3-adic Properties of Hecke Traces of Singular Moduli

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ABSTRACT

3-adic Properties of Hecke Traces of Singular Moduli

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As shown by Zagier, singular moduli can be represented by the coefficients of a certain half integer weight modular form. Congruences for singular moduli modulo arbitrary primes have been proved by Ahlgren and Ono, Edixhoven, and Jenkins. Computation suggests that stronger congruences hold for small primes \( p \in \{2, 3, 5, 7, 11\} \). Boylan proved stronger congruences hold in the case where \( p = 2 \). We conjecture congruences for singular moduli modulo powers of \( p \in \{3, 5, 7, 11\} \). In particular, we study the case where \( p = 3 \) and reduce the conjecture to a congruence for a simpler modular form.

Keywords: modular forms, congruences, singular moduli
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Chapter 1. Introduction

Let $\Gamma_0(N)$ be the congruence subgroup of $\text{SL}_2(\mathbb{Z})$

$$\Gamma_0(N) = \{(a \ b \ c \ d) \in \text{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N}\}.$$ 

A modular form of weight $k$ and level $N$ is a function $f: \mathcal{H} \to \mathbb{C}$, where $\mathcal{H}$ is the upper half plane, that satisfies

$$f \left( \frac{az+b}{cz+d} \right) = (cz+d)^k f(z) \text{ for all } (a \ b \ c \ d) \in \Gamma_0(N)$$

and is holomorphic on $\mathcal{H}$ and at the cusps of $\Gamma_0(N)$. We can define a weakly holomorphic modular form by relaxing the condition that $f$ be holomorphic at the cusps of $\Gamma_0(N)$ and requiring $f$ to be only meromorphic at the cusps. We denote the space of holomorphic modular forms of weight $k$ and level $N$ by $M_k(\Gamma_0(N))$ and the space of weakly holomorphic modular forms of weight $k$ and level $N$ by $M^I_k(\Gamma_0(N))$. In addition we denote the space of modular forms with nontrivial character $\chi$ as $M_k(\Gamma_0(N), \chi)$. However, when considering spaces of modular forms with trivial character, we will just use the notation $M_k(\Gamma_0(N))$ or $M^I_k(\Gamma_0(N))$. These modular forms have Fourier expansions $f(z) = \sum_{n=-\infty}^{\infty} a(n)q^n$ where $q = e^{2\pi iz}$ and $a(n)$ is the $n$th Fourier coefficient.

The Fourier coefficients of many modular forms satisfy interesting congruences. For example, the modular $j$-invariant

$$j(z) = \frac{1}{q} + 744 + \sum_{n=1}^{\infty} c(n)q^n \in M^I_0(\text{SL}_2(\mathbb{Z}))$$
satisfies the congruences

\[
\begin{align*}
    c(2^an) &\equiv 0 \pmod{2^{3a+8}} \\
    c(3^an) &\equiv 0 \pmod{3^{2a+3}} \\
    c(5^an) &\equiv 0 \pmod{5^{a+1}} \\
    c(7^an) &\equiv 0 \pmod{7^a} \\
    c(11^an) &\equiv 0 \pmod{11^a}
\end{align*}
\]

as proved by Lehner in [1, 2] and Atkin in [3].

In addition, congruences have been proved for canonical bases of certain spaces of integer weight modular forms. Following Duke and Jenkins as in [4], write \(k = 12\ell + k'\) where \(k' \in \{4, 6, 8, 10, 14\}\). For \(m \geq -\ell\) there is a unique modular form

\[
f_{k,m} = q^{-m} \sum_{n \geq \ell + 1} a_k(m, n)q^n \in M_k^!(\text{SL}_2(\mathbb{Z})).
\]

Duke and Jenkins ([4], Theorem 3) showed that if \(k \in \{4, 6, 8, 10, 14\}\) and if \(p \nmid m\), then

\[
a_k(m, p^r n) \equiv 0 \pmod{p^{(k-1)r}}.
\]

Doud and Jenkins observed that these congruences are actually stronger than the above result for the small primes \(p = 2, 3, 5\). They proved in [5] that for \(p \in \{2, 3, 5\}\), if \(p \nmid m\), then

\[
a_k(m, p^r n) \equiv 0 \pmod{p^{(k-1)r+a}}
\]

where

\[
a = \begin{cases} 
7 & \text{if } p = 2, \\
2 & \text{if } p = 3, \\
1 & \text{if } p = 5.
\end{cases}
\]
We consider bases for spaces of half integer weight modular forms. These are of particular interest because they are related to singular moduli.

Let $d \equiv 0, 3 \pmod{4}$ and define $Q_d$ as the set of positive definite binary quadratic forms $Q(x, y) = ax^2 + bxy + cy^2$ with discriminant $d = b^2 - 4ac$ under the usual action of $\Gamma = \text{PSL}_2(\mathbb{Z})$. Let $\alpha_Q$ be the unique root of $Q(x, 1)$ in $\mathbb{H}$. Then $j(\alpha_Q)$ for $Q \in Q_d$ are algebraic integers called singular moduli. Note that $j(\alpha_Q)$ depends only on the $\Gamma$-equivalence class of $Q$. In addition, let $h(-d)$ be the class number of $-d$ or the number of $\Gamma$-equivalence classes of primitive quadratic forms in $Q_d$. As $Q$ runs through a complete set of $h(-d)$ representatives of $Q_d/\Gamma$, the corresponding $h(-d)$ values of $j(\alpha_Q)$ form a complete set of algebraic conjugates. The sum of all the $h(-d)$ values of $j(\alpha_Q)$ is the trace.

Following Zagier [6], we make some adjustments to the formula above. We instead consider $J(z) = j(z) - 744$ and sum over all quadratic forms, not just primitive quadratic forms. In addition, define

$$\omega_Q = \begin{cases} 
3 & \text{if } Q \sim_{\Gamma} [a, a, a], \\
2 & \text{if } Q \sim_{\Gamma} [a, 0, a], \\
1 & \text{otherwise}.
\end{cases}$$

Then we define the Hurwitz-Kronecker class number

$$H(-d) = \sum_{Q \in Q_d/\Gamma} \frac{1}{\omega_Q}$$

and the trace function

$$t_1(d) = \sum_{Q \in Q_d/\Gamma} \frac{1}{\omega_Q} J(\alpha_Q).$$

Now let $\eta(z)$ denote the function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$
which is a common building block for many modular forms. Also let
\[ E_4(z) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n \]
be the weight 4 Eisenstein series. Consider the modular form
\[ g_1(z) = \frac{\eta^2(z)}{\eta(2z)} \cdot \frac{E_4(4z)}{\eta^6(4z)} \]
which is a basis element for \( M_2^!(\Gamma_0(4)) \). The Fourier expansion of \( g_1(z) \) is
\[ g_1(z) = q^{-1} - 2 + \sum_{d \equiv 0, 3 \mod 4} B(d)q^d = q^{-1} - 2 + 248q^3 - 492q^4 + 4119q^7 - 7256q^8 + O(q^{11}). \]

Zagier ([6], Theorem 1) showed that \( t_1(d) = -B(d) \). Thus we can understand the divisibilities of traces of singular moduli by studying the congruences for the coefficients of \( g_1(z) \).

We can also define Hecke traces of singular moduli. Let \( k = \lambda + \frac{1}{2}, \lambda \in \mathbb{Z} \), and let \( T_k(m^2) \) denote the half integer weight Hecke operator. For \( p \) prime, \( T_k(p^2) \) acts on the Fourier coefficients of a modular form \( f = \sum a(n)q^n \in M_k(\Gamma_0(4N), \chi) \) by
\[ f \mid T_k(p^2) = \sum \left( a(p^2n) + \left( \frac{(-1)^{\lambda n}}{p} \right) \chi(p)p^{\lambda-1}a(n) + \chi(p^2)p^{2\lambda-1}a(n/p^2) \right) q^n \]
where \( a(n/p^2) = 0 \) if \( p^2 \mid n \) and \( \chi(p) = 0 \) if \( p \mid N \).

Again following Zagier ([6], section 6), define \( J_m \) to be the unique holomorphic function on \( \mathcal{H}/\text{PSL}_2(\mathbb{Z}) \) with Fourier expansion \( q^{-m} + O(q) \). For \( m = 1 \), this is just the function \( J(z) = j(z) - 744 \) as defined above. The beginnings of the Fourier expansions for \( J_m \) for \( m = 2, 3, 4 \) are
Then we define the Hecke trace as

$$t_m(d) = \sum_{Q \in \mathcal{Q}/\Gamma} \frac{1}{\omega_\mathcal{Q}} J_m(\alpha_\mathcal{Q}).$$

Now, following the notation of Boylan in [7], let $B_m(d)$ be the Fourier coefficient of $q^d$ in $g_m(z) = g_1(z) | T_3^2(m^2)$. Zagier ([6], Theorem 5) also showed that $t_m(d) = -B_m(d)$. This again means that we can prove congruences for $t_m(d)$ by proving congruences for $g_m(z)$.

