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Invariant Lattices of Several Elliptic K3 Surfaces

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ABSTRACT

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This work is concerned with computing the invariant lattices of purely non-symplectic automorphisms of special elliptic K3 surfaces. Brandhorst gave a collection of K3 surfaces admitting purely non-symplectic automorphisms that are uniquely determined up to isomorphism by certain invariants. For many of these surfaces, the automorphism is also unique or the automorphism group of the surface is finite and with a nice isomorphism class. Understanding the invariant lattices of these automorphisms and surfaces is interesting because of these uniqueness properties and because it is possible to give explicit generators for the Picard and invariant lattices. We use the methods given by Comparin, Priddis and Sarti to describe the Picard lattice in terms of certain special curves from the fibration of the surface. We use symmetries of the Picard lattice and fixed-point theory to compute the invariant lattices explicitly. This is done for all of Brandhorst's elliptic K3 surfaces having trivial Mordell-Weil group.

Keywords: elliptic surfaces, purely non-symplectic automorphisms, k3 surfaces

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Chapter 1. Introduction

This work is concerned with computing the invariant lattices of certain automorphisms of special elliptic K3 surfaces. By the Torelli theorem, a K3 surface is essentially determined by the Picard lattice. The invariant lattice is the sublattice fixed by the induced action on the Picard lattice. The invariant lattice of automorphisms is useful in the study of mirror symmetry and in the classification of these automorphisms.

In a recent paper, Brandhorst gave a large collection of K3 surfaces admitting purely non-symplectic automorphisms that are uniquely determined up to isomorphism by certain invariants determined by a purely non-symplectic automorphism. These surfaces are determined by the tuple of invariants (n, d) where n is the order of the automorphism and d the determinant of the Picard lattice of the surface. In fact, the determinant of the Picard lattice is determined by the action of the non-symplectic automorphism on the action of the second cohomology group. A natural question is can we determine the invariant lattice of these surfaces?

In [9] the authors showed it is possible to understand the invariant lattice by considering how the automorphism acts on certain curves on the surface. It is of interest to see if this can be accomplish the same thing for the surfaces given by Brandhorst. The majority of these surfaces admit an elliptic fibration with Weierstrass model. We exploit this to obtain a configuration of curves on the surface from the reducible fibers of the fibration which are generators of the Picard lattice. We then employ a variety of different techniques to compute a primitive sublattice of the Picard lattice known as the invariant lattice. The tools discussed in this work are not sufficient to compute the Picard lattices of every elliptic K3 surface in Brandhorst's catalog. Where they fall short, we mention what techniques will be necessary to study these surfaces.

Chapters 2-4 will introduce the necessary theory to accomplish our computation. In Chapter 2, we will introduce lattice theory culminating in Nikulin's theorem on the uniqueness of lattices via the discriminant quadratic form. Chapter 3 will introduce briefly the theory of elliptic surfaces and the theory of divisors to sufficient depth for our purposes. Chapter 4 will overview the fixed-point theory necessary to convince ourselves we have the correct lattices.

Chapter 5 will introduce the surfaces and automorphisms of interest and try to motivate their importance in the program of classifying purely non-symplectic automorphisms of K3 surfaces. We finish with a presentation of our theorem in Chapter 6, which gives the invariant lattices of the automorphisms and surfaces considered. The details of how these were computed are given in the final section as well as the explicit generators of the lattices. We will also survey future work in the area.

To begin, we wish to make clear what we mean by a lattice and mention some properties that will be useful for us.

Definition 2.1. When we refer to **lattices**, we mean a free finitely generated Abelian group Λ equipped with a symmetric, non-degenerate bilinear form

$$\langle\cdot,\cdot\rangle : \Lambda \times \Lambda \to \mathbb{Q},$$

where non-degenerate means that for every $x \in \Lambda$ there exists some $y \in \Lambda$ such that $\langle x, y\rangle \neq 0$.

We primarily concern ourselves with integral lattices, or lattices with bilinear forms taking values in the integers, i.e. $\langle,\rangle : \Lambda \times \Lambda \to \mathbb{Z}$. We say an integral lattice Λ is **even** if for every $x \in \Lambda$ it is the case that $\langle x, x\rangle \in 2\mathbb{Z}$.

It is frequently convenient to think of the bilinear form as being given by

$$\langle x, y\rangle = x^T By$$

for some matrix B. This prompts the following definition.

Definition 2.2. The **Gram matrix** of a lattice is defined as

$$G = [\langle x_i, x_j\rangle]$$

where $\{x_i\}$ is a minimal generating set for Λ. The determinant (or discriminant) of a lattice Λ is defined as the determinant of its Gram matrix.

Given an integral lattice Λ, the bilinear form induces an embedding $\Lambda \hookrightarrow \Lambda^*$, where $\Lambda^* = \mathrm{Hom}\,(\Lambda, \mathbb{Z})$.

Definition 2.3. We define the **discriminant group** of a lattice Λ to be $A_\Lambda = \Lambda^*/\Lambda$, where we identify Λ as the embedded sublattice of Λ^*. We refer to the minimal number of generators of A_Λ as the **length** of A_Λ and write it $\ell(A)$.

The following proposition is well known.

Proposition 2.4. *If Λ is an even lattice, then $\Lambda \subseteq \Lambda^*$ and the discriminant group A_Λ is a finite abelian group. In particular, $|\Lambda^*/\Lambda| = \det B$ where B is a Gram matrix for Λ.*

Proof. We have defined Λ^* to be the collection of linear functionals from Λ to $\mathbb{Z}$. If $\{e_i\}$ is a basis for the lattice Λ, then we can make an identification $e_i \mapsto \langle e_i, - \rangle$. Because Λ is integral, this ensures $\langle e_i, - \rangle$ is a functional from the lattice to the integers. Under this identification $\Lambda \subseteq \Lambda^*$.

We can make the observation that $\Lambda \subset V$ where V is some complex vector space and $\Lambda^* \subset V^*$. Because we can require V be finite dimensional, there is a natural identification of V^* with V and $\Lambda \subseteq \Lambda^* \subset V$. If we think of lattices as dividing V into polyhedra of equal volume, then we can find a **fundamental region** for each of Λ and Λ^* which is a parallelpiped that tiles the ambient vector space. As $\Lambda \subseteq \Lambda^*$, we know that an integer number of copies of the fundamental region of Λ^* tile the fundamental region of Λ. As the lattice points at each of these copies represent the cosets of Λ in the quotient group, we conclude that $|\Lambda^*/\Lambda| < \infty$.

The final part of the proposition follows from the fact that $\det \Lambda^* = \frac{1}{\det \Lambda}$. □

Definition 2.5. The **signature** of a lattice Λ is the signature of the Gram matrix of Λ. In particular, because these are nondegenerate, we can write the signature as (t_+, t_-) where t_+ is the number of $+1$ entries in its diagonalized form and t_- is the number of -1 entries.

An embedding $\Lambda_1 \hookrightarrow \Lambda_2$ is **primitive** if Λ_2/Λ_1 is free. This idea will be particularly useful in arguing that we've found the eigenspace of a given lattice isometry.

Definition 2.6. A **finite symmetric bilinear form** is a bilinear form $b : A \times A \to \mathbb{Q}/\mathbb{Z}$ where A is a finite abelian group. A **finite quadratic form** is a map $q : A \to \mathbb{Q}/2\mathbb{Z}$ such that for all $n \in \mathbb{Z}$ and $a, a' \in A$, $q(na) = n^2 q(a)$ and $q(a + a') - q(a) - q(a') \equiv 2b(a, a') (\text{mod } 2\mathbb{Z})$ for some finite symmetric bilinear form b and finite group A.

A case of finite quadratic forms we will find particularly useful is the **discriminant quadratic form** of a lattice Λ, which is the natural finite quadratic form defined on the discriminant group Λ^*/Λ. The discriminant quadratic form is a powerful invariant that will allow us to establish the uniqueness of lattices.

We'll now give notation for some important lattices and their bilinear forms.

The lattice U is the unimodular, rank 2 hyperbolic lattice and has the following bilinear form.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

For $p \equiv 1 (\text{mod } 4)$, the lattice H_p is the lattice having bilinear form

$$\begin{pmatrix} \frac{p+1}{2} & 1 \\ 1 & 2 \end{pmatrix}$$

and discriminant group $\mathbb{Z}/p\mathbb{Z}$.

When we wish to notate the lattice with bilinear form that is a scalar multiple of a standard bilinear form, we write the standard form of the lattice and enclose the scalar afterward in parentheses. For example, the lattice $U(2)$ has bilinear form

$$\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$$

as we would expect.

In addition there are three useful families of lattice we'll find helpful A_n, D_n and E_n. Each of these is an even, negative definite lattice corresponding to the associated Dynkin diagram. We give the Dynkin diagrams for each family.

Figure 2.1: Dynkin Diagram of A_n

In the case of A_1 we would have just a single point in the diagram.

Figure 2.2: Dynkin Diagram of D_n

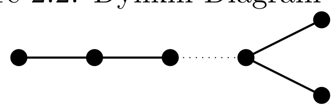
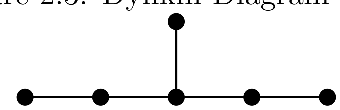


Figure 2.3: Dynkin Diagram of E_6



These diagrams describe lattices as follows. Each node of the diagram describes a basis element of the lattice. The matrix B describing the bilinear form of the lattice has -2 for the diagonal entries and the entry $B_{ij} = 1$ if there is an edge connecting the i and j nodes and zero otherwise.

The bilinear form for the E_8 lattice is given by the following matrix. Note how it can be constructed from the diagram and the rules given previously.

$$\begin{pmatrix} -2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix}$$

An important result for us is when a sublattice of finite index is actually equal to the lattice. To this end, we make the following observation. If we know that $\Lambda \hookrightarrow \Lambda'$, then we

Figure 2.4: Dynkin Diagram of E_7

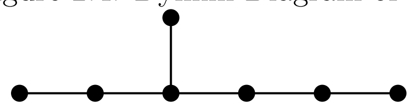
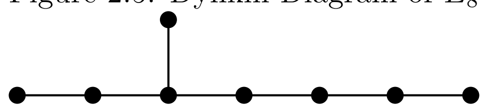


Figure 2.5: Dynkin Diagram of E_8



have the chain of embeddings $\Lambda \hookrightarrow \Lambda' \hookrightarrow (\Lambda')^* \hookrightarrow \Lambda^*$. This follows from the properties of the dual lattice.

We define $H_{\Lambda'} = \Lambda'/\Lambda$. Then we know that $H_{\Lambda'} \subset (\Lambda')^*/\Lambda \subset \Lambda^*/\Lambda = A_\Lambda$. We also make the observation that $(\Lambda')^*/\Lambda/H_{\Lambda'} \cong A_{\Lambda'}$. This leads us to the following immediate proposition.

Proposition 2.7. *If $\Lambda \subset \Lambda'$ and both lattices have the same rank and discriminant group, then the inclusion $\Lambda \hookrightarrow \Lambda'$ is an isomorphism.*

We're similarly interested in the orthogonality of lattices, which is characterized by the following.

Proposition 2.8 (Nikulin, 1980)**.** *Given a lattice Λ and sublattices L, K, it is the case that L is orthogonal to K if and only if $q_L = -q_K$.*

Theorem 2.9 (Nikulin, 1980)**.** *The set of finite quadratic forms is a semigroup under the $\oplus$ operation. This semigroup is generated by the collection of forms $w^\epsilon_{p,k}, u_k, v_k$, which we define subsequently.*

We now give the definitions of each of these generators listed in Theorem 2.9.

