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Zeros of a Two-Parameter Family of Harmonic Trinomials

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ABSTRACT

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This thesis studies complex harmonic polynomials of the form $f(z) = az^n + b\bar{z}^k + z$ where $n, k \in \mathbb{Z}$ with $n > k$ and $a, b > 0$. We show that the sum of the orders of the zeros of such functions is n and investigate the locations of the zeros, including whether the zeros are in the sense-preserving or sense-reversing region and a set of conditions under which zeros have the same modulus. We also show that the number of zeros ranges from n to $n + 2k + 2$ as long as certain criteria are met.

Keywords: harmonic polynomials, zeros, Fundamental Theorem of Algebra

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Contents

List of Figures

Chapter 1. Introduction

Many problems in mathematics center around knowing about the zeros of functions. These range from simple situations like finding how many of an item must be sold to break even to much more complicated problems like the famous Riemann hypothesis, which concerns the locations of the zeros of the Riemann zeta function.

Here, we will be investigating the zeros of continuous complex-valued harmonic polynomials. These can be expressed as $f = h + \bar{g}$, where h and g are analytic polynomials. There are some familiar facts about analytic polynomials that do not quite carry over to complex-valued harmonic polynomials. One comes from the Fundamental Theorem of Algebra, with which one can show that an analytic polynomial of degree n has no more than n distinct roots. To show how this does not hold for complex-valued harmonic polynomials, consider the polynomial $z^4 + \bar{z}^2$. One can verify that the seven numbers $0, \pm i, \pm\frac{\sqrt{3}}{2} \pm \frac{i}{2}$ all satisfy the equation $z^4 + \bar{z}^2 = 0$, showing that the number of roots of this polynomial is greater than $n = 4$.

Another such fact is that analytic functions are sense-preserving at all points at which the derivative is nonzero. We call a function "sense-preserving" in a given domain if the function preserves the orientation of a curve in that domain, that is, a positively-oriented curve is still positively oriented after applying the function to it, and a negatively-oriented curve is still negatively oriented after applying the function to it. On the other hand, we call a function "sense-reversing" in a given domain if the function changes the orientation of a curve in that domain. In the case of complex-valued harmonic functions, we find that such functions may be sense-preserving in some regions and sense-reversing in others. Lewy's Theorem [Lew36] implies that $f = h + \bar{g}$ is locally univalent and sense-preserving if and only if $h'(z) \neq 0$ and the dilatation function $\omega(z) := g'(z)/h'(z)$ satisfies $|\omega(z)| < 1$. We note that, if f is analytic, $g(z) \equiv 0$, so $\omega(z) \equiv 0$ as well. The modulus of the dilatation function can thus be viewed as a measure of how far f is from being analytic.

Many similarities exist between analytic and harmonic functions. In particular, many familiar results for analytic functions hold for complex-valued harmonic functions with only slight modifications. We will be making use of the following harmonic analog of Rouché's theorem as shown by Duren, Hengartner, and Laugesen: [DHL96]

Theorem 1.1 (Rouché's Theorem for Harmonic Functions). *If p and $p+q$ are harmonic functions in a Jordan domain D with boundary C, are continuous in $\bar{D}$, and $|q(z)| < |p(z)|$ on C, then p and $p+q$ have the same number of zeros inside D counted according to their multiplicity as long as none of the zeros are singular.*

It should be noted that by "singular" we mean that the value of the dilatation function has modulus 1 at that point. Throughout this paper we will assume that $p_{a,b}(z)$ has no singular zeros so that we may use Rouché's theorem.

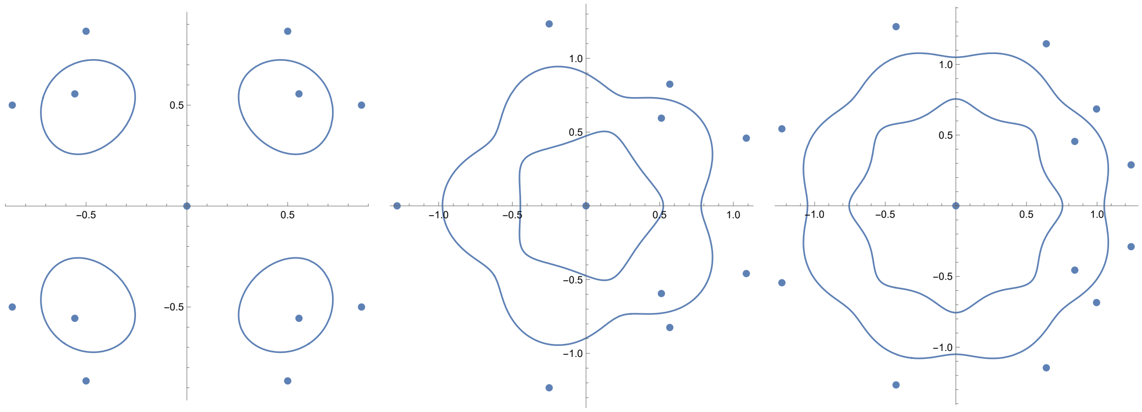
To define the multiplicity of a zero z_0 of a complex harmonic function, we look at its power series expansion about the zero. Let

$$f(z) = h(z) + \overline{g(z)} = a_0 + \sum_{j=r}^{\infty} a_j (z - z_0)^j + \overline{b_0 + \sum_{j=s}^{\infty} b_j (z - z_0)^j}, \tag{1.1}$$

where $a_r \neq 0$ and $b_s \neq 0$. If z_0 is in the sense-preserving region then $r \leq s$ and the order of the zero is r. If z_0 is in the sense-reversing region, then $s \leq r$ and the order of the zero is $-s$. If a zero is on a boundary between the sense-preserving region and the sense-reversing region, i.e., if the zero is singular, then the order of the zero is undefined. We call this boundary the *critical curve.*

Deducing information about the number and locations of zeros of a complex-valued harmonic polynomial is a task that is certainly nontrivial. Recently, Melman [Mel12] investigated trinomials of the form $q(z) = z^n - az^k - 1$, where $1 \leq k \leq n-1, n \geq 3, a \in \mathbb{C}$, and $\gcd(n,k) = 1$ and gleaned information relating to the location of the zeros of q. Similarly, Brilleslyper and Schaubroeck [BS14] [BS18] considered the family of trinomials $p(z) = z^n + z^k - 1$, where $1 \leq k \leq n-1$, $n \geq 2$, and derived a formula for the number of

Figure 1.1: Example plots of zeros and critical curves



(a) $f_1(z) = z^5 + \bar{z}^3 + z$ (b) $f_2(z) = z^6 + 1.5\bar{z}^3 + z$ (c) $f_3(z) = 0.5z^9 + \bar{z}^6 + z$

zeros of p located on the unit circle. Howell and Kyle [HK18] then were able to determine the number of zeros of this same trinomial p in the interior and exterior of the unit circle. Brilleslyper, Brooks, Dorff, Howell, and Schaubroeck [BBD+20] derived formulas for certain cases of the trinomials $p_c(z) = z^n + c\bar{z}^k - 1$. Here, we work with the family of harmonic trimonials

$$p_{a,b}(z) = az^n + b\bar{z}^k + z,$$

where $1 < k < n$ and $a, b \in \mathbb{R}^+$. Figure 1.1 contains some examples of the locations of the zeros of certain specific trinomials, also showing the critical curve for each.