Many results on the congruences of the $t_m$ have already been proven. Ahlgren and Ono [8] proved that for $p$ an odd prime, if $p$ splits in $\mathbb{Q}(\sqrt{-d})$ then

$$t_m(p^2d) \equiv 0 \pmod{p}.$$

Edixhoven [9] extended this result and showed that if $\left(\frac{-d}{p}\right) = 1$, then

$$t_m(p^{2n}d) \equiv 0 \pmod{p^n}.$$

In addition, Jenkins [10] obtained an exact formula for $t_1(p^{2n}d)$, using arguments on duality and formulas for the Hecke operator in half integral weight, from which Edixhoven’s result in the case where $m = 1$ follows as a corollary.

Guerzhoy [11] comments on an observation by Ono that when $p \leq 11$ the maximum congruence modulus of $t_1(d)$ exceeds $p^n$. Note the parallel to the stronger congruences for small primes in the integer weight modular forms. Without proving specific congruence
moduli, Guerzhoy [11] proved that if \( p \leq 11 \) and \( \left( \frac{d}{p} \right) = 1 \) then the \( p \)-adic limit

\[
\lim_{n \to \infty} p^{-n}t_1(p^{2n}d) = 0
\]

which explains Ono’s observation. Boylan was able to prove a stronger congruence in the case where \( p = 2 \) in [7]. He showed for odd \( m \) if \( d \equiv 7 \pmod{8} \), then \( t_m(2^{2n}d) \equiv 0 \pmod{2^{4n+1}} \).

In an attempt to further investigate Ono’s observations, we examine congruences of traces and Hecke traces of singular moduli for the primes \( p = 3, 5, 7, \) and 11.

Define the modular forms

\[
F(z) = \sum_{n=0}^{\infty} \sigma(2n + 1)q^{2n+1} \in M_2(\Gamma_0(4)),
\]

\[
\Theta(z) = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \in M_{\frac{3}{2}}(\Gamma_0(4)),
\]

and the cusp form

\[
\Delta(z) = \frac{E_4(z)^3 - E_6(z)^2}{1728} \in S_{12}(\text{SL}_2(\mathbb{Z})).
\]

Here, \( E_4 \) is the Eisenstein series defined previously and \( E_6 \) is the weight 6 Eisenstein series,

\[
E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n)q^n.
\]

Note that we can also write \( g_1(z) = f_{\frac{E_4}{2}}(z)/\Delta(4z) \) where

\[
f_{\frac{E_4}{2}}(z) = F(z)^3\Theta(z)^{15} - 32F(z)^4\Theta(z)^{11} + 272F(z)^5\Theta(z)^7 - 256F(z)^6\Theta(z)^3
\]

as defined in the appendix of [12]. Computing up to 100,000 coefficients of \( g_1(z) \) using this formula and examining patterns in divisibility, we make the following conjecture regarding congruences that hold for \( t_1(d) \) for the primes 3, 5, 7, 11:
Conjecture 1.1. Let $p = 3, 5, 7$ or $11$. If $d \equiv -k^2 \pmod{4p}$ where $1 \leq k \leq p - 1$, then $t_1(p^{2n}d) \equiv 0 \pmod{p^a}$ where

$$a = \begin{cases} 
3n + 3 & \text{if } p = 3, \\
2n + 1 & \text{if } p = 5, \\
2n & \text{if } p = 7, \\
2n & \text{if } p = 11.
\end{cases}$$

Through the rest of this paper, we will focus on the case where $p = 3$. It should be noted that our computation suggests that $t_1(9^nd) \equiv 0 \pmod{3^{3n+3}}$ if $d \equiv 8, 11 \pmod{12}$; however, we can prove a slightly weaker congruence by reducing the problem to congruences of another modular form. In addition, we can generalize this weaker congruence to include some Hecke traces. The revised conjecture is as follows.

Conjecture 1.2. Let $m \in \mathbb{Z}$ with $(m, 6) = 1$. If $d \equiv 8$ or $11 \pmod{12}$, then $t_m(9^nd) \equiv 0 \pmod{3^{3n+2}}$.

As a further remark, a similar discrepancy appears to exist between the maximum congruence modulus for $p = 2$ based on our computations and what Boylan proved. Our computations suggest that if $d \equiv 7 \pmod{8}$ then $t_1(2^{2n}d) \equiv 0 \pmod{2^{4n+8}}$ compared to $2^{4n+1}$ in Boylan’s Theorem.

The rest of the paper is as follows: In Chapter 2 we examine some preliminary facts and propositions. In Chapter 3 we state a reduction and prove that it implies Conjecture 1.2. We also explore facts about the forms in the reduction. In Chapter 4 we explore possible avenues to prove the reduction.
First we define the slash operator $|_k \gamma$ on a modular form $f \in M_k(\Gamma)$ for a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z})$ as

$$f |_k \gamma = (\det \gamma)^{k/2}(cz + d)^{-k}f \left(\frac{az + b}{cz + d}\right)$$

where if $k$ is a half integer, we define the square root taking the principal branch.

A key operator in proving the above Conjecture is the $U_p$ operator. For $p$ a prime, the action of the $U_p$ operator on $f \in M_k^!(\Gamma_0(N))$ is defined by

$$f | U_p = \frac{1}{p} \sum_{i=0}^{p-1} f | \left( \begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix} \right).$$

For modular forms in integer weight $k$, if $p \mid N$ then $U_p$ maps $M_k^!(\Gamma_0(N))$ to itself, and if $p^2 \mid N$ then $U_p$ maps $M_k^!(\Gamma_0(N))$ to $M_k^!(\Gamma_0(N/p))$ [13].

Now consider modular forms of half integer weight $k$ and level $N$ and character $\chi$. Then for a prime $p$, if $4p \mid N$ then $U_p : M_k^!(\Gamma_0(N), \chi) \rightarrow M_k^!(\Gamma_0(N), \chi^p)$ where $\chi^p$ is as defined in Section 1.2 of [14]. So in half integer weight, $U_p$ introduces a new character $\chi^p$ (see [14]). However, the $U_p^2 = U_{p^2}$ operator preserves spaces in half integer weight similar to the $U_p$ operator in integer weight. For $k$ a half integer, if $p \mid N$ then $U_{p^2}$ maps $M_k^!(\Gamma_0(N))$ to itself, and if $p^2 \mid N$ then $U_{p^2}$ maps $M_k^!(\Gamma_0(N))$ to $M_k^!(\Gamma_0(N/p))$ (see [14]). Furthermore, if $p \nmid N$, then $U_{p^2} : M_k^!(\Gamma_0(N)) \rightarrow M_k^!(\Gamma_0(Np))$. This follows from the previous remark and the fact that $M_k^!(N) \subset M_k!(Np)$.

The action of the $U_p$ operator on the Fourier expansion of a modular form $f = \sum a(n)q^n$ gives

$$f | U_p = \sum a(pm)q^n.$$ 

This operator is multiplicative so that $U_nU_m = U_{nm}$ for integers $n$ and $m$. It is also commutative and a linear operator.
As we are interested in the coefficients \( t_1(9^n d) \) of \( g_1(z) \), we will want to consider
\[
g_1 \mid U_{9^n} = \sum_{d \equiv 0, 3(4)} t_1(9^n d)q^d.
\]

It is also useful to consider the \( U_9 = U_3^2 = U_{3^2} \) operator rather than the \( U_3 \) operator so
that we avoid introducing a character. Recall that \( g_1(z) \in M_1^0(\Gamma_0(4)) \). Since \( 3 \nmid 4 \), acting on
forms in this space with \( U_9 \) gives forms in \( M_1^0(\Gamma_0(12)) \). It will be helpful for us to understand
\( \Gamma_0(12) \) as the forms we are interested in examining, \( g_1 \mid U_{9^n} \), are modular for \( \Gamma_0(12) \).