Given a prime $p \neq 2$, an integer $k \geq 1$ and $\epsilon \in \{\pm 1\}$, let a be the smallest even integer having ϵ as a quadratic residue modulo p. Then the finite quadratic form $w^\epsilon_{p,k} : \mathbb{Z}/p^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$ is defined by

$$w^\epsilon_{p,k}(1) = ap^{-k}.$$

If $p=2$, with $k \geq 1$ and $\epsilon \in \{\pm 1, \pm 5\}$, we define $w^{\epsilon}_{2,k} : \mathbb{Z}/2^k\mathbb{Z} \to \mathbb{Q}/2\mathbb{Z}$ by

$$w^{\epsilon}_{2,k}(1) = \epsilon \cdot 2^{-k}.$$

For $k \geq 1$ an integer, we define the forms u_k, v_k on $\mathbb{Z}/2^k\mathbb{Z} \times \mathbb{Z}/2^k\mathbb{Z}$ by the matrices

$$u_k = \begin{pmatrix} 0 & 2^{-k} \\ 2^{-k} & 0 \end{pmatrix}, v_k = 2^{-k} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

given a lattice Λ, we can compute the collection of invariants (t_+, t_-, q). It is an interesting question when these invariants are sufficient to distinguish lattices up to isomorphism. To that end we have the following powerful result from Nikulin.

Theorem 2.10. *An even lattice Σ having invariants (t_+, t_-, q) is unique if, simultaneously,*

- $t_+ \geq 1, t_- \geq 1, t_+ + t_- \geq 3$
- *for each $p \neq 2$, either rank $\Sigma \geq 2 + l((A_q)_p)$ or $q_p \cong w^{\epsilon}_{p,k} \oplus w^{\epsilon'}_{p,k} \oplus q'_p$*
- *for $p = 2$, either rank $\Sigma \geq 2 + l((A_q)_2)$ or one of the following holds: $q_2 \cong u_k \oplus q'_2, q_2 \cong v_k \oplus q'_p, q_2 \cong w^{\epsilon}_{2,k} \oplus w^{\epsilon'}_{2,k} \oplus q'_2$.*

We will make use of this theorem in Chapter 6 when we each invariant lattice abstractly in terms of the lattices defined here.

It is now time for us to define our other principal object of study. A K3 surface X is a complex projective variety of dimension two such that $K_X = \Omega_X^2$ is a trivial bundle (or alternatively that there exists a nowhere vanishing holomorphic 2-form) and that the dimension of the cohomology group $H^1(\mathcal{O}_X)$ is zero.

These surfaces have a variety of properties we will find useful. One of the foremost is that we can endow the second cohomology group $H^2(X, \mathbb{Z})$ with the structure of an integral lattice. In fact $H^2(X, \mathbb{Z}) \cong E_8^{\oplus 2} \oplus U^{\oplus 3}$ where U is the hyperbolic lattice of rank 2 and E_8 refers to the E_8 lattice as in the previous section. This lattice is the unique unimodular lattice of signature $(3, 19)$ and is known as the **K3 lattice** in the literature.

The other fact we will make use of regards the **Picard lattice**. The Picard group of an algebraic K3 surface is the group of line bundles on the surface under the tensor product. This can be endowed with an even symmetric bilinear form such that it agrees with the restriction of the intersection form on $H^2(X, \mathbb{Z})$. As a consequence of the Hodge Index Theorem, this lattice has signature $(1, \rho - 1)$ where ρ is the **Picard number**. This is convenient because we know the Picard number for all of these surfaces we consider, so combined with other data, we can quickly constrain the Picard lattice with Nikulin's result.

3.1 Elliptic Fibrations and Singular Fibers

We're not just discussing K3 surfaces, but elliptic K3 surfaces. A surface S is called an **elliptic surface** if it possesses a surjective morphism $\pi : S \to B$ where for almost every $b \in B$, the fiber $\pi^{-1}(b)$ is a smooth elliptic curve (or smooth of genus 1). This structure is known as an **elliptic fibration**. It is a fact that complex elliptic K3 surfaces, as we're interested in here, are always fibered over the complex projective line, i.e. $B = \mathbb{P}^1$. An elliptic surface then is the data (X, π) where π is the fibration of interest.

A useful and important structure associated to fibrations is a section.

Definition 3.1. Given a fibration $\pi : X \to C$ where X is an elliptic surface and C is the base curve, a **section** is a map $\pi^* : C \to X$ such that the composition $\pi \circ \pi^* : C \to C$ is the identity.

Given certain assumptions, it is possible to give a **Weierstrass model** of the fibration. This is an equation of the form $y^2 = x^3 + A(t)x + B(t)$ where $t \in \mathbb{P}^1$; putting in t gives the equation of an elliptic curve that is the fiber over t. This is the data that we will use in our work so in later chapters when we discuss specific elliptic K3 surfaces we will give the Weierstrass model only. This model is useful to us because we can see when the fiber is not an elliptic curve by examining the points of vanishing of the discriminant $4A^3 + 27B^2$. A discussion of this is given in more depth in Appendix A. This leads us to an observation, a definition, and a proposition.

Having sections and Weierstrass models are not always guaranteed when considering an algebraic or K3 surface. To make matters worse, some authors will require the existence of a section in their definition while others will define a fibration with section to be a **Jacobian fibration**. All of the surfaces we consider have both sections and a Weierstrass model so this makes no difference for our work.

Definition 3.2. A **singular fiber** of an elliptic fibration π, is a fiber $\pi^{-1}(t)$ that is not a smooth elliptic curve.

Proposition 3.3. *Every elliptic K3 surface has a finite non-zero number of singular fibers.*

Proof. It can be shown that the topological Euler characteristic of any K3 surface is 24 and that the topological Euler characteristic of a smooth elliptic curve is 0. Thus there must be singular fibers on the surface. As every singular fiber has positive Euler characteristic, there can only be finitely many (see [14][15]). □

Theorem 3.4 (Kodaira, [14] and [15])**.** *Over fields of characteristic not equal to 2 or 3, the possible singular fibers come in the families described in Figure 3.1.*

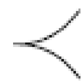
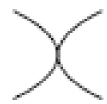
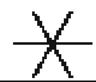
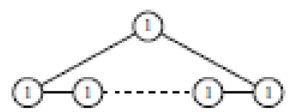
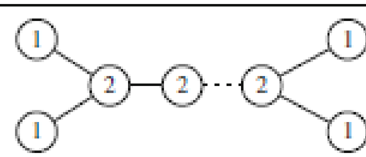
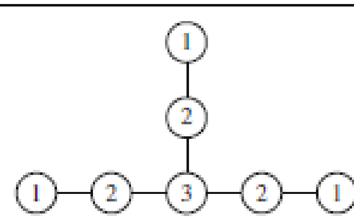
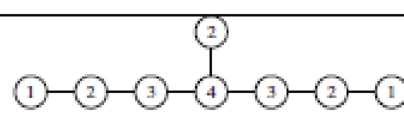
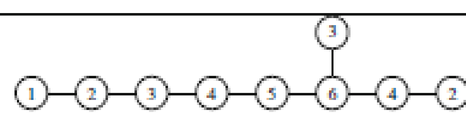
Type	$v(c_4)$	$v(c_6)$	$v(\Delta)$	j	Monodromy	Fiber	Dual Graph
I_0	≥ 0	≥ 0	0	$\mathbb{C}$	I_2	Smooth Elliptic Curve	-
I_1	0	0	1	∞	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	 (curve of arithmetic genus 1 with a nodal singularity)	$\tilde{A}_0$
II	≥ 1	1	2	0	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	 (curve of arithmetic genus 1 with a cuspidal singularity)	$\tilde{A}_0$
III	1	≥ 2	3	1728	$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$	 Two rational curves intersecting at a double point	$\tilde{A}_1$
IV	≥ 2	2	4	0	$\begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$		$\tilde{A}_2$
I_n	0	0	$n > 1$	∞	$\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$	 n nodes	$\tilde{A}_{n-1}$
I_n^*	2	≥ 3	$n+6$	∞	$\begin{pmatrix} -1 & -n \\ 0 & -1 \end{pmatrix}$	 $n+5$ nodes	$\tilde{D}_{n+4}$
	≥ 2	3	$n+6$				
IV^*	≥ 3	4	8	0	$\begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}$		$\tilde{E}_6$
III^*	3	≥ 5	9	1728	$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$		$\tilde{E}_7$
II^*	≥ 4	5	10	0	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$		$\tilde{E}_8$

Figure 3.1: Singular Fibers in the Kodaira classification

In the table of singular fibers, the nodes of the diagrams for the fibers of type I_n, I_n^*, IV^*, III^* and II^* are all smooth rational curves on the surface. The edges then describe intersections of multiplicity 1 between the two curves and for these **reducible fibers** each component curve has self-intersection -2 (see [20]). Thus the fact that we can find a finite family of singular fibers on a given elliptic K3 surface means we can parley the Weierstrass model into a configuration of rational curves on the surface. Why this is useful for probing the Picard lattice is the topic of the next section.

Another definition we want to introduce while we are thinking of configurations of curves is the idea of an **intersection matrix**. Note the similarity this bears to the convention we introduced for describing a lattice with a Dynkin diagram in chapter 2.

Definition 3.5. The **intersection matrix** for a configuration of curves on a surface gives us a matrix in the following way. If the matrix B is the intersection matrix, then the element B_{ii} is the self-intersection of the i-th curve in the configuration and the elements B_{ij}, B_{ji} are equal to the multiplicity of the intersection of the i and j curves.

Every curve we get from the singular fibers of the fibration will have self-intersection -2 and the intersection multiplicity is always 0 or 1. When we take linear combinations of curves, the self-intersection and intersection multiplicities will always be a linear combination of those of the original curves. See Theorem 1.1 of chapter V of [12] for a discussion of this fact. The intersection matrix clearly gives us the matrix of a symmetric bilinear form. Because it agrees with the intersection form of the surface, we will see this is the symmetric bilinear form of $\operatorname{Pic} X$ when several conditions are met.

3.2 The Curve-Line Bundle Dictionary

The Weierstrass model of an elliptic K3 surface gives us a configuration of distinct curves in the surface. This prompts us to consider two different notions of a divisor of a surface.

Definition 3.6. A **prime divisor** of a K3 surface X is an irreducible subvariety of codimension 1. A **Weil divisor** is a formal sum $\sum_Z n_Z Z$ where Z ranges over the irreducible codimension 1 subvarieties.

In particular, the free abelian group of Weil divisors of the surface X is denoted $\operatorname{Div} X$.

Another notion of divisor is that of a Cartier divisor. This type of divisor is closely related to line bundles, which is made precise by the definition.

Definition 3.7. A **Cartier divisor** is defined in one of two equivalent ways.

The first is as a global section of the sheaf $\mathcal{M}_X^\times/\mathcal{O}_X^\times$ where $\mathcal{M}_X$ is the sheaf of rational functions and $\mathcal{O}_X$ the sheaf of regular functions.

A second way to define them is as an open cover $\{U_i\}_{i\in I}$ together with a collection of rational functions $\{f_i\}_{i\in I}$ such that f_i/f_j has no zeros or poles on $U_i \cap U_j$.

We say that a Cartier divisor is **principal** if it is given by a single rational function f on X.

The second definition is much more obviously tied to the Picard group. This is because the data of a cover together with the rational functions f_i/f_j gives us the local trivializations and transition functions of a line bundle. If it were the case every line bundle $\mathcal{L}$ of an algebraic surface X had an associated Cartier divisor, we would have nearly every piece of information we need to begin computing the Picard lattice of an elliptic K3 surface. The following proposition assures us that this is the case.

Proposition 3.8. *Every line bundle $\mathcal{L}$ of an algebraic surface X has an associated Cartier divisor D.*

The sketch of the proof is as follows. Let $\{U_i\}_{i\in I}$ be an open cover of the surface by local trivializations. We may choose some anchor neighborhood U_0 and define $f_0 = 1$. Then, for each U_i we give $f_i = \Phi_{0i}(\text{or } \Phi_{i0})$ where Φ denotes the transition functions of the bundle. This gives us the data necessary for the second definition of a Cartier divisor.

Two final questions remain. The first is that the relationship between the abelian group of Cartier divisors and line bundles is in fact homomorphic. Luckily, for an algebraic surface we have the following exact sequence.

$$0 \to \mathcal{O}_X^\times \to \mathcal{M}_X^\times \to \mathcal{M}_X^\times/\mathcal{O}_X^\times \to 0$$

This gives an exact sequence on sheaf cohomology. Additionally, because we can define $\operatorname{Pic} X$ to be $H^1(X, \mathcal{O}_X^\times)$, we get the following identification.

$$H^0(X, \mathcal{M}_X^\times) \to H^0(X, \mathcal{M}_X^\times/\mathcal{O}_X^\times) \to H^1(X, \mathcal{O}_X^\times) = \operatorname{Pic} X$$

This informs us that there is a homomorphic relation between the Picard group and the group of Cartier divisors modulo the principal Cartier divisors. This quotient is referred to as **Cartier divisors modulo linear equivalence**.

