we have \( h^g \in A \tilde{h} \) for some \( \tilde{h} \in F \). Then by Lemma 2.3,

\[
(b, h)^g = (b^g, h^g) = (b^g, \tilde{h}),
\]

which is contained in \( K \) by Lemma 2.2. Thus any conjugate of an element of \( K \) is in \( K \), proving \( K \trianglelefteq G \).

To prove the second statement we now assume \( G/A \) is abelian. Let \( a \in A \) so \( (a, f) \) is an arbitrary element of \( K_f \) by Lemma 2.2. By Lemma 2.4, a conjugate of a commutator \( (a, f)^h \) can be written as \( (a^h, f) \) which is also in \( K_f \), thus \( K_f \trianglelefteq G \).
**Lemma 2.6.** Let $G$ be a group with normal abelian subgroup $A$. Let $f \in G$. Any commutator of the form $(a, f)$ with $a \in A$ can be written as $(b, f^{-1})$ where $b = (af)^{-1}$. This implies that $K_f = K_{f^{-1}}$.

**Proof.** We have

\[
(a, f) = f^{-1} (a^{-1} f^{-1} af) \\
= f^{-1} (fa^{-1} f^{-1} a) f \\
= (f^{-1}, a)^f \\
= (f^{-1}, af) \\
= (f^{-1}, b^{-1}) \quad \text{because } a^f = b^{-1} \\
= (b^{-1}, f^{-1})^{-1} \quad \text{by Eq. (2.1)} \\
= (b, f^{-1}) \quad \text{by Lemma 2.1.}
\]

It follows by Lemma 2.2 that $K_f = K_{f^{-1}}$. □

If $H \leq G$ then $C_G(H)$ will denote the centralizer of $H$ in $G$.

**Lemma 2.7.** Let $G$ be a group with normal abelian subgroup $A$. Assume that $G/A$ is abelian. Let $f, h \in G$ and let $a \in A$. Then

\[
((a, f), h) = ((a, h), f).
\]

This implies that $h \in C_G(K_f)$ if and only if $f \in C_G(K_h)$.

**Proof.** We have

\[
((a, f), h) = (a, f)^{-1} (a, f)^h \\
= (a^{-1}, f) (a^h, f) \quad \text{by Lemmas 2.1 and 2.4} \\
= (a^{-1} a^h, f) \quad \text{by Lemma 2.1} \\
= ((a, h), f).
\]

The second statement can be deduced by using the definition of $K_f$ found in Lemma 2.2 and noting that if $((a, f), h) = 1$ for all $a \in A$ then $((a, h), f) = 1$ for all $a \in A$. The converse
follows from a similar argument.

**Lemma 2.8.** Let $G$ be a group with normal abelian subgroup $A$. Assume that $G/A$ is abelian. Let $f \in G$. Let $j, k \in \mathbb{N}$. If $j \mid k$ then $K_{f^j} \geq K_{f^k}$.

**Proof.** We proceed by proving these statements:

(i) For $h \in G$, $K_{fh} \leq K_f K_h$.

(ii) For $n \in \mathbb{N}$, $K_{f^n} \leq K_f K_{f^{n-1}}$.

(iii) For $n \in \mathbb{N}$, $K_{f^n} \leq K_f$.

Recall that by Lemma 2.2 an arbitrary element of $K_g$ is $(a, g)$ for $a \in A$. We will use this throughout the proof.

We first prove (i). Since $G/A$ is abelian we have $hf \in Afh$, thus by Lemma 2.3 and Eq. (2.3) we have

$$(a, fh) = (a, hf) = (a, f)(a, h)^f.$$  

By Lemma 2.5 $K_h$ is normal thus $(a, h)^f \in K_h$. This shows that $(a, fh)$ (i.e. an arbitrary element of $K_{fh}$) is a product of a commutator in $K_f$ and a commutator in $K_h$, proving (i).

To prove (ii) we apply (i) with $h = f^{n-1}$. We prove (iii) by induction. When $n = 1$ there is nothing to prove. We assume inductively that it is true for $n = k - 1$, i.e. $K_{f^{k-1}} \leq K_f$. Thus we have $K_{f^k} \leq K_f K_{f^{k-1}} \leq K_f K_f = K_f$ where the first containment follows from (ii) and the second containment follows by our inductive assumption. Thus (iii) is true for $n = k$.

Now to prove Lemma 2.8 we apply (iii), supposing that $n = k/j$ (so $k = nj$) and replacing $f$ in (iii) with $f^j$. This gives

$$K_{(f^j)^n} \leq K_{f^j} \text{ i.e. } K_{f^{jn}} \leq K_{f^j} \text{ i.e. } K_{f^k} \leq K_{f^j}.$$  

**Lemma 2.9.** Let $G$ be a group with normal abelian subgroup $A$. Assume $G/A$ be abelian. Let $f \in F, f \notin A$, satisfy $f^n \in A$ for some $n \in \mathbb{N}$. (Assume $f^i \notin A$ for all $1 \leq i < n$.) Let $m \in \mathbb{N}$. If $\gcd(m, n) = d$ then $K_{f^m} = K_{f^d}$.

**Proof.** We have $\gcd(m, n) = d$, thus (by Bezout’s identity) there exists $m' \in \mathbb{Z}$ such that $mm' \equiv d \mod n$. It follows that $K_{f^{mm'}} = K_{f^d}$. Now we apply Lemma 2.8 twice, (noting that
\( d \mid m \): 

\[
K_{fd} \geq K_{fm} \geq K_{fmm'} = K_{fd}.
\]

We conclude that \( K_{fd} = K_{fm} \).

**Lemma 2.10.** Let \( G \) be a group with normal abelian subgroup \( A \). Let \( g_1, g_2 \in G \) and suppose \( g_1, g_2 \in Af \) for some \( f \in F \). Then \( g_1g_2^{-1} \in K_f \) if and only if there exists \( \alpha \in A \) such that \( g_2 = g_1^\alpha \).

**Proof.** Let \( g_1 = af \) and \( g_2 = bf \) for some \( a, b \in A \). Note that \( g_1g_2^{-1} = (af) \cdot (bf)^{-1} = ab^{-1} \). Then by Lemma 2.6 and Lemma 2.2,

\[
g_1g_2^{-1} = ab^{-1} \in K_f \iff a^{-1}b \in K_{f^{-1}}
\]

\[
\iff \text{there exists } \alpha \in A \text{ such that } a^{-1}b = (\alpha, f^{-1})
\]

\[
\iff \text{there exists } \alpha \in A \text{ such that }
\]

\[
bf = a(a^{-1}b)f = a(\alpha^{-1}f^{-1}f)f = af^\alpha = (af)^\alpha.
\]

**Lemma 2.11.** Let \( G' = [G, G] \) denote the commutator subgroup of a group \( G \) and let \( \varphi \in \mathcal{W}(G) \). If \( c \in G' \) is a commutator, then \( \varphi(c) \) is also a commutator. In particular, \( \varphi(G') = G' \).

In other words, \( a \in G' \) if and only if \( \varphi(a) \in G' \). Also, \( \varphi(K) = K \).

**Proof.** Let \( c \in G' \); thus \( c = g^{-1}g^h \) for some \( g, h \in G \). Then since \( \varphi \) preserves conjugacy classes, there exists some \( f \in G \) such that

\[
\varphi(c) = \varphi(g^{-1} \cdot g^h) \sim \varphi(g^{-1})\varphi(g^h) = \varphi(g)^{-1}\varphi(g)^f = (\varphi(g), f).
\]

(2.6)

This shows that \( \varphi(c) \) is conjugate to a commutator. Since the conjugate of a commutator is a commutator, \( \varphi(c) \) is a commutator. To prove the converse, we note that \( \varphi^{-1} \in \mathcal{W}(G) \); thus \( \varphi^{-1} \) will also map a commutator to a commutator. Thus if we assume \( \varphi(c) \in G' \), then \( \varphi^{-1}(\varphi(c)) = c \in G' \). We have shown that \( \varphi(G') = G' \).

By Lemma 2.2 we have \( K = \langle (a, h) : a \in A, h \in F \rangle \). Then using Eq. (2.6) with \( g = a \in A \) we see that \( \varphi \) sends any generator \( (a, h) \) of \( K \) to the commutator \( (\varphi(a), f) \) for some \( f \in G \).
By Lemma (2.3) we may assume \( f \in F \). Since \( \varphi \) is a bijection, the set \( \{(a, h) : a \in A, h \in F\} \) is mapped to itself. It follows that \( \varphi(K) = K \). \qed

## 2.2 Maps defined using \( C \) and \( F \)

The non-trivial wcts described in Theorems 3.3, 5.1, 5.2, 5.3, 5.4, 5.7, 5.8, and 5.9 may be defined as bijections that conjugate group elements in certain cosets of \( G/A \) by a fixed element but fix elements in other cosets. This motivates the following definitions:

**Definitions**  For \( \lambda \in G \) and \( C \subseteq G/A \), we define a map \( \tau(\lambda, C) : G \to G \) as follows:

\[
\tau(\lambda, C) : \begin{cases} 
g \mapsto g^\lambda & \text{if } Ag \in C, 
g \mapsto g & \text{if } Ag \in F, \end{cases}
\]

where \( F = (G/A) \setminus C \). When \( C \) has been made clear we may simply write \( \tau_\lambda \) to denote the function. We define \( \mathcal{P} \subseteq G/A \) to be

\[
\mathcal{P} = \{Af \in G/A : Af = Af_1 \cdot Af_2 \text{ for some } Af_1 \in C, Af_2 \in F\}.
\]

Lastly we define \( I = \bigcap_{Af \in \mathcal{P}} K_f \). We note that \( \mathcal{P} \) and \( I \) are determined by \( C \), thus we may write \( \mathcal{P}(C) \) and \( I(C) \) when we want to distinguish between \( \mathcal{P} \) subsets (or \( I \) subgroups) that correspond to different \( C \) subsets.

The non-trivial wcts defined in this thesis are all \( \tau(\lambda, C) \) maps such that the cosets of \( G/A \) in either \( C \) or \( F \) form a proper subgroup \( N \leq G/A \). Note that by composing \( \tau(\lambda, C) \in \mathcal{W} \) with the inner automorphism \( I_{\lambda^{-1}} : g \mapsto g^{\lambda^{-1}} \) we effectively interchange \( C \) and \( F \). Thus, it is somewhat of an arbitrary choice whether to put \( N = C \) or \( N = F \). However, for most of the \( \tau_\lambda \) maps we define in this thesis we found it advantageous to put \( N = F \), as this made it easier to prove that \( \tau_\lambda \in \mathcal{W} \) or to prove that \( \tau_\lambda \in \mathcal{W} \setminus \mathcal{W}_0 \).

The following theorem gives sufficient conditions for a \( \tau(\lambda, C) \) map to be a wct.

**Theorem 2.12.** Let \( G \) be a group with an abelian normal subgroup \( A \). Suppose \( \tau(\lambda, C) : G \to G \) is a map of the form given in Eq. (2.7) and that \( \tau_\lambda(Af) = Af \) for all \( Af \in G/A \). (In
Lemma 2.13 we show that the latter requirement is satisfied if \( G/A \) is abelian or if \( \lambda \in A \).

If we have \((\lambda, g^{-1}) \in \mathcal{I} \) for every \( g \in G \) such that \( Ag \in \mathcal{C} \), then \( \tau_\lambda \in \mathcal{W}(G) \).

**Proof.** Note that since \( \tau_\lambda \) acts by conjugation, for any conjugacy class \( C \) in \( G \) we have \( \tau_\lambda(C) = C \). Therefore it is clear that \( \tau_\lambda \) preserves conjugacy classes. By assumption \( \tau_\lambda(Af) = Af \) for all \( Af \in G/A \). This implies that \( \tau_\lambda(\mathcal{C}) = \mathcal{C} \). It follows that \( \tau_\lambda \) is a bijection. Then to prove that \( \tau(\lambda, \mathcal{C}) \in \mathcal{W}(G) \) it remains to show that for any \( g_1, g_2 \in G \) we have \( \tau_\lambda(g_1)\tau_\lambda(g_2) \sim \tau_\lambda(g_1g_2) \).

**Case 1:** \( Ag_1, Ag_2 \in \mathcal{C} \) : Then

\[
\tau_\lambda(g_1)\tau_\lambda(g_2) = g_1^\lambda g_2^\lambda = (g_1g_2)^\lambda \begin{cases} = \tau_\lambda(g_1g_2) & \text{if } Ag_1g_2 \in \mathcal{C} \\ \sim \tau_\lambda(g_1g_2) & \text{if } Ag_1g_2 \in \mathcal{F}. \end{cases}
\]

**Case 2:** \( Ag_1, Ag_2 \in \mathcal{F} \) : Then

\[
\tau_\lambda(g_1)\tau_\lambda(g_2) = g_1g_2 \begin{cases} = \tau_\lambda(g_1g_2) & \text{if } Ag_1g_2 \in \mathcal{F} \\ \sim \tau_\lambda(g_1g_2) & \text{if } Ag_1g_2 \in \mathcal{C}. \end{cases}
\]

**Case 3:** \( Ag_1 \in \mathcal{C} \) and \( Ag_2 \in \mathcal{F} \) :

We have \( \tau_\lambda(g_1)\tau_\lambda(g_2) = g_1^\lambda g_2^\lambda \) and we wish to show this is conjugate to \( g_1g_2 \). (Whether \( Ag_1g_2 \) is in \( \mathcal{C} \) or \( \mathcal{F} \), this suffices.) By hypothesis \( \tau_\lambda(Af) = Af \) for all \( Af \in G/A \), thus \( \tau_\lambda(g_1) \in Ag_1 \) so \( g_1^\lambda g_2^\lambda \in Ag_1g_2 \). Lemma 2.10 asserts that to show \( g_1^\lambda g_2^\lambda \sim g_1g_2 \) it suffices to show that \( (g_1^\lambda g_2^\lambda)(g_1g_2)^{-1} \in K_{g_1g_2} \). We simplify this product:

\[
(g_1^\lambda g_2^\lambda)(g_1g_2)^{-1} = g_1^\lambda g_2^\lambda \cdot g_2^{-1} g_1^{-1} = g_1^\lambda \cdot (g_2^{-1} g_2^{-1}) \cdot g_1^{-1} = \lambda^{-1} g_1 \lambda g_1^{-1} = (\lambda, g_1^{-1}).
\]

We are assuming that \((\lambda, g_1^{-1}) \in \mathcal{I} \) which, by the definitions of \( \mathcal{I} \) and \( \mathcal{P} \), implies (since \( Ag_1g_2 \in \mathcal{P} \)) that \((\lambda, g_1^{-1}) \in K_{g_1g_2} \). Thus by Lemma 2.10 we conclude that \( g_1g_2 \sim g_1^\lambda g_2^\lambda \) which proves this case.

**Case 4:** \( g_1 \) satisfies \( Ag_1 \in \mathcal{F} \) while \( g_2 \) satisfies \( Ag_2 \in \mathcal{C} \) :

Since \( \tau(g_1)\tau(g_2) = g_1g_2^\lambda \sim g_2^\lambda g_1 \) this case reduces to Case 3 so we are done. \( \square \)
Lemma 2.13. Let $G$ be a group with subgroup $A \trianglelefteq G$. Let $f, \lambda \in G$. If $G/A$ is abelian or if $\lambda \in A$, then $f^\lambda \in Af$.

Proof. Let $g = f^\lambda$. Then $Ag = Af^A\lambda$. If $G/A$ is abelian or if $A\lambda = A$, then $Ag = Af$, i.e. $g = f^\lambda \in Af$. \hfill \Box

When $I \trianglelefteq G$, the key hypothesis in Theorem 2.12 that $(\lambda, g^{-1}) \in I$ can be rephrased more simply as $(\lambda, g) \in I$. (By Lemma 2.5, $I \trianglelefteq G$ when $G/A$ is abelian.) The equivalence of the two is shown by the following result:

Lemma 2.14. Let $G$ be a group with an abelian subgroup $A \trianglelefteq G$. Let $g \in G$ and $a \in A$. Let $N \trianglelefteq G$. Then $(a, g) \in N$ if and only if $(a, g^{-1}) \in N$.

Proof. We have

$$(a, g) = ((a^g)^{-1}, g^{-1})$$ by Lemma 2.6

$$= (a^g, g^{-1})^{-1}$$ by Lemma 2.1

$$= ((a, g^{-1})^g)^{-1}. \hfill \Box$$

Every group $G$ we consider in this thesis has an abelian normal subgroup $A$ such that $G/A$ is abelian. Thus by Theorem 2.12 and Lemma 2.13, to show that $\tau(\lambda, \mathcal{C}) \in \mathcal{W}(G)$, it suffices to show that $(\lambda, g^{-1}) \in I$ for every $g \in G$ such that $Ag \in \mathcal{C}$. For the groups we are studying, there are infinitely many such group elements $g$, so it is not practical to check each $(\lambda, g)$ one by one. The following two corollaries give us sufficient conditions to satisfy $(\lambda, g^{-1}) \in I$ that can be verified easily. Corollary 2.15 (ii) and Corollary 2.16 (ii) will be applied repeatedly in Ch. 5.

Corollary 2.15. Let $G$ be a group with an abelian normal subgroup $A$ and let $\lambda \in A$. Let $\tau(\lambda, \mathcal{C}) : G \to G$ where $\tau_\lambda(Af) = Af$ for all $Af \in G/A$.

(i) If $(\lambda, f^{-1}) \in I$ for all $f \in F$ such that $Af \in \mathcal{C}$, then $\tau(\lambda, \mathcal{C}) \in \mathcal{W}(G)$. (Rather than verifying that $(\lambda, g^{-1}) \in I$ for every $g \in Af \in \mathcal{C}$ (as required by Theorem 2.12), it suffices to show $(\lambda, f^{-1}) \in I$ where $f \in F$ satisfies $Af \in \mathcal{C}$.)
(ii) If $\lambda \in \mathcal{I}$ and $\mathcal{I} \trianglelefteq G$ then $\tau(\lambda, \mathcal{C}) \in \mathcal{W}(G)$.

Proof. Lemma 2.3 states that when $\lambda \in A$, for any $g \in Af$ we have $(\lambda, g^{-1}) = (\lambda, f^{-1})$. Thus $(\lambda, f^{-1}) \in \mathcal{I}$ implies $(\lambda, g^{-1}) \in \mathcal{I}$ for all $g \in Af \in \mathcal{C}$. Statement (i) then follows from Theorem 2.12.

Now to prove (ii), assume $\lambda \in \mathcal{I}$. Then for any $g \in G$ (since $\mathcal{I}$ is normal by hypothesis) we have $\lambda^{-1}\lambda g^{-1} = (\lambda, g^{-1}) \in \mathcal{I}$ as well. This holds in particular when $g \in Ag \in \mathcal{C}$, as required for Theorem 2.12. Then by that result we have $\tau_{\lambda} \in \mathcal{W}(G)$. □

Corollary 2.16. Let $G$ be a group with an abelian normal subgroup $A$ and let $\tau(\lambda, \mathcal{C}) : G \rightarrow G$ be a map of the form given in Eq. (2.7). Suppose that $\lambda$ commutes with every element $f \in F$ such that $Af \in \mathcal{C}$.

(i) If $(\lambda, a) \in \mathcal{I}$ for all $a \in A$ then $\tau(\lambda, \mathcal{C}) \in \mathcal{W}(G)$. Equivalently, $(\lambda, x_i) \in \mathcal{I}$ for all generators $x_i$ of $A$, $1 \leq i \leq n$, implies $\tau(\lambda, \mathcal{C}) \in \mathcal{W}(G)$.

(ii) If we have $K_{\lambda} \subseteq \mathcal{I}$ then $\tau(\lambda, \mathcal{C}) \in \mathcal{W}(G)$.

Proof. This follows from Theorem 2.12. Note that since $\lambda$ commutes with every $f \in F$ such that $Af \in \mathcal{C}$, we have $\tau_{\lambda}(Af) = Af$ for all $Af \in G/A$. Thus to show $\tau_{\lambda} \in \mathcal{W}(G)$ is suffices to show that $(\lambda, g^{-1}) \in \mathcal{I}$ for every $g \in G$ such that $Ag \in \mathcal{C}$.

We first prove (i). Let $f \in F$ satisfy $Af \in \mathcal{C}$. For $g \in Af$ we can write $g = a^{-1}f$ for some $a \in A$. By hypothesis $\lambda$ commutes with $f$, thus $(\lambda, f^{-1}) = 1$. Then using Eq. (2.2) we have

\[
(\lambda, g^{-1}) = (\lambda, f^{-1}a) = (\lambda, a) \cdot (\lambda, f^{-1}) \cdot ((\lambda, f^{-1}), a) = (\lambda, a).
\]

By assumption $(\lambda, a) \in \mathcal{I}$ for all $a \in A$ therefore $(\lambda, g^{-1}) \in \mathcal{I}$ for all $g \in Af \in \mathcal{C}$. By Theorem 2.12, $\tau_{\lambda} \in \mathcal{W}(G)$. The second statement in (i) follows from Lemma 2.1.
To prove \((ii)\) we will show that \((\lambda, a) \in J\) for \(a \in A\) and then our result follows from \((i)\). Recall \(K_{\lambda} = \langle (a, \lambda) : a \in A \rangle\) by Lemma 2.2, thus an arbitrary element of \(K_{\lambda}\) can be written as \((a, \lambda)\) for \(a \in A\). We are assuming \(K_{\lambda} \leq J\) thus we have \((a, \lambda)\) and its inverse \((\lambda, a)\) contained in \(J\), so we are done.

\(\square\)
In this chapter we include results that may be applied to groups that are not necessarily crystallographic.

3.1 Sufficient conditions for a WCT to be non-trivial

Theorem 3.1. Let $G$ be a group with an abelian normal subgroup $A$. Assume that the conjugation action of $G/A$ on $A$ defined by $a \cdot (Ag) = a^\lambda$ is faithful. Let $\tau(\lambda, C) \in W(G)$ be a wct of the form given in Eq. (2.7). Also assume that $\mathcal{F}$ and $\mathcal{C}$ are not empty.

(i) If $\lambda \not\in A$ then $\tau(\lambda, C)$ is not an automorphism, i.e. $\tau(\lambda, C) \in W(G) \setminus \text{Aut}(G)$.

(ii) If $|G/A| \geq 3$ then $\tau(\lambda, C)$ is not an anti-automorphism, i.e. either $\tau(\lambda, C) \in \text{Aut}(G)$ or $\tau(\lambda, C) \in W(G) \setminus W_0(G)$.

Proof. We prove (i) by contradiction. Suppose that $\tau_\lambda$ is an automorphism, and let $a \in A$ satisfy $(a, \lambda) \neq 1$. (Since the conjugation action of $G/A$ is faithful and $\lambda \not\in A$ we know such an $a$ exists.) Let $\tau_\lambda = \tau(\lambda, C)$. We consider two cases:

Case 1: $A \in \mathcal{F}$: Then for $f \in Af \in \mathcal{C}$ we have

$$af^\lambda = \tau_\lambda(a)\tau_\lambda(f) = \tau_\lambda(a \cdot f) = (af)^\lambda = a^\lambda f^\lambda.$$ 

Case 2: $A \in \mathcal{C}$: Then for $f \in Af \in \mathcal{F}$ we have

$$a^\lambda f = \tau_\lambda(a)\tau_\lambda(f) = \tau_\lambda(a \cdot f) = af.$$

In either case we arrive at $a = a^\lambda$, contradicting $(a, \lambda) \neq 1$. We conclude that $\tau_\lambda$ is not an automorphism, proving (i).

To prove (ii) we assume that $|G/A| \geq 3$ thus (since $\mathcal{C}$ and $\mathcal{F}$ each contain at least one coset) $\mathcal{C}$ or $\mathcal{F}$ contains at least two cosets. We consider two cases with $A \in \mathcal{F}$ and two cases with $A \in \mathcal{C}$. In all four cases we assume by way of contradiction that $\tau(\lambda, \mathcal{C})$ is an
anti-automorphism and use the fact that the action of $G/A$ on $A$ is faithful (i.e. $a^f = a$ for all $a \in A$ implies $f \in A$) to arrive at a contradiction.

**Case 1:** $\mathcal{F} = \{A\}$: Let $a \in A$ and let $f \in G$ satisfy $Af \in \mathcal{C}$. Note $Af^{-1} \in \mathcal{C}$. (We do not assume $Af$ and $Af^{-1}$ are distinct.) Since we assume $\tau_\lambda$ is an anti-automorphism. Then for $a \in A$,

$$a = \tau_\lambda(a) = \tau_\lambda(a f \cdot f^{-1}) = \tau_\lambda(f^{-1})\tau_\lambda(a f) = (f^{-1})^\lambda(a f) = (f^{-1}a f)^\lambda = a^{f\lambda}.$$  

This implies that $f\lambda$ commutes with $a$ for all $a \in A$. Since the action of $G/A$ is faithful we conclude $f\lambda \in A$, i.e. $\lambda \in Af^{-1}$. This must hold for all $Af \in \mathcal{C}$, which is a contradiction since $|\mathcal{C}| \geq 2$ in this case.

**Case 2:** $A \in \mathcal{F}$ and $|\mathcal{F}| \geq 2$ : Let $g \in G$ satisfy $g \notin A, Ag \in \mathcal{F}$. Let $a \in A$. Assuming that $\tau_\lambda$ is an anti-automorphism, we have

$$ag = \tau_\lambda(a \cdot g) = \tau_\lambda(g)\tau_\lambda(a) = g \cdot a.$$  

This shows that $g$ commutes with any $a \in A$, which by the faithfulness of the action of $G/A$ on $A$, indicates $g \in A$, a contradiction.

**Case 3:** $\mathcal{C} = \{A\}$ : Let $a \in A$ and let $h \in G$ satisfy $Ah \in \mathcal{F}$. Note $Ah^{-1} \in \mathcal{F}$. Again we assume $\tau_\lambda$ is an anti-automorphism, and so we have

$$a^\lambda = \tau_\lambda(a) = \tau_\lambda(ah \cdot h^{-1}) = \tau_\lambda(h^{-1})\tau_\lambda(ah) = h^{-1} \cdot ah = a^h.$$  

We have $a^\lambda = a^h$ for arbitrary $a \in A$. The faithful action of $G/A$ tells us that $A\lambda = Ah$ so $\lambda \in Ah$. This must be true for all $Ah \in \mathcal{F}$, which is a contradiction since we have $|\mathcal{F}| \geq 2$ in this case.

**Case 4:** $A \in \mathcal{C}$ and $|\mathcal{C}| \geq 2$ : Let $a \in A$. Let $f \in G, f \notin A$ satisfy $Af \in \mathcal{C}$. Assuming that $\tau_\lambda$ is an anti-automorphism gives

$$(af)^\lambda = \tau_\lambda(a \cdot f) = \tau_\lambda(f)\tau_\lambda(a) = f^\lambda a^\lambda = (fa)^\lambda.$$  

This shows that $f$ commutes with any $a \in A$ thus we must have $f \in A$, a contradiction. 

}\[\square\]
Theorem 3.2. Let $N$ be a normal subgroup in a group $G$. Assume that $\tau \in \mathcal{W}(G)$ satisfies $\tau(N) = N$ and suppose it induces the map $\tau : G/N \to G/N$. If $\tau$ is a non-trivial wct then $\tau$ is also a non-trivial wct.

Proof. We prove the contrapositive: If $\tau$ is a trivial wct then $\overline{\tau}$ is also a trivial wct. We consider two cases: either $\tau$ is an automorphism or $\tau$ is an anti-automorphism.

First we suppose that $\tau \in \text{Aut}(G)$, i.e. for $g_1, g_2 \in G$ we have $\tau(g_1g_2) = \tau(g_1)\tau(g_2)$. It follows that

$$\tau(g_1N \cdot g_2N) = \tau(g_1g_2N) = \tau(g_1N)\tau(g_2N),$$

thus $\tau$ is an automorphism.

The second case follows from a similar argument. If $\tau$ is an anti-automorphism then for $g_1, g_2 \in G$ we have $\tau(g_1g_2) = \tau(g_2)\tau(g_1)$. In this case,

$$\tau(g_1N \cdot g_2N) = \tau(g_1g_2N) = \tau(g_2N)\tau(g_1N),$$

thus $\tau$ is an anti-automorphism. In either case, $\tau$ is a trivial wct. \qed

3.2 Groups with cyclic quotients

Theorem 3.3. Let $G$ be a group with an abelian normal subgroup $A$ such that $G/A \cong \mathbb{C}_n$ where $2 < n \in \mathbb{N}$. Assume that the conjugation action of $G/A$ on $A$ defined by $a \cdot (Ag) = a^g$ is faithful. Then $\mathcal{W}(G) \neq \mathcal{W}_0(G)$.

Proof. Suppose that $G/A = \langle A\rho \rangle$ where $\rho \in G$ satisfies $\rho^n \in A$. Without loss, we put $F = \{\rho^j : 0 < j \leq n\}$. Accordingly we will write $G/A = \{A\rho^j : 0 < j \leq n\}$.

Let $q$ be a prime divisor of $n$ and let $m \in \mathbb{N}$ satisfy $q^m \mid n$ and $q^{m+1} \nmid n$. Let $c = n/q^m$ (thus $c$ is relatively prime to $q$ and $n = cq^m$). Fix $1 \leq k \leq m$. Note that there exists an index $q^k$ subgroup $\langle A\rho^{q^k} \rangle \leq G/A$. Let $N = n/q^{m-k+1} = cq^{k-1}$. Let $\mathcal{F} = \langle A\rho^{q^k} \rangle$ and let $\mathcal{E} = (G/A) \setminus \mathcal{F}$. We will show that $\tau(\rho^N, \mathcal{E}) \in \mathcal{W}(G) \setminus \mathcal{W}_0(G)$. To prove $\tau(\rho^N, \mathcal{E}) \in \mathcal{W}(G)$ we
will use Corollary 2.16 (ii). It is clear that $\rho^N$ commutes with every element of $F$. By the corollary it suffices to show that $K_{\rho^N} \leq J$.

Since $\mathcal{F}$ is a subgroup of $G/A$, we have $\mathcal{P}(\mathcal{C}) = \mathcal{C}$ thus

$$J = \bigcap_{A_f \in \mathcal{C}} K_f.$$  

Note that $\mathcal{F} = \{A\rho^j \in G/A : q^k | j \}$. Therefore its complement

$$\mathcal{C} = \{A\rho^j \in G/A : q^k \nmid j \}$$

$$= \{A\rho^j \in G/A : \gcd(j, q^k) \in \{1, q, q^2, \ldots, q^{k-1}\}\} \quad \text{since } q \text{ is prime}$$

$$= \{A\rho^j \in G/A : \gcd(j, q^m) \in \{1, q, q^2, \ldots, q^{k-1}\}\} \quad \text{since } k \leq m$$

$$= \{A\rho^j \in G/A : \gcd(j, q^m) | q^{k-1}\}$$

$$= \{A\rho^j \in G/A : \gcd(j, n) | cq^{k-1}\} \quad \text{since } n = cq^m$$

$$= \{A\rho^j \in G/A : \gcd(j, n) | N\} \quad \text{since } N = cq^{k-1}$$

$$= \{A\rho^j \in G/A : K_{\rho \gcd(j, n)} \geq K_{\rho^N}\} \quad \text{by Lemma 2.8}$$

$$= \{A\rho^j \in G/A : K_{\rho} \geq K_{\rho^N}\} \quad \text{by Lemma 2.9.}$$

This shows that $K_{\rho^N}$ is contained in $K_{\rho}$ for all $A\rho^j \in \mathcal{P}$, thus $K_{\rho^N} \leq J$ as desired. By Corollary 2.16 (ii), $\tau_{\rho^N}$ is a wct. By Theorem 3.1 this map is a non-trivial wct. 

The next result uses notation for crystallographic groups that will be explained in §4.2.

**Corollary 3.4.** The following seventeen space groups have non-trivial wct groups: $G_{143}, G_{144}, G_{146}, G_{75}, G_{76}, G_{77}, G_{79}, G_{80}, G_{81}, G_{82}, G_{147}, G_{148}, G_{168}, G_{169}, G_{171}, G_{173},$ and $G_{174}$.

**Proof.** Groups $G_{143}, G_{144},$ and $G_{146}$ satisfy $G/A \cong C_3$.

Groups $G_{75}, G_{76}, G_{77}, G_{79}, G_{80}, G_{81},$ and $G_{82}$ satisfy $G/A \cong C_4$.

Groups $G_{147}, G_{148}, G_{168}, G_{169}, G_{171}, G_{173},$ and $G_{174}$ satisfy $G/A \cong C_6$.

The result follows from Theorem 3.3. 

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A lattice in $n$-dimensional Euclidean space $\mathbb{E}^n$ is a subgroup $\mathfrak{L}$ of $\mathbb{E}^n$ isomorphic to $\mathbb{Z}^n$ [E, p. 25]. The set of translation symmetries of $\mathfrak{L}$ is a group (under composition) that is isomorphic to $\mathbb{Z}^n$. We will denote this group as $A$. An $n$-dimensional crystallographic point group $P$ is a finite group of symmetries of $\mathfrak{L}$ that fix one point in $\mathbb{E}^n$ [E, p. 90]. An $n$-dimensional crystallographic group, or space group $G$ is an extension of $A$ by the point group $P$ [Ja, p.127], [E, p. 155]. In other words, we have the short exact sequence

$$1 \rightarrow A \rightarrow G \rightarrow P \rightarrow 1.$$ 

Thus, $P$ is isomorphic to $G/A$ and $A$ has finite index in $G$ [E, p. 154]. Consequently, a space group element $g \in G$ acts on $v \in \mathbb{E}^n$ in a manner that such that $g(v) = p(v) + t$ for some $p \in P$ and some $t \in \mathbb{E}^n$. [Ja, p. 108], [E, p. 153].

The symmetries in a crystallographic point group of a three-dimensional space group may include inversions, rotations, improper rotations, and reflections [I], [E], [Ja]. Inversions and reflections in a crystallographic point group have order 2. The crystallographic restriction states that for $n \in \{2, 3\}$, the symmetries of a lattice in $\mathbb{E}^n$ must have order 1, 2, 3, 4, or 6 [L]. A consequence of this restriction, together with the fact that only seven distinct lattice structures exist that tile $\mathbb{E}^3$, is that there exist 219 three-dimensional space groups [Ja, p. 119]. The International Union of Crystallography (IUC) has assigned each three-dimensional space group a number from 1 to 230 [I]. (There are 11 isomorphic pairs that are numbered separately.)

### 4.1 Space group elements

There are seven types of elements in three-dimensional space groups. These are: translations, reflections, glide reflections, rotations, screw rotations, inversions, and improper rotations (also known as rotoinversions) [E], [I]. We may think of these group elements as symmetries
of an arrangement of atoms in a crystal in \( \mathbb{E}^3 \). They are also \textit{isometries} of \( \mathbb{E}^3 \), meaning they are bijections that preserve distances.

A \textit{translation} is an orientation-preserving isometry that maps a vector \( \mathbf{v} \) to \( \mathbf{v} + \mathbf{a} \) for some fixed \( \mathbf{a} \in \mathcal{L} \). It is easy to see that any two translations commute, and it is also true that any conjugate of a translation is a translation. Thus the set of translations is a normal abelian subgroup of the crystallographic group [\( \mathbb{E}, \) p. 153]. We will denote the translation subgroup \( A \) because it is abelian. For a three-dimensional space group, \( A \cong \mathbb{Z}^3 \) and there is a standard choice of standard generators which we denote as \( x, y, z \). Note that each non-trivial translation in a crystallographic group has infinite order. There is a one-to-one correspondence between the elements in \( A \) and the points on the lattice \( \mathcal{L} \), and this will be useful in the proofs of Theorems 6.1 and 6.21.

A \textit{reflection} is an order 2, orientation-reversing isometry. Let \( s \) be a reflection in \( \mathbb{E}^3 \), thus it is a reflection across a plane \( \mathcal{P}(s) \), the \textit{mirror plane} of \( s \). Suppose that \( \mathcal{P}(s) \) is a plane that contains the origin and let \( \mathbf{u}_{\mathcal{P}(s)} = \mathbf{u} \) be a unit normal vector to \( \mathcal{P}(s) \). Then for \( \mathbf{v} \in \mathbb{E}^3 \), \( s \) maps \( \mathbf{v} \) to the vector \( \mathbf{v} - 2(\mathbf{v} \cdot \mathbf{u})\mathbf{u} \). Note that for any vector \( \mathbf{w} \in \mathcal{P}(s) \), \( \mathbf{w} \cdot \mathbf{u} = 0 \) so \( s(\mathbf{w}) = \mathbf{w} \). Thus \( s \) fixes every vector in its mirror plane, \( \mathcal{P}(s) \).

A \textit{glide reflection} \( \gamma \) reflects across a plane \( \mathcal{P}(\gamma) \) and then translates in a direction parallel to some vector in \( \mathcal{P}(\gamma) \). A glide reflection fixes \( \mathcal{P}(\gamma) \) but not the vectors in \( \mathcal{P}(\gamma) \). A glide reflection is orientation-reversing. The distance of the translation is half of a unit vector in \( \mathcal{L} \), or some multiple of that distance. The square of a glide reflection is a non-trivial translation. Hence, glide reflections have infinite order. If a reflection \( s \) commutes with a translation \( x \) then their product \( xs \) is one example of a glide reflection [Ja, p. 109-10].

In a three-dimensional space group, a \textit{rotation} \( \rho \) is a finite order, orientation-preserving isometry that fixes every point on a line in \( \mathbb{E}^3 \). This line is the \textit{axis of rotation} and if \( \rho \neq \text{Id} \) then this axis is unique. Suppose \( \rho \) is rotation about a line \( \ell \) that contains the origin, with \textit{turning angle} \( \theta \). (If \( |\rho| = m \) then \( \theta = 2\pi k/m \) where \( \gcd(k, m) = 1 \).) Then \( \rho(\mathbf{v}) = \mathbf{w} \) implies that \( |\mathbf{v}| = |\mathbf{w}| \) and also that the distance from \( \mathbf{v} \) to the line \( \ell \) is equal to the distance from
w to ℓ. By the crystallographic restriction, ρ must have order 1, 2, 3, 4, or 6.

A screw rotation r is a rotation about an axis ℓ together with translation in the direction of ℓ. Thus it fixes the line ℓ but it does not fix the vectors in ℓ. Like translations and rotations, a screw rotation is orientation-preserving. If θ is the turning angle of a screw rotation r in \( \mathbb{E}^3 \), then by the crystallographic restriction, \( \theta \in \{ \pi/3, \pi/2, 2\pi/3, \pi, 2\pi \} \) and \( r^{2\pi/\theta} \) is a nontrivial translation in A in the direction of ℓ, [Ja, p. 109-10]. Hence, screw rotations have infinite order. If a rotation ρ commutes with a translation x then their product \( x\rho \) is one example of a screw rotation.

An inversion through a point \( p \) is an order 2, orientation-reversing isometry that maps a vector \( v \) to \( 2p - v \). It follows that \( p \) is the only point fixed by the inversion. Inversions are also known as point inversions or point reflections. Inversions that fix the origin are known as central inversions [Ja, p. 71].

Lastly we have improper rotations, which are also referred to as rotoreflections or rotoinversions. The latter two designations are indicative of the fact that there are two ways to think of this symmetry element. The first is to consider it a rotation with turning angle \( \theta \) through an axis ℓ, followed by a reflection across a plane \( P \) perpendicular to ℓ. The second is to consider it a rotation with turning angle \( \theta \pm \pi \) through an axis ℓ, followed by an inversion through the point where ℓ intersects \( P \). By the crystallographic restriction, \( \theta \in \{ \pi/3, \pi/2, \pi \} \), however when \( \theta = \pi \) then this symmetry could be regarded as a point inversion. Thus an improper rotation that is not an inversion will have order 4 or order 6 [Ja, p. 70].

We prove a lemma about the cardinality of conjugacy classes in a crystallographic group.

**Lemma 4.1.** Let G be a crystallographic group with translation subgroup A.

(i) If \( a \in A \) then \( |a^G| \leq |G/A| \).

(ii) If \( h \in G \setminus A \) then \( |h^G| = \infty \).
Proof. Recall $A \triangleleft G$. To prove (i) we note that every $g \in G$ can be written $g = bf$ for some $b \in A, f \in F$. Therefore,

$$a^G = \{a^g : g \in G\} = \{a^{bf} : b \in A, f \in F\} = \{a^f : f \in F\}.$$ 

Since there are at most $|G/A|$ elements in $F$, we have $|a^G| \leq |G/A|$.

To prove (ii) we note that the conjugation action of $G/A$ on elements of $A$ is faithful. Thus for fixed $h \in G$ there exists some $b \in A$ (depending on $h$) such that $(h, b) \neq 1$. Note that for $k \in \mathbb{Z}$, $h^{bk} = hh^{-1}b^{-k}hb^k = h(h, b^k)$. Thus

$$h^G = \{h^g : g \in G\} \supseteq \{h^{bk} : k \in \mathbb{Z}\} = \{h(h, b^k) : k \in \mathbb{Z}\} = \{h(h, b)^k : k \in \mathbb{Z}\},$$

where the last equality follows from Lemma 2.1. Since $1 \neq (h, b) \in A$ and therefore has infinite order, we see that $h^G$ contains an infinite number of elements in the $Ah$ coset. \qed

4.2 Naming conventions

There are at least ten different naming systems or naming conventions commonly used to identify space groups and their point groups. In our previous work [HP] we used Hermann-Mauguin notation to represent the seventeen wallpaper groups. Here we will continue to use Hermann-Mauguin notation when referring to a wallpaper group and we will also use it when identifying the crystallographic point group of a space group. However, when referring to a three-dimensional space group we will identify it by the number assigned to it by The International Union of Crystallography. (Recall that IUC has assigned each three-dimensional space group a number from 1 to 230.) We will write the group represented by IUC number $n$ as “$G_n$.”

4.3 The four wallpaper groups

In [HP] we find that the wallpaper groups that have non-trivial wct groups are $p3$, $p4$, $p6$, and $p2mm$. In this section we will briefly describe these four groups. A wallpaper group is
a 2-dimensional crystallographic group. We think of a wallpaper pattern corresponding to a wallpaper group $G$ as a subset of $\mathbb{E}^2$. The action of the group elements on $\mathbb{E}^2$ leaves the pattern unchanged. The translation subgroup $A = \langle x, y \rangle$ is a finite index normal subgroup $A \cong \mathbb{Z}^2$. We think of $x$ as horizontal translation (to the right) and $y$ as either vertical translation upwards for groups $p4$ or $p2mm$ or translation at an angle of $\pi/3$ from the horizontal for groups $p3$ and $p6$. The other generators of $G$ will be denoted $\rho$ (for a rotation) and $\sigma$ (for a reflection).

Each of these four groups are a semi-direct products, $G = A \rtimes P$ where $P$ is the corresponding crystallographic point group. For $p3$, the point group contains only order 3 rotations and is isomorphic to $C_3$. For $p4$, the point group contains only two order 4 rotations and one order 2 rotation. It is isomorphic to $C_4$. Similarly, $p6$ has point group isomorphic to $C_6$ and contains rotations of order 6, 3, and 2. The point group for $p2mm$ contains two reflections and their product, an order 2 rotation. It is isomorphic to $C_3^2$.

The group presentations we used as we studied these groups are:

- $p3 : \langle x, y, \rho \mid (x, y), x^\rho = x^{-1}y, y^\rho = x^{-1}, \rho^3 \rangle$;
- $p4 : \langle x, y, \rho \mid (x, y), x^\rho = y, y^\rho = x^{-1}, \rho^4 \rangle$;
- $p6 : \langle x, y, \rho \mid (x, y), x^\rho = y, y^\rho = x^{-1}y, \rho^6 \rangle$;
- $p2mm : \langle x, y, \rho, \sigma \mid (x, y), \rho^2, \sigma^2, (\rho, \sigma), x^\rho = x^{-1}, y^\rho = y^{-1}, x^\sigma = x, y^\sigma = y^{-1} \rangle$.

### 4.4 The Fifty-Six Groups

If $G$ is a group that acts on $\mathbb{E}^n$, then a **fibration of $G$** is determined by a decomposition of $\mathbb{E}^n$ as a direct product: $\mathbb{E}^n = \mathbb{E}^{n-1} \times \mathbb{E}^1$ such that for all $x \in \mathbb{E}^{n-1}, g \in G$, there exists an $x' \in \mathbb{E}^{n-1}$ such that $g(x) \times x^1 = \{x'\} \times \mathbb{E}^1$. In the case where $n = 3$, the action of $G$ on the subspace $\mathbb{E}^2$ corresponds to the action of some wallpaper group on $\mathbb{E}^2$. In this situation, [CFHT] say that the space group $G$ can be obtained as a fibration over the corresponding wallpaper group.
In [CFHT] Conway et al. show which three-dimensional space groups may be obtained as a fibration over some wallpaper group, i.e. a two-dimensional crystallographic group or \emph{plane group}. For each plane group they list all fibrations that exist over it and the space group that corresponds to each fibration. Accordingly, we focus our attention on the fifty-six space groups that may be obtained as a fibration over one of these four wallpaper groups, i.e. the wallpaper groups described in the previous section. These fifty-six groups are listed in Tables 4.1, 4.2, 4.3, 4.4, and 4.5.

4.5 \textbf{TWELVE CRYSTALLOGRAPHIC POINT GROUPS}

This section has been taken from Table 10.1.2 which begins on page 752 of [I]. It will be convenient to partition the fifty-six space groups according to their crystallographic point group. There exist thirty-two crystallographic point groups in \(E^3\) but only twelve of those will be of interest to us in this thesis. These twelve groups are (using Hermann-Mauguin notation):

\[ \frac{2}{m}, \quad 222, \quad \text{mm2}, \quad \frac{2}{m_m_2}, \quad 4, \quad \overline{4}, \quad \frac{4}{m}, \quad 3, \quad \overline{3}, \quad 6, \quad \overline{6}, \quad \text{and} \quad \frac{6}{m}. \]

Here we will give a description of the non-identity elements that are contained in each of these point groups.

The point groups \(\frac{2}{m}, \quad 222, \quad \text{and mm2}\) are isomorphic to \(C_2^3\). The point group \(\frac{2}{m}\) contains a rotation, a reflection, and an inversion. The point group \(222\) contains three rotations. The point group \(\text{mm2}\) contains one rotation and two reflections. The point group \(\frac{2}{m_m_2}\) is isomorphic to \(C_2^3\). This point group contains one inversion, three rotations, and three reflections. These four point groups correspond to space groups that may be obtained from the wallpaper group \(p2mm\).