There are 6 nonequivalent cusps in \( \Gamma_0(12) \) which are \( \infty, 0, \frac{1}{3}, \frac{1}{6}, \frac{1}{2}, \) and \( \frac{1}{4} \). Thus the poles
of forms in \( M_1^0(\Gamma_0(12)) \) will be at these cusps. Although we specifically want to act on forms
by \( U_9 \) since we are in half integer weight, we can find out what happens at the cusps when
acting on a form by \( U_{9^n} \) by considering what happens at cusps when acting on a form by
\( U_3 \). Using the slash operator definition for \( U_p \), we have
\[
3f \mid U_3 = f \mid \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} + f \mid \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} + f \mid \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}.
\]

If we want to consider what happens at each cusp, we will also slash by another matrix. For
the cusp at 0, we slash by \( \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and get
\[
3f \mid U_3 \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = f \mid \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + f \mid \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + f \mid \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = f \mid \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \mid \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix} + f \mid \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 1 & -1 \\ 0 & 3 \end{pmatrix} + f \mid \begin{pmatrix} 2 & -1 \\ 3 & -1 \end{pmatrix} \mid \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}.
\]

For the cusp at \( \frac{1}{2} \) we have
\[
3f \mid U_3 \mid \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = f \mid \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + f \mid \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} + f \mid \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} = f \mid \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 0 & 1 \\ 0 & 3 \end{pmatrix} + f \mid \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix} + f \mid \begin{pmatrix} 0 & -1 \\ 0 & 3 \end{pmatrix} \mid \begin{pmatrix} 1 & 1 \\ 0 & 3 \end{pmatrix}.
\]
For the cusp at $\frac{1}{4}$ we have

$$3f \mid U_3 \mid \left(\frac{1}{4} \ 0\right) = f \mid \left(\frac{1}{0} \ 0\right) \mid \left(\frac{1}{4} \ 0\right) + f \mid \left(\frac{1}{0} \ 1\right) \mid \left(\frac{1}{0} \ 0\right) + f \mid \left(\frac{1}{0} \ 2\right) \mid \left(\frac{1}{0} \ 0\right) = f \mid \left(\frac{1}{12} \ 0\right) \mid \left(\frac{1}{0} \ 0\right) + f \mid \left(\frac{5}{12} \ 2\right) \mid \left(\frac{1}{0} \ 1\right) + f \mid \left(\frac{3}{4} \ -1\right) \mid \left(\frac{3}{0} \ 1\right).$$

For the cusp at $\frac{1}{3}$ we have

$$3f \mid U_3 \mid \left(\frac{1}{3} \ 0\right) = f \mid \left(\frac{1}{0} \ 0\right) \mid \left(\frac{1}{3} \ 0\right) + f \mid \left(\frac{1}{0} \ 1\right) \mid \left(\frac{1}{3} \ 0\right) + f \mid \left(\frac{1}{0} \ 2\right) \mid \left(\frac{1}{3} \ 0\right) = f \mid \left(\frac{1}{9} \ 0\right) \mid \left(\frac{1}{0} \ 0\right) + f \mid \left(\frac{4}{9} \ -1\right) \mid \left(\frac{1}{0} \ 1\right) + f \mid \left(\frac{7}{3} \ -1\right) \mid \left(\frac{1}{0} \ -1\right).$$

Finally, for the cusp at $\frac{1}{6}$ we have

$$3f \mid U_3 \mid \left(\frac{1}{6} \ 0\right) = f \mid \left(\frac{1}{0} \ 0\right) \mid \left(\frac{1}{6} \ 0\right) + f \mid \left(\frac{1}{0} \ 1\right) \mid \left(\frac{1}{6} \ 0\right) + f \mid \left(\frac{1}{0} \ 2\right) \mid \left(\frac{1}{6} \ 0\right) = f \mid \left(\frac{1}{18} \ 0\right) \mid \left(\frac{1}{0} \ 0\right) + f \mid \left(\frac{7}{18} \ -2\right) \mid \left(\frac{1}{0} \ 1\right) + f \mid \left(\frac{13}{18} \ -8\right) \mid \left(\frac{1}{0} \ -1\right).$$

Note that the rational numbers on the right of the list below are equivalent to the cusp on the left under the action of $\Gamma_0(12)$:

$$\begin{align*}
\frac{1}{3} & : \quad \frac{2}{3}, \frac{1}{6}, \frac{4}{9}, \frac{7}{9} \\
\frac{1}{6} & : \quad \frac{5}{6}, \frac{1}{18}, \frac{7}{18}, \frac{13}{18} \\
\frac{1}{4} & : \quad \frac{3}{4} \\
\infty & : \quad \frac{1}{12}, \frac{5}{12}.
\end{align*}$$

Since slashing with a matrix of the form $\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right)$ does not change holomorphicity at a cusp, then the holomorphicity of $3f \mid U_3$ at the cusps 0, $\frac{1}{2}$, and $\frac{1}{4}$ depends on the holomorphicity of $f$ at the cusps 0 and $\frac{1}{3}$, at the cusps $\frac{1}{6}$ and $\frac{1}{2}$, and at the cusps $\infty$ and $\frac{1}{4}$ respectively.

Additionally, the holomorphicity of $f \mid U_3$ at the cusps $\frac{1}{3}$, $\frac{1}{6}$ and $\infty$ depend only on the holomorphicity of $f$ at those same cusps. This implies that if $f$ is holomorphic at 0 and $\frac{1}{3}$,
then \( f \mid U_{9^a} \) will also be holomorphic at 0 and \( \frac{1}{3} \). If \( f \) is holomorphic at \( \frac{1}{2} \) and \( \frac{1}{6} \) then \( f \mid U_{9^a} \) will be holomorphic at \( \frac{1}{2} \) and \( \frac{1}{6} \), and if \( f \) is holomorphic at \( \frac{1}{4} \) and \( \infty \) then \( f \mid U_{9^a} \) will also be holomorphic at \( \frac{1}{4} \) and \( \infty \). We will return to these facts in Chapter 3.

It is also worthwhile to note that \( g_1(z) \) is in the Kohnen plus space, meaning that the \( d \)th Fourier coefficient is equal to 0 whenever \( d \) is not equivalent to 0, 3 (mod 4). It is notated \( M_+^k(\Gamma) \). Thus \( g_1(z) \in M_+^{15}(\Gamma_0(4)) \). Note that a form in the plus space acted on by \( U_{9^n} \) remains in the plus space since if \( d \equiv 0, 3 \pmod{4} \) and \( 9 \mid d \), then \( \frac{d}{9} \equiv 0, 3 \pmod{4} \).

By following the process listed in Section 3 of [15], we can construct a partial basis for \( M_+^{15}(\Gamma_0(12)) \). This begins with first getting a basis for \( M_{15}^{12}(\Gamma_0(12)) \) which we computed in MAGMA. The forms from this basis which we will need are

\[
\begin{align*}
f_{\frac{15}{2}, 7} &= q^7 + 141q^{15} - 50q^{16} - 360q^{17} + 432q^{18} + 628q^{19} - 1188q^{20} + O(q^{21}), \\
f_{\frac{15}{2}, 8} &= q^8 + 8q^{15} + 215q^{16} - 48q^{17} - 928q^{18} + 496q^{19} + 2666q^{20} + O(q^{21}), \\
f_{\frac{15}{2}, 11} &= q^{11} - 4q^{15} + 14q^{16} + 24q^{17} - 64q^{18} - 5q^{19} + 196q^{20} + O(q^{21}), \\
f_{\frac{15}{2}, 12} &= q^{12} + 2q^{15} - 15q^{16} + 88q^{18} - 30q^{19} - 234q^{20} + O(q^{21}).
\end{align*}
\]

A basis for \( M_+^{15}(\Gamma_0(12)) \) can be found by taking linear combinations of the basis elements for \( M_+^{15}(\Gamma_0(12)) \) and row reducing. We can create two forms, \( g_7, g_8 \in M_+^{15}(\Gamma_0(12)) \) by taking linear combinations of the forms above:

\[
\begin{align*}
g_7 &= f_{\frac{15}{2}, 7} + 15f_{\frac{15}{2}, 11} + 6f_{\frac{15}{2}, 12} = q^7 + 15q^{11} + 6q^{12} + 93q^{15} + 70q^{16} + O(q^{19}), \\
g_8 &= f_{\frac{15}{2}, 8} + 2f_{\frac{15}{2}, 11} + 12f_{\frac{15}{2}, 12} = q^8 + 2q^{11} + 12q^{12} + 24q^{15} + 63q^{16} + O(q^{19}).
\end{align*}
\]

In order to get forms in weight \( \frac{3}{2} \) rather than \( \frac{15}{2} \), we take the two forms \( g_7 \) and \( g_8 \) for weight 15/2 and divide by the form \( C_{6,3}(4z)\Phi_3(4z) \in M_6(\Gamma_0(12)) \) where \( C_{6,3}(z) \) is the normalized cusp form of weight 6 and level 3 with Fourier expansion

\[
C_{6,3}(z) = q - 6q^2 + 9q^3 + 4q^4 + 6q^5 + O(q^6),
\]

11
and

$$\Phi_3(z) = \left(\frac{\eta(3z)}{\eta(z)}\right)^{12} \in M_0^!(\Gamma_0(3)).$$

Note that $\Phi_3(z)$ has a simple pole at the cusp 0 and a simple zero at the cusp $\infty$. From this, we get two forms in $M^!_{3/2}(\Gamma_0(12))$

$$f_{-1} = \frac{g_7}{C_{6,3}(4z)\Phi_3(4z)} = q^{-1} + 9q^3 + 6q^4 + 12q^7 + O(q^8),$$

$$f_0 = \frac{g_8}{C_{6,3}(4z)\Phi_3(4z)} = 1 + 2q^3 + 6q^4 + 12q^7 + O(q^{12}).$$