The second question is the relationship between Weil and Cartier divisors. Weil divisors are a very immediate leap from the configuration of curves given by the fibration. Meanwhile, Cartier divisors are strongly tied to the line bundles making up the Picard lattice. It turns out that every Cartier divisor is in fact also a Weil divisor. This is done by considering the multiplicty of the zeros and poles of the f_i. In particular, we can think of the corresponding Weil divisors as being given by the formal difference of the zeros and the poles of the regular functions on U_i.

It is a more general fact that for a smooth surface the groups of Weil divisors and Cartier divisors are isomorphic, but we don't need this fact for our purposes.

Given collections of divisors, we can impose broader and broader equivalence relations on them. Three principal examples of these equivalence relations are given.

- **Linear Equivalence**: For $C, D \in \operatorname{Div} X$, we say that $C \sim D$ under linear equivalence if $C = D + (f)$ where f is a rational function defined on X.

- **Algebraic Equivalence**: For $C, D \in \operatorname{Div} X$, we say that $C \sim D$ under algebraic equivalence if there is some connected curve T, closed points 0,1$\in T$ and a divisor E of $X \times T$ such that E is flat over T and $E|_{X\times 0} - E|_{X\times 1} = C - D$.

- **Numerical Equivalence**: For $C, D \in \operatorname{Div} X$, we say that $C \sim D$ under numerical equivalence if for every $E \in \operatorname{Div} X$ it is the case that the intersection forms $\langle C, E \rangle = \langle D, E \rangle$.

We have the following hierarchy of implications.

$$\text{Linear Equivalence} \implies \text{Algebraic Equivalence} \implies \text{Numerical Equivalence}$$

If we denote the group of Weil divisors modulo algebraic equivalence as $\operatorname{NS} X$ (which is an abbreviation of Néron-Severi) and the group of Weil divisors modulo numerical equivalence as $\operatorname{Num} X$, we get the following natural surjective group homomorphisms.

$$\operatorname{Pic} X \to \operatorname{NS} X \to \operatorname{Num} X$$

Importantly, for all algebraic K3 surfaces, it is the case that all of these homomorphisms are isomorphisms. This follows from the Riemann-Roch theorem for algebraic surfaces (see [13]). This very nice fact means that we're free to make use of a result of Shioda and Tate without concern. And because of the relationship between divisors and line bundles, this will allow us to generate at least a sublattice of $\operatorname{Pic} X$.

3.3 Shioda-Tate and Mordell-Weil Lattices

Before we can introduce the Shioda-Tate formula, we need to go over several definitions.

Definition 3.9. A **vertical divisor** is any of the irreducible curves making up the singular fibers of the fibration or a smooth genus one curve.

A **horizontal divisor** is any irreducible curve meeting meeting every fiber with a fixed multiplicity. Sections of the fibration are examples.

The portion of the Néron-Severi lattice generated by curves of the singular fibers together with an irreducible fiber and the zero section is sometimes called the **Trivial lattice** or $\operatorname{Triv} X$. It has rank equal to $2 + \sum_t (n_t - 1)$.

Definition 3.10. The **Mordell-Weil lattice** is the torsion-free part of the quotient $\operatorname{Pic} X / \operatorname{Triv} X$.

With all of this in mind, we are now able to relate the components of the singular fibers, the rank of the Mordell-Weil lattice and the Picard number. This is done by way of the following.

Theorem 3.11. *(Shioda-Tate Formula)*

For an elliptic K3 surface X with fibration π, we can relate the ranks of several groups to the Picard number ρ. In particular if n_t is the number of irreducible curves in the singular fiber t we have

$$\rho = 2 + \sum_t (n_t - 1) + \operatorname{rank} MW(X)$$

where MW denotes the Mordell-Weil lattice. In particular, if $\operatorname{rank} MW = 0$ then a subset of the curves from the fibration generate a finite index sublattice of the Picard lattice.

The Shioda-Tate formula is developed in general for elliptic surfaces in [21],[22],[8]. In [6] it is stated specifically for K3 surfaces. In this formula, the 2 comes from the zero section and an irreducible fiber of the fibration. In practice we usually see this as the sublattice U.

One reason the isomorphism between $\operatorname{NS} X$ and $\operatorname{Pic} X$ is important is because Shioda-Tate was originally proved with respect to divisors modulo algebraic equivalence (the Néron-Severi lattice). Some of the literature will work by considering the Néron-Severi lattice while others will work with respect to the Picard lattice. For K3 surfaces these are the same, but this is not true for general surfaces.

For most of the surfaces we're interested in, it will be the case that $\rho = 2 + \sum_t (n_t - 1)$, i.e. $\operatorname{rank} \mathrm{MW} = 0$. For these surfaces, the data of the fibration and some lattice theory are all that's necessary to compute the Picard lattice. If this equality doesn't hold, one must examine the Mordell-Weil lattice for the remainder of the generators. In the case where $\operatorname{rank}\operatorname{Triv} X < \rho$, we will compute the rank of the Mordell-Weil lattice. These are given in Table 5.3.

3.4 Finite Index Sublattices and the Discriminant Form

Suppose X is a K3 surface with $\operatorname{rank}\mathrm{MW} = 0$. The challenge here is that the Shioda-Tate formula only tells us that the Trivial lattice is a finite index sublattice of $\operatorname{Pic} X$. We would really prefer if we could arrive at a collection of generators for $\operatorname{Pic} X$, if possible. Lattice theory helps us confirm that we have a set of generators for the Trivial lattice.

We know that for each singular fiber t we get a collection of curves in $\operatorname{Pic} X$ and by their arrangement in the Dynkin diagram we learn the bilinear form of the sublattice generated by the curves. Table 3.4 gives the correspondence between fiber type, the lattice that fiber generates and the signature and discriminant form of the corresponding lattice.

Singular Fiber	I_2, III	I_3, IV	I_0^*	IV^*	III^*	II^*
Lattice	A_1	A_2	D_4	E_6	E_7	E_8
Discriminant Form	$w_{2,1}^{-1}$	$w_{3,1}^{1}$	v	$w_{3,1}^{-1}$	$w_{2,1}^{1}$	trivial
Discriminant group	$\mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$	$\mathbb{Z}/3\mathbb{Z}$	$\mathbb{Z}/2\mathbb{Z}$	Trivial Group
Signature	(0,1)	(0,2)	(0,4)	(0,6)	(0,7)	(0,8)

Table 3.1: Singular fibers together with their corresponding lattice invariants

Given the data of the elliptic fibration, if Shioda-Tate tells us we have "enough" curves, then we can obtain the form of the Trivial lattice from Table 3.4. We use the code in appendix C to compute the discriminant form of our candidate generators and conclude by Nikulin's theorem that we really have given generators for $\operatorname{Triv} X$. In order to show $\operatorname{Triv} X \cong \operatorname{Pic} X$, in this cases, we need the following:

Theorem 3.12 (Miranda, [19]). *If it is the case that* $\operatorname{rank} MW(X) = 0$ *then we have the following exact sequence.*

$$0 \to \operatorname{Triv} X \to \operatorname{Pic} X \to \operatorname{MW} X \to 0$$

If $|\operatorname{MW} X| = 1$*, then* $\operatorname{Triv} X \cong \operatorname{Pic} X$*.*

There are a number of ways to bound the cardinality of the Mordell-Weil lattice. The following, due to Shioda, is an example.

Theorem 3.13 (Shioda, [21])**.** *Let* X *be an elliptic surface and* $\{F_i\}$ *the collection of singular fibers. Then it is the case that*

$$\frac{\det \operatorname{Pic} X}{|\operatorname{Pic} X_{torsion}|^2} = \frac{\det (s_{ij}) \prod_{F_i} \det F_i}{|MW|^2}$$

where $\det F_i$ *refers to the determinant of the matrix describing the bilinear form of the fiber. The matrix* (s_{ij}) *is the intersection matrix of non-torsion sections of the fibration.*

Corollary 3.14. *For* X *a K3 surface, if* $\operatorname{rank} \operatorname{MW} X = 0$ *we have*

$$|\operatorname{MW} X|^2 \det \operatorname{Pic} X = \prod_{F_i} \det F_i$$

using the same convention as the previous theorem.

This is a very useful result, because for all of the surfaces we consider the determinant of the transcendental lattice $T(X)$ is known. But another result gives a nice corollary which makes it very easy to see when $\operatorname{Triv} X \cong \operatorname{Pic} X$.

Theorem 3.15 (Miranda, [19])**.** *If the Mordell-Weil group is finite, then there is an embedding* $\operatorname{MW} X \hookrightarrow F^*/F$ *where* F *is the lattice corresponding to a singular fiber of the fibration.*

Corollary 3.16. *If the Mordell-Weil group is finite, then* $|\operatorname{MW} X| \leq \gcd \{\det F_i\}$*.*

If the fibration has a II^* *fiber or two fibers with discriminant groups of relatively prime order, it is the case that* $\operatorname{Triv} X \cong \operatorname{Pic} X$*.*

This corollary in particular is what allows us to conclude that we have successfully computed the Picard lattice for all but a few surfaces. From here, we move on to address a few odds and ends that will be useful for our computations.

3.5 Purely Non-symplectic Automorphisms and Useful Miscellany

We're principally interested in this work in computing the invariant lattices of several purely non-symplectic automorphisms of K3 surfaces. We make precise what that means here.

Definition 3.17. An automorphism σ of a K3 surface having order n is **symplectic** if the induced action on $H^0(X, \Omega_X^2)$ is equal to the identity. If this is not the case, we say the automorphism is **non-symplectic**.

If every non-trivial power of σ is also non-symplectic, we say σ is **purely non-symplectic**. Equivalently, we can say that if ω is any non-vanishing 2-form then a purely non-symplectic σ has the induced action $\sigma^*\omega = \zeta_n \omega$ where ζ_n is a primitive n-th root of unity.

It turns out that having a purely non-symplectic automorphism imparts a great deal of structure that we can work with. The fixed-point theory of these symmetries turn out to be particularly nice. Combined with the structure of an elliptic fibration there is a great deal that can be said about these surfaces by way of a range of different ideas. This section serves to highlight these ideas.

It bears making explicit what we mean by an automorphism of an elliptic K3 surface first. We require automorphisms of elliptic K3 surfaces to respect the structure of the fibration. For our purposes, this in particular entails that singular fibers of the surface must be mapped to other singular fibers of the same type and curves contained within a reducible fiber must be sent to curves with the same intersection properties. Finally, the automorphism always possesses a corresponding action on the base $\mathbb{P}^1$.

This means that any automorphism of a K3 surface X translates into an automorphism of the graph representing the Picard lattice. We can then exploit symmetries of the graph to tell us about the eigenspaces of the action of the automorphism on $H^2(X,\mathbb{Z})$.

Any fixed-point $p \in X$ with X a K3 surface, it is the case that the action of the automorphism on the tangent space about p is given by $\begin{pmatrix} \zeta_n^i & 0 \\ 0 & \zeta_n^j \end{pmatrix}$ where $i+j=n+1$. We say this point is of type (i,j). When examining fixed points of multiple different automorphisms of different powers, it is common to give the type of the point as $\frac{1}{n}(i,j)$ where n is the order of the automorphism associated to the point. If the order of the automorphism will not be confused from context, we denote it (i,j).