The next three point groups listed, \(4, \overline{4}, \text{and} \frac{4}{m}\) correspond to space groups that may be obtained from the wallpaper group \(p4\). They are isomorphic to \(C_4, \overline{C_4}, \text{and} \ C_4 \times C_2\), respectively. The point group \(4\) contains three rotations. The point group \(\overline{4}\) contains two
improper rotations as well as one rotation. The point group $\frac{4}{m}$ contains three rotations, two improper rotations, one reflection and one inversion.

Point groups $3$ and $\overline{6}$ correspond to space groups that may be obtained from $p3$. They are isomorphic to $C_3$ and $C_6$, respectively. The point group $3$ contains two rotations. The point group $\overline{6}$ consists of two rotations, two improper rotations, and one reflection.

The point groups $\overline{3}, 6$, and $\frac{6}{m}$ correspond to space groups that may be obtained from $p6$. The first two are isomorphic to $C_6$. The point group $\overline{3}$ consists of two improper rotations, two rotations, and one inversion. The point group $6$ contains five rotations. Lastly we have the point group $\frac{6}{m}$ which is isomorphic to $C_6 \times C_2$. This point group contains five rotations, four improper rotations, one reflection, and one inversion.

4.6 Choosing generators for group presentations

From [SHC, SCC] we have irreducible representations for each of the 219 three-dimensional space groups. These are $4 \times 4$ orthogonal matrices over $\mathbb{Q}$. For each space group, [SHC] gives a sequence of matrices that generates a group isomorphic to that space group. Using Magma, [BCP] we use these matrices to find presentations for the fifty-six space groups of interest to us. For twenty-nine of the fifty-six groups we examined, the matrices we use to define group generators for the presentations are simply the set of matrices given in the file. (Such groups will not be mentioned in the discussion below.) However for twenty-seven space groups there are advantages to using other generators, which will be explained here.

We denote the $i$th matrix in the set corresponding to a space group as $M_i$.

For Groups 75, 76, and 77 we define: $\rho = M_2; \ x = M_3; \ y = M_4; \ z = M_5$.

For Groups 79 and 80 we define: $\rho = M_2; \ x = M_3; \ y = M_3^{-1}M_4; \ z = M_5$.

For Group 81 we define: $\rho = M_2^{-1}; \ x = M_3; \ y = M_4; \ z = M_5$.

For Group 82 we define: $\rho = M_2; \ x = M_3; \ y = M_3M_4^{-1}; \ z = M_5$.

For Groups 83, 84, 85, and 86 we define: $\rho = M_2; \ t = M_3; \ x = M_4; \ y = M_5; \ z = M_6$.

For Groups 87 and 88 we define: $\rho = M_2; \ t = M_3; \ x = M_4; \ y = M_4^{-1}M_5; \ z = M_6$. 

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For Groups 147, 168, 169, and 173, define: \( \rho = M_1^{-1}M_2; \ x = M_3; \ y = M_3M_4; \ z = M_5. \)

For Group 148 we define: \( \rho = M_1^{-1}M_2; \ x = M_3; \ y = M_3^{-1}M_4; \ z = M_5. \)

For Group 171 we define: \( \rho = M_1^{-1}M_2M_5; \ x = M_3; \ y = M_3M_4; \ z = M_5. \)

For Groups 175 and 176, define: \( \rho = M_1^{-1}M_2; \ t = M_3; \ x = M_4; \ y = M_4M_5; \ z = M_6. \)

For Groups 143 and 144 we define: \( \rho = M_1; \ x = M_2; \ y = M_2M_3; \ z = M_4. \)

For Group 146 we define: \( \rho = M_1; \ x = M_2; \ y = M_2^{-1}M_3; \ z = M_4. \)

For Group 174 we define: \( \rho = M_1; \ s = M_2; \ x = M_3; \ y = M_3M_4; \ z = M_5. \)

In the first twenty-one groups listed above (including Groups 175 and 176), we use the single generator \( \rho \) in our presentations instead of the two generators \( M_1 \) and \( M_2 \). By doing so we reduce the number of generators in the presentation, a helpful simplification. We will show that in all cases, this does not yield a proper subgroup of the space group, therefore nothing is lost in this simplification. It suffices to show that \( M_1 \) and \( M_2 \) are contained in \( \langle \rho, x, y, z \rangle \). (Note that for Groups numbered 88 or lower we have \( M_2 = \rho^{\pm 1} \) thus we need only show that \( M_1 \in \langle \rho, x, y, z \rangle \).)

In Groups 75, 76, 79, 81, 82, 83, 85, and 87 we have \( M_1 = \rho^2 \). In Groups 77 and 84 we have \( M_1 = z^{-1}\rho^2 \). In Group 80 we have \( M_1 = x^{-2}yz\rho^2 \). In Group 86 we have \( M_1 = x^{-1}z^{-1}\rho^2 \). In Group 88 we have \( M_1 = y\rho^2 \). In Groups 147, 148, 168, 169, and 175 we have \( M_1 = \rho^2 \) and \( M_2 = \rho^3 \). In Group 171 we have \( M_1 = \rho^2 \) and \( M_2 = z^{-1}\rho^3 \); in Groups 173 and 176 we have \( M_1 = z^{-1}\rho^2 \) and \( M_2 = z^{-1}\rho^3 \).

In most of the twenty-five groups listed above we have defined \( y \) as a product of matrices. This is so that specific relations would be included in the group presentations. (This is also the reason \( \rho \) is defined as \( M_2^{-1} \) in Group 81.) For spacegroups numbered between 75 and 88, these relations are \( x^\rho = yz^\delta \) for \( \delta \in \{-1, 0, 1\} \) and \( y^\rho = x^{-1} \). For spacegroups numbered between 147 and 176 (except 174) these relations are \( x^\rho = y \) and \( y^\rho = x^{-1}yz^\delta \), for \( \delta \in \{0, 2\} \). For spacegroups numbered between 143 and 146 and also for group 174 these relations are \( x^\rho = x^{-1}y \) and \( y^\rho = x^{-1} \). Taking a quotient of the space group mod \( \langle z \rangle \) (this subgroup is normal in all twenty-five of these space groups) these relations are identical to the relations
found in the presentation of wallpaper groups of type \( p3, p4 \) or \( p6 \) in our previous work, [HP]. This consistency will be particularly helpful when applying Theorem 3.2.

For Group 13 we define: \( r = M_1; \ t = M_2; \ x = M_3; \ y = M_4; \ z = M_5^{-1} \). For merely aesthetic reasons we chose to use the inverse of \( M_5 \) instead of \( M_5 \). This affects only one change in the group presentation: it gives \((rt)^2 = z\) rather than \((rt)^2 = z^{-1}\).

For Group 39 we define: \( p = M_4^{-1}M_5M_1; \ s = M_2; \ x = M_3; \ y = M_4; \ z = M_5 \). The simplest way to define \( p \) would be to choose \( p = M_1 \). This would result in the group presentation for Group 39 including the relations \((p,s) = (ps)^2 = y^2z^{-1}\). The relation \((ps)^2 = y^2z^{-1} \notin \mathbb{Z}(G) = \langle z \rangle\) is problematic because it does not satisfy the hypotheses of Proposition 6.13.

By defining \( p = M_4^{-1}M_5M_1 \) we have the nicer group relations \((p,s) = 1\) and \((ps)^2 = z\). With \((ps)^2\) in the center of the group the proposition is applicable, which is a significant advantage.

### 4.7 The Group Presentations

According to Table 1 in [CFHT], there are thirty-one space groups that can be obtained as a fibration over a wallpaper group of type \( p2nm \). These thirty-one groups are listed in Tables 4.1 and 4.2.

The three groups \( G_{10}, G_{12}, \) and \( G_{13} \) have presentations of the form

\[
G = \langle x, y, z, r, t \mid (x, y), (y, z), (x, z), r^2, t^2, (rt)^2 = \alpha_{rt}, \ 
\quad x^r = x^{-1}y^\delta, (y, r), z^r = z^{-1}, (xt)^2, (yt)^2, (zt)^2 \rangle,
\]

where \( \alpha_{rt} \in A = \langle z \rangle \) and \( \delta \in \{0, 1\} \).

The four groups \( G_{16}, G_{17}, G_{21}, \) and \( G_{22} \) have presentations of the form

\[
G = \langle x, y, z, p, r \mid (x, y), (y, z), (x, z), p^2 = \alpha_p, r^2, (pr)^2, \ 
\quad x^p = x^{-1}z^\delta, y^p = y^{-1}z^\delta, (z, p), x^r = x^{-1}y^\gamma, y^r = yz^{-\delta}, z^r = z^{-1} \rangle,
\]

where \( \alpha_p \in A = \langle z \rangle \) and \( \delta, \gamma \in \{0, 1\} \).
The six groups $G_{25}, G_{26}, G_{27}, G_{38}, G_{39},$ and $G_{42}$ have presentations of the form

$$G = \langle x, y, z, p, s \mid (x, y), (y, z), (x, z), p^2 = \alpha_p, s^2 = \alpha_s, (p, s), \rangle$$

$$x^p = x^{-1}z^\delta, y^p = y^{-1}z^\gamma, (z, p), (x, s), y^s = y^{-1}z^\gamma, (z, s),$$

where $\alpha_p, \alpha_s \in A = \langle x, y, z \rangle$ and $\delta, \gamma \in \{0, 1\}$.

Table 4.1: Parameters in the presentations of some space groups $G$ with point groups $\frac{2}{m}$, $222$, or $\text{mm}2$

<table>
<thead>
<tr>
<th>Point group</th>
<th>Group number</th>
<th>Equation number</th>
<th>$\alpha_{rt}$</th>
<th>$\alpha_p$</th>
<th>$\alpha_s$</th>
<th>$\delta$</th>
<th>$\gamma$</th>
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<tbody>
<tr>
<td>$2/m$</td>
<td>10</td>
<td>(4.1)</td>
<td>1</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$2/m$</td>
<td>12</td>
<td>(4.1)</td>
<td>1</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$2/m$</td>
<td>13</td>
<td>(4.1)</td>
<td>$z$</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$222$</td>
<td>16</td>
<td>(4.2)</td>
<td>1</td>
<td></td>
<td></td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$222$</td>
<td>17</td>
<td>(4.2)</td>
<td>$z$</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$222$</td>
<td>21</td>
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<td>1</td>
<td></td>
<td></td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$222$</td>
<td>22</td>
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<td>1</td>
<td></td>
<td></td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\text{mm}2$</td>
<td>25</td>
<td>(4.3)</td>
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<td>0</td>
<td></td>
</tr>
<tr>
<td>$\text{mm}2$</td>
<td>26</td>
<td>(4.3)</td>
<td>$z$</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\text{mm}2$</td>
<td>27</td>
<td>(4.3)</td>
<td>$z$</td>
<td></td>
<td></td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>$\text{mm}2$</td>
<td>38</td>
<td>(4.3)</td>
<td>1</td>
<td>1</td>
<td></td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$\text{mm}2$</td>
<td>39</td>
<td>(4.3)</td>
<td>$z$</td>
<td></td>
<td></td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>$\text{mm}2$</td>
<td>42</td>
<td>(4.3)</td>
<td>1</td>
<td>1</td>
<td></td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The fourteen groups $G_{47}, G_{49}, G_{50}, G_{51}, G_{53}, G_{54}, G_{55}, G_{57}, G_{63}, G_{64}, G_{65}, G_{66}, G_{67},$ and $G_{68}$ have presentations of the form

$$G = \langle x, y, z, p, r, t \mid (x, y), (y, z), (x, z), p^2 = \alpha_p, r^2 = \alpha_r, t^2 = (pr)^2 = \alpha_{pr}, (pt)^2 = \alpha_{pt}, \rangle$$

$$\begin{align*}
(rt)^2 &= \alpha_{rt}, x^p = x^{-1}, y^p = y^{-1}, z^\delta, (z, p), x^r = x^{-1}y^\delta, (y, r), z^r = z^{-1}, (xt)^2, (yt)^2, (zt)^2,
\end{align*}$$

where $\alpha_p \in \langle z \rangle, \alpha_r \in \langle y \rangle, \alpha_{pr}, \alpha_{pt}, \alpha_{rt} \in A = \langle x, y, z \rangle$ and $\delta \in \{0, 1\}$.

The four groups $G_{69}, G_{72}, G_{73},$ and $G_{74}$ have presentations of the form

$$G = \langle x, y, z, p, r, t \mid (x, y), (y, z), (x, z), p^2 = \alpha_p, r^2 = \alpha_r, t^2 = (pr)^2 = \alpha_{pr}, (pt)^2 = \alpha_{pt}, \rangle$$

$$\begin{align*}
(rt)^2 &= \alpha_{rt}, x^p = x^{-1}, y^p = y^{-1}, z^\delta, (z, p), x^r = x^{-1}y^\delta, y^r = yz^{\delta-1}, z^r z, (xt)^2, (yt)^2, (zt)^2,
\end{align*}$$

where $\alpha_p \in \langle z \rangle, \alpha_r \in \langle y \rangle, \alpha_{pr}, \alpha_{pt}, \alpha_{rt} \in A = \langle x, y, z \rangle$ and $\delta \in \{0, 1\}$.
In [CFHT] they find that there are thirteen space groups that can be obtained as a fibration over a wallpaper group of type \( p4 \). These thirteen groups are listed in Table 4.3.

The seven groups \( G_{75}, G_{76}, G_{77}, G_{79}, G_{80}, G_{81}, \) and \( G_{82} \) have presentations of the form

\[
G = \langle x, y, z, \rho \mid (x,y), (y,z), (x,z), \rho^4 = \alpha_{\rho}, x^\rho = yz^\delta, y^\rho = x^{-1}, z^\rho = z^\gamma \rangle,
\]

where \( \alpha_{\rho} \in A = \langle z \rangle, \delta \in \{-1, 0, 1\}, \) and \( \gamma \in \{-1, 1\} \).

The six groups \( G_{83}, G_{84}, G_{85}, G_{86}, G_{87}, \) and \( G_{88} \) have presentations of the form

\[
G = \langle x, y, z, \rho, t \mid (x,y), (y,z), (x,z), \rho^4 = \alpha_{\rho}, t^2, (\rho, t) = \eta, \rho^4 = \alpha_{\rho}, x^\rho = yz^\delta, y^\rho = x^{-1}, (z, \rho), (xt)^2, (yt)^2, (zt)^2 \rangle,
\]

where \( \alpha_{\rho} \in \langle z \rangle, \eta \in A = \langle x, y, z \rangle \) and \( \delta \in \{0, 1\} \).

There are four space groups that can be obtained from a wallpaper group of type \( p3 \) [CFHT]. These are groups \( G_{143}, G_{144}, G_{146}, \) and \( G_{174} \). The first three have a point group \( 3 \).
Table 4.3: Parameters in the presentations of groups with point groups $4, \bar{4}, \text{ or } \frac{4}{m}$

<table>
<thead>
<tr>
<th>Point group</th>
<th>Group number</th>
<th>Equation number</th>
<th>$\alpha_\rho$</th>
<th>$\eta$</th>
<th>$\delta$</th>
<th>$\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$4$</td>
<td>75</td>
<td>(4.6)</td>
<td>$1$</td>
<td>$0$</td>
<td>$1$</td>
<td>$1$</td>
</tr>
<tr>
<td>$4$</td>
<td>76</td>
<td>(4.6)</td>
<td>$z$</td>
<td>$0$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$4$</td>
<td>77</td>
<td>(4.6)</td>
<td>$z^2$</td>
<td>$0$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$4$</td>
<td>79</td>
<td>(4.6)</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$4$</td>
<td>80</td>
<td>(4.6)</td>
<td>$z$</td>
<td>$1$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$\bar{4}$</td>
<td>81</td>
<td>(4.6)</td>
<td>$1$</td>
<td>$0$</td>
<td>$-1$</td>
<td></td>
</tr>
<tr>
<td>$\bar{4}$</td>
<td>82</td>
<td>(4.6)</td>
<td>$1$</td>
<td>$-1$</td>
<td>$-1$</td>
<td></td>
</tr>
<tr>
<td>$4/m$</td>
<td>83</td>
<td>(4.7)</td>
<td>$1$</td>
<td>$1$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$4/m$</td>
<td>84</td>
<td>(4.7)</td>
<td>$z^2$</td>
<td>$z^{-1}$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$4/m$</td>
<td>85</td>
<td>(4.7)</td>
<td>$1$</td>
<td>$x^{-1}$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$4/m$</td>
<td>86</td>
<td>(4.7)</td>
<td>$z^2$</td>
<td>$y^{-1}z^{-1}$</td>
<td>$0$</td>
<td></td>
</tr>
<tr>
<td>$4/m$</td>
<td>87</td>
<td>(4.7)</td>
<td>$1$</td>
<td>$1$</td>
<td>$1$</td>
<td></td>
</tr>
<tr>
<td>$4/m$</td>
<td>88</td>
<td>(4.7)</td>
<td>$z$</td>
<td>$x^{-2}yz$</td>
<td>$1$</td>
<td></td>
</tr>
</tbody>
</table>

Group $G_{174}$ has point group $\bar{6}$. The space groups $G_{143}, G_{144}$, and $G_{146}$ have presentations of the form

$$G = \langle x, y, z, \rho \mid (x, y), (y, z), (x, z), \rho^3 = \alpha_\rho, x^\rho = x^{-1}yz^\delta, y^\rho = x^{-1}, (z, \rho) \rangle,$$  \hspace{1cm} (4.8)

where $\alpha_\rho \in \langle z \rangle$ and $\delta \in \{0, 2\}$.

Table 4.4: Parameters in the presentations of groups with point group $3$

<table>
<thead>
<tr>
<th>Group number</th>
<th>Equation number</th>
<th>$\alpha_\rho$</th>
<th>$\delta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>143</td>
<td>(4.8)</td>
<td>$1$</td>
<td>$0$</td>
</tr>
<tr>
<td>144</td>
<td>(4.8)</td>
<td>$z$</td>
<td>$0$</td>
</tr>
<tr>
<td>146</td>
<td>(4.8)</td>
<td>$1$</td>
<td>$2$</td>
</tr>
</tbody>
</table>

The group $G_{174}$ has presentation

$$G = \langle x, y, z, \rho, s \mid (x, y), (y, z), (x, z), \rho^3, s^2, (\rho, s),$$

$$x^\rho = x^{-1}y, y^\rho = x^{-1}, (z, \rho), (x, s), (y, s), z^s = z^{-1} \rangle.$$  \hspace{1cm} (4.9)

We again refer to [CFHT] to determine which space groups may be obtained from the wallpaper group of type $p6\bar{6}$. According to Table 1 there are eight such groups. These are
listed in Table 4.5. \( G_{168}, G_{169}, G_{171}, G_{173} \) which have point group \( 6 \cong C_6 \).

The six groups \( G_{147}, G_{148}, G_{168}, G_{169}, G_{171}, \) and \( G_{173} \) have presentations of the form

\[
G = \langle x, y, z, \rho | (x, y), (y, z), (x, z), \rho^6 = \alpha_\rho, x^\rho = y, y^\rho = x^{-1}yz^\delta, z^\rho = z^\gamma \rangle, \tag{4.10}
\]

where \( \alpha_\rho \in A = \langle z \rangle, \delta \in \{0, 2\}, \) and \( \gamma \in \{-1, 1\} \).

The two groups \( G_{175} \) and \( G_{176} \) have presentations of the form

\[
G = \langle x, y, z, \rho, t | (x, y), (y, z), (x, z), \rho^6 = \alpha_\rho, t^2, (\rho, t) = \eta, x^\rho = y, y^\rho = x^{-1}y, (\rho, z), (xt)^2, (yt)^2, (zt)^2 \rangle, \tag{4.11}
\]

where \( \alpha_\rho, \eta \in \langle z \rangle \).

<table>
<thead>
<tr>
<th>Point group</th>
<th>Group number</th>
<th>Equation number</th>
<th>( \alpha_\rho )</th>
<th>( \eta )</th>
<th>( \delta )</th>
<th>( \gamma )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \bar{3} )</td>
<td>147</td>
<td>(4.10)</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>( \bar{3} )</td>
<td>148</td>
<td>(4.10)</td>
<td>1</td>
<td>2</td>
<td>-1</td>
<td></td>
</tr>
<tr>
<td>( 6 )</td>
<td>168</td>
<td>(4.10)</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( 6 )</td>
<td>169</td>
<td>(4.10)</td>
<td>( z )</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( 6 )</td>
<td>171</td>
<td>(4.10)</td>
<td>( z^2 )</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( 6 )</td>
<td>173</td>
<td>(4.10)</td>
<td>( z^3 )</td>
<td>0</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( 6/m )</td>
<td>175</td>
<td>(4.11)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>( 6/m )</td>
<td>176</td>
<td>(4.11)</td>
<td>( z^3 )</td>
<td>( z^{-1} )</td>
<td>1</td>
<td></td>
</tr>
</tbody>
</table>
Chapter 5. Space groups with non-trivial WCT groups

5.1 Groups having point group 3 or $\bar{6}$

The following applies to $G_{143}, G_{144},$ and $G_{146}$.

**Theorem 5.1.** Let $G$ be a group with presentation of the form given in Eq. (4.8). Then $\tau(\rho, C)$, with $C = \{A\rho, A\rho^2\}$, is a non-trivial wct map.

**Proof.** For this map we have $P = \{A\rho, A\rho^2\}$ and so by Lemma 2.6 we have $I = K_\rho$. It is clear that $\rho$ commutes with every element of $F$, thus we may apply Corollary 2.16 (ii). Since $I = K_\rho$, we see that $\tau_\rho$ is a wct map.

We may use Theorem 3.2 to verify that $\tau_\rho$ is a non-trivial wct. Note that $Z(G) = \langle z \rangle \trianglelefteq G$.

The quotient $G/Z(G) = \langle x, y, \rho | (x, y), \rho^3, x^\rho = x^{-1}y, y^\rho = x^{-1} \rangle$, is a wallpaper group of type $p3$. Now $\tau_\rho$ induces a map on the quotient $G/Z(G)$:

$$
\tau_\rho : g \mapsto \begin{cases} 
  g' & \text{if } Ag \in C = \{A\rho, A\rho^2\}, \\
  g & \text{if } Ag \in F = \{A\}.
\end{cases}
$$

This is precisely the non-trivial wct that exists in the wallpaper groups of type $p3$ [HP]. It follows by Theorem 3.2 that $\tau_\rho \in W(G) \setminus W_0(G)$. \hfill $\square$

The following applies to $G_{174}$.

**Theorem 5.2.** Let $G$ be a group with presentation of the form given in Eq. (4.9). The following are non-trivial wct maps:

(i) $\tau(\rho, C_1), C_1 = \{A\rho, A\rho^2, A\rho s, A\rho^2 s\}$; and

(ii) $\tau(s, C_2), C_2 = \{As, A\rho s, A\rho^2 s\}$. 

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Proof. For the map $\tau_\rho$ we have $P(C_1) = \{A\rho, A\rho^2, A\rho s, A\rho^2 s\}$ and applying Lemma 2.6 we have $J(C_1) = K_\rho \cap K_{\rho s}$. Now the relations in the presentation of $G$ give $K_\rho = \langle x^2 y^{-1}, xy \rangle$ and $K_{\rho s} = \langle x^2 y^{-1}, xy, z^2 \rangle$, therefore $J(C_1) = K_\rho$. Since $F = \langle \rho, s \rangle$ is an abelian subgroup of $G$, we see that $\rho$ commutes with every element of $F$. By Corollary 2.16 ($ii$) we conclude that $\tau_\rho \in \mathcal{W}(G)$.

Now $\langle z, s \rangle \trianglelefteq G$. (In fact, $G = \langle x, y, \rho \rangle \times \langle z, s \rangle$.) Consider the quotient $G/\langle z, s \rangle$:

$$G/\langle z, s \rangle = \langle x, y, \rho \mid (x, y), \rho x = x^{-1} y, y^\rho = x^{-1} \rangle.$$  

This is the wallpaper group $p3$. The map $\tau_\rho$ induces a map on the quotient $\overline{\tau_\rho} : G/\langle z, s \rangle \to G/\langle z, s \rangle$ defined as

$$\overline{\tau_\rho} : \begin{cases} g \mapsto g^\rho & \text{if } \overline{Ag} \in \overline{C} = \{\overline{A\rho}, \overline{A\rho^3}\}, \\ g \mapsto g & \text{if } \overline{Ag} \in \overline{F} = \{\overline{A}\} \end{cases}$$

which we know to be a non-trivial map [HP]. It follows by Theorem 3.2 that $\tau(\rho, C)$ is also a non-trivial wct.

We now consider the map $\tau_s$. We will show that this is a wct by applying Corollary 2.16 ($ii$). We have $F = \langle \rho, s \rangle$, an abelian subgroup of $G$, thus $s$ commutes with every element of $F$. For this map we have $P = K_s \cap K_{\rho s} \cap K_{\rho s^2}$. The relations in the group presentation give us $K_{\rho s} = K_{\rho s^2} = \langle x^2 y^{-1}, xy, z^2 \rangle$ and $K_s = \langle z^2 \rangle$. Therefore $J(C_2) = K_s$. By Corollary 2.16 ($ii$) we conclude that $\tau_s \in \mathcal{W}(G)$. By Theorem 3.1 $\tau_s \in \mathcal{W}(G) \setminus \mathcal{W}_0(G)$.

\[ \square \]

5.2 Groups having point group $4, \overline{4}$, or $\frac{4}{m}$

The following two theorems apply to space groups with point group $4$ or $\overline{4}$.

**Theorem 5.3.** Let $G$ be a group with a presentation of the form given in Eq. (4.7). Then for $\lambda \in \langle x, y, \rho \rangle$, the map $\tau(\lambda, \{A\rho, A\rho^3\})$ is a non-trivial wct.

**Proof.** For this map $P = \{A\rho, A\rho^3\}$. By Lemma 2.6 $K_\rho = K_{\rho^3}$, thus we have $J = K_\rho$. For any $\lambda \in \langle x, y \rangle \leq A$ we clearly have $(\lambda, \rho)$ and $(\lambda, \rho^3)$ contained in $K_\rho = K_{\rho^3} = J$. Thus by
Lemma 2.14 and Corollary 2.15 (i), $\tau_\lambda$ is a wct.

For $\lambda = \rho$ we may apply Corollary 2.16 (ii) since $\rho$ commutes with every element of $F$. Clearly we have $K_\lambda \leq I$; thus $\tau_\rho$ is also a wct.

Notice that $Z = \langle z \rangle \unlhd G$; thus we may consider the quotient

$$G/Z = \langle x, y, \rho \mid (x, y), \rho^4, x^\rho = y, y^\rho = x^{-1} \rangle.$$  

This is a wallpaper group of type $p4$. The map $\tau_\lambda$ induces a map on the quotient, $\tau_\lambda : G/Z \to G/Z$ defined as

$$\tau_\lambda : (g) = \begin{cases} 
g^\lambda & \text{if } Ag \in \{A\rho, A\rho^3\}, 
g & \text{if } Ag \in \{A, A\rho^2\}, \end{cases}$$

which we know to be a non-trivial wct [HP]. It follows by Theorem 3.2 that $\tau_\lambda$ is also a non-trivial wct. 

\[\square\]

**Theorem 5.4.** Let $G$ be a group with a presentation of the form given in Eq. (4.7). For these groups we have $K_\rho = \langle xy, x^{-1}yz^\delta \rangle$, with $\delta \in \{0, 1\}$. Then for $\lambda \in K_\rho$ the map $\tau(\lambda, \mathcal{C}), \mathcal{C} = \{A\rho, A\rho^3, A\rho t, A\rho^3 t\}$ is a non-trivial wct.

**Proof.** From the relations in the presentation of the group one can check that $K_\rho = \langle xy, x^{-1}yz^\delta \rangle$.

Now for this map we have $\mathcal{P} = \{A\rho, A\rho^3, A\rho t, A\rho^3 t\}$ and so

$$J = K_\rho \cap K_\rho^t \cap K_\rho t \cap K_\rho^3 t = K_\rho \cap K_\rho t,$$

the last equality being justified by Lemma 2.6. Note also that $J \unlhd G$ since it is the intersection of normal subgroups by Lemma 2.5.

We will use Corollary 2.15 (i) to show that $\tau_\lambda$ is a wct. Accordingly, we need to show that $\{(\lambda, f^{-1}) \mid f \in \{\rho, \rho^3, \rho t, \rho^3 t\}\} \subseteq J$. By Lemmas 2.3 and 2.14, it suffices to show that $(\lambda, \rho)$ and $(\lambda, \rho t)$ are contained in $J = K_\rho \cap K_\rho t$. Obviously $(\lambda, \rho)$ and $(\lambda, \rho t) \in K_\rho$. 

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It is also easy to see that \((\lambda, \rho t) \in K_\rho\) since \(\lambda \in K_\rho\) and by Lemma 2.5 \(K_\rho\) is normal, thus \((\lambda, \rho t) = \lambda^{-1} \lambda^{\rho t}\) is a product of two elements in \(K_\rho\). It remains to show that \((\lambda, \rho) \in K_{\rho t}\).

First we prove a lemma.

**Lemma 5.5.**

\[
\text{For } b \in A, \quad (b^{\rho^2}, \rho) = (b, \rho)^{-1}. \tag{5.1}
\]

**Proof.** From the relations in the group presentation we have \(K_{\rho^2 t} = \langle z^\delta, z^2 \rangle\). Since \((z, \rho) = 1\) this implies that \(\rho \in C_G(K_{\rho^2 t})\). It follows by Lemma 2.7 that \(\rho^2 t \in C_G(K_\rho)\). In other words, for \(b \in A, (b, \rho)^{\rho^2 t} = (b, \rho)\). By Lemma 2.4 the left hand side is \((b^{\rho^2}, \rho)^t\). Recall that for \(\alpha \in A\) we have \(\alpha^t = \alpha^{-1}\) and so we have \((b^{\rho^2}, \rho)^{-1} = (b, \rho)\), which proves the lemma.

Now we will show that \((\lambda, \rho) \in K_{\rho t}\) by showing that \((\lambda, \rho)\) can be written as a commutator \((c, \rho t)\) for some \(c \in A\). Let \(\lambda = (a, \rho)\) for some \(a \in A\). Let \(b = a^{\rho^{-1}}\) so that \(\lambda = (b^\rho, \rho)\). Then

\[
(\lambda, \rho) = ((a, \rho), \rho) = (a^\rho \cdot a^{-1}, \rho)
\]

\[
= (a^\rho, \rho)(a^{-1}, \rho) \quad \text{by Lemma 2.1}
\]

\[
= (b^{\rho^2}, \rho)(b^{\rho t}, \rho) \quad \text{since } a^\rho = b^{\rho^2} \text{ and } a^{-1} = b^{\rho t}
\]

\[
= (b, \rho)^{-1}(b^{\rho t}, \rho) \quad \text{using Eq. (5.1)}
\]

\[
= (b, \rho)^{-1}(b, \rho)^{\rho t} \quad \text{by Lemma 2.4}
\]

\[
= ((b, \rho), \rho t).
\]

We conclude that \(\tau_\lambda\) is a wct.

Now we show that \(\tau_{xy}\) and \(\tau_{x^{-1}yz^\delta}\) are non-trivial wcts:

\[
\tau_{xy}(\rho) \cdot \tau_{xy}(\rho) = x^{-2} y^{-2} \rho^2 \neq \rho^2 = \tau_{xy}(\rho^2).
\]

\[
\tau_{x^{-1}yz^\delta}(\rho) \tau_{x^{-1}yz^\delta}(\rho) = x^2 y^{-2} z^{-2\delta} \rho^2 = (x^2 z^{-\delta})^2 \neq \rho^2 = \tau_{x^{-1}yz^\delta}(\rho^2). \quad \square
\]
5.3 Groups having point group $\overline{3}, 6$ or $\frac{6}{m}$

**Proposition 5.6.** We define $\alpha = xy$, and $\beta = x^{-2}y$. For groups having a presentation of the form given in Eq. (4.10) (and assuming that $\delta = 0$ or $\gamma = -1$) we have

$$K_\rho = K_{\rho^3} = \langle x, y, z^{1-\gamma} \rangle;$$
$$K_{\rho^2} = K_{\rho^1} = \langle \alpha, \beta z^\delta \rangle;$$
$$K_{\rho^3} = \langle x^2, y^2, z^{1-\gamma} \rangle.$$

For groups having a presentation of the form given in Eq. (4.11) we have

$$K_\rho = \langle x, y \rangle;$$
$$K_{\rho^2} = K_{\rho^1} = \langle \alpha, \beta \rangle;$$
$$K_{\rho^3} = \langle x^2, y^2 \rangle;$$
$$K_t = \langle x^2, y^2, z^2 \rangle$$
$$K_{\rho t} = K_{\rho^1 t} = \langle \alpha, \beta, z^2 \rangle;$$
$$K_{\rho^1 t} = \langle x, y, z^2 \rangle;$$
$$K_{\rho^3 t} = \langle z^2 \rangle.$$

Each of these subgroups is normal.

**Proof.** From the given presentations and the definition of the $K_f$ subgroups one can calculate the $K_f$ subgroups to be as given above. Lemma 2.6 may also be applied. Since $G/A$ is abelian, we know by Lemma 2.5 that the $K_f$ subgroups are normal. \qed

The following applies to $G_{147}, G_{148}, G_{168}, G_{169}, G_{171},$ and $G_{173}.$

**Theorem 5.7.** Assume $\lambda \neq 1.$ Let $\alpha = xy$, and $\beta = x^{-2}y.$ Groups with presentations of the form described in Eq. (4.10) with $\delta = 0$ or $\gamma = -1$ have the following non-trivial wct maps:

(i) For $\lambda \in \langle x^2, y^2, \rho^3 \rangle$, $\tau(\lambda, C_1), C_1 = \{A, A\rho^2, A\rho^1\}$; and

(ii) For $\lambda \in \langle \alpha, \beta z^\delta, \rho^2 \rangle$, $\tau(\lambda, C_2), C_2 = \{A, A\rho^3\}.$
Proof. We will use Corollaries 2.15 and 2.16 to show that these two maps are wcts. For the first map $\tau(\lambda, C_1)$ we have $P(C_1) = \{A\rho, A\rho^3, A\rho^5\}$. By Proposition 5.6, $K_{\rho^3} \subseteq K_{\rho} = K_{\rho^3}$, thus $I(C_1) = K_{\rho^3}$. Since $x^2$ and $y^2$ are contained in $K_{\rho^3} = I(C_1)$, by Corollary 2.15 (ii) $\tau(x^2, C_1)$ and $\tau(y^2, C_1)$ are wct maps. Now clearly $\rho^3$ commutes with every element of $F$ and $K_{\rho^3}$ is contained in $I(C_1)$. Then by Corollary 2.16 (ii), $\tau(\rho^3, C_1)$ is also a wct map.

For the second map $\tau(\lambda, C_2)$ we have $P(C_2) = \{A\rho, A\rho^2, A\rho^4, A\rho^5\}$. By Proposition 5.6 $K_{\rho^2} = K_{\rho^4} \subseteq K_{\rho} = K_{\rho^2}$, thus $I(C_2) = K_{\rho^2}$. Since $\alpha$ and $\beta z^\delta$ are in $K_{\rho^2} = I(C_2)$, by Corollary 2.15 (ii) $\tau(\alpha, C_2)$ and $\tau(\beta z^\delta, C_2)$ are wct maps. Of course $\rho^2$ commutes with every element of $F$ and $K_{\rho^2}$ is contained in $I(C_2)$, so by Corollary 2.16 (ii) $\tau(\rho^2, C_2)$ is also a wct map.

Note that $\langle z \rangle \trianglelefteq G$. The image of these maps in $G/\langle z \rangle$ are the non-trivial wcts in wallpaper group $p6$. By Theorem 3.2 these maps are also non-trivial wcts.

The following applies to $G_{175}$ and $G_{176}$.

**Theorem 5.8.** Assume $\lambda \neq 1$. Let $\alpha = xy$, $\beta = x^{-2}y$. Groups with presentations of the form described in Eq. (4.11) have following non-trivial wcts maps:

(i) For $\lambda \in \langle x^2, y^2 \rangle$, $\tau(\lambda, C_1), C_1 = \{A, A\rho^2, A\rho^4, A\rho^5t, A\rho^3t\}$; and

(ii) For $\lambda \in \langle \alpha, \beta \rangle$, $\tau(\lambda, C_2), C_2 = \{A, A\rho^3, A\rho^4t\}$.

Additionally, if $\eta = 1$ (which is the case for $G_{175}$), then $\tau(\rho^3, C_1)$ and $\tau(\rho^2, C_2)$ are non-trivial wcts.

Proof. We will use Corollaries 2.15 and 2.16 to show that these two maps are wcts. For the first map $\tau(\lambda, C_1)$ we have $P(C_1) = \{A\rho, A\rho^3, A\rho^5, At, A\rho^3t, A\rho^4t\}$ thus by Proposition 5.6

$$I(C_1) = \langle x, y \rangle \cap \langle x^2, y^2 \rangle \cap \langle x^2, y^2, z^2 \rangle \cap \langle x, y, z^2 \rangle = \langle x^2, y^2 \rangle = K_{\rho^3}.$$ 

Since $x^2$ and $y^2$ are contained in $K_{\rho^3} = I(C_1)$, by Corollary 2.15 (ii), $\tau(x^2, C_1)$ and $\tau(y^2, C_1)$ are wcts.
For the second map, $\tau_\lambda$ we have $\mathcal{P}(\mathbb{C}_2) = (G/A) \setminus \{ A, A\rho^3, At, A\rho^3 t \}$; thus by Proposition 5.6

$$\mathcal{I}(\mathbb{C}_2) = (x, y) \cap (\alpha, \beta) \cap (\alpha, \beta, z^2) \cap (x, y, z^2) = (\alpha, \beta) = K_{\rho^2}.$$ 

Since $\alpha$ and $\beta$ are contained in $K_{\rho^2} = \mathcal{I}(\mathbb{C}_2)$, by Corollary 2.15 (ii) $\tau(\alpha, \mathbb{C}_2)$ and $\tau(\beta, \mathbb{C}_2)$ are wcts.

Now we consider the maps $\tau(\rho^3, \mathbb{C}_1)$ and $\tau(\rho^2, \mathbb{C}_2)$. Here $F$ consists of elements $\rho^j \rho^k$ for $j \in \{ 0, 1, 2, 3, 4, 5 \}$ and $k \in \{ 0, 1 \}$. Since we are assuming $\eta = 1$ it is clear that $\rho^2, \rho^3 \in F$ commute with every element of $F$. Thus by Corollary 2.16 (ii), since $\mathcal{I}(\mathbb{C}_1) = K_{\rho^3}$ and $\mathcal{I}(\mathbb{C}_2) = K_{\rho^2}$ we know that $\tau(\rho^3, \mathbb{C}_1)$ and $\tau(\rho^2, \mathbb{C}_2)$ are wct maps.

Next we show that $\tau_{x^2}, \tau_{y^2}, \tau_\alpha$, and $\tau_\beta$ are not homomorphisms:

$$\tau_{x^2}(\rho) \cdot \tau_{x^2}(\rho) = \rho \cdot \rho \neq x^{-2} y^{-2} \rho^2 = \tau_{x^2}(\rho \cdot \rho);$$

$$\tau_{y^2}(\rho) \cdot \tau_{y^2}(\rho) = \rho \cdot \rho \neq x^2 y^4 \rho^2 = \tau_{y^2}(\rho \cdot \rho);$$

$$\tau_\alpha(\rho) \cdot \tau_\alpha(\rho^2) = \rho \cdot \rho^2 \neq \alpha^{-2} \rho^3 = \tau_\alpha(\rho \cdot \rho^2);$$

$$\tau_\beta(\rho) \cdot \tau_\beta(\rho^2) = \rho \cdot \rho^2 \neq \beta^{-2} \rho^3 = \tau_\beta(\rho \cdot \rho^2).$$

It follows by Theorem 3.1 (ii) that these are non-trivial wcts. This theorem also tells us (since $\rho^2, \rho^3 \notin A$) that $\tau_{\rho^2}$ and $\tau_{\rho^3}$ are non-trivial wcts.

5.4 Some groups that have point group $\frac{2}{m}$, $mm2$, or $\frac{2}{m} \frac{2}{m} \frac{2}{m}$

Theorem 5.9. The following are non-trivial wcts:

(i) For $G_{10}$: $\tau(r, \{ Ar, At \})$ and $\tau(rt, \{ Art, At \})$;

(ii) For $G_{25}, G_{26}, G_{27}, G_{38}, G_{39}$, and $G_{42}$: $\tau(ps, \{ Ap, Aps \})$ and $\tau(s, \{ Ap, As \})$;

(iii) For $G_{47}$: $\tau(prt, \{ Aprt, At, Ap, Ar \})$;

(iv) For $G_{47}$ and $G_{51}$: $\tau(rt, \{ Art, At, Apr, Ap \})$;

(v) For $G_{47}, G_{55}$, and $G_{65}$: $\tau(pt, \{ Apt, At, Ar, Apr \})$. 

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Proof. Note that each of these maps are of the form given in Eq. (2.7). To show that such a map (a $\tau(\lambda, C)$ map) is a wct of the indicated space group, Corollary 2.16 (ii) asserts that it suffices to show that two conditions are met. The first is that $(\lambda, f) = 1$ for every $f \in F$ such that $Af \in C$. The second is that $K_\lambda \leq J$. Also note that for each of the maps here, $J = \bigcap_{Af \in C} K_f$. The relations in the presentations of each of these groups show that the first condition is met. Proposition A.1 gives the $K_f$ subgroups for each of these groups and shows that the second condition (the containment of $K_\lambda$) is also met. Thus we conclude that each of these maps are wcts. By Theorem 3.1 they are non-trivial wcts. \qed
Chapter 6. To determine $\mathcal{W}(G)$

6.1 Method

For the thirty-one groups listed in Tables 4.1 and 4.2 we will determine a set of generators of $\mathcal{W}(G)$. We do so in six steps which we also used in [HP]. These are:

**Step one:** For an arbitrary wct $\varphi$, we show that $\varphi|_A$ is an automorphism.

**Step two:** We show that we may compose $\varphi$ with trivial weak Cayley table maps so that $\varphi|_A$ is the identity map on $A$.

**Step three:** We show that we may again compose $\varphi$ with trivial wcts so that $\varphi$ fixes each coset in $G/A$.

**Step four:** We determine elements of $\mathcal{W}(G)$ that we may compose $\varphi$ with so as to have $\varphi(t) = t$ for all $t \in F$.

**Step five:** We show that for $t \in F$ there is an $f \in F$ such that $\varphi(at) = a^ft$ for $a \in A$.

**Step six:** We determine elements of $\mathcal{W}(G)$ that we may compose $\varphi$ with so as to have $\varphi = \text{Id}$.

In our work here we found it advantageous to combine steps two and three. Note that Theorem 6.1 shows that step one is true for any crystallographic group and similarly, Theorem 6.21 shows that step five is true for any crystallographic group. Also note also that in steps two, three, four, and six we frequently will compose with inner automorphisms. We will denote the automorphism that conjugates each group element by $g \in G$ as $I_g$.

6.2 Step one

Here we show that step one is true for any $n$-dimensional crystallographic group.
Theorem 6.1. Let $G$ be a crystallographic group with translation subgroup $A \cong \mathbb{Z}^n$ for $n \in \mathbb{N}$. Let $\varphi \in W(G)$. Then $\varphi|_A$ is an automorphism.

Proof. By Lemma 4.1, the only elements of $G$ that have finite conjugacy classes are in $A$. Since $\varphi$ preserves conjugacy classes and is bijective, $\varphi$ maps finite conjugacy classes to finite conjugacy classes. We therefore have $\varphi(A) = A$. Because $A$ is abelian, in this proof we will denote the group operation for $A$ additively. Also, to denote the conjugation action of $g \in G$ on $a \in A$ we will write $g(a)$.

Let $a, b \in A$. Using the fact that $\varphi$ respects inverses, we have

$$\varphi(a + b) \sim \varphi(a) + \varphi(b); \quad (6.1)$$
$$\varphi(b) = \varphi(a + b - a) \sim \varphi(a + b) - \varphi(a); \quad (6.2)$$
$$\varphi(a) = \varphi(a + b - b) \sim \varphi(a + b) - \varphi(b). \quad (6.3)$$

Eq. (6.1) implies there exists some $g_1 \in G$ such that $\varphi(a + b) = g_1(\varphi(a) + \varphi(b))$. Similarly, Eq. (6.2) implies there exists some $g_2 \in G$ such that $g_2(\varphi(b)) = \varphi(a + b) - \varphi(a)$ and Eq. (6.3) implies there exists some $g_3 \in G$ such that $g_3(\varphi(a)) = \varphi(a + b) - \varphi(b)$. Solving for $\varphi(a + b)$ we have the following three equations:

$$\varphi(a + b) = g_1(\varphi(a) + \varphi(b)); \quad (6.4)$$
$$= \varphi(a) + g_2(\varphi(b)); \quad (6.5)$$
$$= \varphi(b) + g_3(\varphi(a)). \quad (6.6)$$

Recall that every element of $A$ corresponds to a translation in Euclidean $n$–space, $n \in \mathbb{N}$, and thus to a point on the lattice $\mathfrak{L}$. We may think of $a \in A$ translating the lattice $\mathfrak{L}$ by a distance of $|a|$ and so it follows that $c \in a^G$ will also translate by a distance of $|a|$. Thus for $g \in G$, we can think of $g(a)$ as being some point on an $(n-1)$-sphere of radius $|a|$ centered at the origin. From this we see that Eq. (6.4) indicates that $\varphi(a + b)$ lies on an $(n-1)$-sphere of radius $|\varphi(a) + \varphi(b)|$ centered at the origin. Next we see that Eq. (6.5) indicates that $\varphi(a + b)$ lies on an $(n-1)$-sphere of radius $|\varphi(b)|$ centered at $\varphi(a)$. Lastly Eq. (6.6) implies
\( \varphi(a + b) \) lies on an \((n - 1)\)-sphere of radius \(|\varphi(a)|\) centered at \(\varphi(b)\).

Therefore \( \varphi(a + b) \) lies in the intersection of these three spheres. By Lemma 6.2 this intersection is \( \varphi(a) + \varphi(b) \). In other words, \( \varphi(a + b) = \varphi(a) + \varphi(b) \).