From looking at the first of these, it would appear that the nontrivial zeros of $z^5 + \bar{z}^3 + z$ are arranged on two circles, each centered at the origin. We show that this is, indeed, the case in our first theorem:

Theorem 3.2. *If $n - k = 2$ then the nontrivial zeros of $p_{a,b}(z)$ either are all equal in modulus or are split into two disjoint subsets where all the zeros in each subset are equal in modulus. In the former case there are $n - 1$ nontrivial zeros, all of which are in the sense-preserving region, giving a total of n zeros. In the latter case there are $2n - 2$ zeros in one subset, all of which are in the sense-preserving region, and $n - 1$ zeros in the other subset, all of which are in the sense-reversing region, giving a total of $3n - 2$ zeros.*

Our other two theorems show that the number of zeros depends on the parameters a and

b. These theorems assume that $p_{a,b}(z)$ has no singular zeros in order that Rouché's theorem may be used.

Theorem 4.1. *There exists b_0, dependent on $a, n,$ and k, such that, if $b < b_0$ then $p_{a,b}(z)$ has precisely n distinct zeros, all of which are in the sense-preserving region.*

Theorem 5.1. *If $p_{a,b}(z)$ has no zeros in the annuli*

$$(1 + bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}} < |z| < (bk(an)^{\frac{1-k}{n-1}} - 1)^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}} \tag{1.2}$$

or

$$(bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}} < |z| < (1 + bk(an)^{\frac{1-k}{n-1}})^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}} \tag{1.3}$$

then there exists b_N, dependent on $a, n,$ and k, such that, if $b > b_N$ then $p_{a,b}(z)$ has $n+2k+2$ zeros.

Chapter 2. Preliminaries

Before we work towards the main results, there are some preliminary facts that must be established. The first is about the sum of the orders of the zeros of $p_{a,b}(z)$.

Proposition 2.1. *If $p_{a,b}(z)$ has no singular zeros then the sum of the orders of the zeros of $p_{a,b}(z)$ is n.*

Proof. Since $n > k$ there exists $R > 0$ such that $|az^n| > |b\bar{z}^k + z|$ if $|z| > R$. The function $q(z) = az^n$ has a single zero of order n at the origin. Taking $R \to \infty$, it follows by Rouché's Theorem that the sum of the orders of the zeros of $p_{a,b}(z)$ in the complex plane is n. □

Next we show that the zeros of $p_{a,b}(z)$ are all simple; i.e., each zero has order 1 or -1.

Proposition 2.2. *Each zero of $p_{a,b}(z)$ has order 1 or -1.*

Proof. The first part of this argument is taken with some modification from Brilleslyper et al. [BBD+20] The trinomial $p_{a,b}(z)$ can be written as $p_{a,b}(z) = h(z) + \overline{g(z)} = (az^n + z) + \overline{bz^k}$. Let z_0 be a nontrivial zero of $p_{a,b}(z)$. The series expansions of h and g about z_0 are finite series since h and g are polynomials, where in accordance with the notation of equation (1.1), $a_1 = anz_0^{n-1} + 1$ and $b_1 = bkz_0^{k-1}$. The latter of these is nonzero when $z_0 \neq 0$, and the former is only zero if $z_0 = (-an)^{\frac{1}{n-1}}$. In this case

$$\lim_{z \to z_0} |\omega(z)| = \lim_{z \to z_0} \left| \frac{g'(z)}{h'(z)} \right| = \lim_{z \to z_0} \left| \frac{bkz^{k-1}}{anz^{n-1} + 1} \right| = \infty$$

because the denominator goes to 0 while the numerator goes to some finite nonzero number. This places z_0 in the sense-reversing region, so the order of the zero is $-s$. However, we already showed that $b_1 \neq 0$ in this case, so if there is a zero at $(-an)^{\frac{1}{n-1}}$ then its order is -1. This shows that the nontrivial zeros all have order 1 or -1.

Now we look at the trivial zero. Note that $\omega(0) = 0$, so the origin is in the sense-preserving region, which means the order of the zero is r. In this case we have $a_1 = 1 \neq 0$, so the order of this zero is 1. Thus, each zero of $p_{a,b}(z)$ has order 1 or -1. □

Finally, we have this interesting result, which shows that the modulus of the dilatation function $\omega(z)$ has rotational symmetry in the complex plane about the origin with an order that is a multiple of $n - 1$.

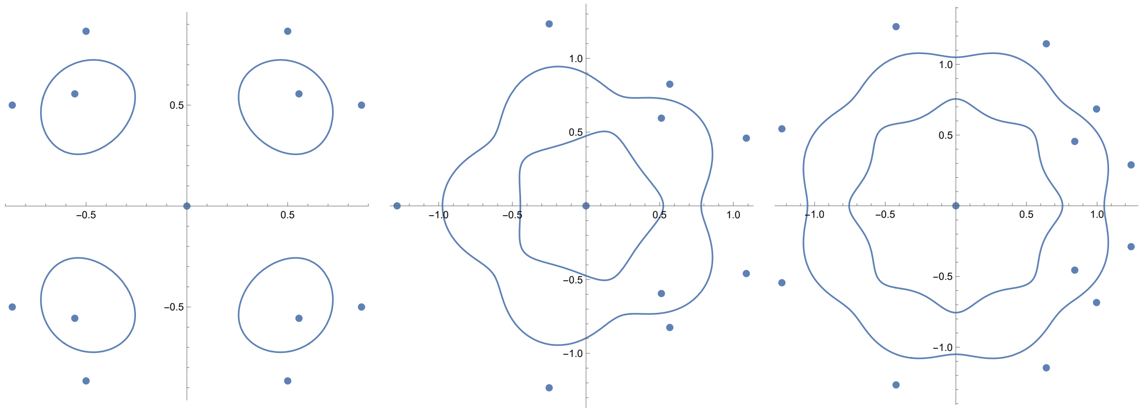
Proposition 2.3. *$|\omega(z)| = |\omega(ze^{\frac{2\pi i}{n-1}})|$ for all $z \in \mathbb{C}$.*

Proof. We have

$$|\omega(ze^{\frac{2\pi i}{n-1}})| = \left| \frac{g'(ze^{\frac{2\pi i}{n-1}})}{h'(ze^{\frac{2\pi i}{n-1}})} \right| = \frac{bk|ze^{\frac{2\pi i}{n-1}}|^{k-1}}{|an(ze^{\frac{2\pi i}{n-1}})^{n-1} + 1|} = \frac{bk|z|^{k-1}|e^{\frac{2\pi i(k-1)}{n-1}}|}{|anz^{n-1}e^{2\pi i} + 1|}$$

$$= \frac{bk|z|^{k-1}}{|anz^{n-1} + 1|} = \left| \frac{g'(z)}{h'(z)} \right| = |\omega(z)|. \quad \square$$

Figure 2.1: Example plots of zeros and critical curves (reproduced)



(a) $f_1(z) = z^5 + \bar{z}^3 + z$ (b) $f_2(z) = z^6 + 1.5\bar{z}^3 + z$ (c) $f_3(z) = 0.5z^9 + \bar{z}^6 + z$

In practice, it appears that the order of the rotational symmetry of $|\omega(z)|$ is exactly $n-1$. For convenience, we reproduce Figure 1.1 here, on which this rotational symmetry can be readily seen by examining the critical curve.