Theorem 3.18 (Dillies, [10])**.** *If we have a tree of rational curves invariant under σ, we know that know that the intersection points of the curves are fixed points. If a particular intersection is a point of type (i,j) we know the types of the other points in the tree.*

If the order of σ is even, these fixed-points occur in the following pattern.

$$..., (1,0), (2, n-1), ..., (\frac{n}{2}, \frac{n}{2}+1), (\frac{n}{2}, \frac{n}{2}+1), (\frac{n}{2}-1, \frac{n}{2}+2)...$$

And if the order is odd, we have the the following pattern.

$$..., (1,0), (2, n-1), ..., (\frac{n+1}{2}, \frac{n+1}{2}), (\frac{n+1}{2}-1, \frac{n+1}{2}+1), ...$$

.

Finally points of type $(1,0)$ are points contained in a curve that is point-wise fixed.

This theorem is important in determining the number of isolated fixed points and point-wise fixed curves and can be found in [10], [24], [23]. This lets us bound from below the number of isolated fixed-points and point-wise fixed curves of an automorphism. Together with some fixed-point theory, it will allow us to compute explicitly the rank of the invariant lattice of the automorphism.

Chapter 4. Computing the Invariant Lattice

The previous chapter of this thesis has concerned itself with finding the Picard lattice of a given K3 surface. However, our principal goal in this work is the computation of the invariant lattice of a K3 surface with respect to certain purely non-symplectic automorphisms. We start with the computation of the rank of the invariant lattice r_σ and then discuss how to parley this into generators of the invariant lattice.

The process of computing the invariant lattice of certain purely nonsymplectic automorphism σ of an elliptic K3 surface comes in 3 steps. First, we compute the rank of the invariant lattice. We can then inspect the configuration of curves from the fibration for symmetries and look for curves that are fixed by the automorphism. Sometimes these symmetries help to inform the computation of r_σ. Then we determine generators of the Picard lattice by the methods outlined in the previous chapter. We examine the symmetries of the Picard lattice and consider the local action of the automorphism at fixed points to see which curves are exchanged and we construct a candidate lattice out of the linear combinations of the orbits of the action. By construction, it is obvious that the candidate lattice embeds primitively into $\operatorname{Pic} X$. To conclude our argument we show our candidate invariant lattice has the right rank, is fixed by σ^* and embeds primitively into the Picard lattice.

4.1 Computing r_σ

The first step, computing r_σ is where most of the mathematical machinery is used. This is accomplished by way of several fixed-point theorems in concert with some general facts about K3 surfaces. That is the topic of this section

4.1.1 The Holomorphic Lefschetz Formula. The first thing we will need is some relations on the isolated fixed points, the number of point-wise fixed curves and the maximal genus of the those fixed curves. This can be accomplished by way of the Holomorphic Lefschetz Formula (see [4],[5]). It is worth noting that the formula given here is a special case of the more general Holomorphic Lefschetz Formula. We are making use of the fact that we are considering a purely non-symplectic automorphism on a K3 surface in this computation. These constraints can be relaxed to work for a holomorphic map (of possibly infinite order) on a compact complex manifold.

Theorem 4.1 (Holomorphic Lefschetz Formula - Atiyah, [4]). *Given a purely non-symplectic automorphism σ of a K3 surface X having isolated fixed points, there is an invariant $L(\sigma)$ defined in the following way.*

$$L(\sigma) = \sum_{i=0}^{2}(-1)^i \, Tr(\sigma^*|_{H^i(X,\mathcal{O}_X)}) \tag{4.1.1}$$

If σ has order n, then $L(\sigma) = 1 + \zeta_n^{n-1}$ where ζ_n is a primitive n-th root of unity. Furthermore,

$$L(\sigma) = \sum_{i+j=n+1, 1<i\leq j<n} \frac{n_{i,j}}{det(I_2 - A_{i,j})} + \alpha\frac{1+\zeta_n}{(1-\zeta_n)^2} \tag{4.1.2}$$

where $n_{i,j}$ is the number of isolated fixed points of type (i, j), $A_{i,j}$ is the linearization of the action on the tangent space at that fixed point, and $\alpha = \sum_{C\subset X^\sigma}(1 - g(C))$.

For an automorphism of a compact complex manifold, the equality between the right hand sides of 4.1.1 and 4.1.2 is known as the holomorphic Lefschetz fixed-point formula. To see why $L(\sigma) = 1 + \zeta_n^{n-1}$, we observe that the cohomology groups on the right hand side of 4.1.1 are the Dolbeault cohomology groups $H^{0,i}_{\bar{\partial}}(X)$ and that by Dolbeault's theorem, there's a natural isomorphism $H^{p,q}_{\bar{\partial}}(X) \cong H^q(X, \Omega^p)$. Thus for $i = 1$ on the right hand side of 4.1.1 we get a zero (by irregularity zero) and for $i = 2$ we're looking at the action on $H^0(X, \mathcal{O}_X(K_X))^\vee$.

It is worth remarking that there is a case where the right hand side of 4.1.2 can be identically zero. If the fixed locus of σ is empty, then it must be the case that $n_{i,j} = 0$ and

$\alpha = 0$. If the order of the automorphism is prime, it can be seen [3] that the fixed locus can also contain two disjoint elliptic curves. In this case, there are no isolated fixed points and $L(\sigma)$ is again zero. Neither of these cases poses an issue because part of the hypothesis of the holomorphic Lefschetz formula is that there be isolated fixed points.

This theorem is widely used in the literature on non-symplectic and purely non-symplectic automorphisms of K3 surfaces to constrain the parameters of the automorphism. Paired with another result, it will allow us to compute the rank of the invariant lattice explicitly.

4.1.2 The Topological Lefschetz Formula.

Once we've arrived at relations on the number of isolated fixed-points and point-wise fixed curves, we'd like to use this to determine the rank of the invariant lattice. To do so, we make use of another powerful fixed-point theorem: the topological Lefschetz fixed-point formula (see [17]).

Theorem 4.2 (Lefschetz, [17])**.** *Let $\sigma : X \to X$ be a continuous map from a compact triangulable topological space to itself. Then $\chi(X^\sigma) = \sum_{i=0}^{4}(-1)^i \operatorname{tr}(\sigma^*|_{H^i(X,\mathbb{Z})})$.*

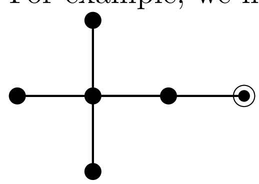
For X a K3 surface, we have that $\chi(X^\sigma) = N + 2\alpha$ with N being the total number of isolated fixed-points and the right hand side is $2 + \operatorname{tr}\sigma^*|_{\operatorname{Pic} X} + \operatorname{tr}\sigma^*|_{T(X)}$. So we obtain the relation $N + 2\alpha = 2 + \operatorname{tr}\sigma^*|_{\operatorname{Pic} X} + \operatorname{tr}\sigma^*|_{T(X)}$. Here we are using the fact that the K3 lattice Because the trace is not sensitive to changes of basis and σ^* is an isometry of the cohomology lattice, we may diagonalize and consider the action on eigenspaces.

As the action has order n each eigenspace has eigenvalue ζ_n^j where ζ_n is a primitive n-th root of unity and $0 \leq j < n$. Because the left hand side consists of integers, this allows us to constrain r_σ in terms of the fixed points and point-wise fixed curves.

4.2 Candidate Invariant Lattice

With the fixed-point theory in place, we want to consider how we propose an invariant lattice from the Picard lattice. In general, we expect the invariant lattice to be generated by the linear combination of the elements of the distinct orbits on the generators. Determining these orbits comes down to examining the Picard lattice for symmetries of order dividing the order of the automorphism.

For example, we might consider the following Picard lattice.



This lattice has a symmetry of order 2 and a symmetry of order 3. If 2 and 3 do not divide the order of the automorphism, we would conclude that the entire lattice is fixed because it must be the case that each of these generators is its own orbit.

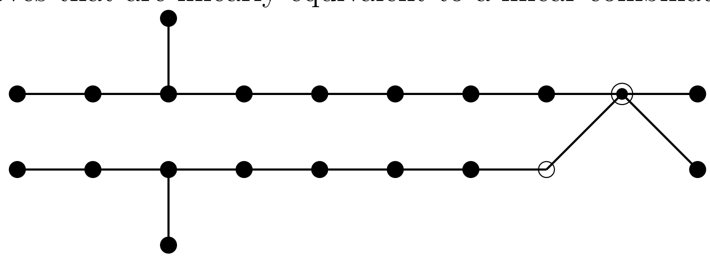
If 2 divided the order of the automorphism, we would conclude that it's likely two of the curves in the lattice are exchanged. Call the curves C_1 and C_2. We can rewrite the sublattice generated by the pair of curves as being generated by $C_1 + C_2$ and $C_1 - C_2$. It is immediate that the generator $C_1 + C_2$ is fixed by the action of the automorphism. Even if the curves are exchanged, this still gives us the same generator.

By using the symmetries we can give a candidate lattice that we believe is the invariant lattice. When we take these linear combinations, the self-intersection and intersection multiplicities are now linear combinations of those of the original curves. So in our example with 2 curves, the generator $C_1 + C_2$ has self-intersection -4 and intersection of multiplicity 2 with the central curve of the D_4 configuration.

4.3 Example Computation

We use this opportunity to string these ideas together to show how the computation works for a surface from start to finish in full detail. We use surface 2 from Brandhorst's catalog for this illustration.

Surface 2 has an automorphism of order 4 and the transcendental lattice has determinant 2^2. For this surface we have the Weierstrass model $y^2 = x^3 + 3t^4x + t^5(t^2 - 1)$. This gives us fibers of type II^* over $t = 0, \infty$ and of type I_2 over $t = \pm 1$. From this we get the following configuration of curves on the surface, where the node with the outer circle represents the section of the fibration and the circles that aren't filled in represent redundant curves, i.e. curves that are linearly equivalent to a linear combination of the other curves.



We would like to argue first that this is indeed the Trivial lattice of the surface, that is the lattice generated by the curves from the reducible fibers of the fibration. Rather than fiddle with writing down an isomorphism, we resort to lattice theory to argue that this must be the Trivial lattice. We know by the Shioda-Tate formula that we have enough of the right generators to generate the Trivial lattice. We can also check that this lattice has the appropriate signature $(1, 19)$ and the discriminant form $w_{2,1}^{-1} \oplus w_{2,1}^{-1}$. Because this has the correct rank and discriminant group, we conclude by proposition 2.7 that this is the Trivial lattice of the surface.

Next, we have to argue that the Trivial lattice is isomorphic to the Picard lattice of the surface. Part of this was done by seeing the rank of the Mordell-Weil lattice was zero. We now wish to show that the Mordell-Weil group is trivial. This is done by way of Corollary 3.16. Recall when the Mordell-Weil group is finite, it embeds into the discriminant group of each fiber. Because the E_8 lattice is unimodular, it has trivial discriminant group. This means there is only one section: the zero section. This means that we have the exact sequence

$$0 \to \operatorname{Triv} X \to \operatorname{Pic} X \to \operatorname{MW} X \to 0$$

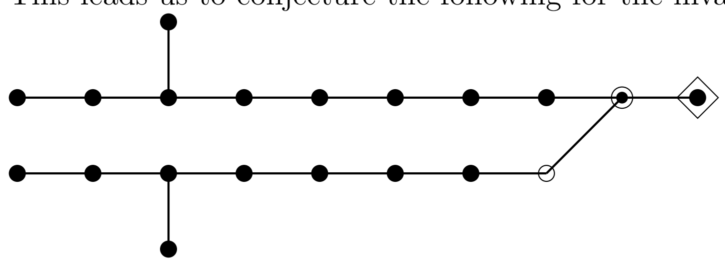
which simplifies to

$$0 \to \operatorname{Triv} X \to \operatorname{Pic} X \to 0$$

meaning these curves actually generate the Picard lattice.

We now turn to the problem of examining how the automorphism acts on this lattice. We know that the automorphism fixes the fibers over $t = 0, \infty$. Because it is an automorphism and the nodes of this lattice represent curves on the surface, it must be the case that curves are sent to curves with the same intersection properties. This means both E_8 configurations are fixed as well as the U configuration containing the section. Because the automorphism acts as an involution on the base, we see that the I_2 fibers over $t = \pm 1$ are exchanged.

This leads us to conjecture the following for the invariant lattice.