**Lemma 6.2.** Let \( a \neq 0 \) and \( b \neq 0 \) be vectors in \( \mathbb{E}^n \). Let \( S_0 \) be an \((n - 1)\)-sphere of radius \(|a + b|\) centered at the origin. Let \( S_a \) be an \((n - 1)\)-sphere of radius \(|b|\) with center \( a \) and similarly let \( S_b \) be an \((n - 1)\)-sphere of radius \(|a|\) with center \( b \). Then the intersection of the three spheres is one point, namely \( a + b \).

**Proof.** Clearly \( a + b \) is contained in each of \( S_a, S_b, \) and \( S_0 \) thus we have \( \{a + b\} \subseteq S_a \cap S_b \cap S_0 \).

It remains to show the reverse containment: \( S_a \cap S_b \cap S_0 \subseteq \{a + b\} \).

We note that if \( a + b = 0 \), then \( S_0 = \{0\} \) so \( S_0 \cap S_a \cap S_b = \{0\} \) is one point, proving this case. So now we assume \( a + b \neq 0 \).

We have \( S_0 = \{x \in \mathbb{E}^n : \|x\|^2 = \|a + b\|^2\} \), \( S_a = \{x \in \mathbb{E}^n : \|(x - a)\|^2 = \|b\|^2\} \), and \( S_b = \{x \in \mathbb{E}^n : \|x - b\|^2 = \|a\|^2\} \). Also, let \( P \) be the hyperplane that is tangent to \( S_0 \) at the point \( a + b \).

![Figure 6.1: The spheres \( S_a, S_b \), and \( S_0 \) with the plane \( P \) in \( \mathbb{E}^2 \)](image-url)
Now any \( x \in S_0 \) must satisfy
\[
x \cdot x - (a + b) \cdot (a + b) = 0. \tag{6.7}
\]
Also, any \( x \in S_a \) must satisfy
\[
(x - a) \cdot (x - a) - b \cdot b = 0. \tag{6.8}
\]
Any \( x \in S_0 \cap S_a \) must satisfy both of these equations. Subtracting Eq. (6.7) from Eq. (6.8) and expanding we have
\[
(x \cdot x - 2a \cdot x + a \cdot a) - b \cdot b - (x \cdot x - (a \cdot a + 2a \cdot b + b \cdot b)) = 0.
\]
Canceling gives
\[
(-2a \cdot x + a \cdot a) + (a \cdot a + 2a \cdot b) = 0.
\]
Simplifying and dividing by 2 gives
\[
a \cdot ((a + b) - x) = 0.
\]
Using \( x \in S_0 \cap S_b \), a similar argument gives
\[
b \cdot ((a + b) - x) = 0.
\]
It follows that any \( x \in S_0 \cap S_a \cap S_b \) satisfies
\[
(a + b) \cdot ((a + b) - x) = 0,
\]
The set of all \( x \) that satisfy this equation is the hyperplane that has normal vector \( a + b \) and contains the point \( a + b \). This is the plane \( P \) that is tangent to \( S_0 \) at the point \( a + b \). Recall that we also have \( x \in S_0 \), thus \( x \in P \cap S_0 \). Of course, this intersection is just one point (the point of tangency), which is \( a + b \).

\[\square\]

6.3 STEPS TWO AND THREE

In this section we will prove results useful for proving step two and step three. These results will be applied in Ch. 7 and Ch. 8 to the thirty-one space groups listed in Tables 4.1 and
Definition For a group $G$ with abelian normal subgroup $A$ we define the set

$$C_2 = C_2(G) = \{ a \in A : a^G = \{a, a^{-1}\} \}.$$

Proposition 6.3. In Table 6.1 we have the $C_2$ subsets for the space groups that have a presentation of the form given in Eqs. (4.1), (4.2), (4.3), (4.4), or (4.5).

<table>
<thead>
<tr>
<th>Group number(s)</th>
<th>$C_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10, 13</td>
<td>$\langle x, z \rangle \cup \langle y \rangle$</td>
</tr>
<tr>
<td>12</td>
<td>$\langle x^2y^{-1}, z \rangle \cup \langle y \rangle$</td>
</tr>
<tr>
<td>16, 17</td>
<td>$\langle x \rangle \cup \langle y \rangle \cup \langle z \rangle$</td>
</tr>
<tr>
<td>21</td>
<td>$\langle x^2y^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle$</td>
</tr>
<tr>
<td>22</td>
<td>$\langle x^2z^{-1} \rangle \cup \langle y^2z^{-1} \rangle \cup \langle z \rangle$</td>
</tr>
<tr>
<td>25, 26, 27</td>
<td>$\langle x \rangle \cup \langle y \rangle$</td>
</tr>
<tr>
<td>38, 39</td>
<td>$\langle x \rangle \cup \langle y^2z^{-1} \rangle$</td>
</tr>
<tr>
<td>42</td>
<td>$\langle x^2z^{-1} \rangle \cup \langle y^2z^{-1} \rangle$</td>
</tr>
<tr>
<td>47-57</td>
<td>$\langle x \rangle \cup \langle y \rangle \cup \langle z \rangle$</td>
</tr>
<tr>
<td>63-68</td>
<td>$\langle x^2y^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle$</td>
</tr>
<tr>
<td>69</td>
<td>$\langle x^2z^{-1} \rangle \cup \langle y^2z^{-1} \rangle \cup \langle z \rangle$</td>
</tr>
<tr>
<td>72-74</td>
<td>$\langle x^2y^{-1}z^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle$</td>
</tr>
</tbody>
</table>

Proof. This follows from the relations found in the presentation of the respective groups. (For brevity we write “47-57” to represent groups $G_{47}, G_{49}, G_{50}, G_{51}, G_{53}, G_{54}, G_{55},$ and $G_{57}$ (note we are not including $G_{48}$ and $G_{56}$); we write “63-68” to represent $G_{63}, G_{64}, G_{65}, G_{66}, G_{67},$ and $G_{68}$.)

Lemma 6.4. Let $G$ be a crystallographic group. Let $\varphi \in W(G)$ and $a \in A$. Let $g \in G$ satisfy $g^2 \in A$. Then $a^g = a^{-1}$ implies that $\varphi(a)^{\varphi(g)} = \varphi(a)^{-1}$.

Proof. By Theorem 6.1 we know that $\varphi|_A$ is a homomorphism thus for $k \in \mathbb{Z}$, $\varphi(a^k) = \varphi(a)^k$. Then $\varphi(a^k \cdot g) \sim \varphi(a)^k \varphi(g)$. Squaring both sides we have

$$\varphi(g^2) = \varphi(a^k \cdot g \cdot a^k g) \sim \varphi(a)^k \varphi(g) \varphi(a)^k \varphi(g) = \varphi(a)^k \varphi(g)^2 (\varphi(a)^k)^{\varphi(g)} = \varphi(g)^2 (\varphi(a) \varphi(a)^{\varphi(g)})^k.$$
Since \( g^2 \in A \) its image \( \varphi(g^2) \) has finitely many conjugates, thus \( \varphi(a) \varphi(a)^{\varphi(g)} = 1 \), in other words, \( \varphi(a)^{\varphi(g)} = \varphi(a)^{-1} \).

**Proposition 6.5.** Let \( G \) be a crystallographic group. Let \( \varphi \in W(G) \). Then

(i) \( Z(G) \cap C_2 = \{1\} \).

(ii) \( \varphi(C_2) = C_2 \).

(iii) \( \varphi(\langle C_2 \rangle) = \langle C_2 \rangle \).

(iv) Let \( j \in \mathbb{N} \). If there exists a set \( B = \{\beta_i\}_{i=1}^j \subseteq C_2 \) such that \( C_2 = \bigcup_{i=1}^j \langle \beta_i \rangle \) and \( \langle C_2 \rangle \) is a free abelian group of rank \( j \) with generating set \( B \), then for \( \beta \in B \) we have \( \varphi(\beta) \in \{a \in A : a \in B \text{ or } a^{-1} \in B\} \).

**Proof.** Statement (i) is clear since \( a^{-1} = a \in A \) is only possible if \( a \) is the identity. Now to prove (ii) we recall that \( \varphi \) maps conjugacy classes to conjugacy classes bijectively, therefore any nontrivial element in \( C_2 \) is mapped to a conjugacy class containing exactly two elements of \( A \). By Proposition 1.1 (ii), \( \varphi \) also respects inverses, thus we conclude that \( \varphi(C_2) = C_2 \).

By Theorem 6.1 \( \varphi|_A \) is an automorphism of \( A \) thus (iii) follows immediately from (ii).

To prove (iv), we will assume there exists \( B = \{\beta_i\}_{i=1}^j \subseteq C_2 \) such that \( C_2 = \bigcup_{i=1}^j \langle \beta_i \rangle \) and \( \langle C_2 \rangle \) is a free abelian group of rank \( j \) with generating set \( B \). Let \( \beta \in B \). As \( \beta \in C_2 \), we have \( \varphi(\beta) \in C_2 \) by (ii). Our assumption that \( C_2 = \bigcup_{i=1}^j \langle \beta_i \rangle \) implies that \( \varphi(\beta) = \beta_i^k \) for some \( 1 \leq i \leq j \) and \( k \in \mathbb{Z} \). Since \( \varphi|_A \) is an automorphism of \( A \), we see that (iii) implies \( \varphi(B) \) must be a free generating set for \( \langle C_2 \rangle \). We therefore must have \( \varphi(\beta) = \beta_i^k \) for \( k \in \{1, -1\} \) and we are done.

The following applies to \( G_{10}, G_{12}, \) and \( G_{13} \). This result is analogous to (i) and (ii) of Proposition 6.9 (which does not apply to these three groups because here \( C_2 \) is not a union of cyclic subgroups).

**Proposition 6.6.** Let \( G \) be a crystallographic group with presentation of the form given in Eq. (4.1). (Therefore \( A = \langle x, y, z \rangle \).) Let \( \varphi \in W(G) \). Then \( \varphi(y) \in \{y, y^{-1}\} \) and \( \varphi(Af) = Af \) for all \( f \in F \).
Proof. The generators used in the group presentation are $x, y, z, r, t$ and we define $F = \{r, rt, t, 1\}$. Every element in the $At$ coset is an involution, and this is not the case for any other coset. This implies $\varphi(At) = At$, and therefore $\varphi(At) \in \{Ar, Art\}$.

By Proposition 6.5 (ii) we know $\varphi(C_2) = C_2$ and here $C_2$ is the union of a free abelian subgroup of rank two and a cyclic subgroup. Since $\varphi|_A$ is a bijective homomorphism, it must preserve this structure and in particular, it must map the cyclic subgroup to itself. The cyclic subgroup contained in $C_2$ is $(y)$. We may therefore assume $\varphi(y) = y^\delta$ for $\delta \in \{-1, 1\}$.

Now $(rt)^2 \in A$ and $y^{rt} = y^{-1}$, thus by Lemma 6.4 we have $\varphi(y)^{\varphi(rt)} = \varphi(y)^{-1}$, i.e. $(y^\delta)^{\varphi(rt)} = y^{-\delta}$. If $\varphi(rt) \in Ar$ we would have a contradiction since $y$ commutes with every element in $Ar$. We conclude that $\varphi(At) = Art$ and thus $\varphi(Ar) = Ar$.

The following corollary applies to $G_{21}, G_{63}, G_{64}, G_{65}, G_{66}, G_{67}$, and $G_{68}$.

**Corollary 6.7.** Let $G$ be a crystallographic group with translation subgroup $A = \langle x, y, z \rangle \cong \mathbb{Z}^3$ and $C_2 = \langle x^2y^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle$. Let $\varphi \in \mathcal{W}(G)$. For $\beta \in \{x^2y^{-1}, y\}$ we have $\varphi(\beta) \in \{(x^2y^{-1})^{\pm 1}, y^{\pm 1}\}$. We also have $\varphi(z) \in \{z, z^{-1}\}$.

**Proof.** By Proposition 6.5 (iv) with $B = \{x^2y^{-1}, y, z\}$ we have

$$\{\varphi(x^2y^{-1}), \varphi(y), \varphi(z)\} \subseteq \{(x^2y^{-1})^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}.$$  

Now since $\varphi|_A$ is a homomorphism,

$$\varphi(x^2y^{-1})\varphi(y) = \varphi(x^2y^{-1} \cdot y) = \varphi(x^2) = \varphi(x)^2$$

must be a square of an element in $A$. The bijectivity of $\varphi$ requires $\varphi(x^2y^{-1})\varphi(y)$ to be a product of two distinct elements in $\{(x^2y^{-1})^{\pm 1}, y^{\pm 1}, z^{\pm 1}\}$. In order for this product to be a square, one of the two factors must be in $\{(x^2y^{-1})^{\pm 1}\}$ and the other must be in $\{y^{\pm 1}\}$. In other words, $\varphi(x^2y^{-1})\varphi(y)$ cannot be a square if either of $\varphi(x^2y^{-1})$ or $\varphi(y)$ is in $\{z^{\pm 1}\}$. Therefore by bijectivity we must have $\varphi(z) \in \{z^{\pm 1}\}$. 

The following applies to $G_{25}, G_{26}, G_{27}, G_{38}, G_{39}$, and $G_{42}$.  

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Lemma 6.8. Let $G$ be a group with presentation of the form given in Eq. (4.3). We define $F = \{1, p, s, ps\}$. Let $\varphi \in \mathcal{W}(G)$. Then

(i) $Z(G) = \langle z \rangle$.

(ii) $a \in C_2$ if and only if both $a^p = a^{-1}$ and $a \in \{a^s, a^{ps}\}$.

(iii) $C_2$ is the union of two cyclic groups.

(iv) $\langle C_2 \rangle \cap Z(G) = 1$.

(v) $\varphi(z) \in \{z^{\pm 1}\}$.

(vi) There exists an outer automorphism $\psi : G \to G$ such that

$$
\psi : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, s^{-1}).
$$

Proof. The relations given in the group presentation indicate that for $i, j, k \in \mathbb{Z}$ we have

$$(x^i y^j z^k)^p = x^{-i} y^{-j} z^u$$

where $u \in \{k, j + k, i + j + k\}$. This shows that for $a \in A,$

$$
\text{if } a^p = a, \text{ then } a \in \langle z \rangle.
$$

(6.9)

It follows that $Z(G) \subseteq \{a \in A : (a, p) = 1\} = \langle z \rangle$. The reverse inclusion is clear since the relations in the presentation of $G$ state that $z$ commutes with $x, y, p$ and $s$. This proves (i).

Recall that $C_2$ is the set of elements $a \in G$ such that $a^G = \{a, a^{-1}\}$. Therefore, $1 \neq a \in C_2$ is inverted by the action of two of the elements of $F$ and fixed by the action of the other two elements of $F$, one of which is of course the identity. Eq. (6.9) shows that $p$ cannot be the non-identity element of $F$ that fixes $a \in C_2$, since any element fixed by $p$ is central and by Proposition 6.5 (i), the only central element contained in $C_2$ is the identity. Hence we have (ii).

Statement (iii) follows from (ii). The set of elements fixed by $s$ is a cyclic subgroup and the set of elements fixed by $ps$ is another cyclic subgroup. Any element in one of these two
cyclic subgroups is inverted by $p$. Therefore, $C_2$ is comprised of the union of these two cyclic subgroups. The relations in the presentation of $G$ determine which elements generate the two cyclic subgroups. (Refer to Table 6.1.)

To prove (iv) we use the facts given in Proposition 6.3, specifically, the generators of the cyclic subgroups that comprise $C_2$. Thus we see that for groups $G_{25}, G_{26},$ and $G_{27}$, we have $\langle C_2 \rangle = \langle x, y \rangle$. For groups $G_{38}$ and $G_{39}$, we have $\langle C_2 \rangle = \langle x, y^2 z^{-1} \rangle$. For group $G_{42}$, we have $\langle C_2 \rangle = \langle x^2 z^{-1}, y^2 z^{-1} \rangle$. Each of these subgroups intersects the cyclic subgroup $\langle z \rangle = Z(G)$ trivially.

We will show that $\varphi(z) \in \{z, z^{-1}\}$ results from the fact that $\varphi|_A$ is a bijective homomorphism. We have $\varphi(Z(G)) = Z(G)$, by Proposition 1.1 (iv), thus $\varphi(z) \in \langle z \rangle$. Let $\varphi(z) = z^k$ for some $k \in \mathbb{Z}$, then as $\varphi|_A$ is a homomorphism $\langle \varphi(z) \rangle = \langle z^k \rangle$. If $k \notin \{\pm 1\}$ then $\langle \varphi(z) \rangle$ would be a proper subgroup of $\langle z \rangle$, contradicting the surjectivity of $\varphi$. This proves (v).

The map $\psi_i$ satisfies the relations in each of the six presentations of these groups thus it is in $\text{Aut}(G)$. Since $\psi_i(z) = z^{-1}$, and $\varphi(z) \in \{z, z^{-1}\}$ by (v), we therefore have (vi).

**Definition** Let $j \in \mathbb{N}$. Given a set $B = \{\beta_i\}_{i=1}^j \subseteq A$ and $f \in F$ we define

$$\text{Inv}(B, f) = \{\beta \in B : \beta^f = \beta^{-1}\}.$$ When $B$ is understood then we may write $\text{Inv}(B, f)$ simply as $\text{Inv}(f)$.

The following proposition applies to twenty-eight of the thirty-one groups listed in Tables 4.1 and 4.2. (It does not apply to $G_{10}, G_{12},$ and $G_{13}$.) Result (iii) applies to the twenty-four groups that have presentations of the form given in Eqs. (4.3), (4.4), or (4.5). Result (iv) applies only to the eighteen groups listed on Table 4.2.

**Proposition 6.9.** For $m, n \in \mathbb{N}$, such that $2 \leq m \leq n$ let $G$ be a group with normal abelian subgroup $A \cong \mathbb{Z}^n$ and $G/A \cong C_2^m$. Suppose that for some $2 \leq j \leq n$ there exists a set $B = \{\beta_i\}_{i=1}^j \subseteq C_2$ such that $C_2 = \bigcup_{i=1}^j \langle \beta_i \rangle$. Assume also that $B$ is a free generating set for $\langle C_2 \rangle \cong \mathbb{Z}^j$. Lastly we assume that the action of $G/A$ on $B$ defined by $\beta \cdot (Ag) = \beta^a$ is
well-defined and faithful, so that $\beta^{g_1} = \beta^{g_2}$ holds for all $\beta \in B$ if and only if $Ag_1 = Ag_2$. Let $
abla \in W(G)$.

(i) For $f,g \in F$, if $\nabla(Af) = Ag$, then $|\text{Inv}(B,f)| = |\text{Inv}(B,g)|$.

(ii) $\nabla(\beta) \in \beta^G$ for all $\beta \in B$ if and only if $\nabla(Af) = Af$ for all $f \in F$.

(iii) If $j = m$ and $\nabla(\beta) \in \beta^G$ for all $\beta \in B$, then there exists $I_f$ for some $f \in F$ so that $(I_f \circ \nabla)|_{(C_2)} = \text{Id}$.

(iv) If $j = m = n$ and $\nabla(\beta) \in \beta^G$ for all $\beta \in B$, then there exists $I_f$ for some $f \in F$ so that $(I_f \circ \nabla)|_A = \text{Id}$.

Proof. We prove (i) using Lemma 6.4. Let $f,g \in F$. Fix $\beta \in \text{Inv}(f)$, so that $\beta^f = \beta^{-1}$. By the Lemma, $\nabla(\beta)^{g(f)} = \nabla(\beta)^{-1}$. By assumption $\nabla(Af) = Ag$, thus the action of $\nabla(f)$ on $A$ must be equal to the action of $g$ on $A$ so we have $\nabla(\beta)^g = \nabla(\beta)^{-1}$. By Proposition 6.5 (iv) we have $\nabla(\beta) \in \{(\beta')^{\pm 1}\}$ for some $\beta' \in B$. As $\nabla$ respects inverses we see that the statement $\nabla(\beta) \in \{(\beta')^{\pm 1}\}$ implies $(\beta')^g = (\beta')^{-1}$. Thus $\beta' \in \text{Inv}(g)$. Since $\nabla$ is bijective, this shows that for every $\beta \in \text{Inv}(f)$ there exists a $\beta' \in \text{Inv}(g)$ and therefore $|\text{Inv}(f)| \leq |\text{Inv}(g)|$. To prove that $|\text{Inv}(f)| \geq |\text{Inv}(g)|$ we note that $\nabla \in W(G)$ implies $\nabla^{-1} \in W(G)$, $\nabla^{-1}(Ag) = Af$ and apply the previous argument. Therefore $|\text{Inv}(f)| = |\text{Inv}(g)|$, proving (i).

We prove (ii). ($\Rightarrow$) Suppose contrapositively that $\nabla(f_1) \in Af_2$ for some $f_1,f_2 \in F$, with $f_1 \not= f_2$. Since $f_1 \not= f_2$, by interchanging $f_1$ and $f_2$ if necessary we may assume there exists some $\beta \in B$ such that $\beta \in \text{Inv}(f_1), \beta \not\in \text{Inv}(f_2)$; thus $\beta^{f_2} = \beta$ but $\beta^{f_1} = \beta^{-1}$. Using $\nabla(f_1) \in Af_2$ and applying Lemma 6.4 to the statement $\beta^{f_1} = \beta^{-1}$, we have $\nabla(\beta)^{f_2} = \nabla(\beta)^{g(f_1)} = \nabla(\beta)^{-1}$. This shows $\nabla(\beta)$ is inverted by the action of $f_2$, although we know $\beta$ and its inverse are not (because $\beta \not\in \text{Inv}(f_2)$). From this we conclude that $\nabla(\beta) \not\in \{\beta^{\pm 1}\} = \beta^G$.

($\Leftarrow$) Conversely, assume that $\nabla(Af) = Af$ for all $f \in F$ and suppose to the contrary that there exists $\beta \in B$ such that $\nabla(\beta) \not\in \beta^G$. By Proposition 6.5 (iv) we know $\nabla(\beta) \in (\hat{\beta})^G = \{\hat{\beta}^{\pm 1}\}$ for some $\hat{\beta} \in B$. In other words, $\nabla$ permutes the conjugacy classes in the
collection \( \{ \beta^G : \beta \in B \} \). Define the subset \( B' \subseteq B \) to be

\[
B' = \{ \beta \in B : \varphi(\beta^G) \neq \beta^G \},
\]

and define a function \( \lambda : B' \to \mathbb{N} \) to be

\[
\lambda : \beta \mapsto |\{ f \in F : \beta \in \text{Inv}(f) \}|.
\]

(Note that the codomain of \( \lambda \) does not include zero because \( \lambda(\beta) = 0 \) means \( \beta \) commutes with every element of \( F \), which would imply \( \beta \) is central, a contradiction.) Choose \( \beta_1 \in B' \) so that \( \lambda(\beta_1) \) is maximal, i.e. \( \lambda(\beta_1) = \max \{ \lambda(\beta) : \beta \in B' \} \). Let \( \varphi(\beta^G_1) = \beta^G_2 \). Note that \( \beta_2 \in B' \) (because \( \varphi \) is bijective) and thus \( \lambda(\beta_1) \geq \lambda(\beta_2) \). In other words, \( \beta_1 \) is inverted by at least as many elements of \( F \) as \( \beta_2 \) is. This, combined with the fact that the action of \( G/A \) on \( B \) is faithful, ensures that there exists an \( h \in F \) such that \( \beta_1 \in \text{Inv}(h) \) but \( \beta_2 \notin \text{Inv}(h) \).

We will show that \( \beta_1 \in \text{Inv}(h) \) implies that \( \beta_2 \in \text{Inv}(h) \), thus arriving at a contradiction.

We have \( \beta_1 \in \text{Inv}(h) \) i.e. \( \beta^h_1 = \beta_1^{-1} \), and applying Lemma 6.4 to this equation gives \( \varphi(\beta_1)^{\varphi(h)} = \varphi(\beta_1)^{-1} \). Since \( \varphi \) respects inverses, \( \beta^h_2 = \beta_2^{-1} \). We are assuming \( \varphi(Ah) = Ah \) so we know \( \varphi(h) \in Ah \), thus \( \beta^h_2 = \beta_2^{-1} \). This shows that \( \beta_2 \in \text{Inv}(h) \), giving the needed contradiction.

Before we can prove \((iii)\) we will first need to prove that when \( j = m \), the collection \( \{ \text{Inv}(f) \}_{f \in F} \) is \( \mathcal{P}(B) \), the power set of \( B \). By definition \( \{ \text{Inv}(f) \}_{f \in F} \) is a subset of the power set of \( B \), thus it suffices to show that the cardinality of \( \{ \text{Inv}(f) \}_{f \in F} \) is at least \( |\mathcal{P}(B)| \). We will use the fact that \( G/A \) acts faithfully on \( B \). If \( f, g \in F \) and \( f \neq g \), then \( j = m \) shows that there exists some \( \beta \in B \) such that \( \beta^f \neq \beta^g \) thus \( \text{Inv}(f) \neq \text{Inv}(g) \). This shows that

\[
|\{ \text{Inv}(f) : f \in F \}| \geq |F| = |G/A| = 2^m = 2^j = |\mathcal{P}(B)|.
\]

We are now ready to prove \((iii)\). By assumption we have \( j = m \) and \( \varphi(\beta_i) \in \beta^G_i = \{ \beta_i, \beta_i^{-1} \} \) for \( 1 \leq i \leq j \). Thus there exists some subset \( B_{\text{Inv}} = \{ \beta \in B : \varphi(\beta) = \beta^{-1} \} \), but also, \( \varphi \) fixes all elements in \( B \setminus B_{\text{Inv}} \). Since \( \{ \text{Inv}(f) \}_{f \in F} \) is the power set of \( B \), we know there exists an \( f \in F \) such that \( \text{Inv}(f) = B_{\text{Inv}} \). In other words, there exists an \( f \in F \) such that \( (\varphi \circ I_f) : \beta_i \mapsto \beta_i \) for \( 1 \leq i \leq n \). Composing we have \( \varphi|_B = \text{Id} \) and since \( \varphi|_A \) is a
homomorphism and $B$ is a generating set for $\langle C_2 \rangle$, we now have $\varphi|_{\langle C_2 \rangle} = \text{Id}$ and we are done.

Lastly we show that statement $(iv)$ follows from $(iii)$ when $n = j = m$. Recall that both $A$ and $\langle C_2 \rangle$ are free abelian groups and $j = n$ means that $\text{rank}(\langle C_2 \rangle) = \text{rank}(A)$. It follows that $\langle C_2 \rangle$ has finite index in $A$ so we may write $|A : \langle C_2 \rangle| = \ell$ for some $\ell \in \mathbb{N}$. Let $A = \langle x_1, x_2, \ldots, x_n \rangle$. Note that for every generator $x_i$ we have $x_i^\ell \in \langle C_2 \rangle$, and so by $(iii)$ we may assume $\varphi(x_i^\ell) = x_i^\ell$. Since $\varphi|_A$ is a homomorphism, we have

$$\varphi(x_i^\ell) = \varphi(x_i^\ell) = x_i^\ell,$$

which can only be true if $\varphi(x_i) = x_i$, since $A \cong \mathbb{Z}^n$. This is true for all generators $x_i$ of $A$, proving $(iv)$. 

The following is a corollary to Proposition 6.9 and it applies to $G_{25}, G_{26}, G_{27}, G_{38}, G_{39}$, and $G_{42}$.

**Corollary 6.10.** Let $G$ be a group with a presentation of the form given in Eq. (4.3). Define $F = \{1, p, s, ps\}$. Let $\varphi \in \mathcal{W}(G)$. Then


$(ii)$ If $\varphi(Af) = Af$ for all $f \in F$ and $\varphi(z) = z$, then composing with inner automorphisms as necessary we may assume that $\varphi|_A = \text{Id}$.

**Proof.** By Lemma 6.8 $(iii)$ there exist $\beta_1, \beta_2 \in C_2$ so that $C_2 = \langle \beta_1 \rangle \cup \langle \beta_2 \rangle$. To prove $(i)$ we apply Proposition 6.9 $(i)$ with $B = \{\beta_1, \beta_2\}$. For these groups we have $|\text{Inv}(p)| = 2$ but $|\text{Inv}(s)| = |\text{Inv}(ps)| = 1$. Thus $\varphi$ must map the $Ap$ coset to itself.

To prove $(ii)$ we assume that $\varphi(Af) = Af$ for all $f \in F$ and we apply Proposition 6.9 $(ii)$. This gives $\varphi(\beta) \in \beta^G$ for all $\beta \in B$. Then applying Proposition 6.9 $(iii)$ we know there exist inner automorphisms so that composing $\varphi$ with these automorphisms we have $\varphi|_{\langle C_2 \rangle} = \text{Id}$. Now by hypothesis $\varphi(z) = z$ so, since $\varphi|_A$ is an isomorphism, we have $\varphi|_{\langle C_2, z \rangle} = \text{Id}$. We will
show that $\langle C_2, z \rangle$ is a finite index subgroup of $A$. Since $\varphi|_A$ is an isomorphism, this implies that $\varphi|_A = \text{Id}$.

By Lemma 6.8 (i), we have $Z(G) = \langle z \rangle$ and by Proposition 6.5 (i), $\langle z \rangle$ intersects $\langle C_2 \rangle$ trivially. Statement (iii) of Lemma 6.8 indicates that $\langle C_2 \rangle$ has rank two. Thus $\langle C_2, z \rangle$ has rank three, which implies it has finite index in $A$, and we are done.

The following is a corollary to Proposition 6.9 and it applies to the space groups numbered between 47 and 74 which are listed in Table 4.2.

**Corollary 6.11.** Let $G$ be a group with a presentation of the form given in Eqs. (4.4) or (4.5). We write the presentation of $G$ using generators $x, y, z, p, r, t$ and we define $F = \{p, r, pr, prt, rt, pt, t, 1\}$. Let $\varphi \in W(G)$. Then we have

(i) $\varphi(At) = At$.

(ii) For $f \in \{p, r, pr\}$ we have $\varphi(Af) = Ag$ for some $g \in \{p, r, pr\}$.

(iii) For $f \in \{prt, rt, pt\}$ we have $\varphi(Af) = Ag$ for some $g \in \{prt, rt, pt\}$.

**Proof.** For these groups we have

\[|\text{Inv}(t)| = 3; \quad |\text{Inv}(p)| = |\text{Inv}(r)| = |\text{Inv}(pr)| = 2; \quad |\text{Inv}(prt)| = |\text{Inv}(rt)| = |\text{Inv}(pt)| = 1.\]

Applying statement (i) of Proposition 6.9 gives the result.

In step three it is often helpful to consider which cosets contain elements of order 2 since cosets containing involutions may only be mapped to cosets that contain involutions. When $f \in F$ has order 2 it is clear that $Af$ contains order 2 elements. However, it is possible to have $f^2 \neq 1$ and yet have involutions in $Af$. The following Lemma makes it easy to determine whether or not a given coset contains involutions. (Note that it only applies to groups having presentations of the form given in Eqs. (4.1), (4.4), or (4.5).)

**Lemma 6.12.** Let $G$ be a group with an abelian normal subgroup $A$ and suppose $t \in G$ satisfies $t^2 = 1$ and $a^t = a^{-1}$ for any $a \in A$. Let $f \in F$ and assume $f^2 \in A$. Then the $Af$ coset contains elements of order 2 if and only if $f^2 \in K_f$.  

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Proof. Let \( a \in A \). Note that as \( f^2 \in A \), \( a f^{-1} = a^f \). Then
\[
(a f)^2 = 1 \iff a f a f = a a^f(f^2) = 1
\]
\[
\iff f^2 = a^{-1} (a^f)^{-1} = a^{-1} a^{ft} = (a, ft).
\]
This shows that there exists an \( a \in A \) such that \((af)\) has order 2 if and only if there exists an \( a \in A \) such that \( f^2 = (a, ft) \). By Lemma 2.2 this is equivalent to \( f^2 \in K_{ft} \).

6.4 Step Four

In this section we will prove results useful for proving step 4. These results will be applied in Ch. 7 and Ch. 8 to the thirty-one space groups listed in Tables 4.1 and 4.2.

The following applies to \( G_{25}, G_{26}, G_{27}, G_{38}, G_{39}, \) and \( G_{42} \).

Proposition 6.13. Let \( G \) be a group with a presentation of the form given in Eq. (4.3).
We define \( F = \{1, p, s, ps\} \). Let \( \varphi \in W(G) \) and suppose that \( \varphi|_A = \text{Id} \) and \( \varphi(p) = p \). Then \( \varphi(f) = f \) for all \( f \in F \).

Proof. We will prove two results and then use them to prove the proposition.

Claim 1: For \( b \in A \), if \( b^s = b^{ps} = b^{-1} \), then \( b = 1 \).
To prove this we first note that \( b^s = b^{ps} \) implies that \( b \) commutes with \( p \) and therefore \( b^G = \{b, b^{-1}\} \), so \( b \in C_2 \). By Lemma 6.8 (iii) we know that \( C_2 \) is the union of two cyclic subgroups that intersect trivially. By Lemma 6.8 (ii) we know one cyclic subgroup contains all elements of \( C_2 \) that are inverted by the action of \( ps \), and the other contains all elements of \( C_2 \) that are inverted by the action of \( s \). It follows that the only element of \( C_2 \) (and therefore any element of \( A \)) that is inverted by both \( s \) and \( ps \) is the identity. This proves Claim 1.

Claim 2: For \( f \in F \) we have \( f^2 \in Z(G) \).
Now recall that for \( f \in F \) we have defined \( \alpha_f = f^2 \). We have \( Z(G) = \langle z \rangle \) by Lemma 6.8 (i), and the relations in the presentation of \( G \) state that \( \alpha_p, \alpha_s \) and \( \alpha_{ps} = \alpha_p \alpha_s \) are all contained in \( \langle z \rangle \). (See Table 4.1.) This proves Claim 2.
We now show that $\varphi|_A = \text{Id}$ and $\varphi(p) = p$ imply that $\varphi(f) = f$ for all $f \in F$. By Proposition 6.9 (ii) we have $\varphi(Af) = Af$ for all $f \in F$. Therefore $\varphi(ps) = bps$ and $\varphi(s) = cs$ for some $b, c \in A$.

Squaring both sides of $\varphi(ps) = bps$ we have

$$\alpha_{ps} = \varphi(\alpha_{ps}) = \varphi((ps)^2) \sim (bps)^2 = bb^psps = bb^ps\alpha_{ps}.$$  

Since $\alpha_{ps}$ is central, conjugacy implies equality; therefore $\alpha_{ps} = bb^ps\alpha_{ps}$ and so $b^{ps} = b^{-1}$. We also have $\varphi(p^{-1} \cdot ps) \sim p^{-1}bps$ and squaring both sides of this relation gives

$$\alpha_s = \varphi(\alpha_s) = \varphi(s^2) \sim (p^{-1}bps)^2 = (b^p s)^2 = b^psb^ps = b^plp^s s^2 = b^plp^s \alpha_s.$$  

Again, conjugacy implies equality since $\alpha_s$ is central, so we have

$$\alpha_s = b^plp^s \alpha_s \text{ thus } (b^p)^{-1} = b^{ps} \text{ and so } b^{-1} = b^s.$$  

We have shown that $b^s = b^{ps} = b^{-1}$ so we must have $b = 1$, i.e. $\varphi(ps) = ps$.

Similarly, squaring both sides of $\varphi(s) = cs$ we have

$$\alpha_s = \varphi(\alpha_s) = \varphi(s^2) \sim (cs)^2 = cc^ss^2 = cc^s\alpha_s.$$  

Conjugacy again implies equality so $\alpha_s = cc^s\alpha_s$ thus $c^s = c^{-1}$. We also have $\varphi(p \cdot s) \sim pcs = c^plps$ and squaring both sides gives

$$\alpha_{ps} = \varphi((ps)^2) \sim (c^plps)^2 = c^pl(c^plps)ps = c^plc^s\alpha_{ps}.$$  

Again we may assume we have equality so $\alpha_{ps} = c^plc^s\alpha_{ps}$ thus $(c^p)^{-1} = c^s$ and conjugating both sides by $p$ gives $c^{-1} = c^{ps}$. We have shown that $c^s = c^{ps} = c^{-1}$ so $c = 1$, and thus $\varphi(s) = s$.  

The following proposition applies to groups $G_{10}, G_{12}, G_{13}$, and also the eighteen groups listed in Table 4.2. However, this result is not useful for the ten groups where $K = \langle x^2, y^2, z^2 \rangle$. It is only useful when $\langle x^2, y^2, z^2 \rangle \leq K$, which is the case in the eleven groups $G_{12}, G_{63}, G_{64}, G_{65}, G_{66}, G_{67}, G_{68}, G_{69}, G_{72}, G_{73},$ and $G_{74}$.

**Proposition 6.14.** Let $G$ be a group with a normal abelian subgroup $A = \langle x_1, x_2, \ldots \rangle$. 


Suppose there exists \( t \in F \) such that \( t^2 = 1 \) and \( (at)^2 = 1 \) for all \( a \in A \). Let \( f \in F \) and assume \( ft \in F \). Let \( \alpha_f = f^2 \) and similarly \( \alpha_{ft} = (ft)^2 \). Assume \( \alpha_f, \alpha_{ft} \in C_2 \). Let \( \varphi \in \mathcal{W}(G) \) and assume \( \varphi|_A = \text{Id} \) and \( \varphi(Af) = Af \). Given \( c \in A \), if we have \( (c, f) \notin \langle x_1^2, x_2^2, \ldots \rangle \), then \( \varphi(t) \neq ct \).

Proof. We will prove the contrapositive:

If \( \varphi(t) = ct \) for some \( c \in A \), then \( (c, f) \in \langle x_1^2, x_2^2, \ldots \rangle \).

Note that for \( a \in A \) we have \( a^{ft} = a^{tf} \), which is a consequence of the relation \( a^t = a^{-1} \) for all \( a \in A \). Also note that \( a^{f^{-1}} = a^f \) and \( a^{t^{-1}} = a^t \) since \( f^2, t^2 \in A \). We will use these two facts throughout this proof.

Let \( \varphi(f) = bf \) for some \( b \in A \). Squaring both sides gives

\[
\alpha_f = \varphi(\alpha_f) = \varphi(f \cdot f) \sim bfbf = bbf f^2 = bbf \alpha_f.
\]

Since \( \alpha^G_f = \{ \alpha_f^{\pm 1} \} \), we see that \( \alpha_f \sim bbf \alpha_f \) implies that \( bbf = \alpha_f^i \) for \( i \in \{-2, 0\} \). Conjugating by \( t \) and solving for \( b^{ft} \) we have \( b^{ft} = b \alpha_f^{-i} \).

Recall that by assumption \( \varphi(t) = ct \) for some \( c \in A \), thus we have \( \varphi(f \cdot t) \sim bfct \). Squaring both sides of this relation gives

\[
\alpha_{ft} = \varphi(\alpha_{ft}) = \varphi(ft \cdot ft) \sim bfct bfct = bcf b^{ft} c^t (ft)^2 = bcf b^{ft} c \alpha_{ft}.
\]

Since \( \alpha^G_{ft} = \{ \alpha_{ft}^{\pm 1} \} \) we have \( bcf b^{ft} c^t = \alpha_{ft}^j \) for \( j \in \{0, -2\} \). Using \( b^{ft} = b \alpha_f^{-i} \) and algebraically rearranging we have

\[
c^t c^t = b^{-1} (b \alpha_f^{-i})^{-1} \alpha_{ft}^j = b^{-2} \alpha_f^i \alpha_{ft}^j.
\]

Since \( i \) and \( j \) are either 0 or \(-2\), this shows that \( c^t c^t = (c, f) \) is the square of an element in \( A \), i.e. \( (c, f) \in \langle x_1^2, x_2^2, \ldots \rangle \).

The following applies to \( G_{63}, G_{64}, G_{65}, G_{66}, G_{67}, \) and \( G_{68} \).

Corollary 6.15. Let \( G \) be a group with a presentation of the form given in Eq. (4.4) (hence we write the presentation of \( G \) with generators \( x, y, z, p, r, \) and \( t \)), and assume that
$\delta = 1, \alpha_r \in \langle y \rangle$, and $\alpha_{rt} \in \langle z \rangle$. Let $\varphi \in W(G)$ and suppose that $\varphi|_A = \text{Id}$ and $\varphi(Ar) = Ar$. Then $\varphi(t) \neq xt$.

**Proof.** We will use Proposition 6.14 which states that if $(x,r) \not\in \langle x^2, y^2, z^2 \rangle$ then $\varphi(t) \neq xt$. To justify applying the proposition we note that by Proposition 6.3 we have $C_2 = \langle x^2y^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle$, thus $\alpha_r, \alpha_{rt} \in C_2$. We have $(x,r) = x^{-2}y$ which is not a square. Therefore, by the proposition, $\varphi(t) \neq xt$. \hfill $\Box$

The following applies to $G_{72}, G_{73}, \text{and } G_{74}$.

**Corollary 6.16.** Let $G$ be a group with a presentation of the form given in Eq. (4.5) (hence we write the presentation of $G$ with generators $x, y, z, p, r, \text{and } t$), and assume that $\alpha_p \in \langle z \rangle, \alpha_{pt} \in \langle x^2y^{-1}z^{-1} \rangle \cup \langle y \rangle$ and $\delta = 1$. Let $\varphi \in W(G)$ and suppose that $\varphi|_A = \text{Id}$ and $\varphi(Ap) = Ap$. Then $\varphi(t) \neq xt$.

**Proof.** To prove this is a straightforward application of Proposition 6.14. By Proposition 6.3 we have $C_2 = \langle x^2y^{-1}z^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle$. Thus we have $\alpha_p, \alpha_{pt} \in C_2$. Then by the proposition, since $(x,p) = x^{-2}z \not\in \langle x^2, y^2, z^2 \rangle$ we conclude $\varphi(t) \neq xt$. \hfill $\Box$

**Proposition 6.17.** Let $G$ be a group with abelian normal subgroup $A$ and let $|G/A| = n \in \mathbb{N}$. Let $\varphi \in W(G)$ and assume $\varphi|_A = \text{Id}$. Fix $f \in F$ such that $\alpha = f^2 \in A$. Let $\varphi(f) = bf$ for some $b \in A$. Then

(i) $b^t = b^{-1}$.

Furthermore, suppose there exists $t \in F$ such that $a^t = a^{-1}$ for all $a \in A$ and $(ft)^2 \in A$. Assume we have $\varphi(t) = t$. Let $\alpha_{ft} = (ft)^2$. Then

(ii) $\alpha_{ft} \sim b^2\alpha_{ft}$, in other words, $b^2\alpha_{ft} \in \alpha_{ft}^G$.  

From the above we immediately have these two statements:
(iii) If $\alpha_{ft} \in C_2$ (i.e. $\alpha_{ft}^G = \{\alpha_{ft}, \alpha_{ft}^{-1}\}$), then $b \in \{1, \alpha_{ft}^{-1}\}$;

(iv) If $ft$ has order 2, then $b = 1$, i.e. $\varphi(f) = f$.

Proof. For any $k \in \mathbb{Z}$ we have $\varphi(\alpha^k \cdot f) \sim \alpha^k b f$. Note that $\alpha$ commutes with $f$ and thus also with $bf$. Then squaring both sides we have

$$\alpha^{2k+1} = \varphi(\alpha^{2k} f^2) \sim \alpha^{2k} b b f = \alpha^{2k} b (\alpha f^{-1}) b f = \alpha^{2k} b a b f = \alpha^{2k+1} b f. \quad (6.10)$$

Let $\gamma = b b f$ and note that to prove $b f = b^{-1}$ it suffices to show that $\gamma = 1$. Now Eq. (6.10) becomes $\alpha^{2k+1} \sim \alpha^{2k+1} \gamma$. This implies that for any $j \in \mathbb{Z}$,

$$\alpha^{(2k+1)j} \sim (\alpha^{2k+1} \gamma)^j = \alpha^{(2k+1)j} \gamma^j. \quad (6.11)$$

We note that $\alpha$ and $\gamma$ are fixed, but the stated conjugacy must hold for any $j, k \in \mathbb{Z}$. Recall $n = |G/A|$ and put $(2k+1) j = 3^n$. (For each $j \in \{1, 3, 9, \ldots, 3^n\}$ there is a corresponding value of $k$ so as to ensure $(2k+1) j = 3^n$.) Thus we have

$$\alpha^{3^n} \sim \alpha^{3^n} \gamma^j. \quad (6.11)$$

Eq. (6.11) thus yields the following $n + 1$ statements:

$$\alpha^{3^n} \sim \alpha^{3^n} \gamma^1;$$
$$\alpha^{3^n} \sim \alpha^{3^n} \gamma^3;$$
$$\alpha^{3^n} \sim \alpha^{3^n} \gamma^9;$$
$$\vdots$$
$$\alpha^{3^n} \sim \alpha^{3^n} \gamma^{3^n}.$$

Now if $\gamma \neq 1$ then this implies that $|(\alpha^{3^n})^G| \geq n + 1$. Since $\alpha^{3^n} \in A$ and $n = |G/A|$ this is a contradiction. We conclude that $\gamma = 1$ and so $b f = b^{-1}$.

It follows that $b$ commutes with $ft$. Now since $\varphi(t) = t$ we have $\varphi(f \cdot t) \sim b f t$. Squaring both sides gives $(ft)^2 \sim (b ft)^2 = b^2 (ft)^2$, i.e. $\alpha_{ft} \sim b^2 \alpha_{ft}$. \hfill \Box

**Proposition 6.18.** Let $G$ be a group. Let $f_1, f_2, t \in G$ and assume $t^2 = 1$. Let $\alpha_1 = (f_1 t)^2$ and $\alpha_2 = (f_2 t)^2$. Let $\varphi \in \mathcal{W}(G.)$
(i) For \( i \in \{1, 2\} \), if \( \varphi(f_i) = \alpha_i^{-1} f_i \) then \((I_t \circ \iota \circ \varphi)(f_i) = f_i\).

(ii) If \( f_i t \) has order 2 then \( I_t \circ \iota \) acts trivially on \( f_i \), and also on \( t \).

(iii) If \( \varphi(f_1 f_2) \sim f_1 f_2 \sim \alpha_1^{-1} f_1 f_2 \) and \( \varphi(f_2) = f_2 \) then \( \varphi(f_1) \neq \alpha_1^{-1} f_1 \).