Chapter 3. Special Case

In this chapter we investigate the case where $n - k = 2$. The motivation for looking into this case comes from the following proposition:

Proposition 3.1. *Let z_0 be a zero of $p_{a,b}(z)$. If ζ is both an $(n-1)$th root of unity and a $(k+1)$th root of unity or, equivalently, if ζ is a dth root of unity where d is the g.c.d. of $n-1$ and $k+1$, then ζz_0 is also a zero of $p_{a,b}(z)$.*

Proof. Consider the polynomial $a(\zeta z_0)^n + b\overline{\zeta z_0}^k + \zeta z_0$. By hypothesis there exist $m_1, m_2 \in \mathbb{Z}$ such that

$$\zeta = e^{\frac{2\pi i m_1}{n-1}} = e^{\frac{2\pi i m_2}{k+1}}.$$

Substituting these expressions into the first and second instances of ζ in our polynomial, respectively, and distributing the powers n and k gives

$$ae^{\frac{2\pi i m_1 n}{n-1}} z_0^n + be^{-\frac{2\pi i m_2 k}{k+1}} \bar{z}_0{}^k + \zeta z_0.$$

Now we can split each of the expressions involving e to obtain

$$ae^{\frac{2\pi i m_1}{n-1}}e^{\frac{2\pi i m_1(n-1)}{n-1}} + be^{\frac{2\pi i m_2}{k+1}}e^{-\frac{2\pi i m_2(k+1)}{k+1}} + \zeta z_0.$$

Notice that the expressions we split off are both equal to ζ, so we can factor out ζ from the entire expression, simplifying the remaining fractional exponents, to obtain

$$\zeta(ae^{2\pi i m_1}z_0^n + be^{-2\pi i m_2}\bar{z_0}^k + z_0).$$

Now the exponents on e are both integer multiples of $2\pi i$, so taking e to those powers gives 1. We now have

$$\zeta(az_0^n + b\bar{z_0}^k + z_0) = \zeta p_{a,b}(z_0).$$

Since z_0 is a zero of $p_{a,b}(z_0)$, this is equal to $\zeta \cdot 0 = 0$. Thus, ζz_0 is also a zero of $p_{a,b}(z)$. $\square$

The most interesting result of this proposition comes when $n-1 = k+1$, or $n-k = 2$. In this case, every $(n-1)$th root of unity is also a $(k+1)$th root of unity, so any nontrivial zero gives rise to $n-1$ zeros all with the same modulus by multiplying by each root of unity. This fact becomes evident in the proof of our theorem:

Theorem 3.2. *If $n-k=2$ then the nontrivial zeros of $p_{a,b}(z)$ either are all equal in modulus or are split into two disjoint subsets where all the zeros in each subset are equal in modulus. In the former case there are $n-1$ nontrivial zeros, all of which are in the sense-preserving region, giving a total of n zeros. In the latter case there are $2n-2$ zeros in one subset, all of which are in the sense-preserving region, and $n-1$ zeros in the other subset, all of which are in the sense-reversing region, giving a total of $3n-2$ zeros.*

The bulk of this proof rests in using algebra and basic calculus to determine, as far as possible, the possible values of the modulus of any zero of $p_{a,b}(z)$ in this case. We show that there are exactly one or two distinct values for the modulus of a nontrivial zero depending on the parameters of $p_{a,b}(z)$. We then determine the number of zeros corresponding to each of these values.

Proof. Let $z = re^{i\theta}$ be a nontrivial zero of $p_{a,b}(z)$. Since $n - k = 2$ we can then write

$$a(re^{i\theta})^{k+2} + b\overline{re^{i\theta}}^{k} + re^{i\theta} = 0$$

$$\Rightarrow ar^{k+2}e^{i(k+2)\theta} + br^{k}e^{-ik\theta} + re^{i\theta} = 0.$$

Since z is a nontrivial zero, we know $r \neq 0$, so we can multiply this equation by $e^{ik\theta}/r$ to get

$$ar^{k+1}e^{i(2k+2)\theta} + br^{k-1} + e^{i(k+1)\theta} = 0.$$

We note that the left-hand side is a quadratic expression in $e^{i(k+1)\theta}$, so the quadratic formula gives

$$e^{i(k+1)\theta} = \frac{-1 \pm \sqrt{1 - 4abr^{2k}}}{2ar^{k+1}}. \tag{3.1}$$

Taking the modulus of both sides, we then have

$$1 = \left| \frac{-1 \pm \sqrt{1 - 4abr^{2k}}}{2ar^{k+1}} \right|. \tag{3.2}$$

Let D be the discriminant, that is, $D = 1 - 4abr^{2k}$. We now investigate the number of possible values for r based on the sign of D.

First, if $D < 0$, then the two possible expressions inside the modulus on the right-hand side of (3.2) are complex conjugates. In this case, multiplying them together gives the square of the modulus, which the left-hand side tells us is 1. We then have

$$1 = \frac{-1 + \sqrt{1 - 4abr^{2k}}}{2ar^{k+1}} \cdot \frac{-1 - \sqrt{1 - 4abr^{2k}}}{2ar^{k+1}} = \frac{b}{ar^2}$$

after some simplification. Rearranging then gives $r = \sqrt{\frac{b}{a}}$. By substituting this value for r into the discriminant, we get a necessary condition for the discriminant to be negative, which is $b^{k+1} > \frac{a^{k-1}}{4}$. We have also established that there is exactly one possible value of r that can make the discriminant negative, namely $r = \sqrt{\frac{b}{a}}$.

If, on the other hand, $D \geq 0$, then the right-hand side of (3.2) is real. Since the only real numbers with modulus 1 are ± 1, we have

$$1 = \frac{-1 \pm \sqrt{1 - 4abr^{2k}}}{2ar^{k+1}} \text{ or } -1 = \frac{-1 \pm \sqrt{1 - 4abr^{2k}}}{2ar^{k+1}}, \tag{3.3}$$

which, after some simplifying, leads to

$$ar^{k+1} + br^{k-1} + 1 = 0 \text{ or } ar^{k+1} + br^{k-1} - 1 = 0.$$

Taking the derivative with respect to r yields the same result in either case, which is $(k+1)ar^k + (k-1)br^{k-2}$. Since we are only considering positive values of a, b, and k, and k must be an integer, this derivative is positive for all $r > 0$. Substituting 0 for r in these equations gives values of 1 in the former case and -1 in the latter case for the left-hand side. We thus conclude using elementary calculus that the former case gives no positive values of r that satisfy the equation, and the latter case gives exactly one positive value of r that satisfies the equation.

We now address the question of whether this positive value of r actually results in the discriminant being nonnegative. This will happen if and only if $r \leq (4ab)^{-\frac{1}{2k}}$. We will show that this is always the case for r a positive solution of $ar^{k+1} + br^{k-1} - 1 = 0$.

We replace r with the value $(4ab)^{-\frac{1}{2k}}$. The resulting polynomial is

$$a(4ab)^{\frac{-k-1}{2k}} + b(4ab)^{\frac{1-k}{2k}} - 1.$$

Consider this as a function f in b. If we let $b = (\frac{a^{k-1}}{4})^{\frac{1}{k+1}}$ then we obtain (after a lot of messy algebra)

$$f\left(\left(\frac{a^{k-1}}{4}\right)^{\frac{1}{k+1}}\right) = \frac{1}{2} + \frac{1}{2} - 1 = 0.$$

Taking the derivative of f with respect to b and doing more messy algebra yields

$$f'(b) = \frac{(k+1)(4^{\frac{k+1}{2k}} a^{\frac{1-k}{2k}} b^{\frac{k+1}{2k}} - 4^{\frac{k-1}{2k}} a^{\frac{k-1}{2k}} b^{\frac{-k-1}{2k}})}{8kb}.$$