Further forms in $M^!_{3/2}(\Gamma_0(12))$ with higher leading powers can be obtained by multiplying one of these two forms by powers of $\Phi_3(4z)$ (depending on what leading exponent is desired). For example, other forms included in a basis for $M^!_{3/2}(\Gamma_0(12))$ might be

$$f_3 = f_{-1}\Phi_3(4z) = q^3 + ..., $$

$$f_4 = f_0\Phi_3(4z) = q^4 + ..., $$

$$f_7 = f_{-1}\Phi_3(4z)^2 = q^7 + ..., $$

$$f_8 = f_0\Phi_3(4z)^2 = q^8 + ..., $$

and so on. Each of these above basis elements are holomorphic at $\infty$ as given by a positive leading exponent. In $M^!_{3/2}(\Gamma_0(12))$ the holomorphicity of a modular form $f$ at $\infty$ determines the holomorphicity at the cusps $\frac{1}{3}$ and $\frac{1}{6}$ and the holomorphicity of $f$ at 0 determines the holomorphicity of $f$ at the cusps $\frac{1}{4}$ and $\frac{1}{2}$ (see [15]). This implies that $f_3, f_4, f_7, f_8, \ldots$ would only have poles possibly at the cusps 0, $\frac{1}{2}$ and $\frac{1}{4}$. As a reminder, these forms don’t form a complete basis since we can also have forms with arbitrary negative leading exponent and must also account for being able to have arbitrary negative orders of vanishing at each of the 6 cusps in $\Gamma_0(12)$. Our computations above also don’t list a complete basis for $M^!_{3/2}(\Gamma_0(12))$, but rather just the ones that we needed to compute our desired forms in $M^!_{3/2}(\Gamma_0(12))$. The forms $f_{-1}, f_0, f_3, f_4, \ldots$ will be sufficient for what we need through the rest of this paper.
Although $g_1(z)$ is a level 4 modular form, if we view it as a level 12 form, we can write it as a linear combination of the basis elements found above:

$$g_1 = f_{-1} - 2f_0 + 3^5f_3 - 2 \cdot 3^5f_4.$$ 

We now prove some facts about $f_0$ and $f_{-1}$ that will be useful in finding a reduction of Conjecture 1.2.

**Proposition 2.1.** For $f_0 = \sum_{n=0}^{\infty} c(n)q^n$ as described in the basis of $M^+_3(\Gamma_0(12))$ above, and for $m \in \mathbb{Z}$ with $2 \nmid m$, we have $f_0 \mid T_3^2(m^2) = \alpha_m f_0$ with $\alpha_m \in \mathbb{Z}$ and $\alpha_3 = 1$.

*Proof.* Note that the space $M_3^2(\Gamma_0(12))$ has dimension 3 and a basis is

$$1 + 2q^3 + 6q^4 + 12q^7 + O(q^{12}),$$

$$q + q^3 + 2q^4 + 2q^6 + 2q^7 + q^9 + 4q^{10} + O(q^{12}),$$

$$q^2 - q^4 + 2q^5 + q^6 - 2q^7 + q^8 + 2q^9 + 2q^{11} + O(q^{12}).$$

The first of these basis elements is $f_0$ meaning that $f_0 \in M^+_3(\Gamma_0(12))$. Also recall that $T_3^2(m^2) : M_3^2(\Gamma_0(12)) \to M_3^2(\Gamma_0(12))$. Using the formula for the action of $T_3^2(p^2)$ on Fourier expansions, we find that for odd $p > 3$

$$f_0 \mid T_3^2(p^2) = c(0) + pc(0) + c(p^2)q + c(2p^2)q^2 + O(q^3)$$

and for $p = 3$, since $p|12$, we have

$$f_0 \mid T_3^2(p^2) = c(0) + c(p^2)q + c(2p^2)q^2 + O(q^3).$$
In either case, note that since \( p \) is odd, \( p^2 \equiv 1 \pmod{4} \) and since \( f_0 \) is in the plus space, \( c(n) = 0 \) whenever \( n \equiv 1, 2 \pmod{4} \). Thus \( c(p^2) = c(2p^2) = 0 \) giving

\[
f_0 \mid T_{3/2}(p^2) = c(0) + pc(0) + O(q^3)
\]

if \( p > 3 \) and

\[
f_0 \mid T_{3/2}(p^2) = c(0) + O(q^3)
\]

if \( p = 3 \). Because \( f_0 \mid T_{3/2}(p^2) \) must be in \( M_{3/2}(\Gamma_0(12)) \), we can write in in terms of the basis elements listed above which gives

\[
f_0 \mid T_{3/2}(p^2) = (c(0) + pc(0))f_0 = (1 + p)f_0
\]

if \( p > 3 \) and

\[
f_0 \mid T_{3/2}(p^2) = c(0)f_0 = f_0
\]

if \( p = 3 \). This proves the result in the case where \( m \) is prime.

Since \( T_{3/2}(m^2)T_{3/2}(n^2) = T_{3/2}(m^2n^2) \) for \((m, n) = 1\) we only need to consider the case where \( m = p^\ell \). If \( p = 3 \) then

\[
T_{3/2}(3^{2\ell}) = (T_{3/2}(3^2))^\ell = U_{3^{2\ell}}.
\]

If \( p > 3 \), then \( p \nmid 12 \) and

\[
T_{3/2}(p^4) = (T_{3/2}(p^2))^2 - p + p^2
\]

and

\[
T_{3/2}(p^{2\ell+2}) = T_{3/2}(p^2)T_{3/2}(p^{2\ell}) - pT_{3/2}(p^{2\ell-2})
\]

for \( \ell \geq 2 \) (see [16], page 605). Using these formulas, the case where \( m = p \), and a simple induction argument, it follows that \( f_0 \mid T_{3/2}(p^{2\ell}) = \alpha_{p^{2\ell}}f_0 \) where \( \alpha_{p^{2\ell}} \in \mathbb{Z} \). All this together proves that \( f_0 \mid T_{3/2}(m^2) = \alpha_m f_0 \) for \( 2 \nmid m \) where \( \alpha_m \in \mathbb{Z} \). \( \square \)
Corollary 2.2. $f_0 \mid U_{9^n} = f_0$.

Proof. Note that for $\Gamma_0(12)$, $T_2(3^2) = U_9$. Thus we have that $f_0 \mid U_9 = f_0 \mid T_2(9) = f_0$. By induction, we can clearly see that $f_0 \mid U_{9^n} = f_0$. □

Proposition 2.3. For $f_{-1}$, $f_3$, and $f_4$ as described in the basis of $M_{12}^+(\Gamma_0(12))$ above,

$$f_{-1} \mid U_9 = 3^5(f_3 - 2f_4).$$

Proof. Since $f_{-1} \in M_{12}^+(\Gamma_0(12))$ and $3 \mid 12$, we have $f_{-1} \mid U_9 \in M_{12}^+(\Gamma_0(12))$. Writing $f_{-1} \mid U_9$ in terms of basis elements constructed above, we find that $f_{-1} \mid U_9 = 243f_3 - 486f_4$. □

Proposition 2.4. For $f_0 = \sum_{n=0}^{\infty} c(n)q^n$, $c(n) = 0$ if $n \equiv 8, 11 \pmod{12}$.

Proof. Recall that $f_0 = \sum_{n=0}^{\infty} c(n)q^n$ is holomorphic. Note that for $1 \leq t \leq 11$ an integer,

$$\frac{1}{12} \sum_{j=0}^{11} e^{-2\pi i j/12} f_0 \mid \begin{pmatrix} 12 & j \\ 0 & 12 \end{pmatrix} = \sum_{n=0}^{\infty} c(12n + t)q^n.$$

Acting on this form by a matrix $\begin{pmatrix} a & b \\ 12^2 & c \end{pmatrix} \in \Gamma_0(12^2)$ we find it is invariant under the action of $\Gamma_0(12^2)$. Letting $t = 8$ and $t = 11$, we have

$$\sum_{n=0}^{\infty} c(n)q^n \in M_{12}^2(\Gamma_0(12^2))$$

and

$$\sum_{n=0}^{\infty} c(n)q^n \in M_{12}^2(\Gamma_0(12^2)).$$

Using the Sturm bound for $M_{12}^2(\Gamma_0(12^2))$, we see these two forms are identically 0, which implies that $c(n) = 0$ for $n \equiv 8, 11 \pmod{12}$. □
In this section we prove that Conjecture 1.2 follows from

Conjecture 3.1. For \( f_3, f_4 \) as described in the basis of \( M_{\frac{3}{2}}^+(\Gamma_0(12)) \), we have

\[
(f_3 - 2f_4) \mid U_n \equiv 0 \pmod{3^{3n}}.
\]

Theorem 3.2. Conjecture 3.1 implies Conjecture 1.2.