Proof. A straightforward calculation proves (i):

\[
(I_t \circ \iota \circ \varphi)(f_i) = (I_t \circ \iota)(\alpha_i^{-1} f_i) \\
= (I_t \circ \iota)((f_i t)^{-2} f_i) \\
= (I_t \circ \iota)(t f_i^{-1} t f_i^{-1} f_i) \\
= (I_t \circ \iota)((f_i^{-1})^2) \\
= f_i.
\]

The first statement in (ii) follows from (i) with \( \alpha_i = 1 \). It is also clear that \( I_t \circ \iota \) acts trivially on \( t \) since \( t = t^{-1} \) and since \( I_t(t) = t \).

To prove (iii) we assume that \( \varphi(f_1 f_2) \sim f_1 f_2 \) and \( \varphi(f_2) = f_2 \). Now suppose to the contrary that \( \varphi(f_1) = \alpha_1^{-1} f_1 \). Then

\[
f_1 f_2 \sim \varphi(f_1 \cdot f_2) \sim \alpha_1^{-1} f_1 \cdot f_2,
\]

which contradicts \( f_1 f_2 \sim \alpha_1^{-1} f_1 f_2 \) and we are done. \( \square \)
Proposition 6.19. The following maps are automorphisms of the indicated groups. (One can show that these are outer automorphisms.)

\[ G_{10}, G_{13} : \psi_x : (x, y, z, r, t) \mapsto (x, y, z, xr, xt), \]
\[ G_{10}, G_{12}, G_{13} : \psi_y : (x, y, z, r, t) \mapsto (x, y, z, yr, yt), \]
\[ G_{10}, G_{12}, G_{13} : \psi_z : (x, y, z, r, t) \mapsto (x, y, z, zr, zt), \]
\[ G_{16}, G_{17} : \psi_x : (x, y, z, p, r) \mapsto (x, y, z, xp, xr), \]
\[ G_{22} : \psi_x : (x, y, z, p, r) \mapsto (x, y, z, xy^{-1}p, xr), \]
\[ G_{16}, G_{17}, G_{21} : \psi_y : (x, y, z, p, r) \mapsto (x, y, z, yp, r), \]
\[ G_{16}, G_{17}, G_{21}, G_{22} : \psi_z : (x, y, z, p, r) \mapsto (x, y, z, zp, zr), \]
\[ G_{25}, G_{26}, G_{27}, G_{38}, G_{39} : \psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s), \]
\[ G_{25}, G_{26}, G_{27} : \psi_y : (x, y, z, p, s) \mapsto (x, y, z, yp, ys), \]
\[ G_{47}, G_{49}, G_{50}, G_{51}, G_{53}, G_{54}, G_{55}, G_{57} : \psi_x : (x, y, z, p, r, t) \mapsto (x, y, z, xp, xr, xt), \]
\[ All \ groups \ listed \ in \ Table \ 4.2 \ except \ G_{69} : \psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, rt), \]
\[ All \ groups \ listed \ in \ Table \ 4.2 : \psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt). \]

Proof. This follows from the relations in the respective group presentations.

6.5 Step five

Here we prove Theorem 6.21 which shows that step five is true for any \( n \)-dimensional crystallographic group. We begin with a lemma.

Lemma 6.20. Let \( G \) be a group with a normal abelian subgroup \( A \). Let \( \varphi \in W(G) \). Suppose that \( \varphi|_A = \text{Id} \) and \( t \in G \) satisfies \( \varphi(t) = t \). Let \( a, b \in A \). If \( \varphi(at) = bt \) then \( a \sim b \).

Proof. Given the hypotheses we have

\[ a \sim a^t = \varphi(a^t) = \varphi(t^{-1} \cdot at) \sim \varphi(t^{-1}) \cdot \varphi(at) = t^{-1}bt \sim b. \]
Recall that $F$ is a set of coset representatives for $G/A$. In the situation of Lemma 6.20, we will write $\varphi(at) = a^r t$, where $r_a \in F$.

**Theorem 6.21.** Let $G$ be a crystallographic group with translation subgroup $A \cong \mathbb{Z}^n$ for $n \in \mathbb{N}$. Let $\varphi \in W(G)$ and suppose that $\varphi|_A = \text{Id}$ and $t \in F$ satisfies $\varphi(t) = t$. Then there is an $f \in F$ such that $\varphi(at) = a^f t$ for all $a \in A$.

**Proof.** From Proposition 1.1 (vii), we see that $\varphi(At) = At$. Now, for $a,b \in A$, we have

$$ab^{-1} = at(bt)^{-1} = \varphi(at \cdot (bt)^{-1}) \sim a^r t \cdot t^{-1}(b^r)^{-1} = a^r(b^r)^{-1}.$$ 

Thus, there is some $f \in F$ such that $ab^{-1} = (a^r(b^r)^{-1})^f$, so that letting $\alpha = r_a f, \beta = r_b f$, this may be rewritten as $ab^{-1} = a^\alpha (b^{-1})^\beta$. Thus we have

$$a(a^{-1})^\alpha = b(b^{-1})^\beta.$$ \hfill (6.12)

Recall that every element of $A$ corresponds to a translation in Euclidean $n$-space, and thus to a point on the lattice $\mathcal{L}$. Let $v_a$ denote the point on $\mathcal{L}$ that corresponds to $a \in A$. For $a \in A$, let $S_a$ denote the $(n - 1)$-sphere in $\mathbb{E}^n$ that contains the origin and that is centered at $v_a$, and let $T_a = \{v_a(a^{-1})^f : f \in F\}$. Then, $T_a$ consists of $|a^G|$ points that lie on $S_a$. Note that the origin is in $T_a$ for all $a \in A$.

**Lemma 6.22.** (i) If $|T_a \cap T_b| = 1$ then there is a $\delta \in F$ such that $a^\delta = a^r, b^\delta = b^r$.

(ii) If the origin, $v_a$, and $v_b$ are collinear, then there is a $\delta \in F$ such that $a^\delta = a^r, b^\delta = b^r$.

**Proof.**

(i) If $|T_a \cap T_b| = 1$, then $T_a \cap T_b = \{0\}$ and so from Eq. (6.12), we get $a(a^{-1})^\alpha = b(b^{-1})^\beta = 1$, so that $a = a^\alpha, b = b^\beta$. From this we obtain $a^f = a^r, b^f = b^s$, so that we can let $\delta = f^{-1}$.

(ii) We may assume that $a \neq b$. By hypothesis the origin, $v_a$, and $v_b$ are collinear, in other words, the centers of the $(n - 1)$-spheres $S_a, S_b$ are on the line through these points. Since $a \neq b$, we have $S_a \neq S_b$, and this together with collinearity tells us that the
radius of $S_a$ is not equal to the radius of $S_b$. Therefore, $S_a \neq S_b$ implies that $S_a \cap S_b$ consists of just the origin (remembering that the origin is common to $S_a$ and $S_b$.) Thus we have $|T_a \cap T_b| = 1$, as in (i), which then gives the result. \hfill \Box

**Lemma 6.23.** Let $b \in A$ satisfy $|b^G| = |G/A|$. Then, for all $a \in A$ there is a unique $f = f_{a,b} \in F$ such that $\varphi(at) = a^f t$, $\varphi(bt) = b^f t$.

*Proof.* If the origin, $v_a$, and $v_b$ are collinear, then the existence of such a $\delta \in F$ follows from Lemma 6.22 (ii), while the uniqueness follows from $|b^G| = |F|$. Now, assume that the origin, $v_a$, and $v_b$ are not collinear. If $|T_a \cap T_b| = 1$, then by Lemma 6.22 (i), there is a $\delta \in F$ such that $a^{r_a} = a^\delta$, $b^{r_b} = b^\delta$, and the fact that $\delta$ is unique follows from $|b^G| = |F|$. So, now assume that $|T_a \cap T_b| > 1$. Note that for $2 \leq k \in \mathbb{N}$, the origin, $v_b$, and $v_{bk}$ are collinear. Then, by Lemma 6.22 (ii) applied to $b$ and $b^k$, there is some $h \in F$ such that $b^{r_b} = b^h$ and $(b^k)^{r_h} = (b^h)^k$. Since $|b^G| = |F|$ this element $h$ is unique. Since $T_a$ is finite, there is some $1 < k \in \mathbb{N}$ such that $|T_a \cap T_{bk}| = 1$; then by Lemma 6.22 (i) we have $h' \in F$ with $(b^k)^{r_{h'}} = (b^h)^k$, $a^{r_a} = a^{h'}$. Since $|b^G| = |F|$ we again see that $h, h'$ are unique, so that $h = h'$. Thus we have $a^{r_a} = a^h$ and $b^{r_b} = b^h$, as required. \hfill \Box

Now let $a, b, c \in A$, where $b$ satisfies $|b^G| = |F|$. Then, by Lemma 6.23 there are unique $f, h \in F$ such that $\varphi(at) = a^f t$, $\varphi(bt) = b^f t$ and $\varphi(bt) = b^h t$, $\varphi(ct) = c^h t$. Since $f, h$ are unique and $\varphi(bt) = b^f t$, $\varphi(bt) = b^h t$ we must have $f = h$; it follows (by fixing $a$ and varying $c$) that for all $d \in A$ we must have $\varphi(dt) = d^f t$ for this value of $f \in F$ that is completely determined by an element $b$ such that $|b^G| = |F|$. This concludes the proof of Theorem 6.21. \hfill \Box

### 6.6 Step Six

In this section we will prove results useful for proving step six. These results will be applied in Ch. 7 and Ch. 8 to the thirty-one space groups listed in Tables 4.1 and 4.2.

**Lemma 6.24.** Let $G$ be a group with a normal abelian subgroup $A$. Assume that $G/A$ is abelian. Let $n = |G/A|$. Let $a \in A$ and $f \in F$. If $f \sim a^{2k} f$ for all $k \in \mathbb{Z}$ such that $0 \leq k \leq n$,
then \(a^{2m} \in K_f\) for some \(1 \leq m \leq n\).

**Proof.** By Proposition C.2 we have

\[
f^G = \bigcup_{h \in F} K_f f^h = \bigcup_{h \in F} K_f(h, f^{-1}) f.
\]

Thus \(f \sim a^{2k}f\) implies that \(a^{2k} \in K_f(h, f^{-1})\) for some \(h \in F\). In other words, there exists an \(h \in F\) for every \(0 \leq k \leq n\) so that \(a^{2k} \in K_f(h, f^{-1})\). Since there are at most \(n\) choices for \(h\) there must be some \(h \in F\) and some \(j \in \mathbb{Z}\) such that \(a^{2j}, a^{2(j+m)} \in K_f(h, f^{-1})\) for some \(1 \leq m \leq n\). It follows that \(a^{2m} \in K_f\).

**Definition** Let \(G\) be a space group and let \(F\) be a set of coset representatives for \(G/A\). For \(f \in F, f \neq 1\), we define the set

\[
\mathcal{R}_f = \{h \in F : f \sim (h, \alpha)f \text{ for all } \alpha \in C_2\}.
\]

Recall that given a set \(B \subseteq A\) we have the following definition of the subset \(\text{Inv}(f)\):

\[
\text{Inv}(B, f) = \{\beta \in B : \beta f = \beta^{-1}\}.
\]

The following proposition applies to the thirty-one groups listed in Tables 4.1 and 4.2.

**Proposition 6.25.** Let \(j, n \in \mathbb{N}\). Let \(G\) be a group with normal subgroup \(A \cong \mathbb{Z}^n\) where \(G/A\) is abelian and finite. Suppose that there exists \(B = \{\beta_i\}_{i=1}^j \subseteq C_2 \subseteq \langle B \rangle \cong \mathbb{Z}^j\).

Assume that the action of \(G/A\) on \(B\) defined by \(\beta \cdot (Ag) = \beta^g\) is well-defined and faithful. Let \(1 \neq f \in F\) and assume that \(\text{rank}(K_f) = |\text{Inv}(B, f)|\). Then

\[
h \in \mathcal{R}_f \text{ if and only if } \text{Inv}(B, h) \subseteq \text{Inv}(B, f).
\]

**Proof.** (\(\implies\)) We assume \(h \in \mathcal{R}_f\). Note that if \(h = 1\), then \(\text{Inv}(h) = \emptyset\) so \(\text{Inv}(h) \subseteq \text{Inv}(f)\) is satisfied. Thus we may assume that \(h \neq 1\). This implies that \(\text{Inv}(h)\) is not empty because \(G/A\) acts faithfully on \(B\). We will show that \(\beta \in \text{Inv}(h)\) and \(f \sim (h, \alpha)f\) for all \(\alpha \in C_2\) together imply that \(\beta \in \text{Inv}(f)\).

Suppose to the contrary that \(\beta \notin \text{Inv}(f)\). As \(\beta \in C_2\) its only conjugates are itself and its inverse. It follows that \(\beta\) commutes with \(f\). Since \(\beta \in \text{Inv}(h)\) we have \((h, \beta) = \beta^2\).
Then for any \( k \in \mathbb{Z} \) we have \((h, \beta^k) = \beta^{2k}\) and note that \(\beta^k \in \mathcal{C}_2\). We are assuming \(f \sim (h, \alpha)f\) for all \(\alpha \in \mathcal{C}_2\), so in particular we have \(f \sim (h, \beta^k)f = \beta^{2k}f\). By Lemma 6.24 we have \(\beta^{2m} \in K_f\) for some \(m \in \mathbb{N}\). Now \(\beta_i \in \text{Inv}(f)\) implies that \(\beta_i^2 \in K_f\) and therefore \(\{\beta_i^2 : \beta_i \in \text{Inv}(f)\} \subseteq K_f\). However, recall that by assumption, \(\beta\) commutes with \(f\); thus \(\beta^{2m}\) commutes with \(f\) and so \(\beta^{2m} \notin \langle \beta_i^2 : \beta_i \in \text{Inv}(f)\rangle\). The fact that \(\text{rank}(\langle B \rangle) = j\) implies that the subgroup generated by \(\beta^{2m}\) and \(\{\beta_i^2 : \beta_i \in \text{Inv}(f)\}\) would be a subgroup of \(K_f\) with rank \(1 + |\text{Inv}(f)|\), a contradiction.

\((\Leftarrow)\) Assume that \(\text{Inv}(h) \subseteq \text{Inv}(f)\). We need to show that \(f \sim (h, \alpha)f\) for all \(\alpha \in \mathcal{C}_2\). For \(\alpha \in \mathcal{C}_2\) which commutes with \(h\) there is nothing to show so we may assume that \(\alpha^h = \alpha^{-1}\). We have \(\alpha \in \mathcal{C}_2 \subseteq \langle B \rangle\), therefore we may write \(\alpha = \beta_1^{k_1} \beta_2^{k_2} \cdots \beta_j^{k_j}\) for some \(k_1, k_2, \ldots, k_j \in \mathbb{Z}\). Notice that for each \(\beta_i \notin \text{Inv}(h)\) we must have \(k_i = 0\), in order to satisfy \(\alpha^h = \alpha^{-1}\). This is true because \(\langle B \rangle\) is a free group of rank \(j\). This shows that \(\alpha \in \langle \text{Inv}(h) \rangle\). By hypothesis \(\text{Inv}(h) \subseteq \text{Inv}(f)\), so \(\alpha \in \langle \text{Inv}(f) \rangle\). We thus have \(\alpha^f = \alpha^{-1}\) and so \((h, \alpha) = (f, \alpha)\). Then
\[
f \sim f^\alpha f = (f^{-1} \alpha^{-1})f(\alpha f) = (f, \alpha)f = (h, \alpha)f.
\]
This shows \(h \in \mathcal{R}_f\).

**Corollary 6.26.** Let \(G \in \{G_{10}, G_{12}, G_{13}\}\). We write the presentation of \(G\) using generators \(x, y, z, r, t\) and we define \(F = \{r, t, rt, 1\}\). For these groups we have
\[
\mathcal{R}_t = F; \quad \mathcal{R}_r = \{1, r\}; \quad \mathcal{R}_{rt} = \{1, rt\}.
\]

**Proof.** For \(G_{10}\) and \(G_{13}\) we have \(\mathcal{C}_2 = \langle x, z \rangle \cup \langle y \rangle\). Without loss of generality, let \(B = \{x, y, z\}\). Then \(\text{Inv}(t) = B\), \(\text{Inv}(r) = \{x, z\}\), and \(\text{Inv}(rt) = \{y\}\). The result follows from Proposition 6.25.

For \(G_{12}\) we have \(\mathcal{C}_2 = \langle x^2y^{-1}, z \rangle \cup \langle y \rangle\). Without loss of generality, let \(B = \{x^2y^{-1}, y, z\}\). Then \(\text{Inv}(t) = B\), \(\text{Inv}(r) = \{x^2y^{-1}, z\}\), and \(\text{Inv}(rt) = \{y\}\). The result follows from Proposition 6.25.
Corollary 6.27. Let $G \in \{G_{16}, G_{17}, G_{21}, G_{22}\}$. These presentations use generators $x, y, z, p, r$, and we define $F = \{p, r, pr, 1\}$. For these groups we have

$$R_p = \{1, p\}; \quad R_r = \{1, r\}; \quad R_{pr} = \{1, pr\}.$$  

Proof. For $G_{16}$ and $G_{17}$ we have $C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle$. Without loss of generality, let $B = \{x, y, z\}$. Then $\text{Inv}(p) = \{x, y\}, \text{Inv}(r) = \{x, z\}$, and $\text{Inv}(pr) = \{y, z\}$. The result follows from Proposition 6.25.

For $G_{21}$ we have $C_2 = \langle x^2y^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle$. Without loss of generality, let $B = \{x^2y^{-1}, y, z\}$. Then $\text{Inv}(p) = \{x^2y^{-1}, y\}, \text{Inv}(r) = \{x^2y^{-1}, z\}$, and $\text{Inv}(pr) = \{y, z\}$. The result follows from Proposition 6.25.

For $G_{22}$ we have $C_2 = \langle x^2z^{-1} \rangle \cup \langle y^2z^{-1} \rangle \cup \langle z \rangle$. Without loss of generality, let $B = \{x^2z^{-1}, y^2z^{-1}, z\}$. Then $\text{Inv}(p) = \{x^2z^{-1}, y^2z^{-1}\}, \text{Inv}(r) = \{x^2z^{-1}, z\}$, and $\text{Inv}(pr) = \{y^2z^{-1}, z\}$. The result follows from Proposition 6.25.

Corollary 6.28. Let $G \in \{G_{25}, G_{26}, G_{27}, G_{38}, G_{39}, G_{42}\}$. We write the presentation of $G$ using generators $x, y, z, p, s$, and we define $F = \{p, s, ps, 1\}$. For these groups we have

$$R_p = \{1, p, ps, s\}; \quad R_s = \{1, s\}; \quad R_{ps} = \{1, ps\}.$$  

Proof. For $G_{25}, G_{26},$ and $G_{27}$ we have $C_2 = \langle x \rangle \cup \langle y \rangle$. Without loss of generality, let $B = \{x, y\}$.

For $G_{38}$ and $G_{39}$ we have $C_2 = \langle x \rangle \cup \langle y^2z^{-1} \rangle$. Without loss of generality, let $B = \{x, y^2z^{-1}\}$.

For $G_{42}$ we have $C_2 = \langle x^2z^{-1} \rangle \cup \langle y^2z^{-1} \rangle$. Without loss of generality, let $B = \{x^2z^{-1}, y^2z^{-1}\}$.

In all cases, $\text{Inv}(p) = B$. On the other hand, $\text{Inv}(s)$ and $\text{Inv}(ps)$ each contain exactly one element of $B$ and $\text{Inv}(s) \neq \text{Inv}(ps)$. The result follows from Proposition 6.25.

Corollary 6.29. Let $G$ be a group with a presentation of the form given in Eqs. (4.4) or (4.5). These presentations use generators $x, y, z, p, r, t$, and we define $F = \{p, r, pr, prt, rt, pt, t, 1\}$. For these groups we have
\[ R_t = F; \]
\[ R_p = \{1, p, prt, rt\}; \quad R_{pt} = \{1, pt\}; \]
\[ R_r = \{1, r, prt, pt\}; \quad R_{rt} = \{1, rt\}; \]
\[ R_{pr} = \{1, pr, rt, pt\}; \quad R_{prt} = \{1, prt\}. \]

**Proof.** For groups with a presentation of the form given in Eq. (4.4) with \( \delta = 0 \) we have \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). Without loss of generality, let \( B = \{x, y, z\} \).

For groups with a presentation of the form given in Eq. (4.4) with \( \delta = 1 \) we have \( C_2 = \langle x^2y^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle \). Without loss of generality, let \( B = \{x^2y^{-1}, y, z\} \).

For groups with a presentation of the form given in Eq. (4.5) with \( \delta = 0 \) we have \( C_2 = \langle x^2z^{-1} \rangle \cup \langle y^2z^{-1} \rangle \cup \langle z \rangle \). Without loss of generality, let \( B = \{x^2z^{-1}, y^2z^{-1}, z\} \).

For groups with a presentation of the form given in Eq. (4.5) with \( \delta = 1 \) we have \( C_2 = \langle x^2y^{-1}z^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle \). Without loss of generality, let \( B = \{x^2y^{-1}z^{-1}, y, z\} \).

In all cases

- \( \text{Inv}(prt) \) is contained in \( \text{Inv}(p) \cap \text{Inv}(r) \);
- \( \text{Inv}(rt) \) is contained in \( \text{Inv}(p) \cap \text{Inv}(pr) \);
- \( \text{Inv}(pt) \) is contained in \( \text{Inv}(r) \cap \text{Inv}(pr) \);
- for \( f \in F \) we have \( \text{Inv}(f) \subseteq \text{Inv}(t) \).

The result follows from Proposition 6.25. \( \square \)

Here we make a comment that applies to the eight space groups with presentation of the form given in Eq. (4.4) with \( \delta = 0 \). For these groups we have \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). A consequence of this is that the definition of \( R_f \) is equivalent to

\[ \{h \in F : f \sim (h, \alpha)f \text{ for all } \alpha \in A\}, \]
as we now show.

Proof of the equivalence: Note that for any crystallographic group we have $C_2 \subseteq A$. It follows that the set $\{h \in F : f \sim (h, \alpha)f \text{ for all } \alpha \in A\}$ is contained in $R_f$. We will show that the reverse containment holds when $G$ is a group with presentation of the form given in Eq. (4.4) with $\delta = 0$.

$$h \in R_f \implies \text{Inv}(h) \subseteq \text{Inv}(f) \text{ by Proposition 6.25}$$

$$\implies K_h \leq K_f \text{ by Lemma 6.30,}$$

$$\implies \text{for } a \in A, (h, a) \in K_f,$$

$$\implies f \sim (h, a)f \text{ for } a \in A \text{ by Lemma 2.10.}$$

**Lemma 6.30.** Let $G$ be a group with presentation of the form given in Eq. (4.4) with $\delta = 0$. Thus we have $A = \langle x, y, z \rangle$ and $C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle$. Let $B = \{x, y, z\}$. Let $h, f \in F$. If $\text{Inv}(B, h) \subseteq \text{Inv}(B, f)$ then $K_h \leq K_f$.

**Proof.** By definition we know that $K_h$ is generated by commutators of the form $(\beta, h)$ for $\beta \in B$. To show $K_h \leq K_f$ it suffices to show that $(\beta, h) \in K_f$. We have two cases to consider: either $\beta \in \text{Inv}(h)$ or $\beta \notin \text{Inv}(h)$. Note that for $g \in F$ we have $(\beta, g) \in \{\beta^{-2}, 1\}$ and $(\beta, g) = \beta^{-2}$ if and only if $\beta \in \text{Inv}(g)$. Therefore for $\beta \in \text{Inv}(h)$ we have $(\beta, h) = \beta^{-2} = (\beta, f) \in K_f$ since by assumption $\beta \in \text{Inv}(h)$ implies $\beta \in \text{Inv}(f)$. For $\beta \notin \text{Inv}(h)$ we have $(\beta, h) = 1 \in K_f$. \qed

In general however, the set $\{h \in F : f \sim (h, \alpha)f \text{ for all } \alpha \in A\}$ will be a subset of $R_f$.

**Example:** For $G \in \{G_{63}, G_{64}, G_{65}, G_{66}\}$, we have $R_p = \{1, prt, rt, p\}$. On the other hand,

$$\{h \in F : p \sim (h, \alpha)p \text{ for all } \alpha \in A\}$$

does not include $prt$ since $(prt, x) = x^2y^{-1}$ and $p \sim x^2y^{-1}p$.

**Lemma 6.31.** Let $G$ be a crystallographic group and assume $C_2 \neq \{1\}$. Let $\varphi \in W(G)$ and assume that $\varphi|_A = \text{Id}$. Suppose that for some $f \in F$ we have $\varphi(f) = f$. Then there exists some $h \in R_f$ such that $\varphi(af) = ahf$ for all $a \in A$. 

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Proof. We know by Theorem 6.21 that there exists some $h \in F$ such that $\varphi(af) = a^h f$ for all $a \in A$. Then

\[ f = \varphi(a^{-1} \cdot af) \sim a^{-1}(a)^h f = (a, h)f. \]

Since this must hold in particular for $a \in C_2$ we must have $h \in R_f$. \qed

**Definition** For $f \in F, f \neq 1$, we define the set

\[ S_f = \{ h \in F : af \sim a^h f \text{ for all } a \in A \}. \]

**Lemma 6.32.** Let $f, h \in F$ and assume $f \neq 1$.

(i) If $(f, h) \in K_f$, then $h \in S_f$.

(ii) If $K_h \leq K_f$, then $h \in S_f$.

(iii) If $h_1, h_2 \in S_f$, then the element in $F$ representing the coset $Ah_1^{-1}h_2$ is in $S_f$.

(iv) $S_f$ is a set of coset representatives for a subgroup of $G/A$.

**Proof.** Let $a \in A$. Assume $(f, h) \in K_f$. First we show that $af \sim (f, h)a^h f$:

\[ af \sim fa \sim (fa)^h = f^h a^h \sim (f^h a^h)f = f^{-1} f^h a^h f = (f, h)a^h f. \]

By Lemma 2.10, $(f, h) \in K_f$ implies $a^h f \sim (f, h)a^h f$. Thus by transitivity $af \sim a^h f$ and so $h \in S_f$, proving (i).

Next, suppose that $K_h \leq K_f$; thus for $a \in A$, we have $(a, h) \in K_f$. Then by Lemma 2.10,

\[ af \sim (a, h)af = a^{-1}a^h af = a^h f, \]

and so $h \in S_f$, proving (ii).

To prove (iii), suppose that $h_1, h_2 \in S_f$. We will show that there exists $c \in A$ such that $ch_1^{-1}h_2 \in S_f$. By definition of $S_f$ for $a \in A$, we have $af \sim a^{h_1} f$ and $af \sim a^{h_2} f$.

Taking $a = b^{h_1^{-1}}$ these two assertions become respectively

\[ b^{h_1^{-1}} f \sim b^{h_1^{-1}h_1} f = bf \text{ and } b^{h_1^{-1}} f \sim b^{h_1^{-1}h_2} f. \]
Again using transitivity we have \(bf \sim b^{h_1^{-1}h_2}f\), in other words, there exists some \(h \in F\) such that \(h = ch_1^{-1}h_2\) for some \(c \in A\) and \(bf \sim bhf\) for all \(b \in A\). This proves \((iii)\). The fourth statement follows from \((iii)\) and from the fact that \(1 \in S_f\).

**Theorem 6.33.** Let \(G\) be a group with abelian normal subgroup \(A\) such that \(G/A\) abelian. Let \(\varphi \in W(G)\) and assume \(\varphi|_A\) = Id. Let \(f \in F\) and let \(g \in G\) satisfy \(fg \in F\) or \(gf \in F\). Let \(\gamma \in F\) be the coset representative for the \(Af_g\) coset (i.e. \(\gamma = fg\) or \(\gamma = gf\)). Suppose \(\varphi(f) = f, \varphi(g) = g,\) and \(\varphi(\gamma) = \gamma\). (We also assume that \(1 \notin \{f, g, \gamma\}\).)

Then there is an \(h \in R_f \cap S_\gamma\) such that \(\varphi(af) = ahf\) for all \(a \in A\).

**Proof.** We have \(\varphi(f) = f\), so by Lemma 6.31 we know there exists an \(h \in R_f\) such that for \(a \in A\) we have \(\varphi(af) = ahf\). It remains to show that \(h \in S_\gamma\).

**Case 1:** \(\gamma = fg\) : Then for \(a \in A\) we have

\[
a\gamma = \varphi(a)\varphi(\gamma) \sim \varphi(a \cdot \gamma) = \varphi(a \cdot fg) = \varphi(af \cdot g) \sim ahfg = ah\gamma.
\]

**Case 2:** \(\gamma = gf\) : (Here we will use the fact that \(G/A\) is abelian, thus \(a^{gh} = a^{hg}\).) Then for \(a \in A\),

\[
a\gamma \sim \varphi(a \cdot \gamma) = \varphi(a \cdot gf) = \varphi(g \cdot a^f) \sim gahf = ahgf = ah\gamma.
\]

In both cases we arrive at \(a\gamma \sim ah\gamma\) for all \(a \in A\), which shows that \(h \in S_\gamma\).

**Corollary 6.34.** Let \(G\) be a group with presentation of the form given in Eqs. (4.4) or (4.5). These presentations use generators \(x, y, z, p, r, t\), and we define \(F = \{p, r, pr, prt, rt, pt, t, 1\}\). Let \(\varphi \in W(G)\) and suppose that \(\varphi|_A\) = Id and \(\varphi(t) = t\). Fix \(f \in \{p^{rt}, rt, pt\}\) and suppose \(\varphi(f) = f\) and \(\varphi(ft) = ft\). Then \(f \notin S_{ft}\) implies that \(\varphi|_{Af}\) = Id.

**Proof.** Recall that \(t^2 = 1\); thus for \(f \in \{p^{rt}, rt, pt\}\) we have \(ft \in F \setminus \{1\}\). Therefore we may apply Theorem 6.33 which asserts that for \(a \in A\) we have \(\varphi(af) = ahf\) for some \(h \in R_f \cap S_{ft}\). To show \(\varphi|_{Af}\) = Id it suffices to show this intersection is \(\{1\}\). Now Corollary 6.29 indicates that for \(f \in \{p^{rt}, rt, pt\}\) we have \(R_f = \{1, f\}\). Since \(f \notin S_{ft}\) by assumption, we see \(R_f \cap S_{ft} = \{1\}\) so we are done.
Theorem 6.35. Let $G$ be a group with abelian normal subgroup $A$. Let $\varphi \in W(G)$. Assume $\varphi|_A = \text{Id}$ and let $\varphi(f) = f$ for some $f \in F \setminus \{1\}$.

(i) Suppose that $\varphi|_{Ag} = \text{Id}$ for some $Ag' \in (G/A) \setminus \{Af^{-1}\}$. Let $g \in Ag'$ satisfy $fg \in F \setminus \{1\}$. Then there exists $h \in R_f \cap R_{fg}$ so that for $a \in A$ we have $\varphi(af) = ahf$.

(ii) Suppose that for $j \in \mathbb{Z}$, there are cosets $Ag'_1, Ag'_2, \ldots, Ag'_j$ in $G/A$ such that $\varphi|_{Ag'_i} = \text{Id}$ for $1 \leq i \leq j$. Let $g_i \in Ag'_i$ satisfy $fg_i \in F \setminus \{1\}$ for $1 \leq i \leq j$. If $R_f \cap \bigcap_{i=1}^j R_{fg_i} = \{1\}$, then $\varphi|_{Af} = \text{Id}$.

Proof. By Lemma 6.31 we know there exists an $h \in R_f$ such that for all $a \in A$ we have $\varphi(af) = ahf$. Then for all $\alpha \in C_2$ we have $\varphi(g) = g$, since $g \in Ag'$, so

$$fg = \varphi(f)\varphi(g) \sim \varphi(\alpha^{-1}af \cdot g) = \varphi(\alpha^{-1}f \cdot \alpha^f g) \sim (\alpha^{-1})^h f \cdot \alpha^f g = (\alpha^{-1})^h \alpha fg = (h, \alpha)fg;$$

This shows $h \in R_{fg}$, proving (i). The second statement follows from applying the first statement $j$ times.

The following applies to space groups $G_{16}, G_{17}, G_{21},$ and $G_{22}.$

Corollary 6.36. Let $G$ be a group with presentation of the form given in Eq. (4.2). These presentations use generators $x, y, z, p, r,$ and we define $F = \{1, p, r, pr\}$. Let $\varphi \in W(G)$ and assume $\varphi|_A = \text{Id}$. Let $f \in F \setminus \{1\}$. Suppose that for some $g \in G$ such that $fg \in F \setminus \{1, f\}$ we have $\varphi|_{Ag} = \text{Id}$. Then

$$\varphi(f) = f \text{ implies that } \varphi|_{Af} = \text{Id}.$$ 

Proof. We will apply Theorem 6.35 (i) and use the fact that by Corollary 6.27, $R_f \setminus \{1, f\}$ for all $f \in F$. Since $fg \in F \setminus \{1, f\}$ the cosets $Af$ and $Afg$ are distinct, thus $R_f \cap R_{fg} = \{1\}$. The result follows directly.

The following applies to space groups $G_{25}, G_{26}, G_{27}, G_{38}, G_{39},$ and $G_{42}.$

Corollary 6.37. Let $G$ be a group with presentation of the form given in Eq. (4.3). These presentations use generators $x, y, z, p, s,$ and we define $F = \{p, ps, s, 1\}$. Let $\varphi \in W(G)$ and
suppose that $\varphi|_A = \text{Id}$, and $\varphi(f) = f$ for all $f \in F$. Then

$$\varphi|_{Aps} = \varphi|_{As} = \text{Id} \text{ implies that } \varphi = \text{Id}.$$ 

**Proof.** Note that $|G/A| = 4$ and so it suffices to show that $\varphi|_{Ap} = \text{Id}$. We assume that $\varphi|_{Aps} = \varphi|_{As} = \text{Id}$ and apply Theorem 6.35 (ii). Then since $\varphi(p) = p$ and $R_{ps} \cap R_{s} = \{1\}$ by Corollary 6.28, the theorem implies $\varphi|_{Ap} = \text{Id}$. 

**Corollary 6.38.** Let $G$ be a group with presentation of the form given in Eqs. (4.4) or (4.5). These presentations use generators $x, y, z, p, r, t$, and we define $F = \{p, r, pr, prt, rt, pt, t, 1\}$.

Let $\varphi \in \mathcal{W}(G)$ and suppose that $\varphi|_A = \text{Id}$, and $\varphi(f) = f$ for all $f \in F$. Then

(i) $\varphi|_{At} = \text{Id}$ implies that $\varphi = \text{Id}$,

(ii) $\varphi|_{Aprt} = \varphi|_{Art} = \varphi|_{Apt} = \text{Id}$ implies that $\varphi = \text{Id}$.

**Proof.** We will use the following results that follow from Corollary 6.29.

For $f \in F \setminus \{1, t\}$, $R_f \cap R_{ft} = \{1\}$. 

For $f_1, f_2 \in \{prt, rt, pt\}$, $R_{f_1} \cap R_{f_2} = \{1\}$. 

First we prove that $\varphi|_{At} = \text{Id}$ implies that $\varphi = \text{Id}$. It suffices to show that for $f \in \{p, r, pr, prt, rt, pt, t\}$, $\varphi(af) = a^h f$ implies that $h = 1$. We apply Theorem 6.35 (i), using $\varphi|_{At} = \text{Id}$ and $\varphi(f) = f$. Then by the theorem we have $h \in R_f \cap R_{ft}$. By Eq. (6.13) this intersection is $\{1\}$, so we conclude $\varphi|_{Af} = \text{Id}$.

Now assume that $\varphi|_{Aprt} = \varphi|_{Art} = \varphi|_{Apt} = \text{Id}$. To prove this implies $\varphi = \text{Id}$ it suffices to show that $\varphi|_{Af} = \text{Id}$ for $f \in \{p, r, pr, t\}$. We proceed by first showing that this holds for $f \in \{p, pr\}$. We will apply Theorem 6.35 (ii), using $\varphi(p) = p$ and $\varphi|_{Art} = \varphi|_{Aprt} = \text{Id}$. Since $R_{prt} \cap R_{rt} = \{1\}$ by Eq. (6.14), the theorem implies $\varphi|_{Ap} = \text{Id}$. Similarly, by using $\varphi(pr) = pr$ with $\varphi|_{Apt} = \varphi|_{rt} = \text{Id}$, and $R_{pr} \cap R_{rt} \cap R_{pt} = \{1\}$ by Eq. (6.14), Theorem 6.35 (ii) gives $\varphi|_{Apr} = \text{Id}$. 

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Now that we have $\varphi|_{Ap} = \varphi|_{Apr} = \text{Id}$, we use this to apply the theorem a third time. Using $\varphi(t) = t$ and since Eq. (6.14) gives $\mathcal{R}_{pt} \cap \mathcal{R}_{prt} = \{1\}$, we have $\varphi|_{At} = \text{Id}$. From (i), this implies $\varphi = \text{Id}$. \qed
Chapter 7. $\mathcal{W}(G)$ of space groups with point group $\frac{2}{m}, 222,$ or $\text{mm}2$

7.1 Groups 10 through 13

**Lemma 7.1.** The following results will be useful as we determine the step 2 automorphisms which can be found in $G_{10}, G_{12},$ and $G_{13}.$

(i) The group of automorphisms of $\mathbb{Z}^2$ is $\text{GL}(2, \mathbb{Z}).$ It is generated by

\[
\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(ii) The set of all matrices \[
\begin{pmatrix} i & j \\ k & \ell \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \text{ such that } j \in 2\mathbb{Z}
\]

is an index three subgroup of $\text{GL}(2, \mathbb{Z})$ and it is generated by

\[
\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(iii) The set of all matrices \[
\begin{pmatrix} i & j \\ k & \ell \end{pmatrix} \in \text{GL}(2, \mathbb{Z}) \text{ such that } k \in 2\mathbb{Z}
\]

is an index three subgroup of $\text{GL}(2, \mathbb{Z})$ and it is generated by

\[
(a^2)^T = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad b^T = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad c = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

**Proof.** We know $\text{SL}(2, \mathbb{Z})$ is an index two subgroup of $\text{GL}(2, \mathbb{Z})$ and it is generated by $a$ and $b$ [N]. Since $c$ has determinant $-1,$ these three matrices generate $\text{GL}(2, \mathbb{Z}),$ thus we have (i).

Let $H$ denote the subset of interest in (ii), i.e. the subset of matrices \[
\begin{pmatrix} i & j \\ k & \ell \end{pmatrix} \in \text{GL}(2, \mathbb{Z})
\]

such that $j$ is even. We verify that this subset is a subgroup by checking that it contains
inverses and is closed under multiplication. Let \( \epsilon = i \ell - k j \in \{ \pm 1 \} \). Then

\[
\begin{pmatrix}
i & j \\
k & \ell
\end{pmatrix}^{-1} = \epsilon
\begin{pmatrix}
\ell & -j \\
-k & i
\end{pmatrix} \in H.
\]

Also, for some \( m, n, p \in \mathbb{Z} \),

\[
\begin{pmatrix}
i & j \\
k & \ell
\end{pmatrix}
\begin{pmatrix}
i' & j' \\
k' & \ell'
\end{pmatrix} =
\begin{pmatrix}
m & ij' + j\ell' \\
n & p
\end{pmatrix} \in H.
\]

It is apparent that \( a^2, b, c \in H \), so \( \langle a^2, b, c \rangle \leq H \). Using Reidemeister-Schreier’s algorithm as implemented in Magma [BCP],[MKS], (see Appendix B), we verified that \( \langle a^2, b, c \rangle \) is an index three subgroup of \( \text{GL}(2, \mathbb{Z}) \). Since three is prime, evidently \( \langle a^2, b, c \rangle \) is not a proper subgroup of \( H \), proving \((ii)\).

Statement \((iii)\) follows from \((ii)\) and from the fact that for matrices \( B, C \) we have

\[
(B^{-1})^T = (B^T)^{-1} \quad \text{and} \quad (BC)^T = C^T B^T.
\]

We will now apply steps one through six to the group \( G_{10} \) to determine a set of generators of \( \mathcal{W}(G_{10}) \). Let \( \varphi \in \mathcal{W}(G_{10}) \). By Proposition 6.6 we have \( \varphi(A f) = A f \) for all \( f \in F \) and \( \varphi(y) \in \{ y, y^{-1} \} \). If we have \( \varphi(y) = y^{-1} \) then we may compose with \( I_t \) and now we may assume that \( \varphi(y) = y \). Since \( C_2 = \langle x, z \rangle \cup \langle y \rangle \), Proposition 6.5 \((ii)\) gives \( \varphi(\langle x, z \rangle) = \langle x, z \rangle \).

Thus \( \varphi|_{\langle x, z \rangle} \) is an automorphism of a free abelian group of rank 2. The set of all such maps is \( \text{GL}(2, \mathbb{Z}) \). Let \( \xi : \langle x, z \rangle \to \langle x, z \rangle \) be one such automorphism. Then

\[
\psi_\xi : (x, y, z, r, t) \mapsto (\xi(x), y, \xi(z), r, t)
\]

determines an automorphism of \( G_{10} \) as it satisfies all the relations in the presentation of \( G_{10} \). By Lemma 7.1 we know \( \text{GL}(2, \mathbb{Z}) \) can be generated by three matrices, therefore there exist three corresponding automorphisms that generate all the automorphisms that are of
the form described by $\psi_\xi$. These are:

$$\begin{align*}
\psi_1 : (x, y, z, r, t) &\mapsto (x, y, xz, r, t); \\
\psi_2 : (x, y, z, r, t) &\mapsto (xz^{-1}, y, z, r, t); \\
\psi_3 : (x, y, z, r, t) &\mapsto (x^{-1}, y, z, r, t).
\end{align*}$$

Composing with these three outer automorphisms, we can ensure that $\varphi|_{\langle x,z \rangle} = \text{Id}$. Since we also have $\varphi(y) = y$ we now have $\varphi|_A = \text{Id}$.

By Proposition 6.19 we may compose with outer automorphisms $\psi_x, \psi_y,$ and $\psi_z$ as necessary so as to have $\varphi(t) = t$. Since $r^2 = (rt)^2 = 1$, Proposition 6.17 gives $\varphi(r) = r$ and $\varphi(rt) = rt$; thus we have $\varphi(f) = f$ for all $f \in F$.

Applying Theorem 6.21 with Lemma 6.31 we have for all $a \in A, \varphi(art) = a^hrt$ for some $h \in \{1, rt\}$. If $h = 1$, then we are done. If $h = rt$ then we may compose with $I_t \circ \iota$. Since $(rt)^2 = (rt, t) = 1$,

$$(I_t \circ \iota)(a^rt) = I_t(rt(a^{-1})rt) = rta^rt = art,$$

we now have $\varphi|_{Art} = \text{Id}$.

Now by Lemma 6.31, for some $h \in \mathcal{R}_r = \{1, r\}, a \in A$, we have $\varphi(ar) = a^hr$. If $h = r$ we may compose with the non-trivial wct $\tau(r, \{Ar, At\})$ so that $\varphi|_{Ar} = \text{Id}$. (Theorem 5.9 proves $\tau_r$ is non-trivial.) Note that Corollary 6.26 gives $\mathcal{R}_t \cap \mathcal{R}_{rt} \cap \mathcal{R}_r = \{1\}$. We use this to apply Theorem 6.35 ($ii$). We have $\varphi(t) = t$ and $\varphi|_{Art} = \varphi|_{Ar} = \text{Id}$. Then by the theorem we have $\varphi|_A = \text{Id};$ and therefore $\varphi = \text{Id}$.

We have shown that

**Theorem 7.2.** For crystallographic group $G_{10}$, the group $W(G)$ is generated by the inverse
map \iota, the inner automorphisms,

\begin{align*}
\psi_1 &: (x, y, z, r, t) \mapsto (x, y, xz, r, t), \\
\psi_2 &: (x, y, z, r, t) \mapsto (x^{-1}, y, z, r, t), \\
\psi_3 &: (x, y, z, r, t) \mapsto (x^{-1}, y, z, r, t), \\
\psi_x &: (x, y, z, r, t) \mapsto (x, y, x, r, x), \\
\psi_y &: (x, y, z, r, t) \mapsto (x, y, z, r, y), \\
\psi_z &: (x, y, z, r, t) \mapsto (x, y, z, zr, zt),
\end{align*}

and the non-trivial wct \tau_r. Thus we have \(W(G) = \langle W_0(G), \tau_r \rangle\).

Let \(\varphi \in W(G_{12})\). By Proposition 6.6 we have \(\varphi(Af) = Af\) for all \(f \in F\) and \(\varphi(y) \in \{y, y^{-1}\}\). Composing with \(I_t\) if necessary we have \(\varphi(y) = y\). By Proposition 6.5 (iii) we know that \(\varphi((x^2y^{-1}, z)) = (x^2y^{-1}, z)\). In other words, \(\varphi|_{(x^2y^{-1}, z)}\) is an automorphism of a free abelian group of rank 2. We may write \(\varphi(x^2y^{-1}) = (x^2y^{-1})^iz^j\) and \(\varphi(z) = (x^2y^{-1})^kz^\ell\) for \(i, j, k, \ell \in \mathbb{Z}\), and thus \(\varphi|_A\) corresponds to the matrix \(\begin{pmatrix} i & j \\ k & \ell \end{pmatrix}\) \(\in GL(2, \mathbb{Z})\), relative to the basis \(x^2y^{-1}, z\). Now by Proposition A.1,

\[K_r = \langle x^2y^{-1}, z^2 \rangle, \quad K_t = \langle x^2, y^2, z^2 \rangle\quad \text{and} \quad K_{rt} = \langle y \rangle;\]

thus we have \(K = \langle x^2, y, z^2 \rangle\). By Lemma 2.11, \(\varphi(K) = K\). Then since \(x^2y^{-1} \in K\),

\(\varphi(x^2y^{-1}) = (x^2y^{-1})^iz^j \in K = \langle x^2, y, z^2 \rangle,\)

so \(j\) is even.