Letting $b = (\frac{a^{k-1}}{4})^{\frac{1}{k+1}}$ again gives, after another round of messy algebra,

$$f'\left(\left(\frac{a^{k-1}}{4}\right)^{\frac{1}{k+1}}\right) = 0,$$

so $(\frac{a^{k-1}}{4})^{\frac{1}{k+1}}$ is a critical value of f. It is clear that the sign of this derivative depends only on the sign of

$$4^{\frac{k+1}{2k}} a^{\frac{1-k}{2k}} b^{\frac{k+1}{2k}} - 4^{\frac{k-1}{2k}} a^{\frac{k-1}{2k}} b^{\frac{-k-1}{2k}}$$

and that this expression is strictly increasing in b. Hence, we see that $f'(b) < 0$ when $b < (\frac{a^{k-1}}{4})^{\frac{1}{k+1}}$ and $f'(b) > 0$ when $b > (\frac{a^{k-1}}{4})^{\frac{1}{k+1}}$. This shows that 0 is the absolute minimum of $f(b)$, which means $ar^{k+1} + br^{k-1} - 1 \geq 0$ when $r = (4ab)^{-\frac{1}{2k}}$. By the Intermediate Value Theorem, we thus know that the positive root satisfies $0 < r \leq (4ab)^{-\frac{1}{2k}}$. Hence, this root makes the discriminant positive, so for any a, b, k, there is exactly one circle of radius r centered at the origin containing roots of $p_{a,b}(z)$ where the discriminant found earlier is positive.

If $b^{k+1} > \frac{a^{k-1}}{4}$, then there may also exist a circle of radius $\sqrt{\frac{b}{a}}$ centered at the origin containing roots of $p_{a,b}(z)$, where the discriminant from earlier is negative. These two circles are the only possibilities for the locations of nontrivial zeros. We now turn our attention to the individual zeros in each circle.

In the case where the discriminant is nonnegative, we found that the left equation in (3.3) cannot happen, so putting the right equation of (3.3) together with (3.1) tells us that $e^{i(k+1)\theta} = -1$. Thus, the zeros on the circle from the nonnegative discriminant case are all the points with argument equal to $\frac{(2m+1)\pi}{k+1}$ for $m \in \{0, 1, \ldots, k\}$. There are $k + 1 = n - 1$ of these zeros, and by Proposition 2.2 they must all have order 1 or -1. If this is the only circle containing zeros, then there are n total zeros, which means that, in this case, the zeros all have order 1, placing them in the sense-preserving region.

Now we look at the case where the discriminant can be negative. Recall that the zeros from this case have modulus $\sqrt{\frac{b}{a}}$. We now show that the circle $|z| = \sqrt{\frac{b}{a}}$ is entirely within the sense-preserving region. We have

$$|\omega(z)| = \frac{bk|z|^{k-1}}{|anz^{n-1}+1|} \leq \frac{bk|z|^{k-1}}{\big|an|z|^{n-1}-1\big|} = \frac{bk(\frac{b}{a})^{\frac{k-1}{2}}}{|an(\frac{b}{a})^{\frac{n-1}{2}}-1|} = \frac{kb^{\frac{k+1}{2}}}{|(k+2)b^{\frac{k+1}{2}} - a^{\frac{k-1}{2}}|}$$

after doing some simplifying and using the fact that $n-k=2$. Now recall that a necessary condition for this case is that $b^{k+1} > \frac{a^{k-1}}{4} \Leftrightarrow b^{\frac{k+1}{2}} > \frac{a^{\frac{k-1}{2}}}{2}$. This condition certainly makes $(k+2)b^{\frac{k+1}{2}} > a^{\frac{k-1}{2}}$, so we can drop the absolute value signs in the denominator. Note also that

$$b^{\frac{k+1}{2}} > \frac{a^{\frac{k-1}{2}}}{2} \Leftrightarrow 2b^{\frac{k+1}{2}} > a^{\frac{k-1}{2}} \Leftrightarrow (k+2)b^{\frac{k+1}{2}} - a^{\frac{k-1}{2}} > kb^{\frac{k+1}{2}} \Leftrightarrow \frac{kb^{\frac{k+1}{2}}}{(k+2)b^{\frac{k+1}{2}} - a^{\frac{k-1}{2}}} < 1.$$

Thus, the circle $|z| = \sqrt{\frac{b}{a}}$ and, hence, all the zeros on that circle are in the sense-preserving region. Going back to (3.1), the fact that we can use both the plus and the minus in front of the square root in this case, and that the right-hand side is nonreal means that we get $2(k+1) = 2n-2$ zeros on the circle $|z| = \sqrt{\frac{b}{a}}$. These zeros all have order 1, so to make the sum of the orders of all the zeros equal n, it must be that the $n-1$ zeros on the other circle each have order -1. Thus, we get a circle with $2n-2$ zeros in the sense-preserving region and a circle with $n-1$ zeros in the sense-reversing region, giving a total of $3n-2$ zeros after including the trivial zero. $\square$

In this latter case, we can also say that the circle with $n-1$ zeros in the sense-reversing region has a smaller radius than the circle with the $2n-2$ zeros in the sense-preserving region, which has radius $\sqrt{\frac{b}{a}}$. This comes from the following proposition:

Proposition 3.3. *If z_0 is a zero of $p_{a,b}(z)$ that is in the sense-reversing region, then*

$$|z_0| < \left(\frac{b(k+1)}{a(n-1)}\right)^{\frac{1}{n-k}}.$$

Proof. For such a z_0 we have

$$\left|\frac{bkz_0^{k-1}}{anz_0^{n-1}+1}\right| > 1 \Rightarrow \frac{bk|z_0|^k}{|anz_0^n + z_0|} > 1 \Rightarrow \frac{bk|z_0|^k}{|a(n-1)z_0^n - b\bar{z_0}^k|} > 1$$

where we have used the fact that z_0 is a zero of $p_{a,b}(z)$ to substitute in the denominator. The multiplication by $|z_0|/|z_0|$ is justified by the fact that z_0 is a sense-reversing zero, so it cannot be the trivial zero as we already found that the origin is in the sense-preserving region. From this we obtain

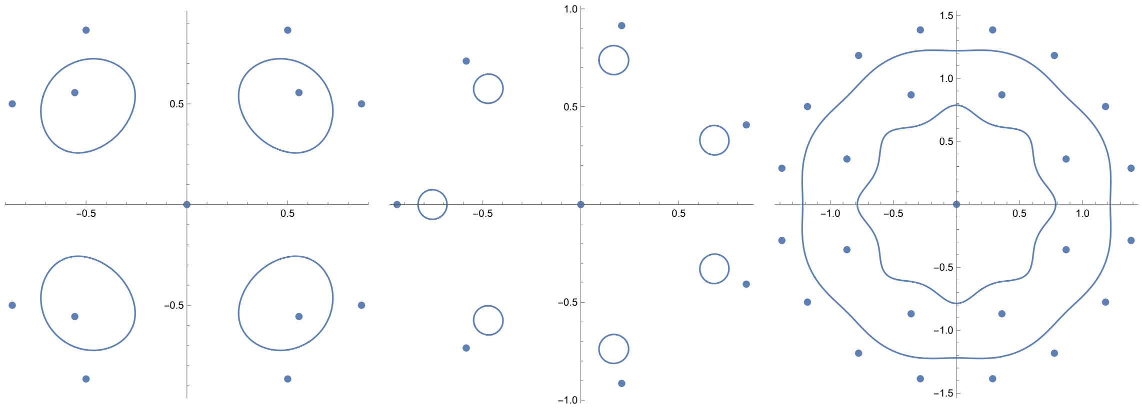
$$bk|z_0|^k > |a(n-1)z_0^n - b\bar{z_0}^k| \Rightarrow bk|z_0|^k > a(n-1)|z_0|^n - b|z_0|^k$$

$$\Rightarrow b(k+1)|z_0|^k > a(n-1)|z_0|^n \Rightarrow |z_0|^{n-k} < \frac{b(k+1)}{a(n-1)}] \Rightarrow |z_0| < \left(\frac{b(k+1)}{a(n-1)}\right)^{\frac{1}{n-k}}. \qquad \square$$

The case being considered in this chapter is $n - k = 2$, and with this condition the expression on the right is equal to $\sqrt{\frac{b}{a}}$, which we found to be the radius of the circle with zeros in the sense-preserving region. Thus, the zeros in the sense-reversing region have smaller modulus than the nontrivial zeros in the sense-preserving region.