Proof. As previously observed, we can write

\[
g_1 = f_{-1} - 2f_0 + 3^5f_3 - 2 \cdot 3^5f_4
\]

as a linear combination of weight \( \frac{3}{2} \) level 12 basis elements. Computing \( g_1 \mid U_n \), we have

\[
g_1 \mid U_n = f_{-1} \mid U_n - 2f_0 \mid U_n + 3^5f_3 \mid U_n - 2 \cdot 3^5f_4 \mid U_n.
\]

By Proposition 2.3 and the Corollary to Proposition 2.1 we have

\[
g_1 \mid U_n = 3^5(f_3 - 2f_4) \mid U_{n-1} - 2f_0 + 3^5(f_3 - 2f_4) \mid U_n.
\]

If we assume Conjecture 3.1 then \((f_3 - 2f_4) \mid U_{n-1} \equiv 0 \pmod{3^{3n-3}}\) and \((f_3 - 2f_4) \mid U_n \equiv 0 \pmod{3^{3n}}\). Thus \( g_1 \mid U_n \equiv -2f_0 \pmod{3^{3n+2}} \).

Now consider \( g_m = g_1 \mid T_2(m^2) \). For \( m \in \mathbb{Z} \) with \( 3 \nmid m \) we have

\[
g_m \mid U_n = \left( g_1 \mid T_2(m^2) \right) \mid U_n = (g_1 \mid U_n) \mid T_2(m^2).
\]

Since \( g_1 \mid U_n \equiv -2f_0 \pmod{3^{3n+2}} \) and for \( m \) with \( 2 \nmid m \), \( f_0 \mid T_2(m^2) = \alpha_m f_0 \), we have \( g_m \mid U_n \equiv -2\alpha_m f_0 \pmod{3^{3n+2}} \) for \( m \) with \( (m,6) = 1 \). Since by Proposition 2.4, the
coefficients of $f_0$ are 0 for powers of $q$ that are 8 or 11 mod 12, then we must have that $t_m(9^n d) \equiv 0 \pmod{3^{n+2}}$ if $d \equiv 8, 11 \pmod{12}$. This is Conjecture 1.2.

For reference, the $q$-expansion of $f_3 - 2f_4$ is

$$f_3 - 2f_4 = q^3 - 2q^4 + 17q^7 - 30q^8 + 138q^{11} - 218q^{12} + 792q^{15} - 1182q^{16} + 3644q^{19} + O(q^{20}).$$

Computation shows that Conjecture 3.1 is true for the first few values of $n$. The beginning of the Fourier expansions for $(f_3 - 2f_4) \mid U_9^n$ for $n = 1$ and $n = 2$ are

$$(f_3 - 2f_4) \mid U_9 = 3^3 \cdot 1873q^3 - 3^3 \cdot 23402q^4 + 3^4 \cdot 10290383q^7 - 3^4 \cdot 19187866q^8 + 3^3 \cdot 191109386q^{11} + O(q^{12})$$

and

$$(f_3 - 2f_4) \mid U_{9^2} = 3^6 \cdot 10475835920081425q^3 - 3^6 \cdot 20438102234886292226q^4 + 3^6 \cdot 1737284483354500114862774372q^7 - 3^7 \cdot 101367032193380120871975026986q^8 + 3^7 \cdot 100153213856669301408220459653144910q^{11} + O(q^{12}).$$

We can clearly see that these first few coefficients of $(f_3 - 2f_4) \mid U_9$ are divisible by at least $3^3$ and that the first few coefficients of $(f_3 - 2f_4) \mid U_{9^2}$ are divisible by at least $3^6$.

Since $(f_3 - 2f_4) \mid U_{9^n} \in M_{1/2}^{1,+}(\Gamma_0(12))$ we can write these forms in terms of a basis for this space. In fact, we will only need the forms for a partial basis described in Chapter 2. This is because $f_3 - 2f_4$ only has poles at the cusps $1/2$ and $1/4$ (see the end of this chapter) and by our work in Chapter 2, acting on this form by $U_9$ will keep the poles at $1/2$ and $1/4$. Thus, we won’t need basis elements with poles of arbitrary order at $\infty$, for example. Using the forms
described in Chapter 2, for \( n = 1 \) and \( n = 2 \) we use SAGE to compute that

\[
(f_3 - 2f_4) \mid U_9 = \sum_{m=1}^{7} a_1(m)f_{-1}(z)\Phi_3(4z)^m + b_1(m)f_0(z)\Phi_3(4z)^m,
\]

\[
(f_3 - 2f_4) \mid U_{9^2} = \sum_{m=1}^{61} a_2(m)f_{-1}(z)\Phi_3(4z)^m + b_2(m)f_0(z)\Phi_3(4z)^m.
\]

The computations show that \( a_1(m), b_1(m) \) are divisible by at least \( 3^3 \) and \( a_2(m), b_2(m) \) are divisible by at least \( 3^6 \). The following table gives the values of \( a_n(m) \), \( b_n(m) \) when \( n = 1 \):

<table>
<thead>
<tr>
<th>( m )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_1(m) )</td>
<td>( 3^3 \cdot 1873 )</td>
<td>( 3^9 \cdot 14126 )</td>
<td>( 3^{14} \cdot 31094 )</td>
<td>( 3^{19} \cdot 6800 )</td>
</tr>
<tr>
<td>( b_1(m) )</td>
<td>( -3^3 \cdot 23402 )</td>
<td>( -3^9 \cdot 78400 )</td>
<td>( -3^{14} \cdot 120580 )</td>
<td>( -3^{19} \cdot 20944 )</td>
</tr>
<tr>
<td>( m )</td>
<td>5</td>
<td>6</td>
<td>7</td>
<td></td>
</tr>
<tr>
<td>( a_1(m) )</td>
<td>( 3^{25} \cdot 608 )</td>
<td>( 3^{31} \cdot 71 )</td>
<td>( 3^{36} \cdot 1 )</td>
<td></td>
</tr>
<tr>
<td>( b_1(m) )</td>
<td>( -3^{25} \cdot 1576 )</td>
<td>( -3^{31} \cdot 160 )</td>
<td>( -3^{36} \cdot 2 )</td>
<td></td>
</tr>
</tbody>
</table>

Divisibility by \( 3^3 \) can clearly be seen in these coefficients. Notice that the \( a_1(m) \) and \( b_1(m) \) are divisible by increasing powers of 3 as \( m \) increases.

As stated before, our computations for the linear combination given for \( n = 2 \) above show that \( a_2(m) \) and \( b_2(m) \) are divisible by at least \( 3^6 \). However, we do not list them here, as there are a lot of them and they are rather large. In addition, the pattern of increasing divisibility observed when \( n = 1 \) also appears in the case where \( n = 2 \).

The observed divisibility of \( a_n(m) \) and \( b_n(m) \) by at least \( 3^3 \) for \( n = 1 \) and at least \( 3^6 \) for \( n = 2 \) proves Conjecture 3.1 for these cases.

For the rest of this chapter we examine the orders of zeros and poles at cusps of the form \( f_3 - 2f_4 \).

Since \( f_3 - 2f_4 \) has zeros at \( \infty \) and the holomorphicity of the cusps at \( \frac{1}{3} \) and \( \frac{1}{6} \) depend on the holomorphicity at the cusp \( \infty \), then it should be the case that \( f_3 - 2f_4 \) has poles only at the cusps \( 0, \frac{1}{2}, \) and \( \frac{1}{4} \). Knowing how the orders of vanishing at the cusps change as we act on \( f_3 - 2f_4 \) with \( U_9 \) is particularly useful if we were to write \((f_3 - 2f_4) \mid U_{9^n}\) as a polynomial.
in another modular form. It also gives insight into and is a potential tool for some avenues of proof for Conjecture 3.1.

Using GP PARI, we calculate orders of vanishing at cusps of $C_{6,3}(4z)$ and the two weight $15/2$ forms $g_7$ and $g_8$ from which $f_0$ and $f_{-1}$ were derived. Note that the widths of the cusps $\infty, \frac{1}{3}, \frac{1}{6}, 0, \frac{1}{2},$ and $\frac{1}{4}$ in $\Gamma_0(12)$ are 1, 4, 1, 12, 3, and 3 respectively. The following tables give the orders of vanishing at each cusp.