If we label the curves in each I_2 fiber as A, B then the node with a diamond about it is the linear combination of the two and has intersection 2 with the section and self-intersection -4. it is clear that $A + B$ is invariant under the action of the automorphism. We also see that $A - B$ is clearly mapped to $B - A$ meaning this is an eigenvector with eigenvalue -1.

It is clear this embeds primitively into the Picard lattice. To confirm this is the invariant lattice, we turn to fixed-point theory to prove the rank of the eigenspace of 1 is 19.

We begin with the holomorphic Lefschetz formula, which relates the number of isolated fixed-points of this automorphism to the number of point-wise fixed rational curves. For the order 4 case, this works out to

$$1 - i = \frac{N}{2(1+i)} + \frac{1+i}{1-i}\alpha$$

which simplifies to $N = 2\alpha + 4$. Now, by examining each E_8 configuration of curves, we see there must be at least 6 isolated fixed points and 2 point-wise fixed curves in each fiber by Theorem 3.18. We see that there are no point-wise fixed curves of positive genus, so $12 = 2(4) + 4$ works out perfectly,i.e. $N = 12, \alpha = 4$. We proceed to the next step.

The topological Lefschetz formula relates the number of fixed-points and point-wise fixed curves to the action on the second cohomology group in the following way.

$$N + 2\alpha = 2 + \operatorname{tr}(\sigma^*|_T) + \operatorname{tr}(\sigma^*|_{\text{Pic}})$$

Because this is an automorphism of order 4, the only possible actions are order 4,2 and 1. The only place an order 4 action is possible is on the Transcendental lattice and we see that the trace of the action the transcendental lattice is $\mu(4) = 0$, where μ is the Möbius function. We arrive that the following simplification, where l denotes the rank of the eigenspace of -1.

$$12 + 2(4) = 2 + r_\sigma - l$$

We make the observation that σ^2 must fix the entire Picard lattice but not the Transcendental lattice, meaning that $20 - r_\sigma = l$. We make this substitution to get

$$38 = 2r_\sigma$$

which completes our proof that this is the invariant lattice of this automorphism.

Chapter 5. Brandhorst's Elliptic K3 Surfaces

We will now consider a special collection of elliptic K3 surfaces. We will place certain restrictions and make certain observations about the role these restrictions play. Let X be an elliptic surface admitting a purely non-symplectic automorphism σ of order n. Recall that $(\operatorname{Pic} X)^{\perp} = T(X)$. We place the restriction that the rank of the Trascendental lattice T be equal to $\varphi(n)$. We also know that the eigenvalues of the eigenspaces of $T(X)$ must be primitive n-th roots of unity. It is this family of surfaces that is classified by Brandhorst in [7] and they enjoy some very important properties.

In Tables 5.1 and 5.2, we have listed all of the surfaces in Brandhorst that are known to be elliptic surfaces. We give the Weierstrass model of the fibration, the determinant of the transcendental lattice T, the order of the automorphism and the coordinate description of the automorphism σ. We additionally supply a number for each surface for ease of reference. The first table gives surfaces such that $\varphi(\text{Order}) \leq 10$, while the second table will cover all of the surfaces with $\varphi(\text{Order}) \geq 12$.

These surfaces are of interest because they are determined up to isomorphism by the tuple (n, d) where n is the order of the automorphism and d the determinant of the Néron-Severi lattice. Of particular interest is the fact that for surfaces 16, 25, 28, 31, 32, 34, 36, 37 and 38 the group of purely non-symplectic automorphisms on these surfaces is cyclic, meaning for 9 of the 38 surfaces every purely non-symplectic automorphism of the surface is a power of some generator. For 12 of these surfaces, we know their purely-nonsymplectic automorphism subgroup or their automorphism group explicitly. These strong uniqueness properties make them important jumping off points for the for the program of classifying purely non-symplectic automorphisms of composite order.

Being able to compute the invariant lattices of these surfaces explicitly is interesting for two different reasons. First, computing the invariant lattice of an automorphism of a surface

Surface Number	Order	det T	X	σ
1	3, 6	3	$y^2 = x^3 - t^5(t-1)^5(t+1)^2$	$(\zeta_3 x, \pm y, t)$
2	4	2^2	$y^2 = x^3 + 3t^4x + t^5(t^2-1)$	$(-x, \zeta_4 y, -t)$
3	5, 10	5	$y^2 = x^3 + t^3x + t^7$	$(\zeta_5^3 x, \pm\zeta_5^2 y, \zeta_5^2 t)$
4	8	2^2	$y^2 = x^3 + tx^2 + t^7$	$(\zeta_8^6 x, \zeta_8 y, \zeta_8^6 t)$
5	12	1	$y^2 = x^3 + t^5(t^2-1)$	$(-\zeta_3 x, \zeta_4 y, -t)$
6	12	$2^2 3^2$	$y^2 = x^3 + t^5(t^2-1)^2$	$(-\zeta_3 x, \zeta_4 y, -t)$
7	12	2^4	$y^2 = x^3 + t^5(t^2-1)^3$	$(-\zeta_3 x, \zeta_4 y, -t)$
8	7, 14	7	$y^2 = x^3 + t^3x + t^8$	$(\zeta_7^3 x, \pm\zeta_7 y, \zeta_7^2 t)$
9	9, 18	3	$y^2 = x^3 + t^5(t^3-1)$	$(\zeta_9^2 x, \pm\zeta_9^3 y, \zeta_9^3 t)$
10	9, 18	3^3	$y^2 = x^3 + t^5(t^3-1)^2$	$(\zeta_9^2 x, \pm y, \zeta_9^3 t)$
11	16	2^2	$y^2 = x^3 + t^2x + t^7$	$(\zeta_{16}^2 x, \zeta_{16}^{11} y, \zeta_{16}^{10} t)$
12	16	2^4	$y^2 = x^3 + t^3(t^4-1)x$	$(\zeta_{16}^6 x, \zeta_{16}^9 y, \zeta_{16}^4 t)$
13	16	2^6	$y^2 = x^3 + x + t^8$	$(-x, iy, \zeta_{16} t)$
14	20	2^4	$y^2 = x^3 + (t^5-1)x$	$(-x, \zeta_4 y, \zeta_5 t$
15	20	$2^4 5^2$	$y^2 = x^3 + 4t^2(t^5-1)x$	$(-x, \zeta_4 y, \zeta_5 t)$
16	24	2^2	$y^2 = x^3 + t^5(t^4+1)$	$(\zeta_3\zeta_8^6 x, \zeta_8 y, \zeta_8^2 t)$
17	24	2^6	$y^2 = x^3 + (t^8+1)$	$(\zeta_3 x, y, \zeta_8 t)$
18	24	$2^2 3^4$	$y^2 = x^3 + t^3(t^4+1)^2$	$(\zeta_3\zeta_8^6 x, \zeta_8 y, \zeta_8^6 t)$
19	24	$2^6 3^4$	$y^2 = x^3 + x + t^1 2$	$(-x, \zeta_{24}^6 y, \zeta_{24} t)$
20	15, 30	5^2	$y^2 = x^3 + 4t^5(t^5+1)$	$(\zeta_3 x, \pm y, \zeta_5 t)$
21	15, 30	3^4	$y^2 = x^3 + t^5x + 1$	$(\zeta_{15}^{10} x, \pm y, \zeta_{15} t)$
22	11, 22	11	$y^2 = x^3 + t^5x + t^2$	$(\zeta_{11}^5 x, \pm\zeta_{11}^2 y, \zeta_{11}^2 t)$

Table 5.1: K3 surfaces admitting purely non-symplectic automorphisms with $\varphi(\sigma) \leq 10$

is a hard task. We are making use of a great deal of interesting machinery to argue that we have given explicit generators of the invariant lattice and that these generators are indeed invariant under the action. We are able to exploit a special configuration of curves derived from the fibration and parley these into generators of the invariant lattice. Second, knowing the invariant lattices of these surfaces will tell tell us how these surfaces fit into an eventual classification of purely non-symplectic automorphisms of composite order.

Some remarks are worthwhile before moving forward. Many surfaces have automorphisms of two different orders and we will adopt the convention that the automorphism of lower order is written σ and the automorphism of greater order will be denoted τ. We also wish to mention several surfaces that we can't address by means of our methods in this work. If we think back to the Shioda-Tate formula in chapter 3, if we don't get enough vertical

Surface Number	Order	det T	X	σ
23	13, 26	13	$y^2 = x^3 + t^5x + t$	$(\zeta_{13}^5 x, \pm\zeta_{13} y, \zeta_{13}^2 t)$
24	26	13	$y^2 = x^3 + t^7x + t^4$	$(\zeta_{13}^{10} x, -\zeta_{13}^2 y, \zeta_{13} t)$
25	21, 42	1	$y^2 = x^3 + t^5(t^7 - 1)$	$(\zeta_{42}^2 x, \zeta_{42}^3 y, \zeta_{42}^{18} t)$
26	21, 42	7^2	$y^2 = x^3 + 4t^4(t^7 - 1)$	$(\zeta_3\zeta_7^6 x, \pm\zeta_7^2 y, \zeta_7 t)$
27	21, 42	7^2	$y^2 = x^3 + t^3(t^7 + 1)$	$(\zeta_3\zeta_7^3 x, \pm\zeta_7 y, \zeta_7^3 t)$
28	28	1	$y^2 = x^3 + x + t^7$	$(-x, \zeta_4, -\zeta_7 t)$
29	28	2^6	$y^2 = x^3 + (t^7 + 1)x$	$(-x, \zeta_4, \zeta_7 t)$
				$(x - (y/x)^2, \zeta_4(y - (y/x)^3), \zeta_7 t)$
30	17, 34	17	$y^2 = x^3 + t^7x + t^2$	$(\zeta_{17}^7 x, \pm\zeta_{17} y, \zeta_{17}^2 t)$
31	32	2^2	$y^2 = x^3 + t^2x + t^11$	$(\zeta_{32}^{18} x, \zeta_{32}^{11} y, \zeta_{32}^2 t)$
32	36	1	$y^2 = x^3 - t^5(t^6 - 1)$	$(\zeta_{36}^2 x, \zeta_{36}^3 y, \zeta_{36}^{30} t)$
33	36	3^4	$y^2 = x^3 + x + t^9$	$(\zeta_{36}^2 x, \zeta_{36}^3 y, \zeta_{36}^{30} t)$
34	48	2^2	$y^2 = x^3 + t(t^8 - 1)$	$(\zeta_{48}^2 x, \zeta_{48}^3 y, \zeta_{48}^6 t)$
35	19, 38	19	$y^2 = x^3t^7x + t$	$(\zeta_{19}^7 x, \pm\zeta_{19} y, \zeta_{19}^2 t)$
36	27, 54	3	$y^2 = x^3 + t(t^9 - 1)$	$(\zeta_{27}^2 x, \zeta_{27}^3 y, \zeta_{27}^6 t)$
37	33, 66	1	$y^2 = x^3 + t(t^11 - 1)$	$(\zeta_{66}^2 x, \zeta_{66}^3 y, \zeta_{66}^6 t)$
38	44	1	$y^2 = x^3 + x + t^11$	$(-x, \zeta_4 y, \zeta_{11} t)$

Table 5.2: K3 surfaces admitting purely non-symplectic automorphisms with $\varphi(\sigma) \geq 12$

curves from our fibration to generate the Picard lattice, we need to examine the Mordell-Weil lattice as well. We give a table of the surfaces with insufficient curves and the rank of the Mordell-Weil lattice.

Each of these fifteen surfaces requires an examination of the Mordell-Weil lattice for a complete description of the Picard lattice and therefore the invariant lattice. The remaining 20 surfaces pose no such difficulty so we will move on to address them.

One other challenge rears its head for two more surfaces. From Brandhorst, we know that surfaces 14 and 29 possess a torsion section. This means that we cannot simply equate the Trivial and Picard lattices for these surfaces. We will not address these surfaces in this work, but the method of performing these computations is laid out in Belcastro's dissertation (see [6]).

For the remaining surfaces, we have everything we need to perform the computation. We move on in the next chapter to presenting the results of our computations.