The results of this chapter can be seen in Figure 3.1.

Figure 3.1: Examples of the case $n - k = 2$



(a) $f_1(z) = z^5 + \bar{z}^3 + z$ (b) $f_2(z) = z^8 + 0.5\bar{z}^6 + z$ (c) $f_3(z) = 0.5z^9 + \bar{z}^7 + z$

Chapter 4. Small b

Now we will begin to investigate what happens in general when we fix a, n, and k and allow b to vary. The examples in Figure 4.1 show that the number of zeros changes as we change b, with there appearing to be a positive correlation between the value of b and the number of zeros.

In this chapter we will prove the following theorem:

Theorem 4.1. *There exists b_0, dependent on a, n, and k, such that, if $b < b_0$ then $p_{a,b}(z)$ has precisely n distinct zeros, all of which are in the sense-preserving region.*

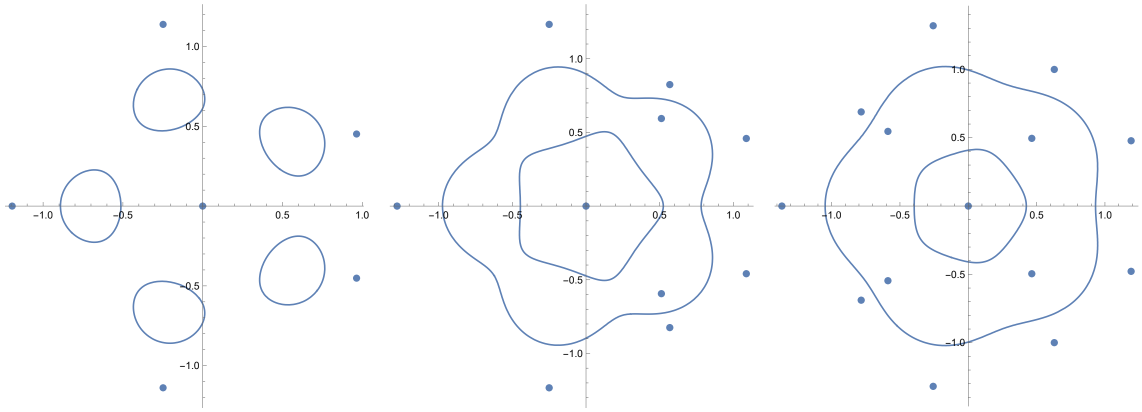
We do this by first establishing the existence of a disk centered at the origin that is entirely within the sense-preserving region. We use this fact to show that there are no nontrivial zeros inside this disk. Finally, we use this lower bound on the modulus of a nontrivial zero to establish that all the zeros of $p_{a,b}(z)$ are in the sense-preserving region for sufficiently small b. We begin with the following lemma:

Lemma 4.2. *The disk*

$$|z| \leq (1 + bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}} \tag{4.1}$$

is entirely contained within the sense-preserving region.

Figure 4.1: Number of zeros increases as b increases



(a) $f_1(z) = z^6 + \bar{z}^3 + z$
6 zeros

(b) $f_2(z) = z^6 + 1.5\bar{z}^3 + z$
10 zeros

(c) $f_3(z) = z^6 + 2\bar{z}^3 + z$
14 zeros

Proof. We will show that

$$|\omega(z)| = \frac{|bkz^{k-1}|}{|anz^{n-1} + 1|} < 1$$

whenever (4.1) is satisfied. Let $d \in \mathbb{R}$ with

$$d \leq (1 + bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}} \tag{4.2}$$

so that $|z| = d(an)^{-\frac{1}{n-1}}$. Since $a, b, k,$ and n are all positive, it is evident from (4.2) that $0 < d < 1$. Taking both sides of (4.2) to the $1 - k$ power and writing the inequality the other way gives

$$1 + bk(an)^{\frac{1-k}{n-1}} \leq d^{1-k}.$$

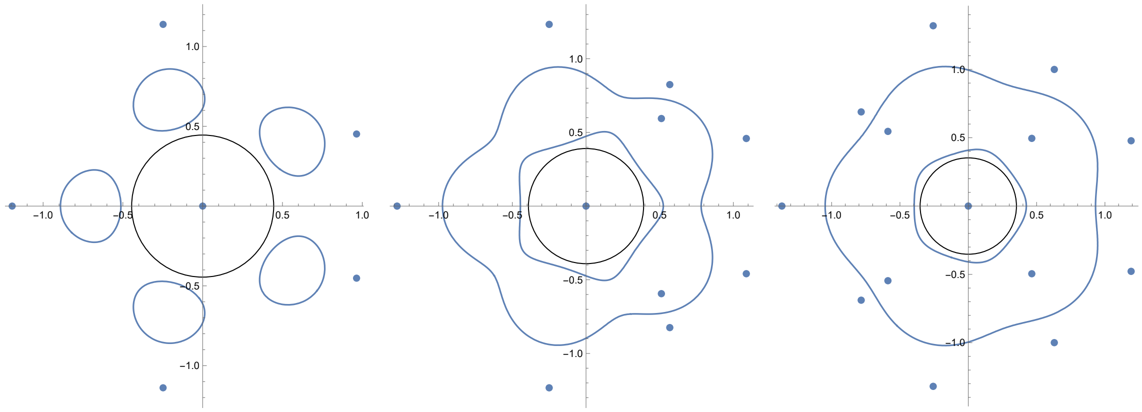
We now replace the 1 on the left-hand side with the smaller quantity d^{n-k} to obtain

$$d^{n-k} + bk(an)^{\frac{1-k}{n-1}} < d^{1-k}.$$

Now, we can multiply both sides by d^{k-1} and move the resulting d^{n-1} to the other side to get

$$bkd^{k-1}(an)^{\frac{1-k}{n-1}} < 1 - d^{n-1}.$$

Figure 4.2: Graphs showing disk in sense-preserving region



(a) $f_1(z) = z^6 + \bar{z}^3 + z$ (b) $f_2(z) = z^6 + 1.5\bar{z}^3 + z$ (c) $f_3(z) = z^6 + 2\bar{z}^3 + z$

Using the fact that $|z| = d(an)^{-\frac{1}{n-1}}$, this is equivalent to

$$bk|z|^{k-1} < 1 - an|z|^{n-1}.$$

Using the triangle inequality, we can then say

$$bk|z|^{k-1} < |1 + anz^{n-1}| \Rightarrow \frac{|bkz^{k-1}|}{|anz^{n-1} + 1|} < 1. \quad \square$$

We have thus established the existence of a disk entirely within the sense-preserving region. In Figure 4.2 we see the graphs from Figure 4.1 again, this time with the circle $|z| = (1 + bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}$ added for reference. It can be seen from these figures that the trivial zero is the only zero inside the circle in each of these cases. We now prove that this is always the case using Rouché's Theorem.