$C_{6,3}(4z)$:

$$
\begin{array}{l}
\infty : \quad (4)(1) = 4 \\
\frac{1}{3} : \quad \left(\frac{1}{4}\right)(4) = 1 \\
\frac{1}{6} : \quad (1)(1) = 1 \\
0 : \quad \left(\frac{1}{12}\right)(12) = 1 \\
\frac{1}{2} : \quad \left(\frac{1}{3}\right)(3) = 1 \\
\frac{1}{4} : \quad \left(\frac{4}{3}\right)(3) = 4 \\
\end{array}
$$

$g_7 \in M_{15/2}(\Gamma_0(12))$:

$$
\begin{array}{l}
\infty : \quad (7)(1) = 7 \\
\frac{1}{3} : \quad \left(\frac{3}{4}\right)(4) = 3 \\
\frac{1}{6} : \quad \left(\frac{7}{4}\right)(1) = \frac{7}{4} \\
0 : \quad (0)(12) = 0 \\
\frac{1}{2} : \quad \left(\frac{1}{12}\right)(3) = \frac{1}{4} \\
\frac{1}{4} : \quad (0)(3) = 0 \\
\end{array}
$$

$g_8 \in M_{15/2}(\Gamma_0(12))$:

$$
\begin{array}{l}
\infty : \quad (8)(1) = 8 \\
\frac{1}{3} : \quad \left(\frac{1}{2}\right)(4) = 2 \\
\frac{1}{6} : \quad \left(\frac{11}{4}\right)(1) = \frac{11}{4} \\
0 : \quad (0)(12) = 0 \\
\frac{1}{2} : \quad \left(\frac{1}{12}\right)(3) = \frac{1}{4} \\
\frac{1}{4} : \quad (0)(3) = 0. \\
\end{array}
$$
Note that
\[ f_3 = f_{-1} \Phi_3(4z) = \left( \frac{g_7}{C_{6,3}(4z) \Phi_3(4z)} \right) \Phi_3(4z) = \frac{g_7}{C_{6,3}(4z)} \]
and
\[ f_4 = f_0 \Phi_3(4z) = \left( \frac{g_8}{C_{6,3}(4z) \Phi_3(4z)} \right) \Phi_3(4z) = \frac{g_8}{C_{6,3}(4z)} \]

This gives us the following orders of vanishing at the cusps for the forms \( f_3, f_4 \in M_{3/2}^+(\Gamma_0(12)) \):

\( f_3 \):

\[ \infty : \quad 7 - 4 = 3 \]
\[ \frac{1}{3} : \quad 3 - 1 = 2 \]
\[ \frac{1}{6} : \quad \frac{7}{4} - 1 = \frac{3}{4} \]
\[ 0 : \quad 0 - 1 = -1 \]
\[ \frac{1}{2} : \quad \frac{1}{4} - 1 = -\frac{3}{4} \]
\[ \frac{1}{4} : \quad 0 - 4 = -4 \]

\( f_4 \):

\[ \infty : \quad 8 - 4 = 4 \]
\[ \frac{1}{3} : \quad 2 - 1 = 1 \]
\[ \frac{1}{6} : \quad \frac{11}{4} - 1 = \frac{7}{4} \]
\[ 0 : \quad 0 - 1 = -1 \]
\[ \frac{1}{2} : \quad \frac{1}{4} - 1 = -\frac{3}{4} \]
\[ \frac{1}{4} : \quad 0 - 4 = -4. \]

This could lead us to guess that the orders of vanishing of \( f_3 - 2f_4 \) may look something like:

\[ \infty : \quad 3 \]
\[ \frac{1}{3} : \quad 1 \]
\[ \frac{1}{6} : \quad \frac{3}{4} \]
\[ 0 : \quad -1 \]
\[ \frac{1}{2} : \quad -\frac{3}{4} \]
\[ \frac{1}{4} : \quad -4. \]
However, this could be different if the linear combination of $f_3 - 2f_4$ produced zeros that $f_3$ and $f_4$ do not have individually.

To check the above as well as the orders of vanishing at cusps for $(f_3 - 2f_4) \mid U_{g^n}$, we will multiply by $\Theta \in M_2(\Gamma_0(4))$ and then a cusp form or otherwise that will eliminate the poles of $(f_3 - 2f_4) \mid U_{g^n} \Theta$ and work backwards to find the orders of the poles and zeros.

Using Ligozat’s theorem (which can be found in [17]) and the fact that $\Theta$ can be written as the eta quotient $\Theta = \frac{\eta(2z)^5}{\eta(z)^2 \eta(4z)^7}$, we find that when considered in $\Gamma_0(12)$, $\Theta$ has the following orders of vanishing at cusps:

\begin{align*}
\infty & : 0 \\
\frac{1}{3} & : 0 \\
\frac{1}{6} & : \frac{1}{4} \\
0 & : 0 \\
\frac{1}{2} & : \frac{3}{4} \\
\frac{1}{4} & : 0.
\end{align*}

We will also consider the form $F(z) = \frac{\eta(4z)^8}{\eta(z)^2} = \sum_{n=0}^{\infty} \sigma(2n+1)q^{2n+1} \in M_4(\Gamma_0(4))$. Again, using Ligozat’s Theorem and considering $F(z)$ to be level 12, we find the following orders of vanishing:

\begin{align*}
\infty & : 1 \\
\frac{1}{3} & : 0 \\
\frac{1}{6} & : 0 \\
0 & : 0 \\
\frac{1}{2} & : 0 \\
\frac{1}{4} & : 3.
\end{align*}

We find that $(f_3 - 2f_4) \Theta F \in M_4(\Gamma_0(12))$ and is holomorphic since it can be written as a linear combination of the basis elements of $M_4(\Gamma_0(12))$. This implies that $f_3 - 2f_4$ must be holomorphic at the cusp 0 and has poles at the cusps $\frac{1}{2}$ and $\frac{1}{4}$ with orders of vanishing not
less than $-\frac{3}{4}$ and $-3$ respectively. This gives that the orders of vanishing of $f_3 - 2f_4$ are as follows:

\[
\begin{align*}
\infty & : 3 \\
\frac{1}{3} & : 1 \\
\frac{1}{6} & : \frac{3}{4} \\
0 & : 0 \\
\frac{1}{2} & : -\frac{3}{4} \\
\frac{1}{4} & : -3.
\end{align*}
\]

Note that this suggests that taking the linear combination $f_3 - 2f_4$ results in a pole being lost at the cusp 0 as well as at the cusp $\frac{1}{4}$ from $f_3$ and $f_4$ individually. From the information about what $U_9$ does to poles in Chapter 2, this implies that if we act on $f_3 - 2f_4$ by $U_9$ repeatedly, then $(f_3 - 2f_4) | U_9^n$ is holomorphic at the cusps 0, $\infty$, $\frac{1}{3}$, and $\frac{1}{6}$ and has poles only at $\frac{1}{2}$ and $\frac{1}{4}$.

If we follow a similar process for $(f_3 - 2f_4) | U_9$, we find that $((f_3 - 2f_4) | U_9)^{\Theta^9 F^9} \in M_{24}(\Gamma_0(12))$ and is holomorphic, again, since it could be written as a linear combination of the basis elements of this space (computed in SAGE). This implies that the orders of vanishing of $(f_3 - 2f_4) | U_9$ at $\frac{1}{2}$ and $\frac{1}{4}$ are $-\frac{27}{4}$ and $-27$ respectively. We also find that $((f_3 - 2f_4) | U_9^2)^{\Theta^{81} F^{81}} \in M_{204}(\Gamma_0(12))$ and so then $(f_3 - 2f_4) | U_9^2$ will have orders of vanishing $-\frac{243}{4}$ and $-243$ at the cusps $\frac{1}{2}$ and $\frac{1}{4}$.

Following the pattern leads us to conjecture that the orders of vanishing at cusps for $(f_3 - 2f_4) | U_9^n$ are

\[
\begin{align*}
\infty & : 3 \\
\frac{1}{3} & : 1 \\
\frac{1}{6} & : \frac{3}{4} \\
0 & : 0 \\
\frac{1}{2} & : -\frac{3}{4}(9^n) \\
\frac{1}{4} & : -3(9^n).
\end{align*}
\]
Now we explore possible approaches to the proof of Conjecture 3.1. The first approach we discuss is similar to Boylan’s work for the case where \( p = 2 \) in [7]. This involves dividing \((f_3 - 2f_4) | U_{9n}\) by a weight \( \frac{3}{2} \) modular form, \( \Theta^3 \cdot h \), where \( h \) is a weight 0 weakly holomorphic modular form, and then writing this quotient as a polynomial or polynomials of one or more weight 0 forms.

In order to strategically pick \( h \), it is useful to know the orders of the zeros and poles of \( \frac{(f_3 - 2f_4) | U_{9n}}{\Theta^3} \). We find that the orders of vanishing at cusps for \( \frac{(f_3 - 2f_4) | U_{9n}}{\Theta^3} \) are

\[
\begin{align*}
\infty : & \quad 3 \\
\frac{1}{3} : & \quad 1 \\
\frac{1}{6} : & \quad 0 \\
0 : & \quad 0 \\
\frac{1}{2} : & \quad \frac{3}{4}(9^n) - \frac{9}{4} \\
\frac{1}{4} : & \quad -3(9^n).
\end{align*}
\]

Note that the order of vanishing at \( \frac{1}{2} \) is in fact an integer.