Let \(\xi : \langle x^2y^{-1}, z \rangle \to \langle x^2y^{-1}, z \rangle\) be the automorphism given by

\[\xi : (x^2y, z) \mapsto ((x^2y^{-1})^iz^j, (x^2y^{-1})^kz^\ell).\]

Thus \(\xi\) corresponds to a matrix \(\begin{pmatrix} i & j \\ k & \ell \end{pmatrix}\) \(\in GL(2, \mathbb{Z})\) of matrices with \(j\)
even. The map

\[ \psi_\xi : (x^2y^{-1}, y, z, r, t) \mapsto (\xi(x^2y^{-1}), y, \xi(z), r, t) \]

\[ = (x, y, z, r, t) \mapsto (x^i y^{(1-i)/2}z^{j/2}, y, x^k z^\ell, r, t) \]

determines an automorphism of \( G_{12} \) because it satisfies the relations in the presentation of \( G_{12} \). By Lemma 7.1 \((ii)\) we know that the set of matrices \( \begin{pmatrix} i & j \\ k & \ell \end{pmatrix} \) in \( \text{GL}(2, \mathbb{Z}) \) with \( j \) even is a subgroup of \( \text{GL}(2, \mathbb{Z}) \) and that it can be generated by three matrices. Therefore there exist three corresponding automorphisms that generate all the automorphisms that are of the form described by \( \psi_\xi \). These are:

\[ \psi_4 : (x, y, z, r, t) \mapsto (xz, y, z, r, t); \]
\[ \psi_5 : (x, y, z, r, t) \mapsto (x, y, x^{-2}yz, r, t); \]
\[ \psi_6 : (x, y, z, r, t) \mapsto (x^{-1}y, y, z, r, t). \]

Composing with these three outer automorphisms gives \( \varphi|_{\langle x^2y^{-1}, z \rangle} = \text{Id} \). Since we also have \( \varphi(y) = y \) and \( \langle y \rangle \times \langle x^2y^{-1}, z \rangle \) has finite index in \( A \), we conclude that \( \varphi|_A = \text{Id} \).

By Proposition 6.19 we may compose with outer automorphisms if necessary so as to have \( \varphi(t) \in \{ t, xt \} \) for some \( i \in \mathbb{Z} \). Composing with \( I_x \) can ensure \( \varphi(t) \in \{ t, xt \} \). However, Proposition 6.14 implies that since \( \varphi(Ar) = Ar \) and \( (x, r) = x^{-2}y \notin \langle x^2, y^2, z^2 \rangle \), we cannot have \( \varphi(t) = xt \). Thus we have \( \varphi(t) = t \).

Since \( r^2 = (rt)^2 = 1 \), Proposition 6.17 gives \( \varphi(r) = r \) and \( \varphi(rt) = rt \); thus we have \( \varphi(f) = f \) for all \( f \in F \).

Applying Theorem 6.21 with Lemma 6.31 for all \( a \in A \), we have \( \varphi(art) = a^h rt \) for some \( h \in \{ 1, rt \} \). If \( h = rt \) then we may compose with \( I_t \circ \iota \). Since

\[ (I_t \circ \iota)(a^h rt) = I_t(rt(a^{-1})rt) = rta^{rt} = art, \]

we now have \( \varphi|_{Ar} = \text{Id} \).

By Lemma 6.31, for all \( a \in A \) we have \( \varphi(ar) = a^h r \) for some \( h \in \mathcal{R}_r \) where \( \mathcal{R}_r = \{ 1, r \} \).
by Corollary 6.26. Suppose that $h = r$. Then

$$x^2y^{-1}t \sim \varphi(x(xy^{-1}) \cdot t) = \varphi(xr \cdot x^{-1}rt) \sim \varphi(xr)\varphi(x^{-1}rt) = x^r r \cdot x^{-1}rt = x^r(x^{-1})^r t = t.$$ However, $x^2y^{-1}t \notin t^G = \langle x^2, y^2, z^2 \rangle t$. This contradiction implies that $h \neq r$; thus $\varphi|_{Ar} = Id$.

Now by Corollary 6.26 we have $R_t \cap R_{rt} \cap R_r = \{1\}$. We use this to apply Theorem 6.35 (ii). Since $\varphi|_{Art} = \varphi|_{Ar} = Id$ and $\varphi(t) = t$, by the theorem it follows that $\varphi|_{At} = Id$ and therefore $\varphi = Id$.

We have shown that

**Theorem 7.3.** For crystallographic group $G_{12}$, the group $W(G)$ is generated by the inverse map $I$, the inner automorphisms and

$$\psi_4 : (x, y, z, r, t) \mapsto (xz, y, z, r, t);$$

$$\psi_5 : (x, y, z, r, t) \mapsto (x, y, x^{-2}yz, r, t);$$

$$\psi_6 : (x, y, z, r, t) \mapsto (x^{-1}y, y, z, r, t);$$

$$\psi_y : (x, y, z, r, t) \mapsto (x, y, z, r, yt);$$

$$\psi_z : (x, y, z, r, t) \mapsto (x, y, zr, zt).$$

Thus we have $W(G) = W_0(G)$. \qed

Let $\varphi \in W(G_{13})$. By Proposition 6.6 we have $\varphi(Af) = Af$ for all $f \in F$ and $\varphi(y) \in \{y, y^{-1}\}$. If we have $\varphi(y) = y^{-1}$ then composing with $I_t$ we may assume that $\varphi(y) = y$. Since $C_2 = \langle x, z \rangle \cup \langle y \rangle$, by Proposition 6.5 (ii) we have $\varphi(\langle x, z \rangle) = \langle x, z \rangle$. In other words, $\varphi|_{\langle x, z \rangle}$ is an automorphism of a free abelian group of rank 2. We may write $\varphi(x) = x^iz^j$ and $\varphi(z) = x^kz^\ell$ for $i, j, k, \ell \in \mathbb{Z}$, and so $\varphi|_A$ corresponds to the matrix $\begin{pmatrix} i & j \\ k & \ell \end{pmatrix} \in GL(2, \mathbb{Z})$. Now for this group we have $G' = \langle x^2, y^2, z \rangle$ and by Lemma 2.11 $\varphi(G') = G'$. Then since $z \in G'$,

$$\varphi(z) = x^kz^\ell \in G' = \langle x^2, y^2, z \rangle$$

so $k$ is even.
Let $\xi : \langle x, z \rangle \to \langle x, z \rangle$ be the automorphism determined by

$$\xi : (x, z) \mapsto (x^i z^j, x^k z^\ell).$$

Thus $\xi$ corresponds to a matrix $\begin{pmatrix} i & j \\ k & \ell \end{pmatrix}$ in the subgroup of $GL(2, \mathbb{Z})$ of matrices with $k$ even. For $a \in A$, the map

$$\psi_{\xi,a} : (x, y, z, r, t) \mapsto (\xi(x), y, \xi(z), ar, t)$$

determines an automorphism of $G_{13}$ if it satisfies the relations in the presentation of $G_{13}$. (The pertinent relations here are $r^2 = 1$ and $(rt)^2 = z$.) Accordingly, we require $(ar)^2 = 1$ (which implies $a^r = a^{-1}$) and $(art)^2 = aar^t z = \xi(z)$, i.e. $a^2 z = x^k z^\ell$. Equivalently, we require that $a = x^{k/2} z^{(\ell-1)/2}$. By Lemma 7.1 (iii) we know that the set of matrices $\begin{pmatrix} i & j \\ k & \ell \end{pmatrix}$ in $GL(2, \mathbb{Z})$ with $k$ even is a subgroup of $GL(2, \mathbb{Z})$ and that it can be generated by three matrices. Therefore there exist three corresponding automorphisms that generate all the automorphisms that are of the form described by $\psi_{\xi}$. These are:

- $\psi_7 : (x, y, z, r, t) \mapsto (x, y, x^2 z, xr, t)$;
- $\psi_8 : (x, y, z, r, t) \mapsto (xz^{-1}, y, z, r, t)$;
- $\psi_9 : (x, y, z, r, t) \mapsto (x^{-1}, y, z, r, t)$.

Composing with these three outer automorphisms, we can ensure that $\varphi|_{\langle x, z \rangle} = \text{Id}$. Since we also have $\varphi(y) = y$ we conclude that $\varphi|_A = \text{Id}$.

By Proposition 6.19 we may compose with outer automorphisms $\psi_x, \psi_y,$ and $\psi_z$ if necessary so as to have $\varphi(t) = t$. Since $r^2 = 1$ and $(rt)^2 = z^{-1}$, Proposition 6.17 gives $\varphi(r) \in \{r, zr\}$ and $\varphi(rt) = rt$. If $\varphi(r) = zr$ then composing with $I_t \circ \iota$ we have $\varphi(r) = r$. (Notice that we still have $\varphi(rt) = rt$ as $(rt)^t = (rt)^{-1}$. Another way to see this is to realize that after composing with $I_t \circ \iota$ the hypotheses of Proposition 6.17 are still met; thus we still have $\varphi(rt) = rt$. ) Thus we now have $\varphi(f) = f$ for all $f \in F$. 

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Let \( i, j, k \in \mathbb{Z} \). For \( G = G_{13} \) we have
\[
(x^iy^jz^kr)^G = (x^2, z^2)x^iy^jz^kr \cup (x^2, z^2)x^{-i}y^jz^{-k-1}r.
\]

From this we see that \( yzr \sim (yz)^tr = yz^{-1}r \); thus \( t \notin S_r \). By Lemma 6.32 (iv) we conclude \( S_r = \{1, r\} \). By Corollary 6.26 \( \mathcal{R}_r = \{1, rt\} \). We use this to apply Theorem 6.33 to the \( \text{Art} \) coset. Since \( \mathcal{R}_r \cap S_r = \{1\} \) we must have \( \varphi|_{\text{Art}} = \text{Id} \).

Next we use a similar argument to show that \( \varphi|_{Ar} = \text{Id} \). We have
\[
(x^iy^jz^kr)^G = (y^2)x^iy^jz^krt \cup (y^2)x^{-i}y^jz^{-k+1}rt.
\]

Therefore \( zrt \sim z^tr = z^{-1}rt \) which shows that \( t \notin S_{rt} \); so by Lemma 6.32 (iv) we have \( S_{rt} = \{1, rt\} \). Corollary 6.26 gives \( \mathcal{R}_r = \{1, r\} \), and so \( \mathcal{R}_r \cap S_{rt} = \{1\} \). By Theorem 6.33 we conclude \( \varphi|_{Ar} = \text{Id} \).

Note that we have \( \mathcal{R}_r \cap \mathcal{R}_{rt} = \{1\} \) by Corollary 6.26. We use this to apply Theorem 6.35 (ii), using the fact that \( \varphi|_{Ar} = \varphi|_{Art} = \text{Id} \) and \( \varphi(t) = t \). Then by the theorem we have \( \varphi|_{At} = \text{Id} \) and therefore \( \varphi = \text{Id} \).

We have shown that

**Theorem 7.4.** For crystallographic group \( G_{13} \), the group \( W(G) \) is generated by the inverse map \( \iota \), the inner automorphisms, and

- \( \psi_7 : (x, y, z, r, t) \mapsto (x, y, x^2z, xr, t) \);
- \( \psi_8 : (x, y, z, r, t) \mapsto (xz^{-1}, y, z, r, t) \);
- \( \psi_9 : (x, y, z, r, t) \mapsto (x^{-1}, y, z, r, t) \);
- \( \psi_x : (x, y, z, r, t) \mapsto (x, y, z, xr, xt) \);
- \( \psi_y : (x, y, z, r, t) \mapsto (x, y, z, r, yt) \);
- \( \psi_z : (x, y, z, r, t) \mapsto (x, y, z, zr, zt) \).

Thus we have \( W(G) = W_0(G) \).
7.2 Groups 16 through 22

Let \( \varphi \in \mathcal{W}(G_{16}) \). By Proposition 6.3 we have \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). By Proposition 6.5 \((iv)\) for \( \beta \in B = \{x, y, z\} \) we have \( \varphi(\beta) \in \{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\} \). Composing \( \varphi \) with either

\[
\psi_1 : (x, y, z, p, r) \mapsto (y, z, x, pr, p) \text{ or } \psi_1^2 : (x, y, z, p, r) \mapsto (z, x, y, pr)
\]

we may assume we have \( \varphi(x) \in \{ x^{\pm 1} \} \). Then composing \( \varphi \) with

\[
\psi_2 : (x, y, z, p, r) \mapsto (x, z, y, r, p)
\]

we have \( \varphi(\beta) \in \beta^G \) for all \( \beta \in \{x, y, z\} \). Composing with \( I_p \) and \( I_{pr} \) if necessary we have \( \varphi(x) = x \) and \( \varphi(y) = y \). To arrive at \( \varphi(z) = z \) we may compose with \( I_p \) composed with

\[
\psi_1 : (x, y, z, p, r) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, r). \tag{7.1}
\]

We may now assume that \( \varphi|_A = \text{Id} \).

By Proposition 6.9 \((ii)\) we have \( \varphi(Af) = Af \) for all \( f \in F \). Thus we have \( \varphi(p) = ap \) for some \( a \in A \). Squaring both sides we have \( 1 = \varphi(p^2) \sim (ap)^2 = aa^p \), thus \( a^p = a^{-1} \). We may therefore assume \( a \in \langle x, y \rangle \). By Proposition 6.19 the following are outer automorphisms:

\[
\psi_x : (x, y, z, p, r) \mapsto (x, y, z, xp, xr) \text{ and } \psi_y : (x, y, z, p, r) \mapsto (x, y, z, yp, r).
\]

Composing \( \varphi \) with these maps we may then assume that we have \( \varphi(p) = p \).

Let \( \varphi(r) = br \). Squaring both sides we have \( 1 = \varphi(r^2) \sim (br)^2 = bb^r \); thus \( b^r = b^{-1} \). We also have \( \varphi(p \cdot r) = pbr \), and squaring both sides we have

\[
1 = \varphi((pr)^2) \sim (pbr)^2 = pbrpbr = b^pb^r = b^pb^{-1},
\]

so that \( b^p = b \), which implies \( b \in \langle z \rangle \). Composing with

\[
\psi_z : (x, y, z, p, r) \mapsto (x, y, z, p, zr),
\]

we have \( \varphi(r) = r \).
We have by Lemma 6.31 and Corollary 6.27 that $\varphi(ap) = a^hp$ for some $h \in \{1, p\}$. Note that $(\psi_i \circ \iota)(ap) = \psi_i(p^{-1}a) = pa = a^hp$, thus if we have $h = p$, then composing with this anti-automorphism we have arranged to have $\varphi|Ap = \text{Id}$. Since $\varphi(r) = r$, Corollary 6.36 gives $\varphi|Ar = \text{Id}$.

Let $\varphi(pr) = cpr$ for some $c \in A$. This gives $\varphi(pr \cdot p) \sim cr$ and $\varphi(pr \cdot r) \sim cp$. Squaring both sides of each of these equations gives the following:

\[
1 \sim (cpr)^2 = cc^{pr} \text{ which implies } c^{pr} = c^{-1};
\]
\[
1 \sim (cr)^2 = cc^r \text{ which implies } c^r = c^{-1};
\]
\[
1 \sim (cp)^2 = cc^p \text{ which implies } c^p = c^{-1}.
\]

This shows $c^{-1} = c^p = c^r = c^{pr}$, thus $c = 1$. We now have $\varphi(pr) = pr$ and therefore by Corollary 6.36 $\varphi|Apr = \text{Id}$, so that $\varphi = \text{Id}$.

We have shown

**Theorem 7.5.** For crystallographic group $G_{16}$, the group $W(G)$ is generated by the inverse map $\iota$, the inner automorphisms, and

$$
\psi_1 : (x, y, z, p, r) \mapsto (y, z, x, pr, p),
\psi_2 : (x, y, z, p, r) \mapsto (x, z, y, r, p),
\psi_3 : (x, y, z, p, r) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, r),
\psi_x : (x, y, z, p, r) \mapsto (x, y, z, xp, xr),
\psi_y : (x, y, z, p, r) \mapsto (x, y, z, yp, r),
\psi_z : (x, y, z, p, r) \mapsto (x, y, z, p, zr).
$$

Thus we have $W(G) = W_0(G)$.

Let $\varphi \in W(G_{17})$. Since $Ar$ and $Apr$ contain elements of order 2 but $Ap$ does not, we must have $\varphi(Ap) = Ap$. If we have $\varphi(Ar) = Apr$ then we may compose $\varphi$ with

$$
\psi_3 : (x, y, z, p, r) \mapsto (y, x, z, p, p^{-1}r), \tag{7.2}
$$

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and now we have $\varphi(Af) = Af$ for all $f \in F$. By Proposition 6.9 (ii) we have $\varphi(\beta) \in \{\beta^\pm 1\}$ for all $\beta \in \{x, y, z\}$. Composing with $I_p$ and $I_{pr}$ if necessary we have $\varphi(x) = x$ and $\varphi(y) = y$.

To arrive at $\varphi(z) = z$ we may compose with $I_p$ composed with $\psi$ : $(x, y, z, p, r) \mapsto (x, y, z, xp, xr)$. (7.3)

We may now assume that $\varphi|_A = \text{Id}$.

Let $\varphi(r) = br$. Squaring both sides we have $1 = \varphi(r^2) \sim (br)^2 = bb^r$, so $b^r = b^{-1}$; thus we have $b \in \langle x, z \rangle$. By Proposition 6.19 the following maps are outer automorphisms:

$$\psi_x : (x, y, z, p, r) \mapsto (x, y, z, xp, xr) \quad \text{and} \quad \psi_z : (x, y, z, p, r) \mapsto (x, y, z, p, zr).$$

Composing $\varphi$ with these maps as necessary we have $\varphi(r) = r$.

Let $\varphi(p) = ap$ for some $a \in A$. Then $\varphi(p \cdot r) \sim apr$, and squaring both sides we have $1 \sim aa^{pr}$. This implies that $a \in \langle y, z \rangle$. Squaring both sides of $\varphi(p) = ap$ we have $z = \varphi(p^2) \sim (ap)^2 = aa^pz$, thus $aa^p \in \{1, z^{-2}\}$. This gives $a \in \langle x, y \rangle \cup \langle x, y \rangle z^{-1}$. Since we also have $a \in \langle y, z \rangle$, we conclude that $a \in \langle y \rangle \cup \langle y \rangle z^{-1}$. Then composing $\varphi$ with $\psi_y : (x, y, z, p, r) \mapsto (x, y, z, yp, r)$, (7.4)

we can arrange to have $\varphi(p) \in \{p, z^{-1}p\}$. Note $zp \sim (zp)^r = z^{-2}p$, thus $\varphi(zp) \sim \varphi(z^{-2}p)$. However, if we have $\varphi(p) = z^{-1}p$, then

$$\varphi(z \cdot p) \sim z \cdot z^{-1}p = p,$$

while

$$\varphi(z^{-2} \cdot p) \sim z^{-2} \cdot z^{-1}p = z^{-3}p.$$ 

Since $p^G = \langle x^2, y^2 \rangle p \cup \langle x^2, y^2 \rangle z^{-1}p$, we see $p \sim z^{-3}p$ which is a contradiction. We therefore must have $\varphi(p) = p$.

We have by Lemma 6.31 and Corollary 6.27 that $\varphi(ap) = a^hp$ for some $h \in \{1, p\}$. Note that $(\psi_1 \circ \iota)(ap) = \psi_1(p^{-1}a^{-1}) = pa = a^hp$, thus if we have $h = p$ then composing with this
anti-automorphism we may assume we have $\varphi|_{Ap} = \text{Id}$. We have $\varphi(r) = r$, thus by Corollary 6.36, $\varphi|_{Ar} = \text{Id}$.

Let $\varphi(pr) = cpr$. Squaring both sides we have $1 \sim (cpr)^2 = cc^{pr}$ so $c^{pr} = c^{-1}$; thus $c \in \langle y, z \rangle$. We also have $r = \varphi(p^{-1} \cdot pr) \sim p^{-1}cpr = c^{pr}r$. Squaring both sides this is $1 \sim (c^{pr})^2 = (cc^{pr})^p$, which implies $c^r = c^{-1}$; thus $c \in \langle x, z \rangle$. Combining these results we have $c \in \langle z \rangle$. Now $\varphi(pr \cdot r) \sim cp$, and squaring both sides we have $z \sim cc^{pr}z = c^2z$, so $c^2 \in \{1, z^{-2}\}$. Thus we have $c \in \{1, z^{-1}\}$. Suppose that $c = z^{-1}$. Then using $\varphi|_A = \varphi|_{Ap} = \varphi|_{Ar} = \text{Id}$ we have

$$xz^{-1}pr = \varphi(x \cdot pr) = \varphi(xp \cdot r) \sim xpr.$$ 

Proposition C.2 gives $\langle xpr \rangle^G = \langle y^2, z^2 \rangle xpr \cup \langle y^2, z^2 \rangle x^{-1}zpr$ which implies $xpr \not\sim xz^{-1}pr$. This is a contradiction. We therefore must have $c = 1$ i.e. $\varphi(pr) = pr$. Corollary 6.36 then gives $\varphi|_{Apr} = \text{Id}$ and thus $\varphi = \text{Id}$.

We have shown

**Theorem 7.6.** For crystallographic group $G = G_{17}$, the group $W(G)$ is generated by the inverse map $\iota$, the inner automorphisms, and

$$\psi_3 : (x, y, z, p, r) \mapsto (y, x, z, p, p^{-1}r);$$

$$\psi_4 : (x, y, z, p, r) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, r);$$

$$\psi_x : (x, y, z, p, r) \mapsto (x, y, z, xp, xr);$$

$$\psi_y : (x, y, z, p, r) \mapsto (x, y, z, yp, r);$$

$$\psi_z : (x, y, z, p, r) \mapsto (x, y, z, p, zr).$$

Thus we have $W(G) = W_0(G)$. 

Let $\varphi \in W(G_{21})$. Proposition 6.3 gives $C_2 = \langle x^2y^{-1} \rangle \cup \langle y \rangle \cup \langle z \rangle$. By Corollary 6.7 we have $\varphi(z) \in z^G$. By Lemma 6.4 it follows that $\varphi(Ap) = Ap$. Composing $\varphi$ with

$$\psi_4 : (x, y, z, p, r) \mapsto (x, x^2y^{-1}, z, p, pr)$$

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we may assume we have \( \varphi(\beta) \in \beta^G \) for all \( \beta \in B = \{ x^2y^{-1}, y, z \} \). Composing \( \varphi \) with \( I_p \) and \( I_{pr} \) if necessary we have \( \varphi(x^2y^{-1}) = x^2y^{-1} \) and \( \varphi(y) = y \). To arrive at \( \varphi(z) = z \) we may compose with \( I_p \) composed with

\[
\psi_i : (x, y, z, p, r) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, r).
\]

We now have \( \varphi(\beta) = \beta \) for \( \beta \in B \), and since \( \langle B \rangle = \langle C_2 \rangle \) and \( \varphi|_A \) is a homomorphism, we have \( \varphi|_{\langle C_2 \rangle} = \text{Id} \). Since \( \langle C_2 \rangle \) has finite index in \( A \) and we conclude that \( \varphi|_A = \text{Id} \). Now by Proposition 6.9 \((ii)\) we have \( \varphi(Af) = Af \) for all \( f \in F \).

Let \( \varphi(pr) = cpr \) for some \( c \in A \). Squaring both sides gives \( 1 \sim cc^{pr} \) i.e. \( c^{pr} = c^{-1} \) thus \( c \in \langle y, z \rangle \). By Proposition 6.19 the maps below define outer automorphisms.

\[
\psi_y : (x, y, z, p, r) \mapsto (x, y, yz, r) \quad \text{and} \quad \psi_z : (x, y, z, p, r) \mapsto (x, y, z, pr).
\]

Composing \( \varphi \) with these automorphisms we may assume we have \( \varphi(pr) = pr \). We have by Lemma 6.31 and Corollary 6.27 that \( \varphi(apr) = a^hpr \) for some \( h \in \{ 1, pr \} \). Note that \( (\psi_i \circ i)(apr) = \psi_i(pra^{-1}) = p = a^{pr}pr \), thus if we have \( h = pr \) then composing \( \varphi \) with this anti-automorphism we may assume we have \( \varphi|_{Apr} = \text{Id} \).

Let \( \varphi(p) = ap \) for some \( a \in A \). Squaring both sides we have \( 1 \sim aa^{p} \) so \( a^{p} = a^{-1} \) thus \( a \in \langle x, y \rangle \). Squaring both sides of \( \varphi(p \cdot pr) \sim ap \cdot pr \) gives \( 1 \sim aa^{r} \), so \( a^{r} = a^{-1} \), thus \( a \in \langle x^2y^{-1}, z \rangle \). Therefore we have \( a \in \langle x^2y^{-1} \rangle \). Composing \( \varphi \) with the inner automorphism

\[
I_x : (x, y, z, p, r) \mapsto (x, y, z, x^{-2}p, x^{-2}yr)
\]

we may assume that we have \( \varphi(p) = p \) and by Corollary 6.36 \( \varphi|_A = \text{Id} \).

Let \( \varphi(r) = br \) for some \( b \in A \). This gives \( \varphi(r \cdot p) \sim bpr \) and \( \varphi(r \cdot pr) \sim bp \). Squaring both
sides of each of these equations gives the following:

\[ 1 \sim (br)^2 = bb^r \text{ which implies } b^r = b^{-1}; \]
\[ 1 \sim (bpr)^2 = bb^{pr} \text{ which implies } b^{pr} = b^{-1}; \]
\[ 1 \sim (bp)^2 = bb^p \text{ which implies } b^p = b^{-1}. \]

This shows \( b^{-1} = b^p = b^r = b^{pr} \) and so we must have \( b = 1 \). We now have \( \varphi(r) = r \) and therefore by Corollary 6.36, \( \varphi|_{Ar} = \text{Id} \) and thus \( \varphi = \text{Id} \).

We have shown that

**Theorem 7.7.** For crystallographic group \( G = G_{21} \), the group \( W(G) \) is generated by the inverse map \( i \), the inner automorphisms, and

\[
\psi_4 : (x, y, z, p, r) \mapsto (x, x^2 y^{-1}, z, p, pr);
\]
\[
\psi_5 : (x, y, z, p, r) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, r);
\]
\[
\psi_6 : (x, y, z, p, r) \mapsto (x, y, z, yp, r);
\]
\[
\psi_7 : (x, y, z, p, r) \mapsto (x, y, z, p, zr).
\]

Thus we have \( W(G) = W_0(G) \). \( \square \)

Let \( \varphi \in W(G_{22}) \). We have \( \varphi(Ap) \in \{Ap, Ar, Apr\} \). Note that the following automorphisms permute these three cosets:

\[
\psi_5 : (x, y, z, p, r) \mapsto (xy^{-1}, x, x^2 z^{-1}, pr, p);
\]
\[
\psi_6 : (x, y, z, p, r) \mapsto (y, x, z, p, pr).
\]

Thus by composing \( \varphi \) with these maps we can assume we have \( \varphi(Af) = Af \) for all \( f \in F \). Note that by Proposition 6.3 we have \( C_2 = \langle x^2 z^{-1} \rangle \cup \langle y^2 z^{-1} \rangle \cup \langle z \rangle \). Then by Proposition 6.9 (ii) we have \( \varphi(\beta) \in \beta^G \) for all \( b \in B = \{x^2 z^{-1}, y^2 z^{-1}, z\} \). Composing \( \varphi \) with \( I_p \) and \( I_r \) as necessary we have \( \varphi(x^2 z^{-1}) = x^2 z^{-1} \) and \( \varphi(y^2 z^{-1}) = y^2 z^{-1} \). Now we have the automorphism

\[
\psi_4 : (x, y, z, p, r) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, r)
\]
and note that

\[ I_p \circ \psi_i : (x^2 z^{-1}, y^2 z^{-1}, z) \mapsto (x^2 z^{-1}, y^2 z^{-1}, z^{-1}). \]

If we have \( \varphi(z) = z^{-1} \) then composing with \( I_p \circ \psi_i \), we have we have \( \varphi(\beta) = \beta \) for all \( \beta \in B \).

Using the fact that \( \varphi|_A \) is a homomorphism gives \( \varphi|_{\langle C_2 \rangle} = \text{Id} \) and since and \( \langle C_2 \rangle \) has finite index in \( A \), it also gives \( \varphi|_A = \text{Id} \).

Let \( \varphi(p) = ap \) for some \( a \in A \). Squaring both sides we have \( 1 \sim aa^p \) thus \( a^p = a^{-1} \). This implies \( a \in \langle xy^{-1}, yz^{-1} \rangle = \langle xy^{-1}, y^2 z^{-1} \rangle \). By Proposition 6.19 the map below defines an outer automorphism:

\[ \psi_x : (x, y, z, p, r) \mapsto (x, y, z, x y^{-1} p, x r). \] (7.5)

Composing \( \varphi \) with \( \psi_x \) and with

\[ I_{y^{-1}} : (x, y, z, p, r) \mapsto (x, y, z, y^2 z^{-1} p, z r), \] (7.6)

we may assume that \( \varphi(p) = p \). Now by Lemma 6.31 and Corollary 6.27 that \( \varphi(ap) = a^h p \) for some \( h \in \{1, p\} \). If we have \( h = p \) then we may compose \( \varphi \) with

\[ \psi_i : (x, y, z, p, r) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, r) \]

and then with \( \iota \). Since \( (\psi_i \circ \iota)(ap) = \psi_i(pa^{-1}) = pa = a^p p \), composing \( \varphi \) with this anti-automorphism we now have \( \varphi|_{Ap} = \text{Id} \).

Let \( \varphi(r) = br \) for some \( b \in A \). Squaring both sides gives \( 1 \sim bb^r \), thus \( b \in \langle x, z \rangle \). We also have \( \varphi(p \cdot r) \sim pbr = b^p pr \) and squaring both sides of this relation gives \( 1 \sim (b^p pr)^2 = b^pb^r \).

This implies \( b \in \langle y, z \rangle \), thus we must have \( \beta \in \langle z \rangle \). Composing \( \varphi \) with the automorphism

\[ \psi_z : (x, y, z, p, r) \mapsto (x, y, z, p, z r) \]

we now have \( \varphi(r) = r \) and so by Corollary 6.36 \( \varphi|_{Ar} = \text{Id} \).

Let \( \varphi(pr) = cpr \) for some \( c \in A \). This gives \( \varphi(pr \cdot p) \sim cr \) and \( \varphi(pr \cdot r) \sim cp \). Squaring
both sides of each of these relations gives the following:

\[ 1 \sim (cpr)^2 = cc^{pr} \text{ which implies } c^{pr} = c^{-1}; \]
\[ 1 \sim (cr)^2 = cc^r \text{ which implies } c^r = c^{-1}; \]
\[ 1 \sim (cp)^2 = cc^p \text{ which implies } c^p = c^{-1}. \]

This shows \( c^{-1} = c^p = c^r = c^{pr}, \) thus \( c = 1. \) We now have \( \varphi(pr) = pr \) and therefore by Corollary 6.36, \( \varphi|_{Apr} = \text{Id} \) and thus \( \varphi = \text{Id}. \)

We have shown that

**Theorem 7.8.** For crystallographic group \( G = G_{22}, \) the group \( W(G) \) is generated by the inverse map \( \iota, \) the inner automorphisms, and

\[
\psi_5 : (x, y, z, p, r) \mapsto (x^{-1}y, x, x^2z^{-1}, pr, p);
\]
\[
\psi_6 : (x, y, z, p, r) \mapsto (y, x, z, p, pr);
\]
\[
\psi_\iota : (x, y, z, p, r) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, r);
\]
\[
\psi_x : (x, y, z, p, r) \mapsto (x, y, z, xy^{-1}p, xr);
\]
\[
\psi_z : (x, y, z, p, r) \mapsto (x, y, z, p, zr).
\]

Thus we have \( W(G) = W_0(G). \) \( \square \)

### 7.3 Groups 25 through 42

Let \( \varphi \in W(G_{25}). \) By Lemma 6.8 (vi) composing \( \varphi \) with \( \psi_\iota \) if necessary we may assume \( \varphi(z) = z. \) By Corollary 6.10 (i) we have \( \varphi(Ap) = Ap. \) If \( \varphi(As) = Aps \) we may compose \( \varphi \) with

\[
\psi_1 : (x, y, z, p, s) \mapsto (y, x, z, p, ps),
\]

and now we have \( \varphi(Af) = Af \) for all \( f \in F. \) Then by Corollary 6.10 (ii), by composing \( \varphi \) with inner automorphisms as necessary we have \( \varphi|_A = \text{Id}. \)
Let \( \varphi(p) = ap \) for some \( a \in A \). Note \( p \) has order 2 thus \( ap \) must have order 2 which implies \( a \in \langle x, y \rangle \). By Proposition 6.19 the maps below determine outer automorphisms:

\[
\psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s) \quad \text{and} \\
\psi_y : (x, y, z, p, s) \mapsto (x, y, z, yp, ys).
\]

Composing \( \varphi \) with these maps we can ensure that \( \varphi(p) = p \). Then by Proposition 6.13 we have \( \varphi(f) = f \) for all \( f \in F \).

Now by Corollary 6.28 we have \( \mathcal{R}_f = \{1, f\} \) for \( f \in \{s, ps\} \). By Lemma 6.31, for all \( a \in A \) we have \( \varphi(as) = a^h s \) for some \( h \in \mathcal{R}_s = \{1, s\} \). Suppose we have \( h = s \), thus \( \varphi(as) = a^s s = sa \). Now composing \( \varphi \) with

\[
\psi_i : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, s^{-1})
\]

and then composing again with \( i \), we now have \( \varphi|_{AuAs} = \text{Id} \). By Lemma 6.31, for all \( a \in A \) we have \( \varphi(aps) = a^h ps \) for some \( h \in \mathcal{R}_{ps} = \{1, ps\} \). Composing \( \varphi \) with the non-trivial wct \( \tau(ps, \{Ap, Aps\}) \) we now have \( \varphi|_{Aps} = \text{Id} \). (Theorem 5.9 proves \( \tau_{ps} \) is non-trivial.)

By Corollary 6.37, since \( \varphi|_{As \cup Aps} = \text{Id} \) we conclude \( \varphi = \text{Id} \).

We have shown that

**Theorem 7.9.** For crystallographic group \( G = G_{25} \), \( \mathcal{W}(G) \) is generated by the inverse map \( i \), the inner automorphisms,

\[
\psi_i : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, s), \\
\psi_1 : (x, y, z, p, s) \mapsto (y, x, z, p, ps), \\
\psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s), \\
\psi_y : (x, y, z, p, s) \mapsto (x, y, z, yp, ys),
\]

and the non-trivial wct \( \tau_{ps} \). Thus we have \( \mathcal{W}(G) = \langle \mathcal{W}_0(G), \tau_{ps} \rangle \).

Let \( \varphi \in \mathcal{W}(G_{26}) \). By Lemma 6.8(vi), composing \( \varphi \) with \( \psi_i \) if necessary we have \( \varphi(z) = z \).

By Corollary 6.10 (i) we have \( \varphi(Ap) = Ap \). Next we note that the \( As \) coset contains no
involutions but there are involutions in $Aps$. Thus we see that $\varphi(As) \neq Aps$. Therefore we have $\varphi(Af) = Af$ for all $f \in F$. By Corollary 6.10 (ii), by composing $\varphi$ with inner automorphisms as necessary we have $\varphi|_A = \text{Id}$.

Let $\varphi(p) = ap$ for some $a \in A$. Squaring both sides gives (since $z \in Z(G)$) we have $z = (ap)^2 = aa^p z$, thus $a^p = a^{-1}$. This implies $a \in \langle x,y \rangle$. By Proposition 6.19 the following maps define outer automorphisms:

$$\psi_x : (x,y,z,p,s) \mapsto (x,y,z,xp,s) \text{ and } \psi_y : (x,y,z,p,s) \mapsto (x,y,z,yp,ys).$$

By composing with these maps we can ensure that $\varphi(p) = p$. Then by Proposition 6.13 we have $\varphi(f) = f$ for all $f \in F$.

Now by Corollary 6.28 we have $R_f = \{1, f\}$ for $f \in \{s, ps\}$. By Lemma 6.31, for all $a \in A$ we have $\varphi(as) = a^h s$ for some $h \in R_s = \{1, s\}$. Suppose we have $h = s$, thus $\varphi(as) = a^s s = sa$. Composing $\varphi$ with

$$\psi_t : (x,y,z,p,s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, s^{-1}),$$

and then with $\iota$, we now have $\varphi|_{A \cup As} = \text{Id}$. By Lemma 6.31, for all $a \in A$ we have $\varphi(aps) = a^h ps$ for some $h \in R_{ps} = \{1, ps\}$. Composing $\varphi$ with the non-trivial wct $\tau(ps, \{Ap, Aps\})$ we now have $\varphi|_{Aps} = \text{Id}$. (Theorem 5.9 proves $\tau_{ps}$ is non-trivial.) By Corollary 6.37, since $\varphi|_{As \cup Aps} = \text{Id}$ we conclude $\varphi = \text{Id}$.

We have shown that

**Theorem 7.10.** For crystallographic group $G = G_{26}$ $W(G)$ is generated by the inverse map $\iota$, the inner automorphisms,

$$\psi_t : (x,y,z,p,s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, s^{-1}),$$

$$\psi_x : (x,y,z,p,s) \mapsto (x,y,z,xp,s),$$

$$\psi_y : (x,y,z,p,s) \mapsto (x,y,z,yp,ys),$$

and the non-trivial wct $\tau_{ps}$. Thus we have $W(G) = \langle W_0(G), \tau_{ps} \rangle$. □
Let $\varphi \in \mathcal{W}(G_{27})$. By Lemma 6.8 (vi) composing $\varphi$ with $\psi_i$, if necessary we may assume $\varphi(z) = z$. By Corollary 6.10 (i) we have $\varphi( Ap) = Ap$. If $\varphi( As) = Aps$ we may compose $\varphi$ with

$$
\psi_1 : (x, y, z, p, s) \mapsto (y, x, z, p, ps),
$$

and now we have $\varphi( Af) = Af$ for all $f \in F$. Then by Corollary 6.10 (ii), by composing $\varphi$ with inner automorphisms as necessary we have $\varphi|_A = \text{Id}$. Let $\varphi(p) = ap$ for some $a \in A$. Note $p$ has order 2 thus $ap$ must have order 2 which implies $a \in \langle x, y \rangle$. By Proposition 6.19 the maps below determine outer automorphisms:

$$
\psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s) \quad \text{and} \\
\psi_y : (x, y, z, p, s) \mapsto (x, y, z, yp, ys).
$$

By composing $\varphi$ with these maps we can assume that $\varphi(p) = p$. Then by Proposition 6.13 we have $\varphi(f) = f$ for all $f \in F$.

Now by Corollary 6.28 we have $\mathcal{R}_f = \{1, f\}$ for $f \in \{s, ps\}$. By Lemma 6.31, for all $a \in A$ we have $\varphi(as) = ah$ for some $h \in \mathcal{R}_s = \{1, s\}$. Suppose we have $h = s$, thus $\varphi(as) = a^s s = sa$. composing with $\iota$ composed with

$$
\psi_i : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, s^{-1})
$$

we now have $\varphi|_{A \cup As} = \text{Id}$. By Lemma 6.31, for all $a \in A$ we have $\varphi(aps) = ah$ for some $h \in \mathcal{R}_{ps} = \{1, ps\}$. Composing $\varphi$ with the non-trivial wct $\tau( ps, \{Ap, Ap s\})$ we now have $\varphi|_{Ap s} = \text{Id}$. (Theorem 5.9 proves $\tau_{ps}$ is non-trivial.)

We have shown that

**Theorem 7.11.** For crystallographic group $G = G_{27}$, the group $\mathcal{W}(G)$ is generated by the
inverse map \( \iota \), the inner automorphisms,

\[
\psi_\iota : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, s^{-1}),
\]

\[
\psi_1 : (x, y, z, p, s) \mapsto (y, x, z, p, ps),
\]

\[
\psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s),
\]

\[
\psi_y : (x, y, z, p, s) \mapsto (x, y, z, yp, ys),
\]

and the non-trivial wct \( \tau_{ps} \). Thus we have \( W(G) = \langle W_0(G), \tau_{ps} \rangle \).

Let \( \varphi \in W(G_{38}) \). By Lemma 6.8 \((vi)\), composing \( \varphi \) with \( \psi_\iota \) if necessary we have \( \varphi(z) = z \).

By Proposition 6.3, \( C_2 = \langle x \rangle \cup \langle y^2z^{-1} \rangle \), thus by Proposition 6.5 \((iv)\) we have \( \varphi(y^2z^{-1}) \in \{x^\pm 1\}, (y^2z^{-1})^\pm 1 \}. \) Now consider that

\[
\varphi(y)^2 = \varphi(y^2) = \varphi(y^2z^{-1} \cdot z) = \varphi(y^2z^{-1})\varphi(z) = \varphi(y^2z^{-1})z.
\]

The left hand side is a square but if \( \varphi(y^2z^{-1}) \in \{x, x^{-1} \} \) then the right hand side would not be a square. We conclude \( \varphi(y^2z^{-1}) \in (y^2z^{-1})^G \). Then by Proposition 6.9 \((ii)\) we have \( \varphi(Af) = Af \) for all \( f \in F \) and by Corollary 6.10 \((ii)\), composing with inner automorphisms if necessary, we have \( \varphi|_A = \text{Id} \).

Let \( \varphi(p) = ap \) for some \( a \in A \). Note \( p \) has order 2 thus \( ap \) must have order 2 which implies \( a \in \langle x, y^2z^{-1} \rangle \). By composing \( \varphi \) with the automorphisms below we can ensure that \( \varphi(p) = p \).

\[
\psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s) \text{ and }
\]

\[
I_y : (x, y, z, p, s) \mapsto (x, y, z, y^{-2}zp, y^{-2}zs).
\]

(Note \( \psi_x \) is an outer autmorphism by Proposition 6.19.) Then by Proposition 6.13 we have \( \varphi(f) = f \) for all \( f \in F \).

Now by Corollary 6.28 we have \( R_f = \{1, f\} \) for \( f \in \{s, ps\} \). By Lemma 6.31, for all \( a \in A \) we have \( \varphi(as) = ah \) for some \( h \in R_s = \{1, s\} \). Suppose we have \( h = s \), thus
\( \varphi(as) = a^s s = sa. \) composing with \( \iota \) composed with

\[
\psi_\iota : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, s)
\]

we now have \( \varphi|_{A \cup As} = \text{Id}. \) By Lemma 6.31, for all \( a \in A \) we have \( \varphi(aps) = a^h ps \) for some \( h \in R_{ps} = \{1, ps\} \). Composing \( \varphi \) with the non-trivial wct \( \tau_{ps} \{Ap, Aps\} \) we now have \( \varphi|_{Aps} = \text{Id}. \) (Theorem 5.9 proves \( \tau_{ps} \) is non-trivial.) By Corollary 6.37, since \( \varphi|_{As \cup Aps} = \text{Id} \) we conclude \( \varphi = \text{Id}. \)

We have shown that

**Theorem 7.12.** For crystallographic group \( G = G_{38} \), \( W(G) \) is generated by the inverse map \( \iota \), the inner automorphisms,

\[
\psi_\iota : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p, s),
\]

\[
\psi_1 : (x, y, z, p, s) \mapsto (y, x, z, p, ps),
\]

\[
\psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s),
\]

and the non-trivial wct \( \tau_{ps} \). Thus we have \( W(G) = \langle W_0(G), \tau_{ps} \rangle \). \( \square \)

Let \( \varphi \in W(G_{39}) \). By Lemma 6.8 \((vi)\), composing \( \varphi \) with \( \psi_\iota \) if necessary we have \( \varphi(z) = z \). By Corollary 6.10 \((i)\) we have \( \varphi(Ap) = Ap \). Since \( As \) contains involutions but \( Aps \) does not, \( \varphi(As) = Aps \) is not possible thus we have \( \varphi(Af) = Af \) for all \( f \in F \). Then by Corollary 6.10 \((ii)\), composing with inner automorphisms if necessary, we have \( \varphi|_A = \text{Id}. \)

Let \( \varphi(p) = ap \) for some \( a \in A \). Squaring both sides of this equation we have

\[
z = \varphi(z) = \varphi(p^2) \sim (ap)^2 = aa^p p^2 = aa^p z.
\]

Since \( z \in Z(G) \) we have equality, i.e. \( z = aa^p z \) thus \( a^p = a^{-1}. \) This tells us \( a \in \langle x, y^2 z^{-1} \rangle \).

By composing with the maps below we can ensure that \( \varphi(p) = p \).

\[
\psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s) \text{ and }
\]

\[
I_y : (x, y, z, p, s) \mapsto (x, y, z, y^{-2} zp, y^{-2} zs).
\]
(By Proposition 6.19 \(\psi_x\) is an outer automorphism.) Then by Proposition 6.13 we have \(\varphi(f) = f\) for all \(f \in F\).

Now by Corollary 6.28 we have \(R_f = \{1, f\}\) for \(f \in \{s, ps\}\). By Lemma 6.31, for all \(a \in A\) we have \(\varphi(as) = a^h s\) for some \(h \in R_s = \{1, s\}\). If \(h = s\) then we may compose with the non-trivial wct \(\tau(s, \{Ap, As\})\) and now we have \(\varphi|_{As} = \text{Id}\).

By Lemma 6.31, for all \(a \in A\) we have \(\varphi(aps) = a^h ps\) for some \(h \in R_{ps} = \{1, ps\}\). Composing if necessary with the non-trivial wct \(\tau(ps, \{Ap, Aps\})\) we may assume that \(\varphi|_{Aps} = \text{Id}\). (Theorem 5.9 proves \(\tau_s\) and \(\tau_{ps}\) are non-trivial.) We note that \(\tau_{ps} = \iota \circ \psi_1 \circ \tau_s\), therefore \(\tau_{ps}\) will not be listed as a generator of \(W(G)\). By Corollary 6.37, since \(\varphi|_{As \cup Aps} = \text{Id}\) and \(R_{ps} \cap f = 1\), we conclude \(\varphi|_{Ap} = \text{Id}\).

We have shown that

**Theorem 7.13.** For crystallographic group 39, \(W(G)\) is generated by the inverse map \(\iota\), the inner automorphisms,

\[
\psi_1 : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, s),
\]

\[
\psi_x : (x, y, z, p, s) \mapsto (x, y, z, xp, s),
\]

and the non-trivial wct \(\tau_s\). Thus we have \(W(G) = \langle W_0(G), \tau_s \rangle\).

Let \(\varphi \in W(G_{12})\). By Lemma 6.8 (vi), composing \(\varphi\) with \(\psi_1\) if necessary we may assume \(\varphi(z) = z\). By Corollary 6.10 (i) we have \(\varphi(Ap) = Ap\). If \(\varphi(As) = Aps\) we may compose \(\varphi\) with the outer automorphism

\[
\psi_1 : (x, y, z, p, s) \mapsto (y, x, z, p, ps),
\]

and now we have \(\varphi(Af) = Af\) for all \(f \in F\). Then by Corollary 6.10 (ii), by composing with inner automorphisms as necessary we have \(\varphi|_A = \text{Id}\).