Lemma 4.3. *The trivial zero is the only zero of $p_{a,b}(z)$ in the disk given by (4.1).*

Proof. We will show this by establishing that $|az^n + b\bar{z}^k| < |z|$ on the circle $|z| = (1 + bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}$. The result will then follow from the previous lemma and Rouché's Theorem. For z on the given circle we have

$$|anz^n + b\bar{z}^k| \leq a|z|^n + b|z|^k = a(1 + bk(an)^{\frac{1-k}{n-1}})^{-\frac{n}{k-1}}(an)^{-\frac{n}{n-1}} + b(1 + bk(an)^{\frac{1-k}{n-1}})^{-\frac{k}{k-1}}(an)^{-\frac{k}{n-1}}.$$

If we can show that the right-hand side is smaller than $(1+bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}$ then we will have our result. Through some algebraic manipulation we have

$$a(1+bk(an)^{\frac{1-k}{n-1}})^{-\frac{n}{k-1}}(an)^{-\frac{n}{n-1}}+b(1+bk(an)^{\frac{1-k}{n-1}})^{-\frac{k}{k-1}}(an)^{-\frac{k}{n-1}} < (1+bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}$$

$$\Leftrightarrow a(1+bk(an)^{\frac{1-k}{n-1}})^{\frac{k-n}{k-1}}(an)^{-1}+b(an)^{\frac{1-k}{n-1}} < 1+bk(an)^{\frac{1-k}{n-1}}$$

$$\Leftrightarrow (1+bk(an)^{\frac{1-k}{n-1}})^{\frac{1-n}{k-1}}+\frac{bn(an)^{\frac{1-k}{n-1}}}{1+bk(an)^{\frac{1-k}{n-1}}} < n.$$

Clearly, $(1+bk(an)^{\frac{1-k}{n-1}})^{\frac{1-n}{k-1}} < 1$, so we need only establish that $\frac{bn(an)^{\frac{1-k}{n-1}}}{1+bk(an)^{\frac{1-k}{n-1}}} < n-1$. By some algebraic manipulation, this is equivalent to $b(an)^{\frac{1-k}{n-1}}(n+k-nk) \leq n-1$. Since we are assuming that $n > k > 1$ and that $n, k \in \mathbb{N}$, it follows that $n+k-nk < 0$, which makes our inequality true no matter the (positive) values of a and b.

Having established that $|az^n + b\bar{z}^k| < |z|$ on the circle $|z| = (1+bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}$, we can now use Rouché's Theorem to say that z and $p_{a,b}(z)$ have the same number of zeros inside the circle $|z| = (1+bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}$. Since z has only the trivial zero with multiplicity 1, and $p_{a,b}(z)$ cannot have a zero of negative order in this region by Lemma 4.2, it must be that $p_{a,b}(z)$ has only one zero in this region, which is readily shown to be the trivial zero. □

With this lemma in hand, we are now ready to prove our second major theorem.

Theorem 4.1. *There exists b_0, dependent on $a, n,$ and k, such that, if $b < b_0$ then $p_{a,b}(z)$ has precisely n distinct zeros, all of which are in the sense-preserving region.*

Proof. We will set

$$b_0 = \frac{a(n-1)}{(k+1)2^{\frac{n-k}{k-1}}(an)^{\frac{n-k}{n-1}}}. \tag{4.3}$$

We claim that the right-hand side of (4.3) is smaller than $(an)^{\frac{k-1}{n-1}}/k$. To see that this is true, note that

$$\frac{a(n-1)}{(k+1)2^{\frac{n-k}{k-1}}(an)^{\frac{n-k}{n-1}}} < \frac{an}{k2^{\frac{n-k}{k-1}}(an)^{\frac{n-k}{n-1}}} = \frac{(an)^{\frac{k-1}{n-1}}}{k2^{\frac{n-k}{k-1}}} < \frac{(an)^{\frac{k-1}{n-1}}}{k}.$$

Thus, for $b < b_0 < (an)^{\frac{k-1}{n-1}}/k$ we have

$$b < \frac{a(n-1)}{(k+1)2^{\frac{n-k}{k-1}}(an)^{\frac{n-k}{n-1}}} < \frac{a(n-1)}{(k+1)(1+bk(an)^{\frac{1-k}{n-1}})^{\frac{n-k}{k-1}}(an)^{\frac{n-k}{n-1}}}.$$

If z_0 is any nontrivial zero of $p_{a,b}(z)$ then we know from Lemma 4.3 that

$$|z_0| > (1+bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}.$$

From the previous inequality we can then build the following chain of inequalities:

$$b < \frac{a(n-1)|z_0|^{n-k}}{k+1} \Rightarrow b(k+1)|z_0|^k < a(n-1)|z_0|^n \Rightarrow bk|z_0|^k < a(n-1)|z_0|^n - b|z_0|^k$$

$$bk|z_0|^k < |a(n-1)z_0^n - b\bar{z_0}^k| \Rightarrow \frac{bk|z_0|^k}{|a(n-1)z_0^n - b\bar{z_0}^k|} < 1.$$

We now use the fact that z_0 is a zero of $p_{a,b}(z)$ to substitute in the denominator, giving

$$\frac{bk|z_0|^k}{|anz_0^n + z_0|} < 1 \Rightarrow \frac{|bkz_0^{k-1}|}{|anz_0^{n-1}+1|} < 1$$

after dividing the numerator and denominator by $|z_0|$ at the end. We have thus established that the modulus of the dilatation function is smaller than 1 for any nontrivial zero when $b < b_0$. This, combined with Propositions 2.1 and 2.2, gives us that $p_{a,b}(z)$ has n zeros in this case, all of which are in the sense-preserving region. □

It should be noted that we are not saying that this chosen value of b_0 is a sharp bound. It may be possible to improve this bound.

Chapter 5. Large b

In this last chapter, we establish the following theorem:

Theorem 5.1. *If $p_{a,b}(z)$ has no zeros in the annuli*

$$(1+bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}} < |z| < (bk(an)^{\frac{1-k}{n-1}}-1)^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}$$

or

$$equation(bk(an)^{\frac{1-k}{n-1}}-1)^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}} < |z| < (1+bk(an)^{\frac{1-k}{n-1}})^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}}$$

then there exists b_N, dependent on $a, n,$ and k, such that, if $b > b_N$ then $p_{a,b}(z)$ has $n+2k+2$ zeros.

We begin this chapter with some lemmas that give restrictions on the sense-preserving and sense-reversing regions. We begin by establishing bounds on the sense-preserving and sense-reversing regions. We then use these bounds together with Rouché's theorem to establish the number of zeros.

Lemma 5.2. *If*

$$|z| \geq (1+bk(an)^{\frac{1-k}{n-1}})^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}}$$

then z is in the sense-preserving region.