Surface Number	Mordell-Weil rank
3	1
8	1
14	0 (Torsion Sections)
15	2
17	6
19	12
20	2
21	5
22	1
23	1
24	1
26	2
27	2
29	0 (Torsion Sections)
30	1
33	4
35	1

Table 5.3: Non-trivial Mordell-Weil Groups

Chapter 6. Main Result

The most exciting thing about this project is that it's possible to explicitly compute generators of the invariant lattices of these surfaces and their automorphisms. A wide range of mathematics is necessary to make the computation possible and we're exploiting a great deal of structure on these surfaces to accomplish it. In general, this computation is hard to accomplish and frequently we only learn that the invariant lattice is abstractly isomorphic to some direct sum of lattices. This can be seen in the classification of purely non-symplectic automorphisms of prime order where the invariant lattices are only given abstractly.

Additionally, this computation is interesting because the classification of purely non-symplectic automorphisms of composite order is an on-going program with only a few composite orders being fully classified. So we apply the techniques of [9] to the computation of the invariant lattices of the surfaces classified by Brandhorst. These surfaces possess strong uniqueness properties that make them interesting from the standpoint of the on-going classification. We present the result of our computation in Theorem 6.1. We remark that surfaces 11-13 were computed in [9].

Theorem 6.1. *All of the elliptic K3 surfaces given by Brandhorst having* $\operatorname{rank}\operatorname{Triv}X = \rho$ *and admitting no torsion sections have invariant lattices given by Table 6.1.*

Even with the wealth of machinery on display, the arguments to prove these are indeed the invariant lattices are technical in places. This is especially true when the the order of the automorphism is not square-free or has many prime factors. Difficulty notwithstanding, it is surprising this is possible at all. Because the computations are long and detailed, we first would like to make some remarks about future work.

Surface Number	Invariant Lattice of σ	Invariant Lattice of τ (where applicable)
1	$U \oplus E_8^{\oplus 2} \oplus A_2$	$U \oplus E_8^{\oplus 2} \oplus A_1(2)$
2	$U \oplus E_8^{\oplus 2} \oplus A_1(2)$	
4	$\langle 4 \rangle \oplus E_8^{\oplus 2}$	
5	$U \oplus E_8^{\oplus 2}$	
6	$U \oplus E_8 \oplus A_1(4) \oplus A_2$	
7	$U \oplus D_4 \oplus E_6$	
9	$U \oplus E_6 \oplus E_8$	$U \oplus E_8 \oplus D_4$
10	$U \oplus E_8 \oplus A_2(3)$	$U \oplus E_8 \oplus A_1(6)$
11	See $U \oplus E_8 \oplus A_3$	
12	See $U(2) \oplus D_4 \oplus \langle -8 \rangle$	
13	See $U(2) \oplus D_4 \oplus \langle -8 \rangle$	
16	$U \oplus E_8 \oplus A_2$	
18	$U \oplus A_2 \oplus A_1(8)$	
25	$U \oplus E_8$	$U \oplus E_8$
28	$U \oplus E_8$	
31	$U \oplus A_3$	
32	$U \oplus E_8$	
34	$U \oplus A_2$	
36	$U \oplus A_2$	$U \oplus A_1(2)$
37	U	U
38	U	

Table 6.1: Theorem 6.1

6.1 Future Work

There are a number of avenues of future work in this area. The most obvious is to compute the invariant lattices of the remaining elliptic surfaces in [7] using more advanced techniques such as consideration of intermediate lattices and computation of the Mordell-Weil lattice. Brandhorst also gives several K3 surfaces that are not known to admit an elliptic fibration and the computation of these invariant lattices is another important future step. This may be accomplished by constructing such a fibration or possibly by other methods.

Other work includes attempting to extend the classification of these automorphisms to larger composite orders. The classification of purely non-symplectic automorphisms on K3 surfaces is very much incomplete. With the exception of automorphisms of prime order, most classifications of automorphisms rely on placing some form of constraint on the action of the automorphism on $\operatorname{Pic} X$ or the surface itself. During the course of this project, relations on the fixed-points of purely non-symplectic automorphisms of order 12 were derived. These relations are necessary to setting up a full classification of automorphisms of this order. Because classifications of order 3 and order 4 automorphisms have already been given, we expect the classification of order 12 automorphisms to be a feasible future undertaking.

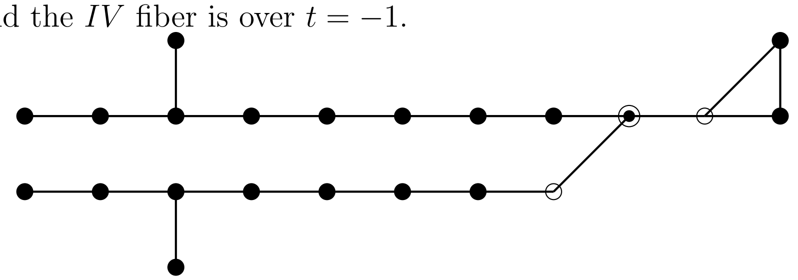
In the final section of this chapter, we present the detailed computations that went into determining the invariant lattice(s) for each surface and automorphism.

6.2 Detailed Computations

Here we present the computations of the Picard and invariant lattices where the computation more closely follows the methods laid out in chapter 4. In the graphs, we always give sections as a circled point. Recall that when giving generators of the Picard lattice, all nodes have self-intersection -2 and edges indicate intersections of multiplicty 1. Curves that are linearly dependent on the others are indicated with a hollow circle, hence the lattice is generated completely by the solid nodes. We use the notation $S(\sigma)$ to denote the invariant lattice of the automorphism σ. Nodes enclosed with triangles or diamonds are used to denote linear combinations of curves.

6.2.1 Surface 1.

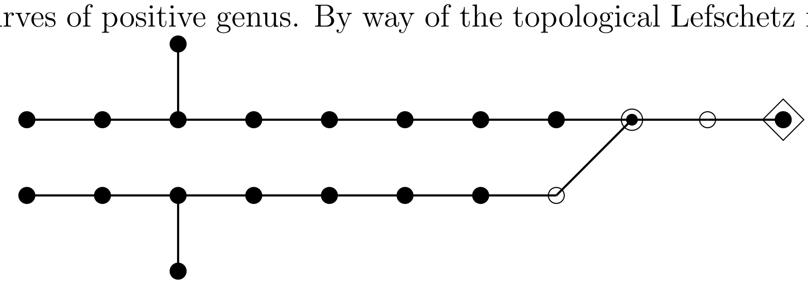
For this surface, we have automorphisms of order 3 and 6. The determinant of the transcendental lattice is 3. From the fibration we get II^* fibers over $t = 0, 1$ and the IV fiber is over $t = -1$.



The above is the Trivial lattice of surface 1. The two E_8 configurations come from the the II^* fibers and the A_2 configuration from the IV fiber. This lattice then is $U \oplus E_8^{\oplus 2} \oplus A_2$. Because we have a II^* fiber, we conclude by Corollary 3.16 this is the Picard lattice.

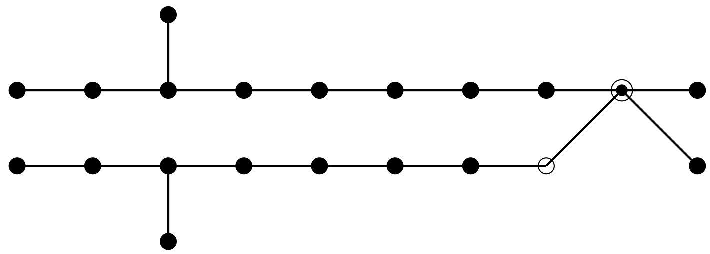
Because it is a surface studied in [16], we know at least one of it's automorphisms has invariant lattice rank 20. Thus this is also the invariant lattice for the automorphism of order 3. Thus $S(\sigma) = \mathrm{Pic}\, X = U \oplus E_8^{\oplus 2} \oplus A_2$ and has discriminant form $w_{3,1}^{1}$.

This surface is one of the cases examined in [10]. We know that there are 9 points of type $\frac{1}{6}(2,5)$ and 6 of type $\frac{1}{6}(3,4)$ and that the automorphism fixes three rational curves and no curves of positive genus. By way of the topological Lefschetz formula we learn that $r_\sigma = 19$.



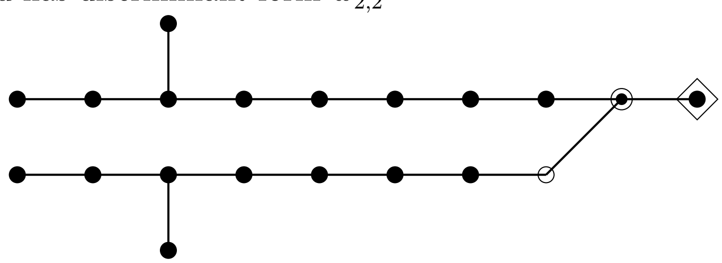
Because this is the invariant lattice of the order 6 automorphism of this surface. In this diagram, the diamond curve is the linear combination of the two extremal curves from the right side of the original lattice. Thus $S(\tau) = U \oplus E_8^{\oplus 2} \oplus A_1(2)$ and has discriminant form $w_{2,2}^{-1}$.

6.2.2 Surface 2. For this surface we have an automorphism of order 4 and the determinant of the Transcendental lattice is 4. We have fibers of type II^* over $t = 0, \infty$ and of type I_2 over ± 1.

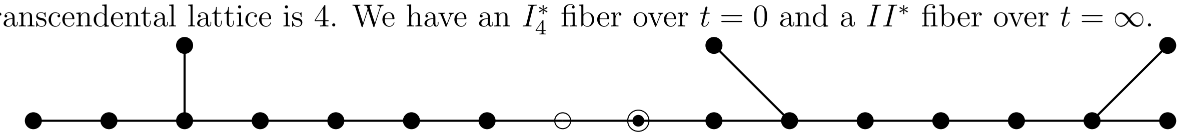


The above is the Trivial lattice of surface 2 and we are able to conclude it is isomorphic to $\operatorname{Pic} X$ by the presence of the II^* fibers (Corollary 3.16). Thus $\operatorname{Pic} X = U \oplus E_8^{\oplus 2} \oplus A_1^{\oplus 2}$.

We know from relations in [2] that the rank of the invariant lattice is 19. As the only place there is room for an order 2 action is the two A_1 sublattices. In fact, from Table 5.1, we see the I_2 fibers are exchanged by σ. We get the following for the invariant lattice where the diamond is the linear combination of the two rightmost curves. Thus $S(\sigma) = U \oplus E_8^{\oplus 2} \oplus A_1(2)$ and has discriminant form $w_{2,2}^{-1}$

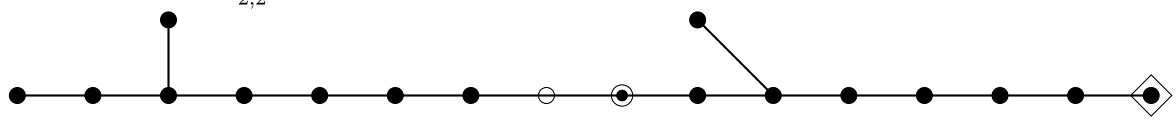


6.2.3 Surface 4. This surface has an automorphism of order 8. The determinant of the transcendental lattice is 4. We have an I_4^* fiber over $t = 0$ and a II^* fiber over $t = \infty$.



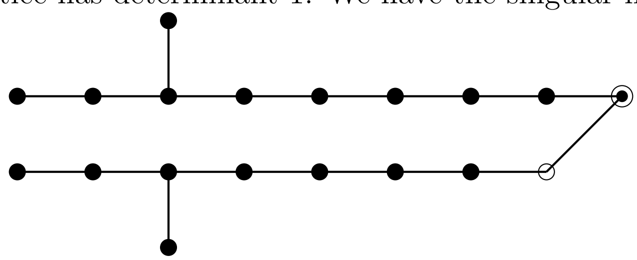
This is the Trivial lattice of X. The D_8 sublattice corresponds to an I_4^* fiber over $t = 0$ and the E_8 to a II^* fiber over $t = \infty$. We conclude by Corollary 3.16 this is the Picard lattice as well. Hence, $\operatorname{Pic} X = U \oplus E_8 \oplus D_8$.