Let \(\varphi(p) = bp\) for some \(b \in A\). Note \(p\) has order 2 thus \(bp\) also has order 2 which implies \(b \in \langle xy^{-1}, xyz^{-1} \rangle\). Note that \(\langle xy^{-1}, xyz^{-1} \rangle\) has index 2 in \(\langle x^2z^{-1}, y^2z^{-1} \rangle\), thus by composing
\( \varphi \) with the inner automorphisms

\[
I_x : (x, y, z, p, s) \mapsto (x, y, z, x^{-2}zp, s)
\]

and \( I_y : (x, y, z, p, s) \mapsto (x, y, z, y^{-2}zp, y^{-2}zs) \),

we may assume that \( \varphi(p) \in \{p, xy^{-1}p\} \). Suppose that \( \varphi(p) = xy^{-1}p \) and let \( \varphi(s) = as \) for some \( a \in A \). Then

\[
\varphi(p \cdot s) \sim xy^{-1}pas = xy^{-1}a^ps.
\]

Since \( s \) and \( ps \) have order 2, \( as \) and \( xy^{-1}a^ps \) also have order 2. We have \( |as| = 2 \) which implies \( a \in \langle y^2z^{-1} \rangle \) and so we have \( a^p = a^{-1} \). Also, \( xy^{-1}a^ps \) has order 2, which implies it is contained in the subgroup \( \langle x^2z^{-1} \rangle \). Thus we conclude that \( xy^{-1}a^p = xy^{-1}a^{-1} \in \langle x^2z^{-1} \rangle \).

This indicates that \( a \in \langle y^2z^{-1} \rangle \cap \langle x^2z^{-1} \rangle xy^{-1} \) which is empty. This is a contradiction thus \( \varphi(p) = xy^{-1}p \) is not possible. We may assume therefore that \( \varphi(p) = p \), and now by Proposition 6.13 we have \( \varphi(f) = f \) for all \( f \in F \).

Now by Corollary 6.28 we have \( \mathcal{R}_f = \{1, f\} \) for \( f \in \{s, ps\} \). By Lemma 6.31, for all \( a \in A \) we have \( \varphi(as) = a^hs \) for some \( h \in \mathcal{R}_s = \{1, s\} \). Suppose we have \( h = s \), thus \( \varphi(as) = a^ss = sa \). Composing \( \varphi \) with \( \iota \) and then with the automorphism

\[
\psi : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, s^{-1})
\]

we now have \( \varphi|_{A \cup As} = \text{Id} \). By Lemma 6.31, for all \( a \in A \) we have \( \varphi(aps) = a^hps \) for some \( h \in \mathcal{R}_{ps} = \{1, ps\} \). Composing \( \varphi \) with the non-trivial wct \( \tau(ps, \{Ap, Aps\}) \) we now have \( \varphi|_{Aps} = \text{Id} \). (Theorem 5.9 proves \( \tau_{ps} \) is non-trivial.) By Corollary 6.37, since \( \varphi|_{As \cup Aps} = \text{Id} \) we conclude \( \varphi = \text{Id} \).

We have shown that

**Theorem 7.14.** For crystallographic group \( G = G_{42} \), the group \( W(G) \) is generated by the inverse map \( \iota \), the inner automorphisms,

\[
\psi : (x, y, z, p, s) \mapsto (x^{-1}, y^{-1}, z^{-1}, p^{-1}, s^{-1})
\]

\[
\psi_1 : (x, y, z, p, s) \mapsto (y, x, z, p, ps),
\]

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and the non-trivial wct $\tau_{ps}$. Thus we have $\mathcal{W}(G) = (\mathcal{W}_0(G), \tau_{ps})$. □
Chapter 8. The WCT Groups of Space Groups

Having point group \( \frac{2}{m} \frac{2}{m} \frac{2}{m} \)

8.1 Groups 47 through 57

Let \( \varphi \in W(G_{47}) \). By Proposition 6.3 \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). Without loss of generality, let \( B = \{ x, y, z \} \). By Proposition 6.5 (iv) we have \( \{ \varphi(x), \varphi(y), \varphi(z) \} \subseteq \{ x^{\pm 1}, y^{\pm 1}, z^{\pm 1} \} \). If \( \varphi(x) \in \{ y^{\pm 1} \} \) then we may compose \( \varphi \) with the automorphism

\[
\psi_1 : (x, y, z, p, r, t) \mapsto (y, x, z, p, pr, t)
\]

so that \( \varphi(x) \in x^G \). Alternatively, if \( \varphi(x) \in \{ z^{\pm 1} \} \) then we may compose \( \varphi \) with the automorphism

\[
\psi_2 : (x, y, z, p, r, t) \mapsto (z, y, x, pr, r, t)
\]

so that \( \varphi(x) \in x^G \). If \( \varphi(y) \in \{ z^{\pm 1} \} \) then we may compose \( \varphi \) with the automorphism

\[
\psi_3 : (x, y, z, p, r, t) \mapsto (x, z, y, r, p, t),
\]

so that we may now assume that \( \varphi(\beta) \in \beta^G \) for all \( \beta \in B \). Applying Proposition 6.9 (iv) we may compose with inner automorphisms so that \( \varphi|_A = \text{Id} \) and by (ii) we also have \( \varphi(Af) = Af \) for all \( f \in F \). By Proposition 6.19 we may compose with automorphisms \( \psi_x, \psi_y, \) and \( \psi_z \) in order to arrange that \( \varphi(t) = t \). Then by Proposition 6.17, since every element of \( F \) has order 2 we have \( \varphi(f) = f \) for all \( f \in F \).

Now by Lemma 6.31 for \( a \in A \) we have \( \varphi(at) = a^ht \) for some \( h \in R_t \). We have \( R_t = F \) by Corollary 6.29. Since \( \langle APr, Art, Apt \rangle = G/A \), we see that composing with the non-trivial wcts \( \tau(\text{prt}, \{ APr, Art, Ap, Ar \}) \), \( \tau(rt, \{ Art, At, Apr, Ap \}) \), and \( \tau(pt, \{ Apt, At, Ar, Apr \}) \), we can assume that we have \( \varphi|_{At} = \text{Id} \). (Theorem 5.9 proves these three functions are non-trivial wcts.) We note that \( \tau_{rt} = \psi_1 \circ \psi_2 \circ \tau_{\text{prt}} \circ \psi_2 \circ \psi_1 \) and \( \tau_{pt} = \psi_1 \circ \psi_3 \circ \tau_{\text{prt}} \circ \psi_3 \circ \psi_1 \) so we will only include \( \tau_{\text{prt}} \) when we list the generators of the wct group. Applying Corollary 6.38 we have \( \varphi = \text{Id} \).
We have shown that

**Theorem 8.1.** For crystallographic group \( G = G_{47} \), the group \( W(G) \) is generated by the inverse map \( \iota \), the inner automorphisms,

\[
\begin{align*}
&\psi_1 : (x, y, z, p, r, t) \mapsto (y, x, z, p, rt, t), \\
&\psi_2 : (x, y, z, p, r, t) \mapsto (z, y, x, pr, r, t), \\
&\psi_3 : (x, y, z, p, r, t) \mapsto (x, z, y, p, r, t), \\
&\psi_x : (x, y, z, p, r, t) \mapsto (x, y, z, x, r, xt), \\
&\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt), \\
&\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, z, r, zt),
\end{align*}
\]

and that \( W(G) = \langle W_0(G), \tau_{pi} \rangle \).

**Proposition 8.2.** For group 49 we have

\[
\begin{align*}
S_p &= S_r = S_{pr} = S_t = F; \\
S_{rt} &= \{1, prt, rt, p\}; \\
S_{pt} &= \{1, pt, p, t\}.
\end{align*}
\]

**Proof.** This follows from Lemma 6.32 and Proposition C.4.

Let \( \varphi \in W(G_{49}) \). By Proposition 6.3 \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). Without loss of generality, let \( B = \{x, y, z\} \). By Proposition 6.5 (iv) we have \( \{\varphi(x), \varphi(y), \varphi(z)\} \subseteq \{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\} \). By Proposition C.4 we have \( G' = \langle x^2, y^2, z \rangle \), thus by applying Lemma 2.11 we conclude \( \varphi(z) \in \{z^{\pm 1}\} \). If \( \varphi(x) \in \{y^{\pm 1}\} \) then we may compose \( \varphi \) with the automorphism \( \psi_1 : (x, y, z, p, r, t) \mapsto (y, x, z, p, pr, t) \) and now \( \varphi(x) \in x^G \) and \( \varphi(y) \in y^G \). Applying Proposition 6.9 (iv) we may compose \( \varphi \) with inner automorphisms so that \( \varphi|_A = \text{Id} \) and by (ii) we have \( \varphi(Af) = Af \) for all \( f \in F \). By Proposition 6.19 we may compose \( \varphi \) with \( \psi_x, \psi_y \), and \( \psi_z \) so that we may have \( \varphi(t) = t \).

The elements of \( F \) that have order 2 (besides \( t \)) are \( p, r, pr, \) and \( pt \). By Proposition 6.17 we must have \( \varphi(pt) = pt, \varphi(rt) = rt, \varphi(prt) = prt, \) and \( \varphi(p) = p \). Recall that \( (prt)^2 = (rt)^2 = 49 \).
so by Proposition 6.17 \( \varphi(pr) \in \{pr, zpr\} \) and \( \varphi(r) \in \{r, zr\} \). If we have \( \varphi(r) = zr \) then by Proposition 6.18 (i) we may compose \( \varphi \) with \( I_t \circ i \) so as to have \( \varphi(r) = r \). Now by (ii) we have \( \varphi(f) = f \) for all \( f \in F - \{pr\} \). Note that \( \varphi(pr \cdot r) = \varphi(p) = p \) and we also have \( p \approx zp \) by Proposition C.4. Therefore by Proposition 6.18 (iii) \( \varphi(r) = r \) implies \( \varphi(pr) \neq zpr \) thus \( \varphi(pr) = pr \). Thus we have \( \varphi(f) = f \) for all \( f \in F \).

We now apply Theorem 6.33 three times. We will use the following facts from Proposition 8.2: \( \text{prt} / \not \in S\text{pt}; \text{rt} / \not \in S\text{pt} \), and \( \text{pt} / \not \in S\text{pr} \). Also recall that by Corollary 6.29 \( R_f = \{1, f\} \) for \( f \in \{\text{prt}, \text{rt}, \text{pt}\} \). Now since \( p \) commutes with \( r \) we have \( r \cdot \text{prt} = \text{pt} \). Then since \( R_{\text{prt}} \cap S\text{pt} = 1 \) Theorem 6.33 implies \( \varphi|_{A\text{prt}} = \text{Id} \). Since \( r^2 = 1 \) we have \( pr \cdot rt = pt \) and \( R_{\text{rt}} \cap S\text{pt} = 1 \) so the theorem implies \( \varphi|_{A\text{rt}} = \text{Id} \). Since \( p \) commutes with \( r \) we have \( r \cdot pt = rt \) so since \( R_{\text{pt}} \cap S\text{rt} = 1 \) we have \( \varphi|_{A\text{pt}} = \text{Id} \). It follows by Corollary 6.38 that \( \varphi = \text{Id} \).

We have shown that

**Theorem 8.3.** For crystallographic group \( G = G_{49} \), the group \( W(G) \) is generated by the inverse map \( i \), the inner automorphisms and

\[
\psi_1 : (x, y, z, p, r, t) \mapsto (y, x, z, p, pr, t), \\
\psi_x : (x, y, z, p, r, t) \mapsto (x, y, z, xp, xr, xt), \\
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt), \\
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt),
\]

and that \( W(G) = W_0(G) \). \( \square \)

**Proposition 8.4.** For group 50 we have

\[
S_p = S_r = S_{pr} = S_t = F; \\
S_{prt} = \{1, pr\}; \\
S_{rt} = \{1, rt\}; \\
S_{pt} = \{1, pt\}.
\]

**Proof.** This follows from Lemma 6.32 and Proposition C.5. \( \square \)

Let \( \varphi \in W(G_{50}) \). By Proposition 6.3, \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). We apply Proposition 6.5 (iv), with \( B = \{x, y, z\} \) and we conclude \( \{\varphi(x), \varphi(y), \varphi(z)\} \subseteq \{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\} \). Proposition
C.5 gives $G' = \langle x, y, z^2 \rangle$ so by Lemma 2.11 we have $\varphi(z) \in \{z^\pm 1\}$. If $\varphi(x) \in \{y^\pm 1\}$ then we may compose with the automorphism

$$\psi_1 : (x, y, z, p, r, t) \mapsto (y, x, z, p, pr, t)$$

and now $\varphi(x) \in x^G$ and $\varphi(y) \in y^G$. By Proposition 6.9 (iv) we may now compose with inner automorphisms so that $\varphi|_A = \text{Id}$ and by (ii) $\varphi(Af) = Af$ for all $f \in F$. By Proposition 6.19 we may compose with $\psi_x, \psi_y, \text{ and } \psi_z$ in order to ensure $\varphi(t) = t$.

We have $(rt)^2 = x^{-1}$, thus by Proposition 6.17 we conclude that $\varphi(r) \in \{r, xr\}$. If we have $\varphi(r) = xr$ we consider the map $I_t$ which maps $r$ to $xr$. Then composing with $I_t \circ i$ we may assume that we have $\varphi(r) = r$.

Since $p, r, \text{ and } pr$ are involutions, we may conclude by Proposition 6.17 that $\varphi(pt) = pt, \varphi(rt) = rt, \text{ and } \varphi(prt) = prt$.

Now let $\varphi(p) = bp$ for some $b \in A$. We have $(pt)^2 = x^{-1}y^{-1}$ thus applying the Proposition again we see that $b^2x^{-1}y^{-1} \in \{xy, xy^{-1}, x^{-1}y, x^{-1}y^{-1}\}$ which gives $b^2 \in \{x^2y^2, x^2, y^2, 1\}$ thus $b \in \{1, x, y, xy\}$. However we also have $bp = \varphi(pt \cdot t) \sim pt \cdot t = p$. Since $p^G = \langle xy, xy^{-1}p \rangle$ we see $b \in \{x, y\}$ would give a contradiction, thus we have $b \in \{1, xy\}$.

Let $\varphi(pr) = cpr$. We again apply Proposition 6.17. Since $(prt)^2 = y^{-1}$ we see that $c \in \{1, y\}$. Now $p \sim bp = \varphi(pr \cdot r) \sim cp$ but $yp \sim p$, therefore we must have $\varphi(pr) = pr$.

Lastly, $pr = \varphi(p \cdot r) \sim bpr$ and $(pr)^G = \langle y, z^2 \rangle pr$ so $b = xy$ gives a contradiction thus $\varphi(p) = p$. We have shown that we can assume $\varphi(f) = f$ for all $f \in F$.

We will now apply Theorem 6.33 twice to the $At$ coset. By the theorem, $\varphi(at) = a^ht$ for some $h \in R_t \cap S_{rt}$ and also $\varphi(at) = a^ht$ for some $h \in R_t \cap S_{pt}$. Combining these we have $h \in R_t \cap S_{rt} \cap S_{pt}$. Proposition 8.4 states that $S_{rt} = \{1, rt\}$ and $S_{pt} = \{1, pt\}$. We conclude that $h = 1$, i.e. $\varphi|_{At} = \text{Id}$. We apply Corollary 6.38 and now we have $\varphi = \text{Id}$.

We have shown that

**Theorem 8.5.** For crystallographic group $G = G_{50}$, the group $W(G)$ is generated by the
inverse map \( \iota \), the inner automorphisms and

\[
\begin{align*}
\psi_1 : (x, y, z, p, r, t) &\mapsto (y, x, z, p, rt, t), \\
\psi_x : (x, y, z, p, r, t) &\mapsto (x, y, z, xp, xr, xt), \\
\psi_y : (x, y, z, p, r, t) &\mapsto (x, y, z, yp, r, yt), \\
\psi_z : (x, y, z, p, r, t) &\mapsto (x, y, z, pr, zr, zt).
\end{align*}
\]

Thus, \( W(G) = W_0(G) \).

\begin{proposition}
For group 51 we have
\[
S_r = S_{rt} = S_t = F; \\
S_p = S_{prt} = \{1, p, rt, prt\}; \\
S_{pr} = S_{pt} = \{1, pr, rt, pt\};
\]
\end{proposition}

\begin{proof}
This follows from Lemma 6.32 and Proposition C.6.
\end{proof}

Let \( \varphi \in W(G_{51}) \). Note that by Lemma 6.12, \( Apr \) and \( Apt \) are the only cosets that do not contain elements of order 2. This together with Corollary 6.11 gives \( \varphi(Apr) = Apr \), \( \varphi(Apt) = Apt \), and \( \varphi(At) = At \). Note that \( \langle Apr, Apt, At \rangle = G/A \). Since \( G/A \) is abelian \( \varphi \) is a homomorphism we conclude that \( \varphi(Af) = Af \) for all \( f \in F \). Then by Proposition 6.9 (ii) and (iv), we may now compose \( \varphi \) with inner automorphisms so that \( \varphi|_A = \text{Id} \). By Proposition 6.19 we may compose with \( \psi_x, \psi_y, \) and \( \psi_z \) so as to have \( \varphi(t) = t \).

We now consider that \( p, r, prt, \) and \( rt \) all have order 2 thus by Proposition 6.17 we have \( \varphi(f) = f \) for all \( f \in \{pt, rt, pr, r\} \). Applying the proposition again, since \( (pt)^2 = (pr)^2 = x^{-1} \) we have \( \varphi(p) \in \{p, xp\} \) and \( \varphi(prt) \in \{prt, xprt\} \). If we have \( \varphi(prt) = xprt \) then by Proposition 6.18 (i), composing with \( \iota \circ \iota \) we have \( \varphi(prt) = prt \). By (ii) we have \( \varphi(f) = f \) for all \( f \in F - \{p\} \). Now by Proposition C.6 we have \( (rt)^G = (y^2)rt \) thus \( rt \sim xrt \). We use this as we apply Proposition 6.18 (iii). Since \( \varphi(p \cdot prt) = p \cdot prt \sim xrt \), it follows by (iii) that \( \varphi(prt) = prt \) implies \( \varphi(p) \neq xp \) therefore we conclude \( \varphi(p) = p \). Thus we now have \( \varphi(f) = f \) for all \( f \in F \).
Since $pt \notin S_p$ and $prt \notin S_{pr}$, we may apply Corollary 6.34 with $f \in \{pt, prt\}$ which gives $\varphi|_{Apt \cup Aprt} = Id$. Now Lemma 6.31 gives $\varphi(art) = a^hrt$ for some $h \in R_{rt} = \{1, rt\}$. If $h = rt$ we may compose with $\tau(rt, \{Art, At, Apr, Ap\})$ so as to have $\varphi|_{Art} = Id$. (Theorem 5.9 proves $\tau_{rt}$ is a non-trivial wct.) We now have $\varphi(g) = g$ for all $g \in Aprt \cup Art \cup Apt$, thus by Corollary 6.38 that $\varphi = Id$.

We have shown that

**Theorem 8.7.** For crystallographic group $G = G_{51}$, the group $W(G)$ is generated by the inverse map $\iota$, the inner automorphisms,

$$
\psi_x : (x, y, z, p, r, t) \mapsto (x, y, z, xp, xr, xt),
$$

$$
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
$$

$$
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt),
$$

and $\tau, \tau_t$. Thus $W(G) = \langle W_0(G), \tau_{rt} \rangle$. 

**Proposition 8.8.** For group 53 we have

$$
S_{pr} = S_t = F; \quad S_p = \{1, p, prt, rt\};
$$

$$
S_r = \{1, r, prt, pt\}; \quad S_{prt} = \{1, pr, prt, t\};
$$

$$
S_{rt} = \{1, rt\}; \quad S_{pt} = \{1, pt\}.
$$

**Proof.** This follows from Lemma 6.32 and Proposition C.7. 

Let $\varphi \in W(G_{53})$. Corollary 6.11 tells us $\varphi(At) = At$ and $\varphi(Ap) \in \{Ap, Ar, Apr\}$. However by Lemma 6.12, $Ap$ contains no elements of order 2 and $Ar$ and $Apr$ do, thus we have $\varphi(Ap) = Ap$. Similarly $Apet$ contains elements of order 2 but $Apt$ and $Art$ do not thus by Corollary 6.11 $\varphi(Aprt) = Aprt$. We know $\varphi$ is a homomorphism as $G/A$ is abelian. Since $\langle Ap, At, Aprt \rangle = G/A$, we conclude $\varphi(Af) = Af$ for all $f \in F$. Now by Proposition 6.9 $(ii)$ and $(iv)$, we may compose with inner automorphisms so that $\varphi|_A = Id$. By Proposition 6.19 we may compose $\varphi$ with $\psi_x, \psi_y$, and $\psi_z$ as necessary so we may now assume $\varphi(t) = t$. 

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We now apply Proposition 6.17 with \( f = p \) using the fact that \((pt)^2 = x^{-1}\) to deduce that \( \varphi(p) \in \{p, xp\} \). If we have \( \varphi(p) = xp \) then we may compose \( \varphi \) with \( I_t \circ \iota \) and now we have \( \varphi(p) = p \).

Again applying the proposition we see that since \( r, pr, \) and \( prt \) have order 2, we must have \( \varphi(rt) = rt, \varphi(prt) =prt, \) and \( \varphi(pr) = pr, \) and since \( p^2 = z \) we must have \( \varphi(pt) \in \{pt, z^{-1}pt\} \). However if \( \varphi(pt) = z^{-1}pt \) then

\[
t = \varphi(p^{-1} \cdot pt) \sim p^{-1}z^{-1}pt = z^{-1}t.
\]

However, \( t^G = \langle xz, x^{-1}z, y^2 \rangle t \) which does not contain \( z^{-1}t \), thus we have a contradiction. We conclude that \( \varphi(pt) = pt \).

Using Proposition 6.17 with \( f = r \) and thus \( \delta = (rt)^2 = x^{-1}z^{-1} \) we see that \( \varphi(r) = br \) implies that \( b \in \{1, x, z, xz\} \). However \( \varphi(r) \) must also satisfy

\[
t = \varphi(r \cdot rt) \sim br \cdot rt = bt,
\]

yet \( t^G = \langle xz, x^{-1}z, y^2 \rangle t \) does not contain \( xt \) nor \( zt \). We also have

\[
pr = \varphi(p \cdot r) = pbr = b^p pr,
\]

but \( (pr)^G = \langle y^2, z \rangle pr \) which does not contain \( (xz)^p pr = x^{-1}zpr \). Thus we must have \( \varphi(r) = r \). We now have \( \varphi(f) = f \) for all \( f \in F \).

We now apply Theorem 6.33 twice to the \( At \) coset. By the theorem, \( \varphi(at) = a^h t \) for some \( h \in R_t \cap S_{rt} \) and also \( \varphi(at) = a^h t \) for some \( h \in R_t \cap S_{pt} \). We conclude \( h \in R_t \cap S_{rt} \cap S_{pt} \). Proposition 8.8 states that \( S_{rt} = \{1, rt\} \) and \( S_{pt} = \{1, pt\} \). We conclude that \( h = 1 \), i.e. \( \varphi|_{At} = \text{Id} \). We apply Corollary 6.38 and now we have \( \varphi = \text{Id} \).

We have shown that

**Theorem 8.9.** For crystallographic group \( G = G_{53} \), the group \( W(G) \) is generated by the
inverse map \( \iota \), the inner automorphisms and

\[
\psi_x : (x, y, z, p, r, t) \mapsto (x, y, z, xp, xr, xt), \\
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt), \\
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).
\]

Thus \( \mathcal{W}(G) = \mathcal{W}_0(G) \).

\[\square\]

**Proposition 8.10.** For group 54 we have

\[
S_{pt} = \{1, pt\}; \quad S_p = S_{prt} = S_{rt} = \{1, p, prt, rt\}; \\
S_r = \{1, r, prt, pt\}; \quad S_{pr} = \{1, pr, rt, pt\}; \\
S_t = F.
\]

**Proof.** This follows from Lemma 6.32 and Proposition C.8.

Let \( \varphi \in \mathcal{W}(G_{54}) \). By Proposition 6.3 \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). We apply Proposition 6.5 \((iv)\), with \( B = \{x, y, z\} \) and we conclude \( \{\varphi(x), \varphi(y), \varphi(z)\} \subseteq \{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\} \). Now by Corollary 6.11 we have \( \varphi(At) = At \) and \( \varphi(Apr) \in \{Ap, Ar, Apr\} \). However, as \( Ap \) and \( Ar \) each contain elements of order 2 but by Lemma 6.12, \( Apr \) does not, we know \( \varphi(Apr) = Apr \). Since \( G/A \) is abelian, \( \varphi \) is a homomorphism thus \( \varphi(Aprt) = Aprt \). By Lemma 6.4 this implies that \( \varphi(x) \in \{x^{\pm 1}\} \) (because \( y \) and \( z \) commute with every element in \( Aprt \).) Now Proposition C.8 gives \( G' = \langle x, y^2, z \rangle \) so by Lemma 2.11 we have \( \varphi(z) \in \{z^{\pm 1}\} \) thus we have \( \varphi(\beta) \in \beta^G \) for all \( \beta \in B \). By Proposition 6.9 \((ii)\) and \((iv)\) we have \( \varphi(Af) = Af \) for all \( f \in F \) and by composing with inner automorphisms we have \( \varphi|_A = Id \). Now by Proposition 6.19 we may compose with \( \psi_x, \psi_y, \) and \( \psi_z \) so that we may assume \( \varphi(t) = t \).

We now apply Proposition 6.17 with \( f = p \) using the fact that \( (pt)^2 = x^{-1} \) to deduce that \( \varphi(p) \in \{p, xp\} \). If we have \( \varphi(p) = xp \) then we may compose with \( I_t \circ \iota \) and now we have \( \varphi(p) = p \).

Again applying the proposition we see that since \( p \) and \( r \) have order 2, we must have \( \varphi(pt) = pt \) and \( \varphi(rt) = rt \). Since \( (pr)^2 = x^{-1} \) we must have \( \varphi(prt) \in \{prt, xprt\} \). However if
\( \varphi(\text{prt}) = x\text{prt} \) then
\[
x\text{prt} = \varphi(p \cdot rt) \sim p \cdot rt,
\]
which contradicts \((\text{prt})^G = \langle x^2 \rangle \text{prt} \cup \langle x^2 \rangle xz\text{prt} \). Therefore \( \varphi(\text{prt}) = \text{prt} \).

Since \((\text{prt})^2 = (rt)^2 = z^{-1} \) Proposition 6.17 indicates we must have \( \varphi(\text{pr}) \in \{\text{pr}, z\text{pr}\} \) and \( \varphi(r) \in \{r, zr\} \). Suppose that \( \varphi(\text{pr}) = z\text{pr} \). Then
\[
\text{prt} = \varphi(\text{pr} \cdot t) \sim z\text{pr} \cdot t
\]
which is a contradiction since \( z\text{prt} \notin (\text{prt})^G \). We conclude that \( \varphi(\text{pr}) = \text{pr} \). Lastly, suppose that \( \varphi(r) = zr \). Then
\[
p = \varphi(p) = \varphi(\text{pr} \cdot r) \sim \text{pr} \cdot zr = z^{-1}p.
\]
This contradicts \((p)^G = \langle x^2, y^2 \rangle p \cup \langle x^2, y^2 \rangle xp \) thus \( \varphi(r) = r \). We now have \( \varphi(f) = f \) for all \( f \in F \).

Note that according to Proposition 8.10 \( \text{prt} \notin S_{\text{pr}}, \text{rt} \notin S_r, \) and \( \text{pt} \notin S_p \). We therefore may apply Corollary 6.34 with \( f \in \{\text{prt}, \text{rt}, \text{pt}\} \) which gives \( \varphi|_{\text{Aprt} \cup \text{Art} \cup \text{Apt}} = \text{Id} \). Then by Corollary 6.38 we see that \( \varphi \) is the identity map on all of \( G \).

We have shown that

**Theorem 8.11.** For crystallographic group \( G = G_{5A} \), the group \( W(G) \) is generated by the inverse map \( \iota \), the inner automorphisms, and
\[
\psi_x : (x, y, z, p, r, t) \mapsto (x, y, z, xp, xr, xt),
\]
\[
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
\]
\[
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).
\]
Therefore \( W(G) = W_0(G) \). \( \square \)
Proposition 8.12. For group 55 we have

\[ S_p = S_{pt} = S_t = F; \]
\[ S_r = S_{prt} = \{1, r, prt, pt\}; \]
\[ S_{pr} = S_{rt} = \{1, pr, rt, pt\}; \]

Proof. This follows from Lemma 6.32 and Proposition C.9. \qed

Let \( \varphi \in W(G_{55}) \). By Corollary 6.11 we have \( \varphi(Apt) \in \{Apt, Art, Aprt\} \). Now by Lemma 6.12, \( Apt \) contains elements of order 2 and \( Art \) and \( Aprt \) do not, thus we know \( \varphi(Apt) = Apt \).

By Proposition 6.3 \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). Without loss of generality, let \( B = \{x, y, z\} \). By Proposition 6.5 (iv) we have \( \{\varphi(x), \varphi(y), \varphi(z)\} \subseteq \{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\} \). Now we apply Lemma 6.4 using \( \varphi(Apt) \in Apt \). This implies that \( \varphi(z) \in z^G \) (because \( z \) is inverted by every element in \( Apt \) but \( x \) and \( y \) commute with every element in \( Apt \)). If \( \varphi(x) \in y^G \) we may compose \( \varphi \) with the outer automorphism

\[ \psi_1 : (x, y, z, p, r, t) \mapsto (y, x, z, p, pr, t) \]

and now we have \( \varphi(\beta) = \beta \) for all \( \beta \in \{x, y, z\} \). By Proposition 6.9 (ii) \( \varphi(Af) = Af \) for all \( f \in F \) and by (iv) we may compose \( \varphi \) with inner automorphisms so that we have \( \varphi|_A = \text{Id} \).

By Proposition 6.19 there exist automorphisms \( \psi_x, \psi_y, \psi_z \) and composing \( \varphi \) with these maps we have \( \varphi(t) = t \).

The squares of the elements in \( F \) are \( p^2 = (pt)^2 = 1, \ r^2 = y, \ (pr)^2 = x, \ (rt)^2 = x^{-1}, \ (prt)^2 = y^{-1}. \) Applying these facts to Proposition 6.17 we have \( \varphi(p) = p, \ \varphi(pt) = pt, \ \varphi(r) \in \{r, xr\}, \ \varphi(pr) \in \{pr, ypr\}, \ \varphi(prt) \in \{pr, x^{-1}prt\}, \) and \( \varphi(rt) \in \{rt, y^{-1}rt\}. \) If we have \( \varphi(rt) = y^{-1}rt, \) then by Proposition 6.18 (i) we may compose \( \varphi \) with \( I_t \circ \iota \) so as to ensure that \( \varphi(rt) = rt. \) Then by (ii) we have \( \varphi(f) = f \) for all \( f \in F - \{r, pr, prt\}. \) We will now Proposition 6.18 (iii) to show that this implies \( \varphi(f) = f \) for all \( f \in \{r, pr, prt\}. \) Accordingly,
we note that by Proposition C.9,

\[ yt \sim xyt \text{ since } (xyt)^G = \langle xy, xy^{-1}, z^2 \rangle t; \]
\[ y^{-1}pt \sim pt \text{ since } (pt)^G = \langle z^2 \rangle pt; \]
\[ xp \sim p \text{ since } (p)^G = \langle xy, xy^{-1} \rangle p. \]

Therefore we have

\[ \varphi(r \cdot rt) = \varphi(y \cdot t) \sim yt \sim xyt; \]
thus by Proposition 6.18 (iii), \( \varphi(r) \neq xr \) and so we have \( \varphi(r) = r \). Similarly we have

\[ \varphi(pr \cdot rt) = \varphi(pyt) = \varphi(y^{-1} \cdot pt) \sim y^{-1}pt \sim pt; \]
thus by the proposition, \( \varphi(pr) \neq ypr \) so \( \varphi(pr) = pr \). Lastly, we have

\[ \varphi(prt \cdot rt) = \varphi(px^{-1}) = \varphi(x \cdot p) \sim xp \sim p; \]
thus by the proposition, \( \varphi(prt) \neq x^{-1}prt \) i.e. \( \varphi(prt) =prt \) and so we now have \( \varphi(f) = f \) for all \( f \in F \).

Note that \( prt \notin S_{pr} \) and \( rt \notin S_r \). Thus we may apply Corollary 6.34 to get \( \varphi|_{A_{prt \cup A_{rt}}} = \text{Id} \). Now by Lemma 6.31, for all \( a \in A \) we have \( \varphi(apt) = a^hpt \) for some \( h \in \{1, pt\} \). If \( h = pt \) then we may compose with the non-trivial wct \( \tau(pt, \{At, Apt, Ar, Apr\}) \) to get \( \varphi|_{A_{pt}} = \text{Id} \). (Theorem 5.9 shows \( \tau_{pt} \) is a non-trivial wct.) By Corollary 6.38 we now have \( \varphi = \text{Id} \) on all of \( G \).

We have shown that

\textbf{Theorem 8.13.} For crystallographic group \( G = G_{55} \), the group \( \mathcal{W}(G) \) is generated by the inverse map \( \iota \), the inner automorphisms,

\[ \psi_1 : (x, y, z, p, r, t) \mapsto (y, x, z, p, pr, t), \]
\[ \psi_x : (x, y, z, p, r, t) \mapsto (x, y, z, xp, xr xt), \]
\[ \psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, ryt), \]
\[ \psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zrzt), \]

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and \( \tau_{pt} \). Therefore \( \mathcal{W}(G) = \langle \mathcal{W}_0(G), \tau_{pt} \rangle \).

**Proposition 8.14.** For group 57 we have

\[
\begin{align*}
S_p &= \{1, p, prt, rt\}; \\
S_r &= S_{prt} = S_{pt} = \{1, r, prt, pt\}; \\
S_{pr} &= \{1, pr, rt, pt\}; \\
S_{rt} &= \{1, rt\}; \\
S_t &= F.
\end{align*}
\]

**Proof.** This follows from Lemma 6.32 and Proposition C.10. □

Let \( \varphi \in \mathcal{W}(G_{57}) \). By Proposition 6.3 \( C_2 = \langle x \rangle \cup \langle y \rangle \cup \langle z \rangle \). We apply Proposition 6.5 (iv), with \( B = \{x, y, z\} \) and we conclude \( \{\varphi(x), \varphi(y), \varphi(z)\} \subseteq \{x^{\pm 1}, y^{\pm 1}, z^{\pm 1}\} \). Proposition C.10 gives \( G' = \langle x^2, y, z \rangle \) so by Lemma 2.11 we have \( \varphi(x) \in xG \). By Corollary 6.11 we have \( \varphi(Apt) \in \{Aprt, Art, Apt\} \). Since \( Apt \) contains elements of order 2 but by Lemma 6.12 \( Aprt \) and \( Art \) do not, we have \( \varphi(Apt) = Apt \). Applying Lemma 6.4 to the relation \( z^{pt} = z^{-1} \) we see that \( \varphi(z) \in yG \) gives a contradiction. We conclude that \( \varphi(z) \in z^G \) and so \( \varphi(y^G) = y^G \) as well. By Proposition 6.9 (ii) and (iv) \( \varphi(Af) = Af \) for all \( f \in F \) and we may compose \( \varphi \) with inner automorphisms so that \( \varphi|_A = \text{Id} \). By Proposition 6.19 we may compose with automorphisms \( \psi_x, \psi_y, \) and \( \psi_z \) as necessary so as to have \( \varphi(t) = t \).

We now apply Proposition 6.17 with \( f = r \) using the fact that \( (rt)^2 = z^{-1} \) to deduce that \( \varphi(r) \in \{r, zr\} \). If we have \( \varphi(r) = zr \) then we may compose with \( I_t \circ \iota \) and now we have \( \varphi(r) = r \).

Again applying the proposition we see that since \( pr \) and \( pt \) have order 2, we must have \( \varphi(prt) = prt \) and \( \varphi(p) = p \). Since \( (prt)^2 = y^{-1} \) we have \( \varphi(pr) \in \{pr, ypr\} \); also, \( r^2 = y \) gives \( \varphi(rt) \in \{rt, y^{-1}rt\} \). However if \( \varphi(pr) = ypr \) then

\[ r = \varphi(p^{-1} \cdot pr) \sim p^{-1} \cdot ypr = y^{-1}r, \]

and if \( \varphi(rt) = y^{-1}rt \) then

\[ r = \varphi(rt \cdot t) \sim y^{-1}rt \cdot t = y^{-1}r. \]
In both cases we arrive at $r \sim y^{-1}r$ which is a contradiction since $r^G = \langle x^2, z^2 \rangle r \cup \langle x^2, z^2 \rangle y^{-1}zr$. Therefore we have $\varphi(pr) = pr$ and $\varphi(rt) = rt$.

Lastly, since $p^2 = z$ the proposition gives $\varphi(pt) \in \{pt, z^{-1}pt\}$. If $\varphi(pt) = z^{-1}pt$ then $yz^{-1} \cdot prt \sim \varphi(yz^{-1} \cdot prt) = \varphi(r \cdot pt) \sim rz^{-1}pt = yprt$, which contradicts $(yprt)^G = \langle x^2 \rangle y^u \cdot p^u, u \in \{0, 1\}$, thus $\varphi(pt) = pt$.

We now have $\varphi(f) = f$ for all $f \in F$.

Note that $prt \notin S_{pr}, rt \notin S_r$, and $pt \notin S_p$, thus by Corollary 6.34 we have $\varphi|_{Aprt \cup Art \cup Apt} = Id$ and so by Corollary 6.38 we now have $\varphi = Id$.

We have shown that

**Theorem 8.15.** For crystallographic group $G = G_{57}$, the group $W(G)$ is generated by the inverse map $\iota$, the inner automorphisms and

$$
\psi_x : (x, y, z, p, r, t) \mapsto (x, y, z, xp, xrt),
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, ryt),
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, przt).
$$

Therefore $W(G) = W_0(G)$. \[\square\]

### 8.2 Groups 63 through 68

The following applies to $G_{63}, G_{64}, G_{65}, G_{66}, G_{67},$ and $G_{68}$.

**Proposition 8.16.** Let $\varphi \in W(G)$ where $G \in \{G_{63}, G_{64}, G_{65}, G_{66}, G_{67}, G_{68}\}$.

(i) $\varphi(z) \in \{z, z^{-1}\}$. Also, for $\alpha \in \{x^2y^{-1}, y\}$ we have $\varphi(\alpha) \in \{(x^2y^{-1})^\pm 1, y^\pm 1\}$.

(ii) For $Af \in \{At, Ap, Apt\}$ we have $\varphi(Af) = Af$.

(iii) For $Af \in \{Ar, Apr\}$ we have $\varphi(Af) \in \{Ar, Apr\}$. Also, for $f \in \{prt, rt\}$ we have $\varphi(Af) \in \{Aprt, Art\}$. Finally, $\varphi(Ar) = Apr$ if and only if $\varphi(Art) = Aprt$.  

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Proof. Statement (i) comes directly from Corollary 6.7.

We prove $\varphi(Apt) = Apt$ by using Corollary 6.11. We have $\varphi(Apt) \in \{Ap, Art, Apt\}$. Applying Lemma 6.4 to the relation $z^{rt} = z^{-1}$ we have $\varphi(z)\varphi(pt) = \varphi(z)^{-1}$. Since both $prt$ and $rt$ commute with $z$ we cannot have $\varphi(pt) \in Ap \cup Art$.

By Corollary 6.11 we have $\varphi(At) = At$. Since $G/A$ is abelian $\varphi$ is a homomorphism, thus $\varphi(Ap) = \varphi(Apt \cdot At) = Apt \cdot At = Ap$, proving (ii). Statement (iii) follows by Corollary 6.11 and the fact that $\varphi$ is a bijective homomorphism.

The following applies to $G_{63}, G_{64}, G_{65}, G_{66}, G_{67},$ and $G_{68}$.

Proposition 8.17. Let $G$ be a group with presentation of the form given in Eq. (4.4) with $\delta = 1$. Let $\varphi \in W(G)$ and suppose that $\varphi|_A = \text{Id}$ and $\varphi(Af) = Af$ for all $f \in F$. Then there exists an automorphism $\psi$ such that $\psi \circ \varphi$ satisfies $(\psi \circ \varphi)(t) = t$.

Proof. Let $\varphi(t) = ct$ for some $c \in A$. By Proposition 6.19 there exist outer automorphisms

$$
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
$$

and $\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt)$.

Composing with these maps as well as with $I_x$, we may assume $c \in \{1, x\}$. By Corollary 6.15 $c = x$ is not possible so we are done.

Proposition 8.18. For group 63 we have

$$
S_t = S_{pr} = S_{prt} = F;
$$
$$
S_p = S_{rt} \{1, p, prt, rt\};
$$
$$
S_r = S_{pt} = \{1, r, prt, pt\}.
$$

Proof. This follows from Lemma 6.32 and Proposition C.11.

Let $\varphi \in W(G_{63})$. Here the $Ap$ coset contains involutions but by Lemma 6.12 the $Art$ coset does not, therefore we cannot have $\varphi(Ap) = Art$. As $G/A$ is abelian, $\varphi$ is a homomorphism and this together with Proposition 8.16 (ii) and (iii) gives $\varphi(Af) = Af$ for
all \( f \in F \). Then by Proposition 6.9 \((iv)\) we may compose with inner automorphisms so that \( \varphi|_A = \text{Id} \). By Proposition 8.17 we may compose with an automorphism so that \( \varphi(t) = t \).

Five of the seven nontrivial elements in \( F \) have order 2, thus by Proposition 6.17 we have \( \varphi(f) = f \) for \( f \in \{p, pr, prt, rt\} \). The proposition also gives \( \varphi(r) \in \{r, zr\} \) and \( \varphi(pt) \in \{pt, z^{-1}pt\} \). If we have \( \varphi(pt) = z^{-1}pt \) then by Proposition 6.18 \((i)\) we may compose with \( I_t \circ I_t \) and now we have \( \varphi(pt) = pt \). Then by \((ii)\) we have \( \varphi(f) = f \) for \( f \in F - \{r\} \). We will use this with Proposition 6.18 \((iii)\) to show that \( \varphi(r) = r \) as well. One can check that \( rpt = z^{-1}prt \) thus \( zrpt = prt \). By Proposition C.11, \((prt)^G = \langle x^2y^{-1}\rangle \), so we have \( prt \sim z^{-1}prt \). Thus we have

\[
\varphi(r \cdot pt) = \varphi(z^{-1} \cdot prt) \sim z^{-1} \cdot prt \sim ptr = zrpt.
\]

By Proposition 6.18 \((iii)\) this implies \( \varphi(r) = r \) and therefore we now have \( \varphi(f) = f \) for all \( f \in F \).

Notice that \( pt \notin S_p \), thus by Corollary 6.34 we have \( \varphi|_{Ap} = \text{Id} \).

By Proposition 8.18 \( S_{pt} = \{1, r, pt, prt\} \) and by Corollary 6.29 \( R_p = \{1, p, rt, prt\} \). Applying Theorem 6.33 to the \( Ap \) coset we see that since \( R_p \cap S_{pt} = \{1, prt\} \) we have \( \varphi(ap) = a^h p \) for all \( a \in A \) where \( h \in \{1, prt\} \). However if \( h = prt \) then

\[
p = \varphi(x^{-1} \cdot xp) \sim x^{-1}xp^{prt} = x^{-2}yp.
\]

Since Proposition C.11 gives \( p^G = \langle x^2, y^2 \rangle p \cup \langle x^2, y^2 \rangle z^{-1}p \) we see that this is a contradiction. Thus we conclude \( \varphi|_{Ap} = \text{Id} \).

Next we apply Theorem 6.35 \((ii)\), using \( \varphi(t) = t \) and \( \varphi|_{Ap} = \varphi|_{Ap} = \text{Id} \). Since \( R_t \cap R_{pt} = \{1\} \) by Corollary 6.29 then by the theorem it follows that \( \varphi|_{At} = \text{Id} \). By Corollary 6.38 \( \varphi = \text{Id} \).

We have shown that

**Theorem 8.19.** For crystallographic group \( G = G_{63} \), the group \( W(G) \) is generated by the
inverse map \( \iota \), the inner automorphisms, and

\[
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
\]
\[
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).
\]

Therefore \( W(G) = W_0(G) \).

**Proposition 8.20.** For group 64 we have

\[
S_t = S_{pr} = F;
\]
\[
S_p = S_{rt} \{1, p,prt, rt\};
\]
\[
S_r = S_{prt} = S_{pt} = \{1, r, prt, pt\};
\]

**Proof.** This follows from Lemma 6.32 and Proposition C.12.

Let \( \varphi \in W(G_{64}) \). The \( Aprt \) coset contains involutions but by Lemma 6.12 the \( Art \) coset does not, therefore we cannot have \( \varphi(Aprt) = Art \). As \( G/A \) is abelian, \( \overline{\varphi} \) is a homomorphism and this together with Proposition 8.16 \((ii)\) and \((iii)\) gives \( \varphi(Af) = Af \) for all \( f \in F \). Then by Proposition 6.9 \((iv)\) we may compose \( \varphi \) with inner automorphisms so that \( \varphi|_A = \text{Id} \). By Proposition 8.17 we may compose \( \varphi \) with an automorphism so that \( \varphi(t) = t \).

Recall that \( p^2 = z, r^2 = y, (pr)^2 = 1, (rt)^2 = z^{-1}, (pt)^2 = y^{-1}, \) and \((prt)^2 = y^{-2}\).