Proof. Similarly to Lemma 4.2, we will show that

$$|\omega(z)| = \frac{|bkz^{k-1}|}{|anz^{n-1}+1|} < 1$$

for such z. Let $q \in \mathbb{R}$ with

$$q \geq (1+bk(an)^{\frac{1-k}{n-1}})^{\frac{1}{n-k}} \tag{5.1}$$

so that $|z| = q(an)^{-\frac{1}{n-1}}$. Since a, b, k, and n are all positive, it is evident from (5.1) that $q > 1$. Through some manipulation of (5.1) we get

$$q^{n-k} \geq 1 + bk(an)^{\frac{1-k}{n-1}}.$$

We now replace the 1 on the right-hand side with the smaller quantity q^{1-k} to obtain

$$q^{n-k} < q^{1-k} + bk(an)^{\frac{1-k}{n-1}}.$$

Now, we can multiply both sides by q^{k-1} and subtract 1 from both sides to get

$$q^{n-1} - 1 > bkq^{k-1}(an)^{\frac{1-k}{n-1}}.$$

Using the fact that $|z| = q(an)^{-\frac{1}{n-1}}$, this is equivalent to

$$an|z|^{n-1} - 1 > bk|z|^{k-1}.$$

Using the triangle inequality, we can then say

$$|1 + anz^{n-1}| > bk|z|^{k-1} \Rightarrow \frac{|bkz^{k-1}|}{|anz^{n-1} + 1|} < 1. \quad \square$$

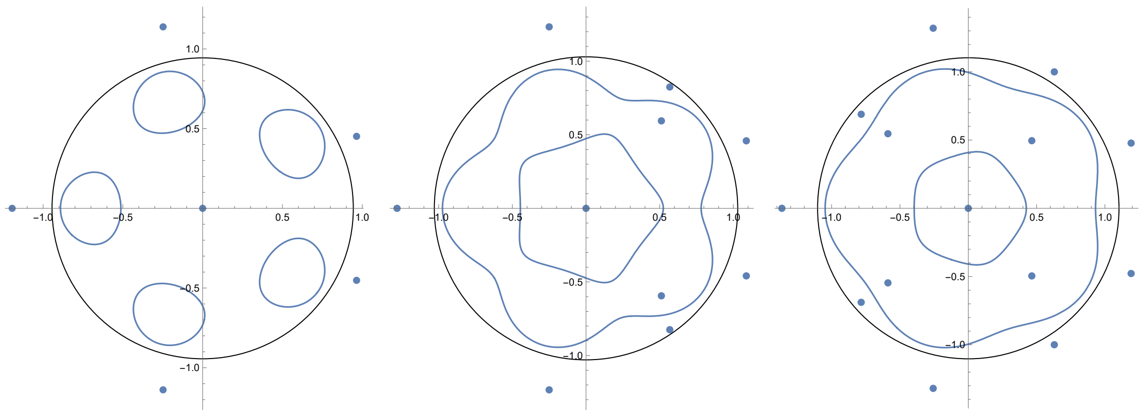
Figure 5.1 gives some graphs with the same polynomials as in Figure 4.2, but this time with this new bound.

Lemma 5.3. *If* $b > \frac{2}{k}(an)^{\frac{k-1}{n-1}}$ *and*

$$(bk(an)^{\frac{1-k}{n-1}} - 1)^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}} < |z| < (bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}} \tag{5.2}$$

then z is in the sense-reversing region.

Figure 5.1: Graphs showing circle for which the outside is sense-preserving



(a) $f_1(z) = z^6 + \bar{z}^3 + z$ (b) $f_2(z) = z^6 + 1.5\bar{z}^3 + z$ (c) $f_3(z) = z^6 + 2\bar{z}^3 + z$

Proof. Again, similarly to previous results, we will show that

$$|\omega(z)| = \frac{|bkz^{k-1}|}{|anz^{n-1}+1|} > 1$$

whenever (5.2) is satisfied. If $b > \frac{2}{k}(an)^{\frac{k-1}{n-1}}$ then

$$(bk(an)^{\frac{1-k}{n-1}} - 1)^{-\frac{1}{k-1}} < 1 < (bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1}{n-k}},$$

so we can let $c \in \mathbb{R}$ with

$$(bk(an)^{\frac{1-k}{n-1}} - 1)^{-\frac{1}{k-1}} < c < (bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1}{n-k}}$$

so that $|z| = c(an)^{-\frac{1}{n-1}}$. We will split this compound inequality into two, taking $c \leq 1$ for the inequality on the left and $c \geq 1$ for the inequality on the right.

From

$$(bk(an)^{\frac{1-k}{n-1}} - 1)^{-\frac{1}{k-1}} < c \leq 1$$

we obtain

$$c^{1-k} < bk(an)^{\frac{1-k}{n-1}} - 1 \Rightarrow c^{1-k} < bk(an)^{\frac{1-k}{n-1}} - c^{n-k} \Rightarrow c^{n-1} + 1 < bkc^{k-1}(an)^{\frac{1-k}{n-1}}.$$

Using the fact that $|z| = c(an)^{-\frac{1}{n-1}}$, this is equivalent to

$$an|z|^{n-1} + 1 < bk|z|^{k-1}.$$

Using the triangle inequality, we can then say

$$|anz^{n-1} + 1| < bk|z|^{k-1} \Rightarrow \frac{|bkz^{k-1}|}{|anz^{n-1} + 1|} > 1.$$

On the other hand, from

$$1 \leq c < (bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1}{n-k}}$$

we obtain

$$c^{n-k} < bk(an)^{\frac{1-k}{n-1}} - 1 \Rightarrow 1 < bk(an)^{\frac{1-k}{n-1}} - c^{n-k} \Rightarrow c^{1-k} < bk(an)^{\frac{1-k}{n-1}} - c^{n-k}.$$

We now have an inequality identical to one from the previous case, so we follow the same argument as before to say $\frac{|bkz^{k-1}|}{|anz^{n-1}+1|} > 1$. Thus, the interior of the annulus described by (5.1) is within the sense-reversing region. $\square$

In Figure 5.2 we see some figures showing cases where this result holds. We now move to the main theorem of this chapter.

Theorem 5.1. *If $p_{a,b}(z)$ has no zeros in the annuli*

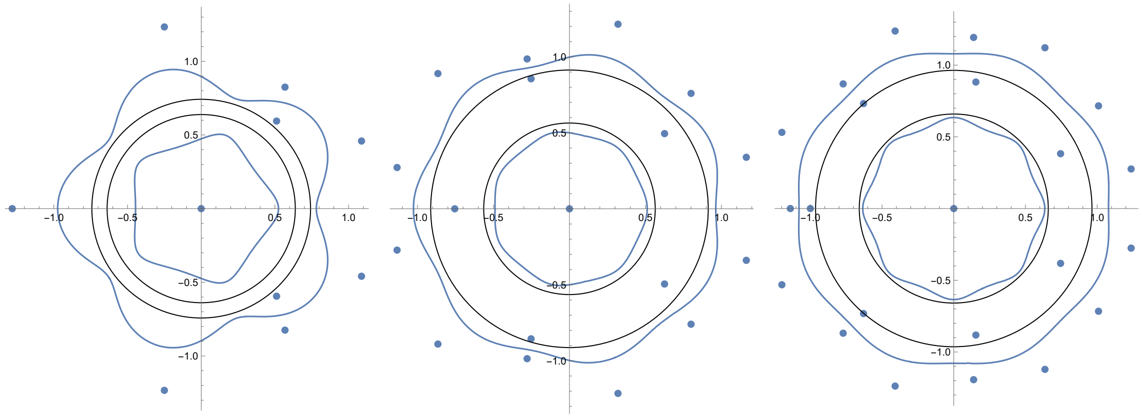
$$(1 + bk(an)^{\frac{1-k}{n-1}})^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}} < |z| < (bk(an)^{\frac{1-k}{n-1}} - 1)^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}} \tag{5.3}$$

or

$$(bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}} < |z| < (1 + bk(an)^{\frac{1-k}{n-1}})^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}} \tag{5.4}$$

then there exists b_N, dependent on $a, n,$ and k, such that, if $b > b_N$ then $p_{a,b}(z)$ has $n+2k+2$ zeros.