Using Ligozat’s theorem and Theorem 5.7 in [18], we can find \( \eta \)-quotients with poles and zeros at the desired cusps. Since we are dealing with zeros at the cusps \( \infty \) and \( \frac{1}{3} \) and poles at the cusps \( \frac{1}{2} \) and \( \frac{1}{4} \), we consider the \( \eta \)-quotients

\[
\begin{align*}
h_{\infty, \frac{1}{4}} & = \frac{\eta(2z)^2\eta(12z)^4}{\eta(4z)^4\eta(6z)^2}, \\
h_{\frac{1}{2}, \frac{1}{4}} & = \frac{\eta(2z)^2\eta(3z)^3\eta(12z)}{\eta(z)\eta(4z)^3\eta(6z)^2}, \\
h_{\infty, \frac{1}{2}} & = \frac{\eta(z)^3\eta(4z)^2\eta(6z)\eta(12z)^2}{\eta(2z)^7\eta(3z)}, \\
h_{\frac{1}{3}, \frac{1}{2}} & = \frac{\eta(z)^2\eta(3z)^2\eta(4z)^3\eta(6z)}{\eta(2z)^7\eta(12z)},
\end{align*}
\]

where \( h_{q,r} \) has a single zero at the cusp \( q \) and a single pole at the cusp \( r \).
If we divide only by $\Theta^3$, meaning we take $h = 1$, we compute

$$
\frac{(f_3 - 2f_4) \mid U_9}{\Theta_3} = \sum_{m=-9}^{27} a_{1,m} h_{\frac{1}{2},\frac{1}{4}}^m
$$

where $h_{\frac{1}{2},\frac{1}{4}} = h_{\infty,\frac{1}{2}}/h_{\infty,\frac{1}{4}}$. The $a_1(m)$ here are not integers and, in fact, have denominators divisible by powers of 3. Since $\Gamma_0(12)$ has genus 0 we should be able to write

$$
\frac{(f_3 - 2f_4) \mid U_9}{\Theta^3} = \sum_{m=-\frac{3(9^n)}{2}}^{\frac{3(9^n)}{2}} a_{n,m} h_{\frac{1}{2},\frac{1}{4}}^m
$$

for all $n$. However, since $a_{1,m}$ clearly do not have the divisibility we desire, this may not be the most effective strategy to proving Conjecture 3.1.

If we instead pick $h$ to be some product or quotient of the $\eta$-quotients above, then we can write $\frac{(f_3 - 2f_4) \mid U_9}{\Theta^3 \cdot h}$ as the sum of two polynomials of $\eta$-quotients written previously in which the coefficients have more promising divisibility. For example, if we pick $h = h_{\infty,\frac{1}{2}}^3$ then we compute

$$
\frac{(f_3 - 2f_4) \mid U_9}{\Theta^3 \cdot h} = \sum_{m=1}^{6} c_{1,m} h_{\frac{1}{2},\frac{1}{2}}^m + \sum_{m=1}^{27} d_{1,m} h_{\frac{1}{2},\frac{1}{4}}^m,
$$

and

$$
\frac{(f_3 - 2f_4) \mid U_9^2}{\Theta^3 \cdot h} = \sum_{m=1}^{60} c_{2,m} h_{\frac{1}{3},\frac{1}{2}}^m + \sum_{m=1}^{243} d_{2,m} h_{\frac{1}{3},\frac{1}{4}}^m.
$$

where $c_{1,m}$ and $d_{1,m}$ are divisible by at least $3^3$ and $c_{2,m}$ and $d_{2,m}$ are divisible by at least $3^6$. It is worthwhile to note that the $c_{1,m}$, $d_{1,m}$, $c_{2,m}$, $d_{2,m}$ are divisible by increasing powers of 3 as $m$ increase, similar to what we saw writing $(f_3 - 2f_4) \mid U_9^n$ in terms of basis elements in Chapter 3. Consider the following conjecture:

**Conjecture 4.1.** We have

$$
\frac{(f_3 - 2f_4) \mid U_9^n}{\Theta^3 \cdot h} = \sum_{m=1}^{\frac{3(9^n-1)}{4}} c_{n,m} h_{\frac{1}{3},\frac{1}{2}}^m + \sum_{m=1}^{\frac{3(9^n)}{4}} d_{n,m} h_{\frac{1}{3},\frac{1}{4}}^m
$$

where $c_{n,m}$ and $d_{n,m}$ are divisible by $3^{3n}$. 

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Since $\Theta^3 \cdot h = q^3 + O(q^4)$, Conjecture 4.1 implies Conjecture 3.1.

Proving Conjecture 4.1 would involve an induction argument and proving a result about the forms

$$\frac{(\Theta^3h \frac{1}{3}, \frac{1}{2})^m}{\Theta^3h} | U_9,$$

and

$$\frac{(\Theta^3h \frac{1}{3}, \frac{1}{2})^m}{\Theta^3h}.$$

Computations suggest that

$$\frac{(\Theta^3h \frac{1}{3}, \frac{1}{2})^m}{\Theta^3h} = a_{m,-2}h_{\frac{1}{3}, \infty}^2 + a_{m,-1}h_{\frac{1}{3}, \infty} + \sum_{s=1}^{9m+6} a_{m,s}h_{\frac{1}{3}, \frac{1}{7}, s}^2$$

and

$$\frac{(\Theta^3h \frac{1}{3}, \frac{1}{2})^m}{\Theta^3h} = b_{m,-2}h_{\frac{1}{3}, \infty}^2 + b_{m,-1}h_{\frac{1}{3}, \infty} + \sum_{s=1}^{6} b_{m,s}h_{\frac{1}{3}, \frac{1}{7}, s}^2 + \sum_{s=1}^{9m} b'_{m,s}h_{\frac{1}{3}, \frac{1}{7}, s},$$

where $h_{\frac{1}{3}, \infty} = h_{\frac{1}{3}, \frac{1}{2}}/h_{\infty, \frac{1}{2}}$ which has a simple pole at $\infty$ and a simple zero at $\frac{1}{3}$.

One challenge that arises from using these conjectured formulas to prove Conjecture 4.1 is that the principal parts will need to zero out, so that we have a zero at $\infty$ in $\frac{(f_3 - 2f_4)}{\Theta^3h}$. Another challenge is that the divisibility of the $a_{m,s}, b_{m,s},$ and $b'_{m,s}$ appears to decrease as $m$ decreases. This may be okay since the divisibility of the $c_{n,m}$ and $d_{n,m}$ increase as $m$ increase. In addition, a challenge to proving the above conjectured formulas is the next step involves working with level 108 forms with zeros or poles at nearly every one of the 18 cusps of $\Gamma_0(108)$.

Other choices for $h$ have led to conjectures similar to Conjecture 4.1 that would imply Conjecture 3.1. However, challenges similar to those described above arise in proving the conjecture in these cases as well.

We now turn to a entirely different approach to proving Conjecture 3.1 which involves more closely examining the coefficients of $(f_3 - 2f_4) | U_{9n}$. Following a procedure similar to Zagier ([6], Section 2), we can find a recurrence relation for the coefficients of $(f_3 - 2f_4) | U_{9n}$. 

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Finding such a relationship between coefficients would mean that we only need to know that the first few coefficients of \((f_3 - 2f_4) \mid U_9\) have the divisibility that we want in order to know that all of the coefficients do.

For ease of notation, let \(f = f_3 - 2f_4\). Let \(\Phi_3(z) \in M^!_0(\Gamma_0(3))\) be defined as in Chapter 2. Let \(E = \frac{1}{240}(E_4(z) - E_4(3z)) \in M_4(\Gamma_0(3))\) where \(E_4\) is the usual weight 4 Eisenstein series. We will also define \(v_3(\frac{m}{n}) = \text{ord}_3(m) - \text{ord}_3(n)\). Also let \(\theta = q \frac{d}{dq} = \frac{1}{2\pi i} \frac{d}{dz}\) be a differential operator. Note that, in general, \(\theta\) does not preserve modularity. However, if \(f\) is a weight 0 modular form, then \(\theta f\) is modular and has weight 2. Lastly, define a normalized Rankin-Cohen bracket on modular forms \(f\) of weight \(k\) and \(g\) of weight \(h\) by

\[
[f, g] = (\theta f) g - 3f (\theta g).
\]

Note this is the same definition Zagier used in Section 2 of [6]. The result of this Rankin-Cohen bracket is a modular form of weight \(k + h + 2\).

We will find a recurrence relation for the coefficients of \(f \mid U_9\) by considering \((f \mid U_9 \Theta) \mid U_4\) and \([f \mid U_9, \Theta] \mid U_4\). Note that \(f \mid U_9 \Theta \in M^!_2(\Gamma_0(12))\) and acting on this form by \(U_4\) drops the level so that \((f \mid U_9 \Theta) \mid U_4 \in M^!_2(\Gamma_0(3))\). Because \(f \mid U_9\) is in the Kohnen plus space, we drop to level 3 rather than level 6. Again, this is parallel to Zagier’s work in Section 2 of [6]. We also have that \([f \mid U_9, \Theta] \in M^!_4(\Gamma_0(12))\) and acting by \(U_4\) again drops the level so that \([f \mid U_9, \Theta] \mid U_4 \in M^!_4(\Gamma_0(3))\). This is desirable since \(\Gamma_0(3)\) has only two cusps, 0 and \(\infty\), compared to the 6 distinct cusps of \(\Gamma_0(12)\).