Applying this information to Proposition 6.17 we have \( \varphi(prt) = prt \) as well as

\[
\varphi(p) \in \{p, yp\}, \varphi(r) \in \{r, zr\}, \varphi(pr) \in \{pr, y^2pr\}, \varphi(rt) \in \{rt, y^{-1}rt\}, \varphi(pt) \in \{pt, z^{-1}pt\}.
\]

By Proposition 6.18 \((i)\), we may compose \( \varphi \) with \( I_t \circ \iota \) and now we may assume \( \varphi(rt) = rt \). Thus by \((ii)\) we now have \( \varphi(f) = f \) for \( f \in \{prt, rt, t\} \). We will use Proposition 6.18 \((iii)\) to show that this implies that \( \varphi(f) = f \) for \( f \in \{p, r, pr, pt\} \). To show the hypotheses of the
proposition are met we note that by Proposition C.12 we have

\[ yt \sim yzt \quad \text{since } (yt)^G = \langle x^2, yz, yz^{-1} \rangle yt; \]

\[ prt \sim yprt \quad \text{since } (prt)^G = \langle x^2 y^{-1} \rangle prt \cup (x^2 y^{-1}) y^2 prt; \]

\[ y^{-1}t \sim y^{-1}z^{-1}t \quad \text{since } (y^{-1}t)^G = \langle x^2, yz, yz^{-1} \rangle y^{-1}t; \]

\[ y^{-1}pt \sim ypt \quad \text{since } (ypt)^G = \langle z^2 \rangle ypt \cup \langle z^2 \rangle zpt. \]

Now we have

\[ \varphi(r \cdot rt) = \varphi(y \cdot t) \sim yt \sim yzt = z(r \cdot rt) \text{ and } \varphi(rt) = rt, \]

thus Proposition 6.18 \((iii)\) implies \(\varphi(r) \neq zr\) and so \(\varphi(r) = r\). Similarly, as we have

\[ \varphi(p \cdot rt) = prt \sim yp \cdot rt \text{ and } \varphi(rt) = rt, \]

by the proposition we have \(\varphi(p) \neq yp\) thus \(\varphi(p) = p\). Now we have (since \(ptp = (pt)^2 t = y^{-1}t\),

\[ \varphi(pt \cdot p) = \varphi(y^{-1} \cdot t) \sim y^{-1}t \sim z^{-1}y^{-1}t = z^{-1}pt \cdot p \text{ and } \varphi(p) = p. \]

By the proposition it follows that \(\varphi(pt) \neq z^{-1}pt\) thus \(\varphi(pt) = pt\). Lastly, we have

\[ \varphi(pr \cdot rt) = \varphi(y^{-1} \cdot pt) \sim y^{-1}pt \sim ypt = y^2(y^{-1}pt) = y^2 pr \cdot rt \text{ and } \varphi(rt) = rt, \]

and so the proposition gives \(\varphi(pr) = pr\). We now have \(\varphi(f) = f\) for all \(f \in F\).

By Proposition 8.20, \(pt \notin S_p\), so Corollary 6.34 gives \(\varphi|_{Ap} = Id\).

Applying Theorem 6.33 to the \(Ap\) coset we have \(\varphi(ap) = a^h p\) for some \(h \in \mathcal{R}_p \cap S_{pt}\). By Corollary 6.29 and Proposition 8.20 this intersection is \(\{1, prt\}\). However if \(h = prt\) then

\[ p = \varphi(x^{-1} \cdot xp) \sim x^{-1}x^{prt} p = x^{-2} yp. \]

Since \(p^G = \langle x^2, y^2 \rangle p \cup \langle x^2, y^2 \rangle yz^{-1} p\) this is a contradiction, thus \(\varphi|_{Ap} = Id\).

Next we apply Theorem 6.35 \((ii)\), using \(\varphi(t) = t\) and \(\varphi|_{Ap} = \varphi|_{Apt} = Id\). Since we have \(\mathcal{R}_{pt} \cap \mathcal{R}_p = \{1\}\) by Corollary 6.29 the theorem gives \(\varphi|_{At} = Id\). Then by Corollary 6.38 \(\varphi = Id\).

We have shown that
Theorem 8.21. For crystallographic group \( G = G_{64} \), the group \( \mathcal{W}(G) \) is generated by the inverse map \( \iota \), the inner automorphisms, and

\[
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
\]
\[
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).
\]

Therefore \( \mathcal{W}(G) = \mathcal{W}_0(G) \). \( \square \)

Proposition 8.22. For group 65 we have \( S_f = F \) for all \( f \in F - \{1\} \).

**Proof.** Here \( F \) is an abelian subgroup of \( G \). It follows that for \( f \in F - \{1\}, h \in F, a \in A, \)

\[
a f \sim (af)^h = a^h f^h = a^h f.
\]

Let \( \varphi \in \mathcal{W}(G_{65}) \). Proposition 8.16 (i) we have \( \varphi(y) \in \{(x^2y^{-1})^{\pm1}, y^{\pm1}\} \). If \( \varphi(y) \notin y^G = \{y^{\pm1}\} \) then we may compose with the outer automorphism

\[
\psi_1 : (x, y, z, p, r, t) \mapsto (x, x^2y^{-1}, z, p, pr, t),
\]

and now we have \( \varphi(\beta) \in \beta^G \) for all \( \beta \in \{x^2y^{-1}, y, z\} \). By Proposition 6.9 (iv) we can compose with inner automorphisms to get \( \varphi|_A = \text{Id} \) and \( \varphi(Af) = Af \) for all \( Af \in G/A \). By Proposition 8.17 we may compose with an automorphism so that \( \varphi(t) = t \).

Now for \( f \in F \) we have \( ft \in F \) and \( (ft)^2 = 1 \). Therefore by Proposition 6.17 we have \( \varphi(f) = f \) for all \( f \in F \).

By Lemma 6.31, for \( a \in A \) we have \( \varphi(art) = a^h rt \) for some \( h \in R_{rt} = \{1, rt\} \). If we have \( \varphi(art) = a^{rt} rt \) then composing with \( I_t \circ \iota \) we have

\[
(I_t \circ \iota \circ \varphi)(art) = (I_t \circ \iota)(a^{rt}rt) = ((a^{rt}rt)^{-1}t = (tr(a^{-1}rt)^{-1}t = (a^{-1}rt)^t = art.
\]

Thus now we may assume we have \( \varphi|_{Art} = \text{Id} \).

By Lemma 6.31, for \( a \in A \) we have \( \varphi(apt) = a^h pt \) for some \( h \in R_{pt} = \{1, pt\} \). If we have \( \varphi(apt) = a^{pt} pt \) then composing with the non-trivial wct \( \tau(pt, \{Apt, At, Ar, Apr\}) \), we can assume we have \( \varphi|_{Art \cup Apt} = \text{Id} \). (Theorem 5.9 shows that \( \tau_{pt} \) is a non-trivial wct.) We apply
Theorem 6.35 (i) twice using $\varphi(t) = t$ and $\varphi|_{Art} = \varphi|_{Apt} = \text{Id}$. Since $R_t \cap R_r \cap R_p = \{1, \text{prt}\}$ by Corollary 6.29, the theorem indicates that for $a \in A, \varphi(at) = a^h t$ for some $h \in \{1, \text{prt}\}$. However, if $h = \text{prt}$ then

$$t = \varphi(x^{-1} \cdot xt) \sim x^{-1} x^{\text{prt}} t = x^{-2} yt,$$

which is a contradiction since Proposition C.13 gives $t^G = \langle x^2, y^2, z^2 \rangle t$, thus $x^{-2} yt \sim t$. It follows that $h = 1$, in other words, $\varphi(at) = at$ for all $a \in A$. By Corollary 6.38 we have $\varphi = \text{Id}$.

We have shown that

**Theorem 8.23.** For crystallographic group $G = G_{65}$, the group $W(G)$ is generated by the inverse map $\iota$, the inner automorphisms,

$$\psi_1 : (x, y, z, p, r, t) \mapsto (x, x^2 y^{-1}, z, p, pr, t),$$
$$\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),$$
$$\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt),$$

and the non-trivial wct $\tau_{\text{pt}}$. Thus we have $W(G) = \langle W_0(G), \tau_{\text{pt}} \rangle$. □

**Proposition 8.24.** For group 66 we have

$$S_p = S_r = S_{pr} = F;$$
$$S_{prt} = S_{rt} = \{1, p, \text{prt}, rt\};$$
$$S_t = S_{pt} = \{1, p, pt, t\}.$$

**Proof.** This follows from Lemma 6.32 and Proposition C.14. □

Let $\varphi \in W(G_{66})$. By Proposition 8.16(i) we have $\varphi(y) \in \{(x^2 y^{-1})^{\pm 1}, y^{\pm 1}\}$. If $\varphi(y) \notin y^G = \{y^{\pm 1}\}$ then we compose with the outer automorphism

$$\psi_1 : (x, y, z, p, r, t) \mapsto (x, x^2 y^{-1}, z, p, pr, t),$$
and now we have $\varphi(\beta) \in \beta^G$ for all $\beta \in \{x^2y^{-1}, y, z\}$. By Proposition 6.9 (iv) we may compose $\varphi$ with inner automorphisms to get $\varphi|_A = \text{Id}$ and $\varphi(Af) = Af$ for all $Af \in G/A$.

By Proposition 8.17, composing $\varphi$ with an automorphism gives $\varphi(t) = t$.

Now for $f \in \{p, p_{rt}, rt, pt\}$ we have $ft \in F$ and $(ft)^2 = 1$. Therefore by Proposition 6.17 we have $\varphi(f) = f$ for all $f \in \{p, p_{rt}, rt, pt\}$. The proposition also gives $\varphi(pr) \in \{pr, zpr\}$ and $\varphi(r) \in \{r, zr\}$. Composing $\varphi$ with $I_t \circ \iota$ if necessary we can arrange to have $\varphi(r) = r$, by Proposition 6.18 (i). By (ii) we now have $\varphi(f) = f$ for $f \in F - \{pr\}$. We will show that $\varphi(pr) = pr$ as well by applying (iii). Now Proposition C.14 gives $p^G = \langle x^2, y^2 \rangle p$, so $p \sim zp$. Thus we have

$$\varphi(pr \cdot r) = \varphi(p) = p \sim zp = zpr \cdot r \text{ and } \varphi(r) = r.$$  

It follows by Proposition 6.18 (iii) that $\varphi(pr) = pr$ and we now have $\varphi(f) = f$ for all $f \in F$.

We now apply Theorem 6.33 three times. We will use the following facts from Proposition 8.24: $pr \notin S_t, rt \notin S_{pt}$; and $pt \notin S_{rt}$. Also recall that by Corollary 6.29 $R_f = \{1, f\}$ for $f \in \{p_{rt}, rt, pt\}$. First we will apply the theorem to the $A_{p_{rt}}$ coset. Since $(pr)^2 = 1$ we have $pr \cdot p_{rt} = t$. Then as $R_{p_{rt}} \cap S_t = 1$ Theorem 6.33 implies $\varphi|_{A_{p_{rt}}} = \text{Id}$. Next we consider the $A_{rt}$ coset. Since $r^2 = 1$ we have $pr \cdot rt = pt$. We also have $R_{rt} \cap S_{pt} = 1$ so the theorem implies $\varphi|_{A_{rt}} = \text{Id}$. Lastly we apply the theorem to the $A_{pt}$ coset. Since $p$ commutes with $r$ we have $pr \cdot pt = rt$. We also have $R_{pt} \cap S_{rt} = 1$. By the theorem we conclude that $\varphi|_{A_{pt}} = \text{Id}$. Now it follows by Corollary 6.38 that $\varphi = \text{Id}$.

We have shown that

**Theorem 8.25.** For crystallographic group $G = G_{66}$, the group $W(G)$ is generated by the
inverse map \( \iota \), the inner automorphisms, and

\[
\psi_1 : (x, y, z, p, r, t) \mapsto (x, x^2y^{-1}, z, p, pr, t),
\]

\[
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
\]

\[
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).
\]

Therefore \( \mathcal{W}(G) = \mathcal{W}_0(G) \).

Proposition 8.26. For group 67 we have

\[
S_t = S_{pr} = S_{rt} = F;
\]

\[
S_p = \{1, p, pr, rt\};
\]

\[
S_r = S_{pt} = S_{prt} \{1, r, prt, pt\}.
\]

Proof. This follows from Lemma 6.32 and Proposition C.15.

Let \( \varphi \in \mathcal{W}(G_{67}) \). By Proposition 8.16 \((i)\) we have \( \varphi(y) \in \{(x^2y^{-1})^{\pm 1}, y^{\pm 1}\} \). If \( \varphi(y) \notin y^G = \{y^{\pm 1}\} \) then we compose \( \varphi \) with the outer automorphism

\[
\psi_2 : (x, y, z, p, r, t) \mapsto (x, x^2y^{-1}, z, p, xpr, xy^{-1}t),
\]

and now we have \( \varphi(\beta) \in \beta^G \) for all \( \beta \in \{x^2y^{-1}, y, z\} \). By Proposition 6.9 \((iv)\) we may compose with inner automorphisms to get \( \varphi|_A = \text{Id} \) and by \((ii)\) we have \( \varphi(Af) = Af \) for all \( Af \in G/A \). We may now compose \( \varphi \) with an automorphism so that \( \varphi(t) = t \), according to Proposition 8.17.

Recall that \( p, pr \) and \( rt \) have order 2, thus by Proposition 6.17 we have \( \varphi(f) = f \) for \( f \in \{r, pr, pt\} \). The proposition also gives \( \varphi(p) \in \{p, yp\}, \varphi(pr) \in \{pr, y^2pr\} \) and \( \varphi(rt) \in \{rt, y^{-1}rt\} \). By Proposition 6.18 \((i)\), composing with \( I_t \circ \iota \) if necessary we may assume \( \varphi(pr) = pr \). We now have \( \varphi(f) = f \) for \( f \in F - \{p, rt\} \), by \((ii)\). Note that Proposition C.15 gives

\[
r^G = \langle x^2y^{-1}, z^2 \rangle r \cup \langle x^2y^{-1}, z^2 \rangle y^{-1}r \text{ and } (ypt)^G = \langle z^2 \rangle pt \cup \langle z^2 \rangle ypt,
\]
therefore \( r \sim yr \) and \( ypt \sim y^2pt \). So we have

\[
\varphi(p \cdot pr) = \varphi(r) = r \sim yr = yp \cdot pr \quad \text{and} \quad \varphi(pr) = pr,
\]

thus by Proposition 6.18 (iii) we have \( \varphi(p) \neq yp \) so \( \varphi(p) = p \). Similarly we have

\[
\varphi(rt \cdot pr) = \varphi(y^2 \cdot pt) \sim y^2pt \sim ypt = y^{-1}(y^2pt) = y^{-1}rt \cdot pr \quad \text{and} \quad \varphi(pr) = pr.
\]

Then by Proposition 6.18 (iii) we have \( \varphi(rt) \neq y^{-1}rt \) thus \( \varphi(rt) = rt \). We now have \( \varphi(f) = f \) for all \( f \in F \).

We apply Theorem 6.33 to the \( Ap \) coset, noting that since \( p \) has order 2, \( p \cdot pr = r \), and thus for \( a \in A \) we have \( \varphi(ap) = a^h p \) for some \( h \in R_p \cap S_r \). By Corollary 6.29 and Proposition 8.26 we see this intersection is \( \{1, prt\} \). However, if \( h = prt \) then

\[
zp \sim \varphi(z \cdot p) = \varphi(x^{-1}z \cdot xp) \sim x^{-1}z \cdot x^{prt}p = x^{-1}z \cdot x^{-1}yp = x^{-2}yzp.
\]

This is a contradiction since according to Proposition C.15 \( (zp)^G = \langle x^2, y^2 \rangle zp \cup \langle x^2, y^2 \rangle yz^{-1}p \).

We conclude that \( \varphi_{Ap} = Id \).

Proposition 8.26 indicates \( pt \not\in S_p \) thus by Corollary 6.34 \( \varphi|_{Ap^t} = Id \). We therefore may apply Theorem 6.35 (ii) using \( \varphi(t) = t, \varphi_{Ap} = \varphi|_{Ap^t} = Id \). Since Corollary 6.29 indicates \( R_{pt} \cap R_p = \{1\} \) the theorem implies that \( \varphi|_{At} = Id \). Then by Corollary 6.38 \( \varphi = Id \).

We have shown that

**Theorem 8.27.** For crystallographic group \( G = G_{67} \), the group \( W(G) \) is generated by the inverse map \( \iota \), the inner automorphisms, and

\[
\begin{align*}
\psi_2 : (x, y, z, p, r, t) &\mapsto (x, x^2y^{-1}, z, p, xpr, xy^{-1}t), \\
\psi_y : (x, y, z, p, r, t) &\mapsto (x, y, z, yp, r, yt), \\
\psi_z : (x, y, z, p, r, t) &\mapsto (x, y, z, p, zr, zt).
\end{align*}
\]

Thus \( W(G) = W_0(G) \).
Proposition 8.28. For group 68 we have

\[ S_p = S_{prt} = S_{rt} = \{1, p, prt, rt\}; \]

\[ S_{pr} = \{1, pr, rt, pt\}; \]

\[ S_r = \{1, pt\}; \]

\[ S_t = F. \]

Proof. This follows from Lemma 6.32 and Proposition C.16.

Let \( \varphi \in W(G_{68}) \). By Proposition 8.16 (i) we have \( \varphi(y) \in \{(x^2y^{-1})^{\pm 1}, y^{\pm 1}\} \). If \( \varphi(y) \notin y^G = \{y^{\pm 1}\} \) then we compose \( \varphi \) with the outer automorphism

\[ \psi_3 : (x, y, z, p, r, t) \mapsto (x, x^2y^{-1}, z, p, xpr, xt), \]

and now we have \( \varphi(\beta) \in \beta^G \) for all \( \beta \in \{x^2y^{-1}, y, z\} \). By Proposition 6.9 (ii) and (iv) we may compose with inner automorphisms to get \( \varphi|_A = \text{Id} \) and \( \varphi(Af) = Af \) for all \( Af \in G/A \). By Proposition 8.17 we may compose with an automorphism so that \( \varphi(t) = t \).

From the presentation of \( G \) we see that \( p \) and \( r \) have order 2, thus by Proposition 6.17 we have \( \varphi(pt) = pt \) and \( \varphi(rt) = rt \). The proposition also gives

\[ \varphi(p) \in \{p, x^2y^{-1}p\} \quad \text{and} \quad \varphi(prt) \in \{prt, x^2y^{-1}prt\}. \]

By Proposition 6.18 (i) if \( \varphi(prt) = x^2y^{-1}prt \) then composing with \( I_t \circ I \) we have \( \varphi(prt) = prt \), and now by (ii) we have \( \varphi(f) = f \) for all \( f \in \{prt, rt, pt, t\} \). Note that by Proposition C.16 we have

\[ (rt)^G = (y)rt \cup (y)zrt; \]

therefore \( rt \sim x^2y^{-1}rt \). Then

\[ \varphi(p \cdot prt) = \varphi(rt) \sim x^2y^{-1}rt = x^2y^{-1}p \cdot prt \quad \text{and} \quad \varphi(prt) = prt, \]

and so by Proposition 6.18 (iii) we have \( \varphi(p) = p \).

Now we have \( \varphi(pt) = pt \) and \( \varphi(p) = p \) so Corollary 6.34 may be applied to the \( A pt \) coset. Since \( pt \notin S_p \) the Corollary gives \( \varphi|_{Apt} = \text{Id} \). Next we apply Theorem 6.33 to the \( A p \) coset. By Corollary 6.29 and Proposition 8.28 we have \( R_p \cap S_{pt} = \{1\} \). The theorem therefore gives
\( \varphi|_{Ap} = \text{Id} \). Using this we may apply Theorem 6.35 (i) using \( \varphi(prt) = prt \) and \( \varphi|_{Ap} = \text{Id} \). Since \( R_{prt} \cap R_{rt} = 1 \) by Corollary 6.29, then according to the theorem \( \varphi|_{Aprt} = \text{Id} \).

Now by Proposition 6.17 we have \( \varphi(r) \in \{r, zr\} \). We will use \( \varphi|_{Aprt} = \text{Id} \) with Proposition 6.18 (iii) to show \( \varphi(r) \neq zr \). By Proposition C.16 we have

\[
(ypt)^G = \langle z^2 \rangle ypt \cup \langle z^2 \rangle y^{-1} zpt \cup \langle z^2 \rangle x^2 y^{-2} pt \cup \langle z^2 \rangle x^2 zpt,
\]

so \( ypt \sim yzp \). Thus we have

\[
\varphi(r \cdot x^2 prt) = \varphi(y \cdot pt) \sim ypt = z(ypt) = zr \cdot x^2 prt \text{ and } \varphi(x^2 prt) = x^2 prt.
\]

By statement (iii) of the proposition we have \( \varphi(r) = r \).

Now by Proposition 6.17 \( \varphi(pr) \in \{pr, zpr\} \). We will again apply Proposition 6.18 (iii) to show \( \varphi(pr) \neq zpr \). Proposition C.16 gives

\[
p^G = \langle x^2, y^2 \rangle p \cup \langle x^2, y^2 \rangle yp,
\]

therefore \( p \sim zp \). Thus we have

\[
\varphi(pr \cdot r) = \varphi(p) = p \sim zp = zpr \cdot r \text{ and } \varphi(r) = r,
\]

therefore by (iii) we have \( \varphi(pr) = pr \). We now have \( \varphi(f) = f \) for all \( f \in F \).

Now that we have \( \varphi(r) = r \) we may apply Theorem 6.33 to the Art coset. Since \( R_{rt} \cap S_t = \{1\} \), we conclude that \( \varphi|_{Art} = \text{Id} \). We have shown that \( \varphi|_{Aprt \cup Art \cup Apt} = \text{Id} \), thus by Corollary 6.38 \( \varphi = \text{Id} \).

We have shown that

**Theorem 8.29.** For crystallographic group \( G = G_{68} \), the group \( W(G) \) is generated by the inverse map \( \iota \), the inner automorphisms, and

\[
\begin{align*}
\psi_3 &: (x, y, z, p, r, t) \mapsto (x, x^2 y^{-1}, z, p, xpr, x^{-1} yt), \\
\psi_y &: (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt), \\
\psi_z &: (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).
\end{align*}
\]
Thus $W(G) = W_0(G)$. 

8.3 Groups 69 Through 74

**Proposition 8.30.** For group $G_{69}$ we have $S_f = F$ for all $f \in F$.

**Proof.** Fix $f \in F$. For any $h \in F$ we have $af \sim (af)^h = a^h f$ since $F$ is an abelian subgroup of $G$. 

Let $\varphi \in W(G_{69})$. By Proposition 6.3 we have $C_2 = \langle x^2 z^{-1} \rangle \cup \langle y^2 z^{-1} \rangle \cup \langle z \rangle$. We apply Proposition 6.5 (iv) with $B = \{x^2 z^{-1}, y^2 z^{-1}, z\}$ which gives $\{\varphi(x^2 z^{-1}), \varphi(y^2 z^{-1}), \varphi(z)\} \subseteq \{(x^2 z^{-1})^\pm 1, (y^2 z^{-1})^\pm 1, z^\pm 1\}$. Now

$$\psi_1 : (x, y, z, p, r, t) \mapsto (xy^{-1}, x, x^2 z^{-1} pr, p, t)$$

and

$$\psi_2 : (x, y, z, p, r, t) \mapsto (y, x, z, p, pr, t)$$

determine automorphisms of $G_{69}$ and note that

$$\psi_1 : x^2 z^{-1} \mapsto y^{-2} z$$

$$\psi_2 : x^2 z^{-1} \mapsto y^2 z^{-1}$$

$$\psi_1 : y^2 z^{-1} \mapsto z$$

$$\psi_2 : y^2 z^{-1} \mapsto x^2 z^{-1}$$

$$\psi_1 : z \mapsto x^2 z^{-1}$$

$$\psi_2 : z \mapsto z.$$ 

In other words, these two automorphisms permute the elements of $B$. Thus composing with $\psi_1$ and $\psi_2$ as needed we have $\varphi(\beta) \in \{\beta^\pm 1\}$ for all $\beta \in B$. Then by Proposition 6.9 (iv) we may compose with inner automorphisms to have $\varphi|_A = \text{Id}$ and by (ii) we have $\varphi(Af) = Af$ for all $f \in F$.

Let $\varphi(t) = ct$ for some $c \in A$. By Proposition 6.19 the following defines an automorphism:

$$\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zrzt).$$

Composing $\varphi$ with this map and also with $I_x$ and $I_y$, as necessary we may assume that $c \in \{1, x, y, xy\}$. 

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We will use Proposition 6.14 to show that $\varphi(t) \notin \{xt, yt, xyt\}$. To justify using the proposition we note that $\alpha_p = \alpha_{pt} = \alpha_r = \alpha_{rt} = 1$ thus they clearly are contained in $C_2$. Note also that we have $\varphi(Ar) = Ar$ and $\varphi(Ap) = Ap$. Then by the proposition,

\[(x, p) = x^{-2}z \notin \langle x^2, y^2, z^2 \rangle \text{ therefore } \varphi(t) \neq xt;\]
\[(y, r) = z^{-1} \notin \langle x^2, y^2, z^2 \rangle \text{ therefore } \varphi(t) \neq yt;\]
\[(xy, r) = x^{-2}z^{-1} \notin \langle x^2, y^2, z^2 \rangle \text{ therefore } \varphi(t) \neq xyt.\]

We conclude that $\varphi(t) = t$. It follows by Proposition 6.17 that since every nontrivial element of $F$ has order 2, $\varphi(f) = f$ for all $f \in F$.

Now by Lemma 6.31 for $a \in A$ we have $\varphi(at) = ah^t$ for some $h \in R_t = F$. We will assume $h \in \{p, pr, prt, pt\}$ and show this leads to a contradiction. Note that $x^h \in \{(xz^{-1})^{\pm 1}\}$. This gives

\[x^2z^{-1}t \sim \varphi(x^2z^{-1} \cdot t) = \varphi(xz^{-1} \cdot xt) \sim xz^{-1} \cdot x^ht = xz^{-1}(xz^{-1})^{\pm 1}t.\]

Since $t^G = \langle x^2, y^2, z^2 \rangle t$ we see $xz^{-1}(xz^{-1})^{\pm 1}t \sim t$ but $x^2z^{-1}t \sim t$. This contradiction leads us to conclude that $\varphi(at) = ah^t$ for some $h \in \{1, r, rt, t\}$. We will give a similar argument to show that $h \notin \{r, rt\}$. Assuming $h \in \{r, rt\}$ we have $y^h \in \{(yz^{-1})^{\pm 1}\}$ thus

\[y^2z^{-1}t \sim \varphi(y^2z^{-1} \cdot t) = \varphi(yz^{-1} \cdot yt) \sim yz^{-1} \cdot y^ht = yz^{-1}(yz^{-1})^{\pm 1}t \sim t,\]

a contradiction. Therefore we have $\varphi(at) = ah^t$ for some $h \in \{1, t\}$. If $h = t$ then we have $\varphi(at) = a^t = (at)^t$. Note that since every element $at \in At$ has order 2, composing with $I_t \circ \iota$ we now $\varphi|_{At} = \Id$ and so by Corollary 6.38, $\varphi = \Id$.

We have shown that

**Theorem 8.31.** For crystallographic group $G = G_{69}$, the group $\mathcal{W}(G)$ is generated by the
inverse map \( \iota \), the inner automorphisms, and

\[
\psi_1 : (x, y, z, p, r, t) \mapsto (xy^{-1}, x, x^2z^{-1}pr, p, t), \\
\psi_2 : (x, y, z, p, r, t) \mapsto (y, x, z, p, pr, t), \\
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zrzt).
\]

Therefore \( W(G) = W_0(G) \).

Proposition 8.32. For group 72 we have

\[
S_t = S_p = S_{pt} = F; \\
S_r = S_{prt} = \{1, r, prt, pt\}; \\
S_{pr} = S_{rt} = \{1, pr, rt, pt\}.
\]

Proof. This follows from Lemma 6.32 and Proposition C.18.

Let \( \varphi \in W(G_{72}) \). We have \( C_2 = \langle x^{-2}yz \rangle \cup \langle y \rangle \cup \langle z \rangle \) by Proposition 6.3. We apply Proposition 6.5 (iv) with \( B = \{x^{-2}yz, y, z\} \) and conclude that for \( \beta \in B \) we have \( \varphi(\beta) \in \{(x^{-2}yz)^{\pm 1}, (y)^{\pm 1}, (z)^{\pm 1}\} \).

Lemma 6.12 indicates that the \( Apt \) coset is unlike \( Art \) and \( Aprt \) in that \( Apt \) contains elements of order 2. This fact together with Corollary 6.11 gives \( \varphi(Apt) = Apt \) and \( \varphi(At) = At \). Since \( G/A \) is abelian \( \varphi \) is a homomorphism thus \( \varphi(Ap) = Ap \). By Lemma 6.4 \( \varphi(x^{-2}yz) \) and \( \varphi(y) \) are inverted by the action of \( p \) therefore \( \{\varphi(x^{-2}yz), \varphi(y)\} \subseteq \{(x^{-1}yz)^{\pm 1}, y^{\pm 1}\} \) and so \( \varphi(z) \in \{z^{\pm 1}\} \). If we have \( \varphi(x^{-2}yz) \in \{y^{\pm 1}\} \) then we may compose \( \varphi \) with

\[
\psi_3 : (x, y, z, p, r, t) \mapsto (x, x^2y^{-1}z^{-1}, z, p, pr, t),
\]

and now we have \( \varphi(x^{-2}yz) \in \{(x^{-2}yz)^{\pm 1}\} \) and \( \varphi(y) \in \{y^{\pm 1}\} \). Then by Proposition 6.9 (ii) and (iv), composing \( \varphi \) with inner automorphisms if necessary we have \( \varphi|_A = \text{Id} \) and \( \varphi(Af) = Af \) for all \( f \in F \).
Let \( \varphi(t) = ct \) for some \( c \in A \). By Proposition 6.19 the maps

\[
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt)
\]

and \( \psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, zr, zt) \)

determine outer automorphisms. Composing \( \varphi \) with these maps and with \( I_x \) we may assume \( c \in \{1, x\} \). By Corollary 6.16 \( c = x \) is not possible so we have \( \varphi(t) = t \).

It follows by Proposition 6.17 that since \( p \) and \( pt \) have order 2, \( \varphi(p) = p \) and \( \varphi(pt) = pt \). The proposition also gives

\[
\varphi(pr) \in \{pr, ypr\}; \quad \varphi(r) \in \{r, x^2y^{-1}z^{-1}r\}; \quad \varphi(rt) \in \{rt, y^{-1}rt\}; \quad \varphi(prt) \in \{prt, x^2yzprt\}.
\]

By Proposition 6.18 (i), if \( \varphi(prt) = x^2yzprt \) then we may compose with \( I_t \circ \iota \) and now \( \varphi(prt) = prt \). It follows by (ii) that we now have \( \varphi(f) = f \) for all \( f \in \{p, prt, pt, t\} \).

Now we will use Proposition 6.18 (iii) three times to show that this implies \( \varphi(f) = f \) for \( f \in \{r, pr, rt\} \) as well. The following facts from Proposition C.18 will be needed to satisfy the requisite hypotheses:

\[
\begin{align*}
p & \sim yp \quad \text{and} \quad p \sim y^{-1}p \quad & \text{since} \quad p^G = \langle x^2z^{-1}, y^2 \rangle p; \\
t & \sim x^2y^{-1}z^{-1}t \quad & \text{since} \quad t^G = \langle x^2, y^2, z \rangle t.
\end{align*}
\]

Now we have

\[
\varphi(rt \cdot (prt)^{-1}) = \varphi(p) = p \sim y^{-1}p = y^{-1}rt \cdot (prt)^{-1} \quad \text{and} \quad \varphi(prt)^{-1} = (prt)^{-1},
\]

therefore Proposition 6.18 (iii) implies \( \varphi(rt) = rt \).

Next we consider that

\[
\varphi(r^{-1} \cdot rt) = \varphi(t) = t \sim x^2y^{-1}z^{-1}t = (x^2y^{-1}z^{-1}r)^{-1} \cdot rt \quad \text{and} \quad \varphi(rt) = rt.
\]

By the proposition we have \( \varphi(r^{-1}) = r^{-1} \) thus \( \varphi(r) = r \). Lastly we have

\[
\varphi(pr \cdot r^{-1}) = \varphi(p) = p \sim yp = ypr \cdot r^{-1} \quad \text{and} \quad \varphi(r^{-1}) = r^{-1},
\]

therefore \( \varphi(pr) = pr \) by the proposition. We now have \( \varphi(f) = f \) for all \( f \in F \).

Now \( rt \notin S_r \) and \( prt \notin S_{pr} \) thus by Corollary 6.34, \( \varphi|_{Art \cup Aprt} = \text{Id} \). Now by Lemma 6.31
and Corollary 6.29 we know that for all \( a \in A, \varphi(apt) = a^hpt \) for some \( h \in \{1, pt\} \). Suppose \( h = pt \). Let \( \gamma = x^2y^{-2}z^{-1} \) and note that \( ptprt = \gamma r \). Then for \( a, b \in A \) we have

\[
ab^pl \gamma r \sim \varphi(ab^pl \gamma \cdot r) = \varphi(ab^lptprt) = \varphi(apt \cdot bprt) \sim a^pl pt \cdot bprt = (ab)^pl ptprt = (ab)^pl \gamma r.
\]

Now if we let \( a = b = x^{-1}y \), the left hand side becomes

\[
ab^pl \gamma r = x^{-1}y(x^{-1}y)^pl \gamma r = x^{-1}y(x^{-1}yz)x^2y^{-2}z^{-1}r = r,
\]

while the right hand side becomes

\[
(ab)^pl \gamma r = (x^{-2}y^2)^pl \gamma r = x^{-2}y^2 z^2 x^2y^{-2}z^{-1}r = zr.
\]

By Proposition C.18, \( r^G = \langle x^2y^{-1}, z^2 \rangle r \cup \langle x^2y^{-1}, z^2 \rangle y^{-1}zr \) therefore \( r \sim zr \). This contradiction indicates that \( \varphi(apt) = a^pl pt \) is only possible if \( h = 1 \). In other words, \( \varphi |_{Apt} = Id \), and now by Corollary 6.38 \( \varphi = Id \).

We have shown that

**Theorem 8.33.** For crystallographic group \( G = G_{72} \), the group \( W(G) \) is generated by the inverse map \( \iota \), the inner automorphisms, and

\[
\psi_3 : (x, y, z, p, r, t) \mapsto (x, x^2y^{-1}z^{-1}, z, p, pr, t),
\]

\[
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
\]

\[
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).
\]

Thus \( W(G) = W_0(G) \).

**Proposition 8.34.** For group 73 we have

\[
S_{rt} = \{1, p, pr, rt\}; \quad S_{prt} = \{1, r, pr, pt\};
\]

\[
S_{pt} = \{1, pr, rt, pt\}; \quad S_p = \{1, p\};
\]

\[
S_r = \{1, r\}; \quad S_{pr} = \{1, pr\};
\]

\[
S_t = F.
\]

**Proof.** This follows from Lemma 6.32 and Proposition C.19.
Let $\varphi \in \mathcal{W}(G_{73})$. We have $C_2 = \langle x^{-2}yz \rangle \cup \langle y \rangle \cup \langle z \rangle$ by Proposition 6.3. We apply Proposition 6.5 (iv) with $B = \{x^{-2}yz, y, z\}$; thus for $\beta \in B$ we have $\varphi(\beta) \in \{(x^{-2}yz)^{\pm 1}, (y)^{\pm 1}, (z)^{\pm 1}\}$.

The following maps determine automorphisms of $G_{73}$:

$$
\psi_4 : (x, y, z, p, r, t) \mapsto (xy^{-1}, z, x^2y^{-1}z^{-1}, x^2z^{-1}pr, p, t), \text{ and }
\psi_5 : (x, y, z, p, r, t) \mapsto (x, z, y, x^{-2}yzr, p, x^{-1}yt).
$$

These two automorphisms permute the elements of $B$ and so composing $\varphi$ with $\psi_4$ and $\psi_5$ we can arrange to have $\varphi(\beta) \in \{\beta^{\pm 1} : \beta \in B\}$. Then by Proposition 6.9 (iv) we can compose $\varphi$ with inner automorphisms to have $\varphi|_A = \text{Id}$ and by (ii) we have $\varphi(Af) = Af$ for all $f \in F$.

Let $\varphi(t) = ct$ for some $c \in A$. By Proposition 6.19 the maps

$$
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
$$

and

$$
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt)
$$

determine outer automorphisms. Composing $\varphi$ with these maps and with $I_x$ we may assume $c \in \{1, x\}$. By Corollary 6.16 $c = x$ is not possible so we have $\varphi(t) = t$.

Now by Proposition 6.17 we have

$$
\varphi(p) \in \{p, x^2y^{-1}z^{-1}p\}; \quad \varphi(r) \in \{r, zr\}; \quad \varphi(pr) \in \{pr, ypr\};
$$

$$
\varphi(prt) \in \{prt, x^2y^{-1}z^{-1}prt\}; \quad \varphi(rt) \in \{rt, y^{-1}rt\}; \quad \varphi(pt) \in \{pt, z^{-1}pt\}.
$$

By Proposition 6.18 (i), composing with $I_t \circ \iota$ if necessary we may assume that we have $\varphi(p) = p$. We will now use Proposition 6.18 (iii) to show that $\varphi(p) = p$ implies $\varphi(prt) = prt$.

Note that

$$(rt)^G = \langle y \rangle rt \cup \langle y \rangle zrt.
$$

This gives two pertinent facts. The first is that $rt \sim y^{-1}rt$ therefore (since $\varphi(rt) \in \{rt, y^{-1}rt\}$), we have $\varphi(rt) \sim rt$. Secondly, we have $rt \sim x^{-2}yzrt$. We are now ready to apply the proposition. We have

$$
\varphi(prt \cdot p^{-1}) = \varphi((rt)^{p^{-1}}) \sim \varphi(rt) \sim x^{-2}yzrt = (x^2y^{-1}z^{-1})^{\varphi(rt)} \sim x^2y^{-1}z^{-1}prt \cdot p^{-1}.
$$
The above, together with \( \varphi(p^{-1}) = p^{-1} \), implies that \( \varphi(prt) = prt \), by Proposition 6.18 (iii).

Next we show that \( \varphi(r) = r \) if and only if \( \varphi(pt) = pt \). Suppose to the contrary that \( \varphi(r) = r \) and \( \varphi(pt) = z^{-1}pt \). Then (using \( rpt = x^2z^{-2}prt \))

\[
x^2z^{-2} \cdot prt \sim \varphi(x^2z^{-2} \cdot prt) = \varphi(r \cdot pt) \sim r \cdot z^{-1}pt = zrpt = x^2z^{-1}prt.
\]

Since \( x^2z^{-1}prt \in (prt)^G = \langle x^2y^{-1}z^{-1} \rangle \)prt \( \cup \langle x^2y^{-1}z^{-1} \rangle yrpt \) by Proposition C.19, this gives a contradiction. Now to prove the converse suppose we have \( \varphi(r) = zr \) and \( \varphi(pt) = pt \). Then

\[
x^2z^{-2} \cdot prt \sim \varphi(x^2z^{-2} \cdot prt) = \varphi(r \cdot pt) \sim zr \cdot pt = x^2z^{-1}prt.
\]

We arrive at the same contradiction which proves the biconditional.

We will use this result to show that \( \varphi\lvert_{Aprt} = Id \). To begin, we recall that by Lemma 6.31 and Corollary 6.29 that \( \varphi(aprt) = ah \) for some \( h \in \{1, prt\} \). Suppose that \( h = prt \). Then (using \( prtr = z^{-1}pt \)) we have

\[
xz^{-1} \cdot \varphi(pt) \sim \varphi(xz^{-1} \cdot pt) = \varphi(xprt \cdot r) \sim x^{prt}prt \cdot \varphi(r) = x^{-1}yzprt \cdot \varphi(r).
\]

We have two cases to consider. If \( \varphi(pt) = pt \) thus \( \varphi(r) = r \), the above is

\[
xz^{-1}pt \sim x^{-1}yzprt \cdot r = x^{-1}ypt.
\]

Alternatively, if \( \varphi(pt) = z^{-1}pt \) thus \( \varphi(r) = zr \), this becomes

\[
xz^{-2}pt \sim x^{-1}yzprt \cdot zr = x^{-1}yzpt.
\]

As \( xz^{-1}pt, xz^{-2}pt \in (xpt)^G = \langle z \rangle xpt \cup \langle z \rangle xy^{-1}pt \) we see that in both cases we have a contradiction. Thus \( h \neq prt \) and so \( \varphi\lvert_{Aprt} = Id \).

We will use this result to show that \( \varphi(rt) = rt \). Suppose to the contrary that \( \varphi(rt) = y^{-1}rt \). Then since \( \varphi \) respects inverses, then

\[
xp \sim \varphi(x \cdot p) = \varphi(xprt \cdot (rt)^{-1}) \sim xprt(rt)^{-1}y = xy^{-1}p.
\]

However \( (xp)^G = \langle x^2z^{-1}, y^2 \rangle xp \cup \langle x^2z^{-1}, y^2 \rangle xyz^{-2}p \) thus \( xp \sim xy^{-1}p \), a contradiction. We conclude that \( \varphi(rt) = rt \).
Now suppose that $\varphi(pr) = ypr$. This gives

$$y^{-1}\varphi(pt) \sim \varphi(y^{-1} \cdot pt) = \varphi(pr \cdot rt) \sim ypr \cdot rt = pt.$$ 

The left hand side is either $y^{-1}pt$ or $y^{-1}z^{-1}pt$, neither of which is contained in $(pt)^G = \langle z \rangle pt \cup \langle z \rangle x^2y^{-1}pt$. This shows that we must have $\varphi(pr) = pr$.

Next we apply Theorem 6.33 to the $Art$ coset. (Recall that we have $\varphi(rt) = rt, \varphi(p) = p$ and $\varphi(prt) = prt$.) Since $R_{rt} \cap S_{prt} = \{1\}$ by Corollary 6.29 and Proposition 8.34, the theorem implies that $\varphi|_{Art} = \text{Id}$.

We use this result to show that $\varphi(pt) = pt$. Suppose to the contrary that $\varphi(pt) = z^{-1}pt$. Then (using $ptrt = yzpr$) we have

$$x^{-1}pr \sim \varphi(x^{-1}pr) = \varphi(x^{-1}y^{-1}z^{-1}ptrt) = \varphi(pt \cdot x^{-1}y^{-1}z^{-1}rt) \sim z^{-1}pt \cdot x^{-1}y^{-1}z^{-1}rt = x^{-1}z^{-1}pr.$$ 

This is a contradiction because according to Proposition C.19, $(x^{-1}pr)^G = \langle yz, yz^{-1} \rangle x^{-1}pr \cup \langle yz, yz^{-1} \rangle x^3zpr$, so now we have $\varphi(pt) = pt$. We have already shown that this implies $\varphi(r) = r$, so now we have $\varphi(f) = f$ for all $f \in F$.

Notice that $pt \notin S_p$, thus by Corollary 6.34 $\varphi|_{Art} = \text{Id}$. Then by Corollary 6.38 $\varphi = \text{Id}$.

We have shown that

**Theorem 8.35.** For crystallographic group $G = G_{73}$, the group $W(G)$ is generated by the inverse map $\iota$, the inner automorphisms, and

$$\psi_4 : (x, y, z, p, r, t) \mapsto (xy^{-1}, z, x^2y^{-1}z^{-1}, x^2z^{-1}pr, p, t),$$

$$\psi_5 : (x, y, z, p, r, t) \mapsto (x, z, y, x^{-2}yzr, p, x^{-1}yt),$$

$$\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),$$

$$\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).$$

Therefore $W(G) = W_0(G)$. □
Proposition 8.36. For group $74$ we have

$$S_{pr} = S_t = \{1, t, pr, prt\}; \quad S_p = \{1, p, prt, rt\};$$
$$S_r = S_{prt} = S_{pt} = \{1, r, prt, pt\}; \quad S_{rt} = F.$$ 

Proof. This follows from Lemma 6.32 and Proposition C.20.

Let $\varphi \in W(G_{74})$. We have $C_2 = \langle x^{-2}yz \rangle \cup \langle y \rangle \cup \langle z \rangle$ by Proposition 6.3. We apply Proposition 6.5 (iv) with $B = \{x^{-2}yz, y, z\}$, which tells us that for $\beta \in B$ we have $\varphi(\beta) \in \{(x^{-2}yz)^{\pm 1}, (y)^{\pm 1}, z^{\pm 1}\}$.

It is clear that the $Art$ coset contains elements of order 2 (since $rt$ has order 2). By Lemma 6.12 we find that the $Aprt$ coset also contains elements of order 2 but $Apt$ does not. This fact together with Corollary 6.11 gives $\varphi(Apt) = Apt$ and $\varphi(At) = At$. Since $G/A$ is abelian $\overline{\varphi}$ is a homomorphism and so $\varphi(Ap) = Ap$. By Lemma 6.4 $\varphi(x^{-2}yz)$ and $\varphi(y)$ are inverted by the action of $p$. It follows that $\{\varphi(x^{-2}yz), \varphi(y)\} \subseteq \{(x^{-1}yz)^{\pm 1}, y^{\pm 1}\}$ and thus $\varphi(z) \in \{z^{\pm 1}\}$. If we have $\varphi(x^{-2}yz) \in \{y^{\pm 1}\}$ then we may compose $\varphi$ with the automorphism

$$\psi_6 : (x, y, z, p, r, t) \mapsto (x^{-1}z, x^{-2}yz, z, p, x^{-1}ypr, x^{-1}t),$$

and now we have $\varphi(x^{-2}yz) \in \{(x^{-2}yz)^{\pm 1}\}$ and $\varphi(y) \in \{y^{\pm 1}\}$. Then by Proposition 6.9 (ii) and (iv), composing with inner automorphisms if necessary we have $\varphi\vert_A = Id$ and $\varphi(Af) = Af$ for $f \in F$.

Let $\varphi(t) = ct$ for some $c \in A$. By Proposition 6.19 the maps

$$\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),$$

and $\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt)$

determine outer automorphisms. Composing with these functions and with $I_x$ we may assume $c \in \{1, x\}$. By Corollary 6.16 $c = x$ is not possible so we have $\varphi(t) = t$.

It follows by Proposition 6.17 that since $p, pr$ and $rt$ are involutions, $\varphi(pt) = pt, \varphi(prt) = prt, \varphi(r) = r$. The proposition also gives

$$\varphi(p) \in \{p, yp\}; \varphi(pr) \in \{pr, y^2pr\}; \varphi(rt) \in \{rt, y^{-1}rt\}.$$
By Proposition 6.18 (i), we may compose $\varphi$ with $I_t \circ \iota$ if necessary so as to have $\varphi(rt) = rt$. Then by (ii) we now have $\varphi(f) = f$ for all $f \in F - \{p, pr\}$. We will apply Proposition 6.18 (iii) twice in order to show that this implies that $\varphi(p) = p$ and $\varphi(pr) = pr$ as well. To justify this we will use the following facts from Proposition C.20:

$$yprt \notin (prt)^G = \langle x^2 y^{-1} z^{-1} \rangle_{prt} \cup \langle x^2 y^{-1} z^{-1} y^2 \rangle_{prt};$$

$$yr \notin r^G = \langle x^2 y^{-1}, z \rangle_r \cup \langle x^2 y^{-1}, z^2 \rangle y^{-1} r.$$

We have

$$\varphi(p \cdot rt) = yprt \sim (yp)(rt) \text{ and } \varphi(rt) = rt,$$

thus by the proposition we conclude $\varphi(p) \neq yp$ i.e. $\varphi(p) = p$.