By the relations given in [23], it is the case that $r_\sigma = 17$. As the only order 2 symmetry consists of exchanging the two curves at the rightmost edge of the D_8, $S(\sigma)$ is given by the following arrangement of divisors. This lattice is abstractly isomorphic to $\langle 4\rangle \oplus E_8^{\oplus 2}$ and has discriminant form $w_{2,2}^1$.



6.2.4 Surface 5.

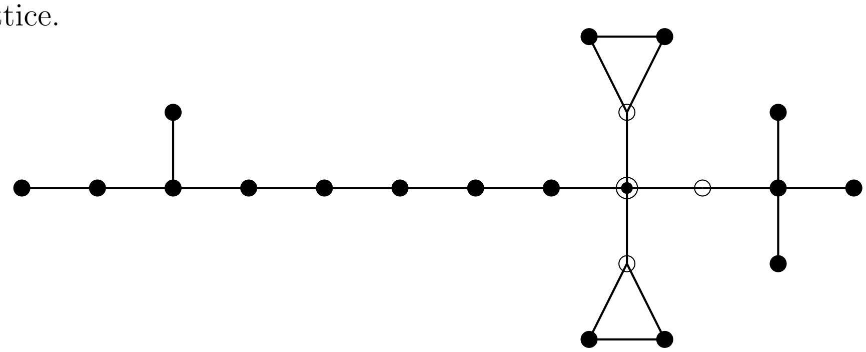
This surface has an automorphism of order 12 and the Transcendental lattice has determinant 1. We have the singular fibers II^* over $t = 0, \infty$.



The above is the Trivial lattice for surface 5. We conclude it is isomorphic to the Picard lattice because we have II^* fibers, by Corollary 3.16. As one of the E_8 trees is the fiber of $t = 0$ and the other over $t = \infty$, they can't be exchanged. Thus there is no order 2 or 3 symmetry in this arrangement. Because the order of the automorphism associated to this surface is 12, this is also the invariant lattice. Because this is one of the surfaces studied by Kondō in [16] we are even more confident in this result. Hence $S(\sigma) = U \oplus E_8^{\oplus 2} = \operatorname{Pic} X$. This lattice is unimodular.

6.2.5 Surface 6.

This surface has an automorphism of order 12 and determinant $2^2 3^2$ for the Transendental lattice. We have reducible fibers of type II^* over $t = 0$, type IV over $t = \pm 1$ and I_0^* over $t = \infty$. This gives us the following configuration of curves for the Trivial lattice.



Because we have a II^* fiber, we conclude that this is also the Picard lattice. (Corollary 3.16) Hence, $\operatorname{Pic} X = U \oplus E_8 \oplus D_4 \oplus 2A_2^{\oplus 2}$.

We make a remark regarding the action on the base. In [7], the action of of the automorphism is given as fixing the base. The challenge with this is that the fixed-point at the intersection of the II^* fiber and the section is of type $(6,7)$ which is not a point contained in a point-wise fixed curve. It can be observed that this action does not fix the Weierstrass model and the action should be $(-\zeta_3 x, \zeta_4 y, -t)$. We know from [10] that the 2-form must be $\frac{dx \wedge dt}{dy}$. Thus the action on the 2-form is given by $\frac{(-\zeta_3)(-1)}{\zeta_4} = \zeta_{12}$ making the action purely non-symplectic as desired.

By examining the configuration, we determine there are 3 fixed-points of type $(2,11)$, 2 of type $(3,10)$, 2 of type $(4,9)$, 2 of type $(5,8)$, 2 of type $(6,7)$ and $\alpha = 1$. These fit the relations for order 12, so we feel good about moving forward in the computation. The following is the topological Lefschetz formula for σ.

$$11 = 3r_\sigma - \frac{3}{2}r_{\sigma^2} - r_{\sigma^3} + \frac{1}{2}r_{\sigma^6}$$

We need to compute the ranks of the different powers of σ to finish the computation. We start with σ^2, which has order 6. We notice that this surface is not one of the cases given in [10]. Because the base is certainly fixed by σ^2, the A_2 configurations over $t = \pm 1$ are invariant under σ^2. We get 7 points of type $\frac{1}{6}(2,5)$ and 4 of type $\frac{1}{6}(3,4)$. From the relations in [10], we know that we have two point-wise fixed rational curves. We get the following topological Lefschetz formula for σ^2.

$$7 + 2\alpha = \frac{5}{2}r_{\sigma^2} - r_{\sigma^4} - \frac{1}{2}r_{\sigma^6}$$

We know that the permutation of the extremal curves of the D_4 configuration is order 3 because one of the fixed-points of the central curve is of type $\frac{1}{6}(2,5)$. Because σ^4 has an order 3 action on the lattice, this means that $r_{\sigma^4} = 16$. This is because the rank of the invariant lattice for an order 3 automorphism must be even and because the three D_4 curves are permuted it is not the case that it fixes the whole lattice, but does fix the curves in the E_8, U and both A_2 configurations.

In considering r_{σ^6}, we observe that the only possible action is order 2 and that the IV fibers are both fixed. Thus $r_{\sigma^6} = 16, 18$ because it must be divisible by 2. To see it is not equal to 18, we observe that if it were equal to 18, then the Lefschetz formula for σ^2 becomes

$$64 + 4\alpha = 5r_{\sigma^2}$$

and the left hand side can not be made divisible by 5 without adding more point-wise fixed rational curves. As this would violate the relations in [10], we conclude that $r_{\sigma^6} = 16$. This means that

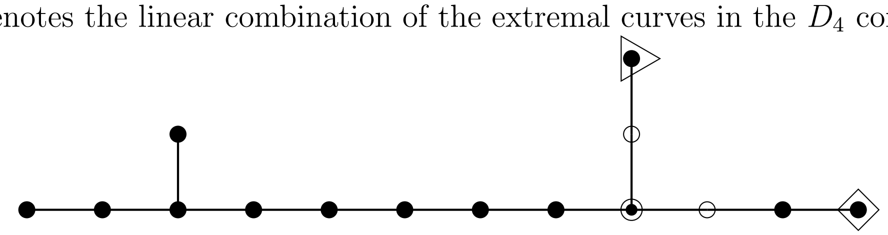
$$62 + 4\alpha = 5r_{\sigma^2}$$

or that $r_\sigma = 14$.

Finally, the fact that $r_{\sigma^6} \neq 18$ means that there is an order 2 action on the curves inside each IV fiber when fixed. This means that $r_{\sigma^3} = 15$. By the topological Lefschetz formula for σ, we see that

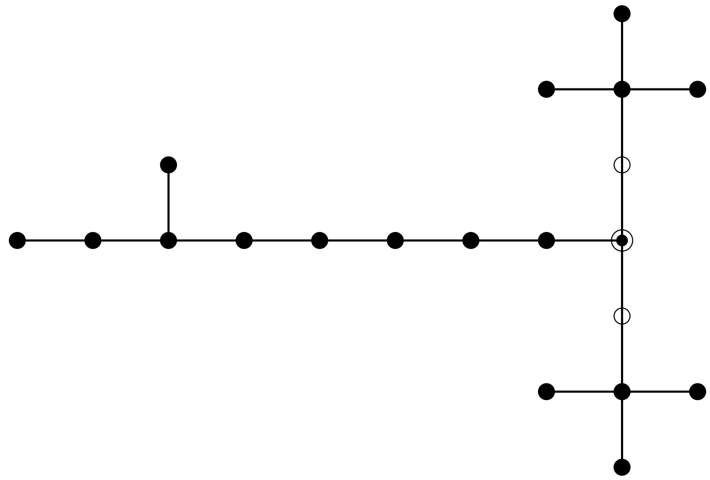
$$11 = 3r_\sigma - 21 - 15 + 8$$

which gives that $r_\sigma = 13$. This lets us conclude that the following is $S(\sigma)$, where the triangle denotes the linear combination of all of the curves in both A_2 configurations and the diamond denotes the linear combination of the extremal curves in the D_4 configuration.



Because this certainly embeds primitively, has the appropriate rank and is clearly fixed by the automorphism, we know this must be $S(\sigma) = U \oplus E_8 \oplus A_1(4) \oplus A_2$. This has discriminant form $w_{2,3}^{-1} \oplus w_{3,1}^{1}$.

6.2.6 Surface 7. This surface has an automorphism of order 12. The Transcendental lattice has determinant 2^4. We have a II^* fiber over $t = 0$ and two I_0^* fibers over $t = \pm 1$.



We see this is the Picard lattice because we have a II^* fiber. (Corollary 3.16) Hence, $\operatorname{Pic} X = U \oplus E_8 \oplus D_4^{\oplus 2}$.

By examining the configuration of curves, we see that we have at least 3 isolated fixed points of type $(2, 11)$, at least 2 of type $(3, 10)$, at least 1 of type $(4, 9)$, at least one of type $(5, 8)$ and at least 1 of type $(6, 7)$. We also have a point-wise fixed curve in the E_8 configuration. We notice that if we have another point of type $(3, 10)$ and another of type $(6, 7)$ we would satisfy the relations on fixed-points for order 12. These extra points make sense because we need isolated fixed-points where the II fiber over $t = \infty$ meets the section and the node of the curve. We turn to the topological Lefschetz formula.

$$10 + 2(1) = 2 + r_\sigma - l + (i + (-i))m_1 - m_2 + m_3 + \mu(12)$$

Here l denotes the rank of the eigenspace of -1, m_1 the rank of the eigenspace of i, m_2 the rank of the eigenspace of the primitive 3rd root of unity and m_3 the rank of the eigenspace of the primitive 6th root of unity. We can make several simplifying observations to get the new expression below.

$$10 = r_\sigma - l - m_2 + m_3$$

We observe that we can write $l = r_{\sigma^2} - r_\sigma$, $m_2 = \frac{1}{2}(r_{\sigma^3} - r_\sigma)$ and $m_3 = \frac{1}{2}(r_{\sigma^6} - r_{\sigma^2} - r_{\sigma^3} + r_\sigma)$. All together this gives the formula

$$10 = 3r_\sigma - \frac{3}{2}r_{\sigma^2} - r_{\sigma^3} + \frac{1}{2}r_{\sigma^6}$$

which means if we can find the rank of the invariant lattices of $\sigma^2, \sigma^3, \sigma^6$, we know the rank of the invariant lattice, $S(\sigma)$. We consider σ^2 first as it is an order 6 automorphism. By inspecting the configuration, we realize that we have at least 6 points of type $\frac{1}{6}(2,5)$, at least 6 of type $\frac{1}{6}(3,4)$ and at least two point-wise fixed rational curves. This fits the relations given in [10] so we proceed for now. The following is the topological Lefschetz formula for σ^2.

$$12 + 2\alpha = 2 + r_{\sigma^2} - (r_{\sigma^4} - r_{\sigma^2}) - \frac{1}{2}(r_{\sigma^6} - r_{\sigma^2}) + 2\mu(6)$$

We make the observation that the one of the fixed points in the central curve of each D_4 configuration is of type $\frac{1}{6}(2,5)$. As this action on the tangent space has order 3, this means the permutation on the curves is order 3 and $r_{\sigma^4} = r_{\sigma^2}$. This tells us additionally that $r_{\sigma^6} = 18$. This gives us the following new formula.

$$34 + 4\alpha = 3r_{\sigma^2}$$

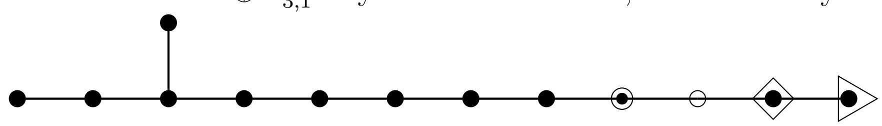
The only choices for α that make the left hand side divisible by 3 are $-1, 2, 5$. We know that $\alpha = -1$ gives us $r_{\sigma^2} = 10$ which is two small and $\alpha = 5$ gives us that $r_\sigma = 18$ which would mean that the automorphism acts trivially on $\operatorname{Pic} X$, which is absurd. This tells us that $r_{\sigma^2} = 14$.