Note that both \((f \mid U_9 \Theta) \mid U_4\) and \([f \mid U_9, \Theta] \mid U_4\) have a zero at \(\infty\), but are not holomorphic, so must have poles at 0. In addition, note that

\[
M^!_2(\Gamma_0(3)) = \mathbb{C}(E_2(z) - 3E_2(3z)) \oplus \frac{d}{dz} M^!_0(\Gamma_0(3))
\]

where \(E_2 = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n\) is the weight 2 Eisenstein series. However, \((f \mid U_9 \Theta) \mid U_4\) does not have a constant term, so it must be that \((f \mid U_9 \Theta) \mid U_4 \in \frac{d}{dz} M^!_0(\Gamma_0(3))\). Since \(\Phi_3(z)\)
is the level 3 Hauptmodul, then we can write \((f \mid U_9 \Theta) \mid U_4\) as the derivative of a polynomial in \(\Phi_3(z)\). We compute that

\[
(f \mid U_9 \Theta) \mid U_4 = \theta(-3^6 \cdot 728\Phi_3(z) - 3^{11} \cdot 2786\Phi_3(z)^2 - 3^{15} \cdot 6488\Phi_3(z)^3 \\
- 3^{23} \cdot 68\Phi_3(z)^4 - 3^{28} \cdot 8\Phi_3(z)^5 - 3^{32}\Phi_3(z)^6).
\]

Notice that each of the coefficients is divisible by at least \(3^6\).

In addition, we find that \([f \mid U_9, \Theta] \mid U_4 \in M_4^!(\Gamma_0(3))\) so it can be written as an Eisenstein series \(E\) times a polynomial in \(\Phi_3(z)\). We compute that

\[
[f \mid U_9, \Theta] \mid U_4 = E(-3^3 \cdot 93608 - 3^{14} \cdot 2128\Phi_3(z) - 3^{18} \cdot 12272\Phi_3(z)^2 - 3^{23} \cdot 7328\Phi_3(z)^3 \\
- 3^{29} \cdot 608\Phi_3(z)^4 - 3^{33} \cdot 200\Phi_3(z)^5 - 3^{38} \cdot 8\Phi_3(z)^6).
\]

Notice that the first coefficient is divisible by \(3^3\) while the rest are divisible by at least \(3^{14}\).

For convenience, we will write

\[
(f \mid U_9 \Theta) \mid U_4 = \theta P(\Phi_3(z)) = \theta \sum c(n)q^n
\]

and

\[
[f \mid U_9, \Theta] \mid U_4 = E \cdot Q(\Phi_3(z)) = E \cdot \sum b(n)q^n
\]

where \(P(x)\) and \(Q(x)\) are degree 6 polynomials with coefficients as above. Let \(f\) have Fourier expansion \(f = \sum_{n \equiv 0,3(4)}^\infty a(n)q^n\). Then \(f \mid U_9 = \sum_{n \equiv 0,3(4)}^\infty a(9n)q^n\). Considering this and the Fourier expansion of \(\Theta = 1 + 2 \sum_{n=1}^\infty q^{n^2}\) and comparing Fourier coefficients of \((f \mid U_9 \Theta) \mid U_4\) and \(\theta P(\Phi_3(z))\) we find that

\[
\sum_{r \in \mathbb{Z}} a(9(4n - r^2)) = a(9(4n)) + 2 \sum_{r > 0} a(9(4n - r^2)) = nc(n).
\]
Doing the same with $[f \mid U_9, \Theta] \mid U_4$ and $E \cdot Q(\Phi_3(z))$ we find that

$$4na(9 \cdot 4n) + 2 \sum_{r>0} (4n - 4r^2)a(9(4n - r^2)) = \sum_{i=1}^{n} (\sigma_3(i) - \sigma_3(i/3))b(n - i).$$

Combining these two formulas, we find that

$$\sum_{r>0} r^2a(9(4n - r^2)) = \frac{1}{2} n^2c(n) - \frac{1}{8} \sum_{i=1}^{n} (\sigma_3(i) - \sigma_3(i/3))b(n - i).$$

These relationships give us the following recurrence relations:

$$a(9(4n)) = nc(n) - 2 \sum_{1 \leq r \leq \sqrt{4n-3}} a(9(4n - r^2))$$

$$a(9(4n - 1)) = \frac{1}{2} n^2c(n) - \frac{1}{8} \sum_{i=1}^{n} (\sigma_3(i) - \sigma_3(i/3))b(n - i) - \sum_{2 \leq r \leq \sqrt{4n-3}} r^2a(9(4n - r^2)).$$

Since $c(n)$ and $b(n)$ have high enough divisibility by powers of 3, as long as the first few coefficients of $f \mid U_9$ satisfy $v_3(a(9n)) \geq 3^3$, this recurrence relation implies that all of the coefficients of $f \mid U_9 = (f_3 - 2f_4) \mid U_9$ satisfy $v_3(a(9n)) \geq 3^3$. We have seen already that the first several $a(n)$ satisfy this property. Thus $(f_3 - 2f_4) \mid U_9$ is in fact divisible by $3^3$ as Conjecture 3.1 suggests.

If we make the following conjecture, then we can find such recurrence relations for $f \mid U_{9k}$ for all $k$ that will allow us to prove Conjecture 3.1.

**Conjecture 4.2.** We have

$$(f \mid U_{9k} \Theta) \mid U_4 = q^{\frac{d}{dq}} P_k(\Phi_3(z)) = q^{\frac{d}{dq}} \sum c_k(n)q^n$$

and

$$[f \mid U_{9k}, \Theta] \mid U_4 = E \cdot Q_k(\Phi_3(z)) = E \cdot \sum b_k(n)q^n$$

where $v_3(c_k(n)) \geq 3^{3k}$ and $v_3(b_k(n)) \geq 3^{3k}$. 

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Assuming Conjecture 4.2 and again comparing Fourier coefficients we can write a recurrence relation for \( f|U_{9k} \) as

\[
a(9^k(4n)) = nc_k(n) - 2 \sum_{1 \leq r \leq \sqrt{4n-3}} a(9^k(4n - r^2))
\]

\[
a(9^k(4n - 1)) = \frac{1}{2} n^2 c_k(n) - \frac{1}{8} \sum_{i=1}^{n} (\sigma_3(i) - \sigma_3(i/3)) b_k(n-i) - \sum_{2 \leq r \leq \sqrt{4n-3}} r^2 a(9^k(4n - r^2)).
\]

which implies if the first few coefficients of \( f|U_{9k} \) have the desired divisibility, then the rest of the coefficients of \( f|U_{9k} \) will also have at least the desired divisibility.

Again, assuming Conjecture 4.2, the first term of \( f|U_{9k} \) is the \( q^3 \) term which has coefficient

\[
a(9^k(3)) = \frac{1}{2} c_k(1) - \frac{1}{8} b_k(0).
\]

Since \( c_k(n) \) and \( b_k(n) \) are divisible by \( 3^{3k} \), then \( v_3(a(9^k(3))) = 3^{3k} \). Now suppose that for all \( \ell < d \), \( v_3(a(9^k \cdot \ell)) \geq 3^{3k} \). Consider \( a(9^kd) \).

If \( d \equiv 0 \pmod{4} \) then \( d = 4n \) for some \( n \) and

\[
a(9^kd) = a(9^k(4n)) = nc_k(n) - 2 \sum_{1 \leq r \leq \sqrt{4n-3}} a(9^k(4n - r^2)).
\]

Since \( 3^{3k} | c_k(n) \) and \( 3^{3k} | a(9^k \cdot \ell) \) for \( \ell < d \) we must have \( v_3(a(9^kd)) \geq 3^{3k} \).

If \( d \equiv 3 \pmod{4} \) then \( d = 4n - 1 \) for some \( n \) and

\[
a(9^kd) = a(9^k(4n-1)) = \frac{1}{2} n^2 c_k(n) - \frac{1}{8} \sum_{i=1}^{n} (\sigma_3(i) - \sigma_3(i/3)) b_k(n-i) - \sum_{2 \leq r \leq \sqrt{4n-3}} r^2 a(9^k(4n - r^2)).
\]

Then we have that \( v_3(a(9^kd)) \geq 3^{3k} \) since \( 3^{3k} \) divides \( b_k(n) \) and \( c_k(n) \) for all \( n \) and since \( v_3(a(9^k \cdot \ell)) \geq 3^{3k} \) for \( \ell < d \). Therefore \( f|U_{9k} \equiv 0 \pmod{3^{3k}} \). This proves that Conjecture 4.2 implies Conjecture 3.1.

The main tasks in proving Conjecture 4.2 are verifying that we can write \( (f|U_{9k}\Theta)|U_4 \) and \( [f|U_{9k},\Theta]|U_4 \) in the proposed forms and proving that \( P_k(\Phi_3(z)) \) and \( Q_k(\Phi_3(z)) \) are
divisible by at least $3^k$. This will mostly likely involve some type of induction argument on $k$.

Beyond proving Conjecture 3.1, further work would involve trying to prove similar results for $p \in \{5, 7, 11\}$. It will be interesting to see if similar methods work for these primes and if it also turns out that a logical reduction proves a slightly weaker congruence than computation suggests.
Bibliography