Similarly, since we have $\varphi(p) = p$ and

$$\varphi(pr \cdot p) = \varphi(r^p) \sim \varphi(r) = r \sim r^p \sim yr = (y^2 pr)(p),$$

the proposition gives $\varphi(pr) = pr$. We now have $\varphi(f) = f$ for all $f \in F$.

Since $pt \notin S_p$ and $rt \notin S_r$, Corollary 6.34 gives $\varphi|_{Apt \cup Art} = Id$. We now apply Theorem 6.35 (i) using $\varphi(p) = p$ and $\varphi|_{Art} = Id$. Since Corollary 6.29 gives $R_p \cap R_{prt} = \{1,prt\}$ we have $\varphi(ap) = a^hp$ for some $h \in \{1,prt\}$. However if $h = prt$ then

$$zp \sim \varphi(z \cdot p) = \varphi(x^{-1}z \cdot xp) = x^{-1}z \cdot x^{prt}p = x^{-1}zx^{-1}yzp = x^{-2}yz^2p,$$

which is a contradiction since Proposition C.20 indicates that $x^{-2}yz^2p \sim yzp \notin (zp)^G = \langle x^2 z^{-1}, y^2 \rangle_{zp} \cup \langle x^2 z^{-1}, y^2 \rangle yz^{-1}$. We conclude that $h = 1$ i.e. $\varphi|_{Ap} = Id$.

Again we apply Theorem 6.35, this time using $\varphi(prt) = prt$ and $\varphi|_{Ap} = Id$. We also use $R_{prt} \cap R_{rt} = 1$ as given by Corollary 6.29. It follows by the theorem that we have $\varphi|_{Aprt} = Id$.

Then by Corollary 6.38 $\varphi = Id$.

We have shown that

**Theorem 8.37.** For crystallographic group $G = G_{74}$, the group $W(G)$ is generated by the
inverse map \( \iota \), the inner automorphisms, and

\[
\psi_6 : (x, y, z, p, r, t) \mapsto (x^{-1}z, x^{-2}yz, z, p, x^{-1}ypr, x^{-1}t),
\]
\[
\psi_y : (x, y, z, p, r, t) \mapsto (x, y, z, yp, r, yt),
\]
\[
\psi_z : (x, y, z, p, r, t) \mapsto (x, y, z, p, zr, zt).
\]

Therefore \( W(G) = W_0(G) \).

Bibliography


Appendix A. $K_f$ subgroups

Proposition A.1. Recall that for each $f \in F$ we define the subgroup $K_f$ to be

$$K_f = \langle (x, f), (y, f), (z, f) \rangle.$$

Below we have the $K_f$ subgroups for groups that have a presentation of the form given in Eqs. (4.1), (4.2), (4.3), (4.4), or (4.5).

Table A.1: $K_f$ subgroups for thirty-one groups

<table>
<thead>
<tr>
<th>Group number(s)</th>
<th>$K_p$</th>
<th>$K_r$</th>
<th>$K_{pr}$</th>
<th>$K_{ps}$ or $K_{prt}$</th>
<th>$K_s$ or $K_{rt}$</th>
<th>$K_{pt}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>10, 13</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>12</td>
<td></td>
<td>$\langle x^2, z^2 \rangle$</td>
<td>$\langle x^2y^{-1}, z^2 \rangle$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>16, 17</td>
<td>$\langle x^2, y^2 \rangle$</td>
<td>$\langle x^2, z^2 \rangle$</td>
<td>$\langle y^2, z^2 \rangle$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>$\langle x^2, y^2 \rangle$</td>
<td>$\langle x^2y^{-1}, z^2 \rangle$</td>
<td>$\langle y, z^2 \rangle$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>$\langle x^2z^{-1}, y^2z^{-1} \rangle$</td>
<td>$\langle x^2, z \rangle$</td>
<td>$\langle y^2, z \rangle$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>25, 26, 27</td>
<td>$\langle x^2, y^2 \rangle$</td>
<td>$\langle x^2, z^2 \rangle$</td>
<td>$\langle x^2, z \rangle$</td>
<td>$\langle y^2 \rangle$</td>
<td>$\langle y^2z^{-1} \rangle$</td>
<td></td>
</tr>
<tr>
<td>38, 39</td>
<td>$\langle x^2, y^2z^{-1} \rangle$</td>
<td>$\langle x^2, z \rangle$</td>
<td>$\langle y^2, z \rangle$</td>
<td>$\langle y^2 \rangle$</td>
<td>$\langle y^2z^{-1} \rangle$</td>
<td></td>
</tr>
<tr>
<td>42</td>
<td>$\langle x^2z^{-1}, y^2z^{-1} \rangle$</td>
<td>$\langle x^2, z \rangle$</td>
<td>$\langle y^2, z \rangle$</td>
<td>$\langle x^2z^{-1} \rangle$</td>
<td>$\langle y^2z^{-1} \rangle$</td>
<td>$\langle z \rangle$</td>
</tr>
<tr>
<td>47-57</td>
<td>$\langle x^2, y^2 \rangle$</td>
<td>$\langle x^2, z^2 \rangle$</td>
<td>$\langle y^2, z^2 \rangle$</td>
<td>$\langle x^2z^{-1} \rangle$</td>
<td>$\langle y^2z^{-1} \rangle$</td>
<td>$\langle z \rangle$</td>
</tr>
<tr>
<td>63-68</td>
<td>$\langle x^2, y^2 \rangle$</td>
<td>$\langle x^2y^{-1}, z^2 \rangle$</td>
<td>$\langle y, z^2 \rangle$</td>
<td>$\langle x^2y^{-1} \rangle$</td>
<td>$\langle y \rangle$</td>
<td>$\langle z^2 \rangle$</td>
</tr>
<tr>
<td>69</td>
<td>$\langle x^2z^{-1}, y^2z^{-1} \rangle$</td>
<td>$\langle x^2, z \rangle$</td>
<td>$\langle y^2, z \rangle$</td>
<td>$\langle x^2z^{-1} \rangle$</td>
<td>$\langle y^2z^{-1} \rangle$</td>
<td>$\langle z \rangle$</td>
</tr>
<tr>
<td>72, 73, 74</td>
<td>$\langle x^2z^{-1}, y^2 \rangle$</td>
<td>$\langle x^2y^{-1}, z^2 \rangle$</td>
<td>$\langle yz, yz^{-1} \rangle$</td>
<td>$\langle x^2y^{-1}z^{-1} \rangle$</td>
<td>$\langle y \rangle$</td>
<td>$\langle z \rangle$</td>
</tr>
</tbody>
</table>

Additionally, for groups having $t \in F$ we have $K_t = \langle x^2, y^2, z^2 \rangle$. (For brevity we write “47-57” to represent groups $G_{47}, G_{49}, G_{50}, G_{51}, G_{53}, G_{54}, G_{55}$, and $G_{57}$ and we write “63-68” to represent $G_{63}, G_{64}, G_{65}, G_{66}, G_{67}$, and $G_{68}$.)

Note that $G_{10}, G_{12}$ and $G_{13}$ do not contain an element $p$ (nor $pr, ps, prt, pt$) therefore we have nothing to write in the cell of the table that corresponds to those $K_f$ subgroups. The next two rows in the table have blank cells in three places for a similar reason; $G_{16}, G_{17}, G_{21}$ and $G_{22}$ do not contain an element $s$ nor $t$, (nor $ps, prt, rt, pt$); $G_{25}, G_{26}, G_{27}, G_{38}, G_{39}$, and $G_{42}$ do not contain an element $r, pr$ nor $pt$.

Proof. This follows from the relations found in the presentation of the respective groups. □
Appendix B. Proof of Lemma 7.1 (Magma)

Below is the code (with its output) that we used to prove that $H = \langle a^2, b, c \rangle$.

\[
\begin{align*}
F\langle a, b, c \rangle & := \text{FreeGroup}(3); \\
G & := \text{quo} < F|\{c^2, a * b * a * (b * a * b)^{-1}, (a * b * a)^4, (a * c)^2, (b * c)^2, \\
& (c * a)^2, (c * b)^2\}>; \\
\text{Index}(G, \text{sub} < G|a, b >); \\
& 2 \\
\text{Index}(G, \text{sub} < G|a^2, b, c >); \\
& 3
\end{align*}
\]
Appendix C. Commutators and conjugacy
classes for eighteen space groups

Here we list the conjugacy classes for the eighteen groups listed in Table 4.2. We do not include the conjugacy classes for the elements in $A$, as those conjugacy classes are not needed as we determine $\mathcal{W}(G)$. Before we list these conjugacy classes, we give some results that were used to determine them.

The following lemma is applicable to the eighteen groups listed in Table 4.2. This lemma will be useful when determining certain commutators which are used when determining the conjugacy classes.

**Lemma C.1.** Let $G$ be a group with an abelian normal subgroup $A$ such that $G/A$ is abelian. Let $p, r, t \in G$ and suppose that $t^2 = 1$ and that $a^t = a^{-1}$ for $a \in A$.

Let $U = (p, r)$, $V = (p, t)$, and $W = (r, t)$. Then we have

\[
\begin{align*}
(p, pr) &= U; & (p, p) &= U^{-1}V; \\
(p, pt) &= (t, pt) = V; & (rt, pt) &= U^{-1}VW^{-1}; \\
(r, rt) &= (t, rt) = W; & (rp, rt) &= (U^{-1}V)^r; \\
(r, pt) &= UW; & (pr, rt) &= (U^{-1}V)^rW; \\
(pr, r) &= U^r; & (pr, t) &= (pr, p) = (t, p) = V^rW; \\
(r, prt) &= U^rW; & (pr, pt) &= UV^rW; \\
(pr, prt) &= U^rW; & (pt, pr) &= UV^rW \cdot V^{-1}.
\end{align*}
\]

**Proof.** Note that $G/A$ abelian implies that every commutator is in $A$. We apply Eqs. (2.1), (2.3) and (2.5) and these results follow.

**Proposition C.2.** Suppose $G/A$ is abelian. Let $a \in A$ and $f \in F$. Then

\[
(a^c f)^G = \bigcup_{\lambda \in (G/A)/\mathcal{F}} K_f a\lambda(f, \lambda)f,
\]

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where $\mathcal{F} \in G/A$ is the image of $f$ in the quotient so we are taking the union over $\lambda \in G$ such that $\bar{\lambda}$ is a coset representative of the cosets in $G/A$ after taking a quotient over $\langle \mathcal{F} \rangle$.

**Proof.** Let $b \in A$. Using 2.3 we have

$$(af)^{-1}b = (a, f) = (f, b) \in K_f,$$

thus $K_f(af) = K_f(a f)^b$. From this we see that

$$(af)^G = \bigcup_{\lambda \in F} K_f(af)^\lambda = \bigcup_{\lambda \in F} K_f a^\lambda f^{-1} f = \bigcup_{\lambda \in F} K_f a^\lambda (\lambda, f^{-1}) f.$$

Next we show that $K_f(\lambda, f^{-1}) = K_f(f, \lambda)$. Let $a = (f^{-1}, \lambda)$. Clearly $(a, f) \in K_f$. Then

$$(a, f) = ((f^{-1}, \lambda), f)$$

$$= (\lambda, f^{-1}) f^{-1} (f^{-1}, \lambda) f$$

$$= \lambda^{-1} f \lambda f^{-1} (f^{-1} f) \lambda^{-1} f^{-1} \lambda f$$

$$= (\lambda, f^{-1})(\lambda, f)$$

$$= (\lambda, f^{-1})(f, \lambda) \in K_f,$$

so that $K_f(\lambda, f^{-1}) = K_f(f, \lambda)$.

Lastly we show that it suffices to take the union over $\lambda \in G$ where $\bar{\lambda}$ is a coset representative of the quotient $(G/A)/\langle \mathcal{F} \rangle$. In other words, we show that if $\lambda_1 f = \lambda_2$ then $K_f a^{\lambda_1}(\lambda_1, f^{-1}) = K_f a^{\lambda_2}(\lambda_2, f^{-1})$. Using the Witt-Hall Identities we have

$$a^{\lambda_1}(\lambda_1, f^{-1}) \cdot (a^{\lambda_2}(\lambda_2, f^{-1}))^{-1} = a^{\lambda_1}(\lambda_1, f^{-1}) \cdot (a^{\lambda_1 f}(\lambda_1 f, f^{-1}))^{-1}$$

$$= a^{\lambda_1}(\lambda_1, f^{-1}) \cdot (f^{-1}, \lambda_1 f)(a^{\lambda_1 f})^{-1}$$

$$= a^{\lambda_1}(a^{-1}) f \lambda_1 (\lambda_1, f^{-1}) \cdot (f^{-1}, \lambda_1 f)$$

$$= (a(a^{-1}) f)^\lambda_1 (\lambda_1, f^{-1}) \cdot (f^{-1}, f)(f^{-1}, \lambda_1)((f^{-1}, \lambda_1), f)$$

$$= (a^{-1}, f)^\lambda_1 ((f^{-1}, \lambda_1), f).$$

Since $G/A$ is abelian, $K_f$ is normal and $(f^{-1}, \lambda) \in A$. Thus $(a(a^{-1}) f)^\lambda_1 ((f^{-1}, \lambda), f)$ is the product of two elements of $K_f$. This shows that the $K_f$ cosets corresponding to $\bar{\lambda_1}(\mathcal{F})$ and $\bar{\lambda_2}(\mathcal{F})$ are the same. $\square$
Proposition C.3. The conjugacy classes in $G_{47}$ are as follows:

For $i, j, k \in \mathbb{Z}$,

\[(x^i y^j z^k p)^G = \langle x^2, y^2 \rangle x^i y^j z^{\pm k} p;\]
\[(x^i y^j z^k r)^G = \langle x^2, z^2 \rangle x^i y^{\pm j} z^k r;\]
\[(x^i y^j z^k pr)^G = \langle y^2, z^2 \rangle x^{\pm i} y^j z^k pr;\]
\[(x^i y^j z^k prt)^G = \langle x^2, y^2 \rangle x^i y^{\pm j} z^{\pm k} prt\]
\[(x^i y^j z^k rt)^G = \langle y^2 \rangle x^i y^{\pm j} z^{\pm k} rt;\]
\[(x^i y^j z^k pt)^G = \langle z^2 \rangle x^{\pm i} y^j z^k pt;\]
\[(x^i y^j z^k t)^G = \langle x^2, y^2, z^2 \rangle x^i y^j z^k t.\]

We also have $G' = K = \langle x^2, y^2, z^2 \rangle$.

Proof. This follows from Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{47}$. The presentation of $G_{47}$ is of the form given in Eq. (4.4), with $\alpha_p = \alpha_r = \alpha_{pr} = \alpha_{pt} = \alpha_{rt} = 1$ and $\delta = 0$.

Proposition C.4. For $G_{49}$ we have the following commutators:

\[(p, r) = (p, t) = (p, rt) = 1;\]
\[(t, r) = (pt, r) = (t, pr) = (pt, pr) = (prt, r) = (rt, pt) = z.\]

Thus $G' = \langle x^2, y^2, z \rangle$. 

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The conjugacy classes in $G_{49}$ are as follows:

For $i, j, k \in \mathbb{Z}$,

$$(x^i y^j z^k p)^G = \langle x^2, y^2 \rangle x^i y^j z^{\pm k} p;$$

$$(x^i y^j z^k r)^G = \langle x^2, z \rangle x^i y^{\pm j} r;$$

$$(x^i y^j z^k pr)^G = \langle y^2, z \rangle x^{\pm i} y^j pr;$$

$$(x^i y^j z^k prt)^G = \langle x^2 \rangle x^i y^{\pm j} z^k \pr \cup \langle x^2 \rangle x^i y^{\pm j} z^{1-k} \pr t;$$

$$(x^i y^j z^k rt)^G = \langle y^2 \rangle x^{\pm i} y^j z^k rt \cup \langle y^2 \rangle x^{\pm i} y^j z^{1-k} rt;$$

$$(x^i y^j z^k pt)^G = \langle z^2 \rangle (x^i y^j)^{\pm 1} z^k \pt \cup \langle z^2 \rangle (x^i y^{-j})^{\pm 1} z^{k+1} \pt;$$

$$(x^i y^j z^k t)^G = \langle x^2, y^2, z \rangle x^i y^j t.$$

**Proof.** This follows from Lemma C.1, Proposition A.1, Proposition C.2, and the relations in the presentation of $G_{49}$. \qed

**Proposition C.5.** For $G_{50}$ we have the following commutators:

$$(p, r) = 1;$$

$$(r, t) = (prt, r) = (pt, r) = x;$$

$$(t, pr) = (pt, pr) = (pt, rt) = y;$$

$$(t, p) = (rt, p) = xy;$$

$$(rt, pr) = xy^{-1}.$$

Thus $G' = \langle x, y, z^2 \rangle$. 

The conjugacy classes in $G_{50}$ are: For $i, j, k \in \mathbb{Z}$,

$$(x^i y^j z^k p)^G = (x^i y^{x^i y^j} z^{x^i y^j}) x^i y^j z^{x^i y^j} p;$$

$$(x^i y^j z^k r)^G = (x, z^2) y^{x^i y^j} z^{x^i y^j} r;$$

$$(x^i y^j z^k pr)^G = (y, z^2) x^{x^i y^j} z^{x^i y^j} pr;$$

$$(x^i y^j z^k prt)^G = (x^2) x^{x^i y^j} z^{x^i y^j} prt \cup (x^2) x^{x^i y^j} z^{x^i y^j} pr t \cup (x^2) x^{x^i y^j} z^{x^i y^j} pr t = (x^2) x^{x^i y^j} z^{x^i y^j} pr t;$$

$$(x^i y^j z^k r) = (y, z^2) x^{x^i y^j} z^{x^i y^j} r;$$

$$(x^i y^j z^k pr) = (y^2, z^2) x^{x^i y^j} z^{x^i y^j} pr;$$

$$(x^i y^j z^k prt) = (x^2) x^{x^i y^j} z^{x^i y^j} prt \cup (x^2) x^{x^i y^j} z^{x^i y^j} pr t;$$

$$(x^i y^j z^k r t) = (y^2, z^2) x^{x^i y^j} z^{x^i y^j} r t;$$

$$(x^i y^j z^k p t) = (z^2) x^{x^i y^j} z^{x^i y^j} p t;$$

$$\langle x^i y^j z^k t \rangle = (x, y^2, z^2) x^{x^i y^j} z^{x^i y^j} t.$$

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{50}$. 

Proposition C.6. For $G_{51}$ we have the following commutators:

$$(r, t) = (p, rt) = (pr, pt) = (rt, pt) = (pr, rt) = 1;$$

$$(r, p) = (t, p) = (p, rt) = (pt, r) = (pr, t) = x.$$ 

Thus $G' = \langle x, y^2, z^2 \rangle$.

The conjugacy classes in $G_{51}$ are: For $i, j, k \in \mathbb{Z}$,

$$(x^i y^j z^k p)^G = (x^2, y^2) x^{x^i y^j} z^{x^i y^j} p;$$

$$(x^i y^j z^k r)^G = (x, z^2) y^{x^i y^j} z^{x^i y^j} r;$$

$$(x^i y^j z^k pr)^G = (y^2, z^2) x^{x^i y^j} z^{x^i y^j} pr;$$

$$(x^i y^j z^k prt)^G = (x^2) x^{x^i y^j} z^{x^i y^j} prt \cup (x^2) x^{x^i y^j} z^{x^i y^j} pr t;$$

$$(x^i y^j z^k r t)^G = (y^2, z^2) x^{x^i y^j} z^{x^i y^j} r t;$$

$$(x^i y^j z^k p t)^G = (z^2) x^{x^i y^j} z^{x^i y^j} p t;$$

$$\langle x^i y^j z^k t \rangle = (x, y^2, z^2) x^{x^i y^j} z^{x^i y^j} t.$$ 

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{51}$. 


Proposition C.7. For $G_{53}$ we have the following commutators:

\[(t, pr) = 1;\]
\[(rt, p) = (pr, t) = x;\]
\[(r, p) = (pt, pr) = (rt, pt) = z;\]
\[(t, p) = (t, r) = xz;\]
\[(pt, r) = xz^2.\]

Thus $G' = \langle x, y^2, z \rangle$.

The conjugacy classes in $G_{53}$ are: For $i, j, k \in \mathbb{Z}$,

\[(x^i y^j z^k p)^G = \langle x, y^2 \rangle y^j z^u p, u \in \{k, -k - 1\};\]
\[(x^i y^j z^k r)^G = \langle x, z^2 \rangle y^j z^k r \cup \langle x, z^2 \rangle y^{-j} z^{k+1} r;\]
\[(x^i y^j z^k pr)^G = \langle y^2, z \rangle x^{\pm i} y^j pr;\]
\[(x^i y^j z^k prt)^G = \langle x^2 \rangle x^{i} (y^j z^k)^{\pm 1}prt \cup \langle x^2 \rangle x^{i+1} (y^j z^k)^{\pm 1}prt;\]
\[(x^i y^j z^k r)^G = \langle y^2 \rangle x^u y^j z^v r, u \in \{i, 1 - i\}, v \in \{k, 1 - k\};\]
\[(x^i y^j z^k pt)^G = \langle z^2 \rangle x^u y^j z^k pt \cup \langle z^2 \rangle x^u y^{-j} z^{k+1} pt, u \in \{i, 1 - i\};\]
\[(x^i y^j z^k t)^G = \langle x^2, y^2, z^2 \rangle x^i y^j z^k t \cup \langle x^2, y^2, z^2 \rangle x^{i+1} y^j z^{k+1} t\]
\[= \langle xz, x^{-1} z, y^2 \rangle x^i y^j z^k t.\]

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{53}$. \qed

Proposition C.8. For $G_{54}$ we have the following commutators:

\[(p, rt) = 1;\]
\[(r, p) = (t, p) = x;\]
\[(t, r) = (rt, pt) = (pt, pr) = z;\]
\[(pt, r) = xz;\]
\[(pr, t) = (r,prt) = xz^{-1}.\]
Thus we have \( G' = \langle x, y^2, z \rangle \).

The conjugacy classes in \( G_{54} \) are: For \( i, j, k \in \mathbb{Z} \),
\[
(x^i y^j z^k p)^G = \langle x^2, y^2 \rangle x^i y^j z^k p \cup \langle x^2, y^2 \rangle x^{i+1} y^j z^{-k} p
\]
\[
(x^i y^j z^k r)^G = \langle xz, xz^{-1} \rangle x^i y^j z^k r \cup \langle xz, xz^{-1} \rangle x^{i+1} y^j z^{k+1} r
\]
\[
(x^i y^j z^k pr)^G = \langle y^2, z \rangle x^i y^j z^k pr, \quad u \in \{i, 1 - i\}
\]
\[
(x^i y^j z^k prt)^G = \langle x^2 \rangle x^i y^j z^k prt \cup \langle x^2 \rangle x^{i+1} y^j z^{1-k} prt
\]
\[
(x^i y^j z^k rt)^G = \langle y^2 \rangle x^i y^j z^k rt, \quad u \in \{k, 1 - k\}
\]
\[
(x^i y^j z^k pt)^G = \langle z^2 \rangle x^i y^j z^k pt, \quad (z^2) x^{1-i} y^j z^{k+1} pt \cup \langle z^2 \rangle x^{1-i} y^j z^k pt
\]
\[
(x^i y^j z^k t)^G = \langle x, y^2, z \rangle y^i t.
\]

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of \( G_{54} \). \( \square \)

Proposition C.9. For \( G_{55} \) we have the following commutators:
\[
(p, t) = (r, pt) = (pr, pt) = (rt, pt) = 1;
\]
\[
(prt, r) = x^2;
\]
\[
(p, r) = (t, r) = (rt, p) = (t, pr) = xy.
\]

It follows that \( G' = \langle xy, xy^{-1}, z^2 \rangle \).

The conjugacy classes in \( G_{55} \) are: For \( i, j, k \in \mathbb{Z} \),
\[
(x^i y^j z^k p)^G = \langle xy, xy^{-1} \rangle x^i y^j z^k p
\]
\[
(x^i y^j z^k r)^G = \langle x^2, z^2 \rangle x^i y^j z^k r \cup \langle x^2, z^2 \rangle x^{i+1} y^{-1-j} z^k r
\]
\[
(x^i y^j z^k pr)^G = \langle y^2, z^2 \rangle x^i y^j z^k pr \cup \langle y^2, z^2 \rangle x^{1-i} y^{j+1} z^k pr
\]
\[
(x^i y^j z^k prt)^G = \langle x^2 \rangle x^i y^j z^k prt \cup \langle x^2 \rangle x^{i+1} y^{-1-j} z^k z^k prt
\]
\[
(x^i y^j z^k rt)^G = \langle y^2 \rangle x^i y^j z^k rt \cup \langle y^2 \rangle x^{1-i} y^{j+1} z^k rt
\]
\[
(x^i y^j z^k pt)^G = \langle z^2 \rangle x^i y^j z^k pt
\]
\[
(x^i y^j z^k t)^G = \langle xy, xy^{-1}, z^2 \rangle x^i y^j z^k t.
\]

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Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{55}$.

Proposition C.10. For $G_{57}$ we have the following commutators:

\[
(p_{rt}, r) = 1; \\
(r_{t}, p) = (t, p_{r}) = y; \\
(t, p) = (p_{t}, p_{r}) = (r_{t}, p_{t}) = z; \\
(p, r) = yz^{-1}; \\
(p_{t}, r) = z^2; \\
(t, r) = yz.
\]

It follows that $G' = \langle x^2, y, z \rangle$.

The conjugacy classes in $G_{57}$ are: For $i, j, k \in \mathbb{Z}$,

\[
(x^i y^j z^k p)^G = \langle x^2, y \rangle x^i z^k p \cup \langle x^2, y \rangle x^i z^{-k-1}p; \\
(x^i y^j z^k r)^G = \langle x^2, z^2 \rangle x^i y^j z^k r \cup \langle x^2, z^2 \rangle x^i y^{-j-1}z^{k+1}r; \\
(x^i y^j z^k p_{rt})^G = \langle y^2, z \rangle x^i y^j p_{rt} \cup \langle y^2, z \rangle x^{-i}y^{j+1}pr; \\
(x^i y^j z^k p_{rt})^G = \langle x^2 \rangle x^i y^j z^k p_{rt}, u \in \{ j, 1 - j \}; \\
(x^i y^j z^k r_{t})^G = \langle y^2 \rangle x^i y^j z^k r_{t} \cup \langle y^2 \rangle x^{-i}y^{j+1}z^{u}r_{t}, u \in \{ k, 1 - k \}; \\
(x^i y^j z^k p_{t})^G = \langle z^2 \rangle x^i y^j z^k p_{t}, \cup \langle z^2 \rangle x^i y^{-j}z^{k+1}p_{t}; \\
(x^i y^j z^k t)^G = \langle x^2, y, z \rangle x^i t.
\]

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{57}$.

Proposition C.11. For $G_{63}$ we have the following commutators:

\[
(p, r_{t}) = (p_{r}, t) = (p_{rt}, r) = 1; \\
(r, p) = (t, r) = (t, p) = (p_{t}, p_{r}) = (r_{t}, p_{t}) = z; \\
(p_{t}, r) = z^2.
\]
It follows that \( G' = \langle x^2, y, z \rangle \).

The conjugacy classes in \( G_{63} \) are as follows:

For \( i, j, k \in \mathbb{Z} \),

\[
(x^i y^j z^k p)^G = \langle x^2, y^2 \rangle x^i y^j z^k p, \quad u \in \{ j, i + j \}, \quad v \in \{ k, -k - 1 \};
\]

\[
(x^i y^j z^k r)^G = \langle x^2 y^{-1}, z^2 \rangle x^i y^j z^k r
\]

\[
\cup \langle x^2 y^{-1}, z^2 \rangle x^{-i} y^{-j} z^{k+1} r;
\]

\[
(x^i y^j z^k p)^G = \langle y, z \rangle x^i y^j z^k p;
\]

\[
(x^i y^j z^k p r t)^G = \langle x^2 y^{-1} \rangle (x^i y^j)^{\pm 1} z^{\pm k} p r t;
\]

\[
(x^i y^j z^k r t)^G = \langle y \rangle x^i z^u r t, \quad u \in \{ k, 1 - k \};
\]

\[
(x^i y^j z^k p t)^G = \langle z^2 \rangle x^i y^j z^{\delta(i+2)} z^{k+\delta} p t, \quad \delta \in \{ 0, 1 \},
\]

\[
\cup \langle z^2 \rangle x^i y^j z^{\delta(i+2)} z^{-k+\delta} p t, \quad \delta \in \{ 0, 1 \};
\]

\[
(x^i y^j z^k t)^G = \langle x^2, y^2, z \rangle x^i y^u t, \quad u \in \{ j, i + j \}.
\]

**Proof.** This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of \( G_{63} \). \(\square\)
Proposition C.12. For $G_{64}$ we have the following commutators:

\[(prt, r) = 1;\]
\[(p, r) = (pt, rt) = yz^{-1};\]
\[(t, p) = (t, r) = (pt, pr) = yz;\]
\[(t, pr) = (rt, p) = y^2;\]
\[(pt, r) = z^2.\]

It follows that $G' = \langle x^2, y, z \rangle$.

The conjugacy classes in $G_{64}$ are as follows:

For $i, j, k \in \mathbb{Z}$,

\[(x^i y^j z^k p)^G = \langle x^2, y^2 \rangle x^i y^u z^k p, u \in \{j, i + j\}\]
\[
\cup \langle x^2, y^2 \rangle x^i y^u z^{-k-1} p, u \in \{j + 1, i + j + 1\};
\]
\[(x^i y^j z^k r)^G = \langle x^2 y^{-1}, z^2 \rangle x^i y^j z^k r\]
\[
\cup \langle x^2 y^{-1}, z^2 \rangle x^i y^{-i-j-1} z^{k+1} r;
\]
\[(x^i y^j z^k pr)^G = \langle y, z \rangle x^{\pm i} pr\]
\[(x^i y^j z^k prt)^G = \langle x^2 y^{-1} \rangle x^i y^u z^{\pm k} prt, u \in \{j, 2 - i - j\};\]
\[(x^i y^j z^k rt)^G = \langle y \rangle x^{\pm i} z^u r t, u \in \{k, 1 - k\};\]
\[(x^i y^j z^k pt)^G = \langle z^2 \rangle x^{(1-2\delta_1) i} y^{j+\delta_1 i} z^{k+1} pt, \delta_1 \in \{0, 1\}\]
\[
\cup \langle z^2 \rangle x^{(2\delta_2-1) i} y^{1-j-\delta_2 i} z^{k+1} pt, \delta_2 \in \{0, 1\};
\]
\[(x^i y^j z^k t)^G = \langle x^2, yz, yz^{-1} \rangle y^j z^k t, \text{ if } i \text{ is even},\]
\[
\langle x^2, y, z \rangle xt \text{ if } i \text{ is odd}.
\]

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{64}$. \qed
Proposition C.13. Let $G = G_{65}$. Here $F$ is an abelian subgroup of $G$, thus $G' = K = \langle x^2, y, z^2 \rangle$.

The conjugacy classes in $G_{65}$ are as follows:

For $i, j, k \in \mathbb{Z}$,

\[
(x^iy^jz^k)G = (x^2, y^2)x^iy^u z^{\pm k}p, \quad u \in \{j, i + j\};
\]

\[
(x^iy^jz^k r)G = (x^2y^{-1}, z^2)(x^iy^j)^{\pm 1}z^k r;
\]

\[
(x^iy^jz^k pr)G = (y, z^2)x^{\pm i} z^k pr;
\]

\[
(x^iy^jz^k pr t)G = (x^2y^{-1})(x^iy^j)^{\pm 1}z^k pr t;
\]

\[
(x^iy^jz^k rt)G = (y)x^{\pm i} z^k rt;
\]

\[
(x^iy^jz^k pt)G = (z^2)(x^iy^u)^{\pm 1}z^k pt, \quad u \in \{j, -i - j\};
\]

\[
(x^iy^jz^k t)G = (x^2, y^2, z^2)x^iy^u z^k t, \quad u \in \{j, i + j\}.
\]

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{65}$. □

Proposition C.14. For $G_{66}$ we have the following commutators:

\[
(p, r) = (p, t) = (p, rt) = 1;
\]

\[
(t, r) = (prt, r) = (rt, pt) = (pt, pr) = (t, pr) = (pt, r) = z.
\]

It follows that $G' = \langle x^2, y, z \rangle$. 

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The conjugacy classes in $G_{66}$ are as follows:

For $i,j,k \in \mathbb{Z}$,

$$(x^i y^j z^k p)^G = (x^2, y^2)x^i y^u z^\pm k p, u \in \{ j, i + j \};$$

$$(x^i y^j z^k r)^G = (x^2 y^{-1}, z)(x^i y^j)^\pm 1 r;$$

$$(x^i y^j z^k pr)^G = (y, z)x^\pm i pr;$$

$$(x^i y^j z^k prt)^G = (x^2 y^{-1})(x^i y^j)^\pm 1 z^n prt, u \in \{ k, 1 - k \};$$

$$(x^i y^j z^k rt)^G = (y)x^\pm i z^u rt, u \in \{ k, 1 - k \};$$

$$(x^i y^j z^k pt)^G = (z^2)(x^i y^j)^\pm 1 z^k pt$$

$$\cup (z^2)(x^{-i} y^{i+j})^\pm 1 z^k+1 pt;$$

$$(x^i y^j z^k t)^G = (x^2, y^2, z^2)x^i y^j z^k t$$

$$\cup (x^2, y^2, z^2)x^i y^{i+j} z^k+1 t.$$ 

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{66}$.

Proposition C.15. For $G_{67}$ we have the following commutators:

$$(r, pt) = (r, prt) = 1;$$

$$(p, r) = (pt, pr) = (pt, rt) = (t, p) = (t, r) = y;$$

$$(rt, p) = (t, pr) = y^2.$$ 

It follows that $G' = K = \langle x^2, y, z^2 \rangle$. 

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The conjugacy classes in $G_{67}$ are:

For $i, j, k \in \mathbb{Z}$,

$$(x^iy^jz^kp)^G = \langle x^2, y^2 \rangle x^y z^kp, u \in \{i, i + j\}$$

$\cup \langle x^2, y^2 \rangle x^y z^{-k}p, u \in \{j + 1, i + j + 1\}$;

$$(x^iy^jz^kr)^G = \langle x^2y^{-1}, z^2 \rangle x^y z^kr, u \in \{j, -i - j - 1\}$$

$$(x^iy^jz^kp)^G = \langle y, z^2 \rangle x^y z^{kpr}$$;

$$(x^iy^jz^kpr)^G = \langle y, z^2 \rangle x^z z^{kpr};$$

$$(x^iy^jz^kpt)^G = \langle z^2 \rangle x^z z^{kpt}, u \in \{j, 1 - i - j\}$$

$\cup \langle z^2 \rangle x^{-i}z^kp, u \in \{i + j, 1 - j\}$$

$$(x^iy^jz^kt)^G = \langle x^2, y, z^2 \rangle x^z z^kt.$$ 

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{67}$. \qed

**Proposition C.16.** For $G_{68}$ we have the following commutators:

$$(p, rt) = 1;$$

$$(t, r) = (rt, pt) = (pt, pr) = z;$$

$$(p, r) = (p, t) = x^{-2}y;$$

$$(r, pt) = x^{-2}yz^{-1};$$

$$(pr, t) = (r, prt) = x^2y^{-1}z^{-1}.$$ 

It follows that $G' = \langle x^2, y, z \rangle$.

The conjugacy classes in $G_{68}$ are:
For $i, j, k \in \mathbb{Z}$,

\[
(x^iy^jz^kp)^G = \langle x^2, y^2 \rangle x^iy^jz^kp, \quad u \in \{j, i + j\}, \\
\cup \langle x^2, y^2 \rangle x^iy^jz^{-k}p, \quad u \in \{j + 1, i + j + 1\}; \\
(x^iy^jz^kr)^G = \langle x^2y^{-1}, z^2 \rangle x^iy^jz^u r, \quad u \in \{j, -i - j\}, \quad v \in \{k, k - 1\}; \\
(x^iy^jz^kp)^G = \langle y, z^2 \rangle x^u z^v pr, \quad u \in \{i, 2 - i\}, \quad v \in \{k, k - 1\}; \\
(x^iy^jz^kpr)^G = \langle x^2y^{-1} \rangle (x^iy^j)^{\pm 1} z^u prt, \quad u \in \{k, 1 - k\}; \\
(x^iy^jz^kr)^G = \langle y \rangle x^{\pm i} z^urt, \quad u \in \{k, 1 - k\}; \\
(x^iy^jz^kpt)^G = \langle z^2 \rangle x^iy^jz^k (y^{i-2j}z)^{\delta} pt, \quad \delta \in \{0, 1\}, \\
\cup \langle z^2 \rangle x^{2-i} y^{-j-1} z^k (y^{i+2j}z)^{\delta} pt, \quad \delta \in \{0, 1\}; \\
(x^iy^jz^kt)^G = \langle x^2, y, z \rangle x^i t.
\]

**Proof.** This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{68}$. \hfill \Box

**Proposition C.17.** The following pertains to $G_{69}$.

Here $F$ is an abelian subgroup of $G$, thus $G' = K = \langle x^2, y^2, z \rangle$.

For $i, j, k \in \mathbb{Z}$,

\[
(x^iy^jz^kp)^G = \langle x^2z^{-1}, y^2z^{-1} \rangle x^iy^jz^up, \quad u \in \{k, -i - j - k\} \\
\cup \langle x^2z^{-1}, y^2z^{-1} \rangle x^iy^jz^u p, \quad u \in \{j + k, -i - k\}; \\
(x^iy^jz^kr)^G = \langle x^2, z \rangle x^iy^jz^urr, \quad u \in \{k, -i - j - k\} \\
\cup \langle x^2, z \rangle x^iy^jz^{-j}z^u r, \quad u \in \{j + k, -i - k\}; \\
(x^iy^jz^kpr)^G = \langle y^2, z \rangle x^{\pm i} y^j pr; \\
(x^iy^jz^kprt)^G = \langle x^2z^{-1} \rangle (x^iy^j)^{\pm 1} z^u prt, \quad u \in \{k, 1 - k\}; \\
(x^iy^jz^krt)^G = \langle y^2z^{-1} \rangle x^{\pm i} z^u r t, \quad u \in \{k, 1 - k\}; \\
(x^iy^jz^kpt)^G = \langle z \rangle x^{\pm i} y^j p t; \\
(x^iy^jz^kt)^G = \langle x^2, y^2, z^2 \rangle x^iy^jz^ut, \quad u \in \{k, j + k, i + k\}.
\]
Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{69}$. 

Proposition C.18. For $G_{72}$ we have the following commutators:

\[
(p, t) = (pt, rt) = (pt, pr) = (pt, r) = (pt, prt) = 1;
\]
\[
(p, r) = (t, r) = (t, pr) = (rt, p) = x^2z^{-1};
\]
\[
(r, pr) = (rt, prt) = x^2y^{-2}z^{-1};
\]
\[
(rt, pr) = y^2.
\]

It follows that $G' = K = \langle x^2, y, z \rangle$.

The conjugacy classes in $G_{72}$ are:

For $i, j, k \in \mathbb{Z}$,

\[
(x^iy^jz^k)^{G'} = \langle x^2z^{-1}, y^2 \rangle (x^iy^jz^k)^{\pm 1} p
\]
\[
\cup \langle x^2z^{-1}, y^2 \rangle (x^iy^{j+k}z^k)^{\pm 1} p;
\]
\[
(x^iy^jz^k)^{G'} = \langle x^2y^{-1}, z^2 \rangle x^iy^jz^k r, u \in \{k, i + k\}
\]
\[
\cup \langle x^2y^{-1}, z^2 \rangle x^{-i}y^{-j}z^{-1}u^r, u \in \{k + 1, i + k + 1\};
\]
\[
(x^iy^jz^k)^{G'} = \langle yz, yz^{-1} \rangle x^iy^jz^k u^r, u \in \{k, i + k\}
\]
\[
\cup \langle yz, yz^{-1} \rangle x^{-i-2}y^jz^k u^r, u \in \{k + 1, i + k + 1\};
\]
\[
(x^iy^jz^k)^{G'} = \langle x^2y^{-1}z^{-1} \rangle x^iy^jz^k u^r, u \in \{k, -i - k\}
\]
\[
\cup \langle x^2y^{-1}z^{-1} \rangle x^{-i}y^{-j}z^k u^r, u \in \{-k, i + k\};
\]
\[
(x^iy^jz^k)^{G'} = \langle y \rangle x^iz^u r, u \in \{k, -i - k\}
\]
\[
\cup \langle y \rangle x^2iz^u r, u \in \{i + k - 1, -k - 1\};
\]
\[
(x^iy^jz^k)^{G'} = \langle z \rangle (x^iy^u)^{\pm 1} p t, u \in \{j, -i - j\};
\]
\[
(x^iy^jz^k)^{G'} = \langle x^2, y^2, z \rangle y^it \text{ if } i \text{ is even,}
\]
\[
or \langle x^2, y, z \rangle xt \text{ if } i \text{ is odd.}
\]

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the
Proposition C.19. For $G_{73}$ we have the following commutators:

\[
(p, r) = x^{-2}y^2; \quad (p, t) = x^{-2}y; \\
(t, r) = yz; \quad (pr, r) = x^2; \\
(rt, p) = (prt, rt) = y; \quad (rt, pr) = y^2z; \\
(pt, r) = x^2y^{-1}z; \quad (pr, t) = x^2y^{-2}z^{-1}; \\
(pt, pr) = (rt, pt) = z; \quad (r, prt) = (pt, prt) = x^2y^{-1}z^{-1}.
\]

Therefore $G' = K = \langle x^2, y, z \rangle$.

The conjugacy classes in $G_{73}$ are:

For $i, j, k \in \mathbb{Z}$,

\[
(x^iy^jz^kp)^G = \langle x^iz^{-1}, y \rangle x^iz^kp \cup \langle x^iz^{-1}, y \rangle x^iz^{-i-k-1}p \text{ if } i \text{ is even,} \\
\langle x^iz^{-1}, y \rangle x^iz^kp \cup \langle x^iz^{-1}, y \rangle x^iz^{-i-k-1}p; \text{ if } i \text{ is odd;} \\
(x^iy^jz^k)^G = \langle x^iy^{-1}, z^2 \rangle x^iy^z r, u \in \{k, i + k + 1\} \\
\cup \langle x^iy^{-1}, z^2 \rangle x^iy^{-i-j}z^ur, u \in \{i + k, 1 + k\}; \\
(x^iy^jz^kpr)^G = \langle y, z \rangle x^u pr, u \in \{i, 2 - i\}, \text{ if } i \text{ is even,} \\
\langle yz, yz^{-1} \rangle x^iy^jz^k(x^2-2i)z^pr, \delta \in \{0, 1\} \text{ if } i \text{ is odd;} \\
(x^iy^jz^kprt)^G = \langle x^iy^{-1}z^{-1} \rangle x^iy^z r, u \in \{j, 1 - i - j\}, v \in \{k, -i - k\}; \\
(x^iy^jz^krt)^G = \langle y \rangle x^iz^ur, u \in \{k, 1 - i - k\} \\
\cup \langle y \rangle x^{-i}z^ur, u \in \{i + k, 1 - k\}; \\
(x^iy^jz^kpt)^G = \langle z \rangle x^i y^u pt, u \in \{j, -i - j\} \\
\cup \langle z \rangle x^{-2}y^u pt, u \in \{i + j - 1, -1 - j\}; \\
(x^iy^jz^kt)^G = \langle x^2, y, z \rangle x^it.
\]

Proof. This follows from Lemma C.1, Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{73}$. 

\[\square\]
Proposition C.20. For $G_{74}$ we have the following commutators:

\[(r, prt) = (r, pt) = (prt, pt) = 1;\]
\[(p, r) = (t, p) = (pr, r) = (pt, pr) = (pt, rt) = y;\]
\[(rt, p) = (t, pr) = (prt, rt) = y^2;\]
\[(rt, pr) = y^3.\]

Therefore $G' = K = \langle x^2, y, z \rangle$.

The conjugacy classes in $G_{74}$ are:

For $i, j, k \in \mathbb{Z}$,

\[(x^i y^j z^k p)^G = \langle x^2 z^{-1}, y^2 \rangle x^i y^u z^k p, u \in \{j, i + j\}\]
\[\cup \langle x^2 z^{-1}, y^2 \rangle x^i y^u z^{-i-k} p, u \in \{i + j + 1, j + 1\};\]
\[(x^i y^j z^k r)^G = \langle x^2 y^{-1}, z^2 \rangle x^i y^u z^v r, u \in \{j, -i - j - 1\}, v \in \{k, i + k\};\]
\[(x^i y^j z^k p r)^G = \langle y z, y z^{-1} \rangle x^{\pm 1} y^j z^k p r\]
\[\cup \langle y z, y z^{-1} \rangle x^{\pm 1} y^{j+1} z^{i+k} p r;\]
\[(x^i y^j z^k p r t)^G = \langle x^2 y^{-1} z^{-1} \rangle x^i y^u z^v p r t, u \in \{j, 2 - j - i\}, v \in \{k, -i - k\};\]
\[(x^i y^j z^k k r t)^G = \langle y \rangle (x^i z^u)^{+1} \rangle r t, u \in \{k, -i - k\};\]
\[(x^i y^j z^k p t)^G = \langle z \rangle x^i y^u p t, u \in \{j, 1 - i - j\}\]
\[\cup \langle z \rangle x^{-i} y^u p t, u \in \{i + j, 1 - j\};\]
\[(x^i y^j z^k t)^G = \langle x^2, y, z^2 \rangle z^k t \text{ if } i \text{ is even,}\]
\[\langle x^2, y z, y z^{-1} \rangle x y^j z^k t \text{ if } i \text{ is odd}.\]

Proof. This follows from Proposition A.1, Proposition C.2, and from the relations in the presentation of $G_{74}$. 

\[\square\]