Figure 5.2: Graphs showing annulus contained in sense-reversing region



(a) $f_1(z) = z^6 + 1.5\bar{z}^3 + z$ (b) $f_2(z) = z^8 + 2\bar{z}^4 + z$ (c) $f_3(z) = z^9 + 2\bar{z}^6 + z$

Proof. Our first step will be to show that $p_{a,b}(z)$ has the same total order of zeros inside the circle $|z| = (bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}}$ as does the polynomial $b\bar{z}^k + z$ using Rouché's Theorem. To do this we must show that $|az^n| < |b\bar{z}^k + z|$ on this circle. We will set $b_N = \frac{n-1}{n-k}(an)^{\frac{k-1}{n-1}}$. We then have

$$b > \frac{n-1}{n-k}(an)^{\frac{k-1}{n-1}} \Rightarrow b(n-k)(an)^{\frac{1-k}{n-1}} > n-1 \Rightarrow bn(an)^{\frac{1-k}{n-1}} - bk(an)^{\frac{1-k}{n-1}} > n-1$$

$$\Rightarrow bk(an)^{\frac{1-k}{n-1}} - 1 < bn(an)^{\frac{1-k}{n-1}} - n.$$

Since $\frac{n-1}{n-k} > \frac{2}{k}$ for all $k \geq 2$, it follows that $b > \frac{2}{k}(an)^{\frac{k-1}{n-1}}$ or, equivalently, $bk(an)^{\frac{1-k}{n-1}} - 1 > 1$, which means $(bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1-k}{n-k}} < 1$, so we can multiply the $-n$ on the right-hand side by $(bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1-k}{n-k}}$ and still preserve the inequality. This leads to

$$bk(an)^{\frac{1-k}{n-1}} - 1 < bn(an)^{\frac{1-k}{n-1}} - n(bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1-k}{n-k}}$$

$$\Rightarrow \frac{1}{an}(abk(an)^{\frac{1-k}{n-1}} - a) < b(an)^{\frac{1-k}{n-1}} - (bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{1-k}{n-k}}$$

after dividing both sides by n and factoring out $1/a$ on the left-hand side. We now multiply both sides by $(bk(an)^{\frac{1-k}{n-1}} - 1)^{\frac{k}{n-k}}(an)^{-\frac{1}{n-1}}$, which is guaranteed to be positive by our choice of b, to obtain

$$a(bk(an)^{\frac{1-k}{n-1}}-1)^{\frac{n}{n-k}}(an)^{-\frac{n}{n-1}} < b(bk(an)^{\frac{1-k}{n-1}}-1)^{\frac{k}{n-k}}(an)^{-\frac{k}{n-1}} - (bk(an)^{\frac{1-k}{n-1}}-1)^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}}$$

$$\Rightarrow a|z|^n < b|z|^k - |z| \Rightarrow |az^n| < |b\bar{z}^k + z|.$$

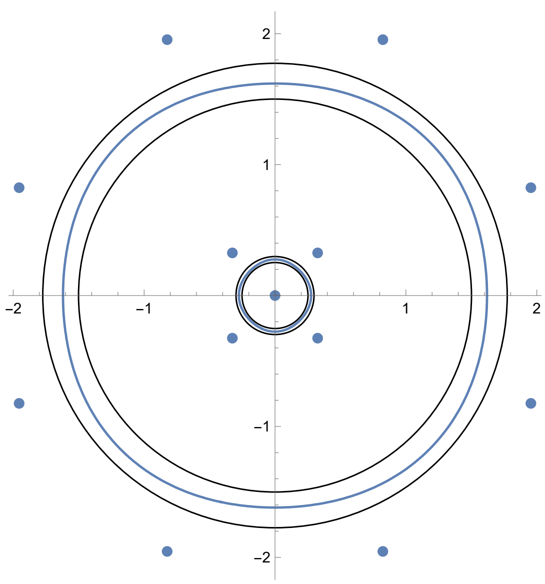
We can now use Rouché's Theorem to say that $p_{a,b}(z)$ and $b\bar{z}^k + z$ have the same total order of zeros inside the circle $|z| = (bk(an)^{\frac{1-k}{n-1}}-1)^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}}$.

Using an argument similar to that of Proposition 2.1, we can show that the sum of the orders of the zeros of $b\bar{z}^k+z$ is $-k$. Doing some simple manipulations on the equation $b\bar{z}^k+z = 0$ we find that the nontrivial zeros of $b\bar{z}^k + z$ have modulus $b^{-\frac{1}{k-1}}$. For our chosen range of b this becomes $\left(\frac{n-1}{n-k}\right)^{-\frac{1}{k-1}}(an)^{-\frac{1}{n-1}}$, which is smaller than $(bk(an)^{\frac{1-k}{n-1}}-1)^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}}$ because $\left(\frac{n-1}{n-k}\right)^{-\frac{1}{k-1}} < 1 < (bk(an)^{\frac{1-k}{n-1}}-1)^{\frac{1}{n-k}}$. Thus, all the zeros of $b\bar{z}^k+z$ are inside our circle. From (5.3), Lemmas 4.2 and 5.2, and Proposition 2.2 we conclude that $p_{a,b}(z)$ must have only the trivial zero (which is sense-preserving) and $k+1$ zeros in the sense-reversing region inside the circle $|z| = (bk(an)^{\frac{1-k}{n-1}}-1)^{\frac{1}{n-k}}(an)^{\frac{1}{n-1}}$. From (5.4) and Lemma 5.1 all the rest of the zeros of $p_{a,b}(z)$ must be in the sense-preserving region outside of $|z| = (1 + bk(an)^{\frac{1-k}{n-1}})^{\frac{1}{n-k}}(an)^{-\frac{1}{n-1}}$. Using Propositions 2.1 and 2.2, we conclude that there must be $n+k$ of them. This gives a total of $n+2k+2$ zeros for $p_{a,b}(z)$. □

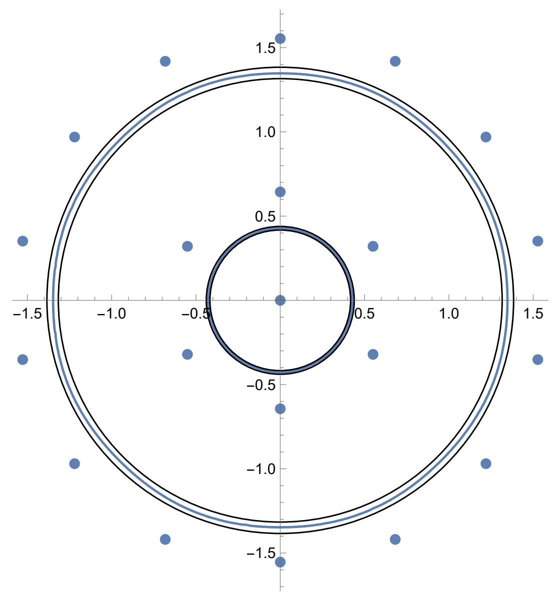
Again, we point out that this chosen value of b_N may not necessarily be a sharp bound.

In this last theorem, we had to assume that $p_{a,b}(z)$ has no zeros in the annuli given in (5.3) and (5.4). From looking at examples like the ones given Figures 5.2 and 5.3, we see that there are sometimes zeros in (5.4), but it appears that there are not zeros there if b is made sufficiently large. There appear to never be any zeros in (5.3). It may be possible to prove that there are no zeros in either of these annuli for sufficiently large b, and until this is done, we do not have a complete determination of the total number of zeros.

Figure 5.3: Graphs showing no zeros in restricted annuli



(a) $f_1(z) = z^5 + 4.5\bar{z}^3 + z$



(b) $f_2(z) = z^9 + 6\bar{z}^5 + z$

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