Now we only need r_{σ^3} to finish our computation. We know that the linear combination of the two D_4 configurations is fixed by σ^3, meaning $r_{\sigma^3} \geq 14$. We also have the following simplification of the topological Lefschetz formula for σ.

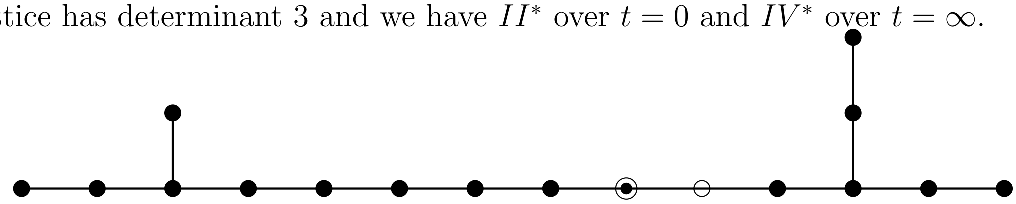
$$22 + r_{\sigma^3} = 3r_\sigma$$

Because the left hand side must be divisible by 3, we conclude that $r_{\sigma^3} = 14, 17$. Because σ^3 is an order 4 automorphism, we merely need to see observe that $r_{\sigma^3} = 17$ is not a case given in the classification of [2]. Thus $r_{\sigma^3} = 14$ and $r_\sigma = 12$.

This means we have the following for $S(\sigma)$, where the diamond represents the linear combination of the central curves of the D_4 configurations and the triangle the linear combination of the six extremal curves of both D_4 configurations. As this is certainly fixed by the action, is primitive and of the correct rank, the lattice below must be $S(\sigma)$ and has discriminant form $v \oplus w_{3,1}^{-1}$. By Nikulin's theorem, it is abstractly isomorphic to $U \oplus D_4 \oplus E_6$.



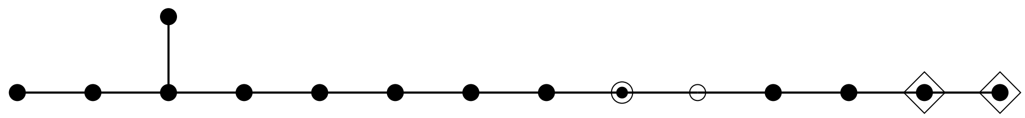
6.2.7 Surface 9. This surface has automorphisms of order 9 and 18. The transcendental lattice has determinant 3 and we have II^* over $t = 0$ and IV^* over $t = \infty$.



This is the Trivial lattice of the surface. Because we have a II^* fiber, we know by Corollary 3.16 that this is the Picard lattice. As σ has order 9, it must fix the whole lattice as there is no order 3 symmetry. We can write this as $S(\sigma) = \operatorname{Pic} X = U \oplus E_6 \oplus E_8$, which has discriminant form $w_{3,1}^{-1}$.

As τ is order 18, we consider τ^3, which has order 6. As there are three II fibers that don't contribute to the Picard lattice, we recognize this as one of the cases in [10]. This tells us there are 5 points of type $(2, 5)$ and 8 of type $(3, 4)$ and 2 point-wise fixed rational curves.

Because the $U \oplus E_8$ curves must be fixed as well as the central curves of the E_6, we conjecture the invariant lattice of τ^3 is the following.



Where the diamond curves are the linear combination of the appropriate curves. To see this is the lattice, we must use fixed-point theory to prove $r_{\tau^3} = 14$. By examining the proposed lattice, it's clear that this lattice is fixed by $(\tau^*)^3$ so $r_{\tau^3} \geq 14$. We know by the topological Lefschetz formula that

$$13 + 2\alpha_{\tau^3} = 2 + \operatorname{tr}((\tau^3)^*|_T) + \operatorname{tr}((\tau^3)^*|_{\mathrm{Pic}})$$

and we point out that by [10] $\alpha = 1, 2$. We consider the trace of the action on the transcendental lattice first. We know that the eigenvalues of σ on the transcendental lattice are $\zeta_{18}, \zeta_{18}^5, \zeta_{18}^7, \zeta_{18}^{11}, \zeta_{18}^{13}, \zeta_{18}^{17}$. This means $\operatorname{tr}((\tau^3)^*|_{\mathrm{Pic}}) = 3(\zeta_6 + \zeta_6^5) = 3$.

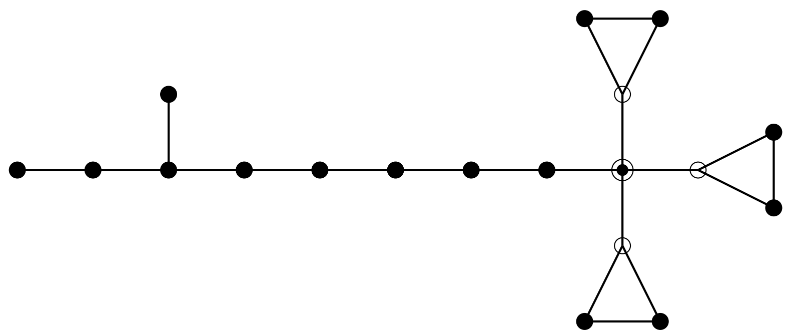
By examining $\operatorname{Pic} X$, we see the only eigenvalues on this sublattice can be ± 1. This means that we can write the trace as $r_{\tau^3} - (16 - r_{\tau^3}) = 2r_{\tau^3} - 16$.

So it is the case that

$$24 + 2\alpha = 2r_{\tau^3}$$

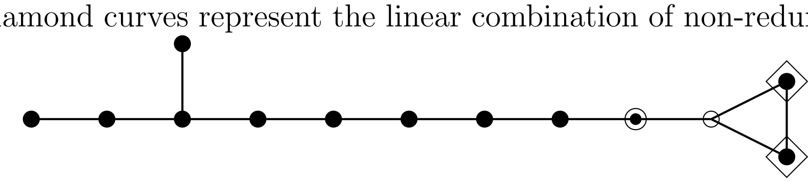
and because $\alpha = 1$ or 2, it must be the case that $r_{\tau^3} = 14$. Because the invariant lattice of τ is contained in the invariant lattice of τ^3 and τ also clearly fixes the invariant lattice of τ^3, the conjectured lattice is the invariant lattice of τ. Thus $S(\tau)$ is given by the second arrangement of divisors above and has discriminant form v. By Nikulin's theorem, $S(\tau)$ is abstractly isomorphic to $U \oplus E_8 \oplus D_4$.

6.2.8 Surface 10. This surface has automorphisms of order 9 and 18. The determinant of the Transcendental lattice is 3^2. We have a II^* fiber over $t = 0$ and three IV fibers over the third roots of unity.



This is the Trivial lattice of the surface. The configuration comes from the combination of the II^* fiber over $t = 0$ and the three IV fibers over the roots of unity. Because we have a II^* fiber, we know this is also the Picard lattice. (Corollary 3.16) So Pic $X = U \oplus E_8 \oplus A_3^{\oplus 3}$.

Here we have to do a little work to see what the invariant lattice of the order 9 automorphism is. We're able to determine there are at least 8 isolated fixed points in the E_8 configuration. We're also able to determine the lattice below is clearly fixed by the automorphism. This is because the automorphism exchanges each of the three IV fibers and the diamond curves represent the linear combination of non-redundant components of the fiber.



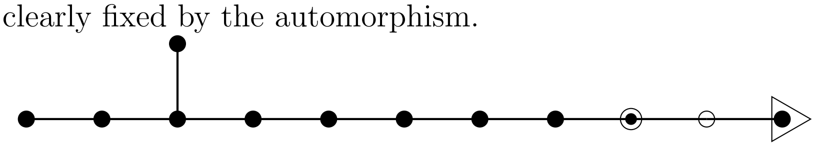
To argue that $r_\sigma = 12$, we turn to the topological Lefschetz formula. We know that the action on the transcendental lattice is given by the primitive 9th roots of unity and because 9 is not square free, this means that $\operatorname{tr}(\sigma^*|_T) = 0$. So we get the following.

$$N + 2\alpha = 2 + r_\sigma - \frac{1}{2}(16 - r_\sigma) = \frac{3}{2} r_\sigma - 6$$

By examining the E_8 configuration, we learn there are at least 3 fixed points of type $(2, 8)$, 2 of type $(3, 7)$, 2 of type $(4, 6)$ and 1 of type $(5, 5)$ together with at least one point-wise fixed rational curve.

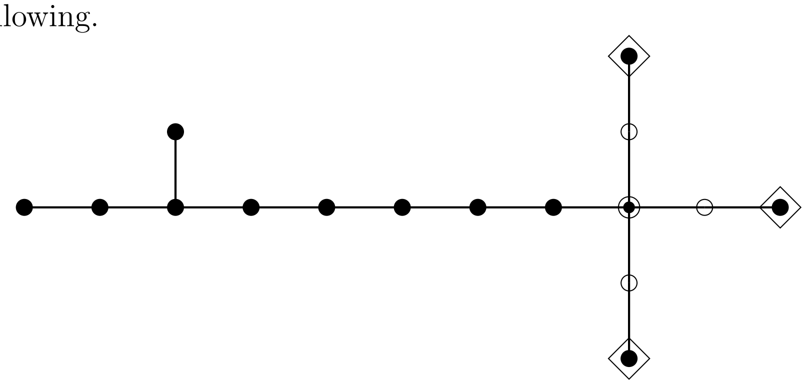
We know from [18] that the relation $n_{3,7} = 2\alpha + 1$ holds. So it must be the case that $\alpha \geq 1$ and $n_{3,7} \geq 3$. From the same we know that $n_{2,8} + n_{5,5} = 3\alpha + 1$ This means that $n_{2,8} + n_{5,5} = 4, 7$ and we have enough of these types of points for the first option. Finally, we know that $n_{4,6} + 3n_{2,8} = 8\alpha + 4$ and so we must have at least one more fixed-point of type $(4, 6)$. In the final tally, this means that $N \geq 10$ and $\alpha \geq 1$. So we get the relation $18 = \frac{3}{2} r_\sigma$ or $r_\sigma = 12$ because adding any more fixed points or point-wise fixed curves would give us too large a rank for the invariant lattice by the same relations. Thus our conjectured lattice is the invariant lattice in question. So $S(\sigma) = U \oplus E_8 \oplus A_2(3)$ and has discriminant form $w^1_{3,1} \oplus w^1_{3,2}$.

Now we consider τ the automorphism of order 18. We observe that the only place for an order two action on the invariant lattice of σ is to exchange the two curves in the $A_2(3)$ configuration. This would give us the lattice below where the curve represented with a triangle is the linear combination of the two diamond curves. It is worth remarking that this is clearly fixed by the automorphism.



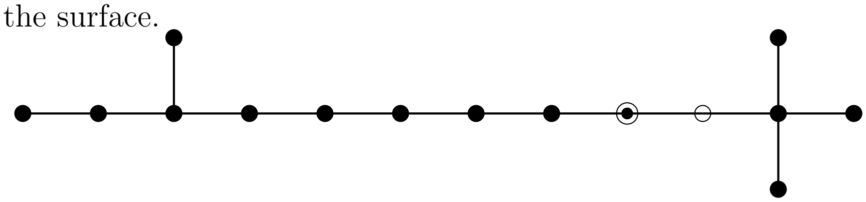
To see this is in fact the invariant lattice, we have to prove that $r_\tau \neq 12$ and we're done. We turn to an examination of τ^3 which has order 6. Because of the additional II fiber over $t = \infty$, we recognize this as one of the cases in [10] and we know that σ^3 has 7 fixed points of type $(2, 5)$, 4 of type $(3, 4)$, 2 point-wise fixed rational curves and no point-wise fixed curves of positive genus.

As before we turn to the topological Lefschetz formula. We again know that $\operatorname{tr}((\tau^3)^*|_T) = 3$. Because each of the IV fibers is fixed by τ^3, the only eigenvalues on $\operatorname{Pic} X$ are ± 1. By a topological Lefschetz argument, we see that $r_{\sigma^3} = 13$ or that the invariant lattice is the following.



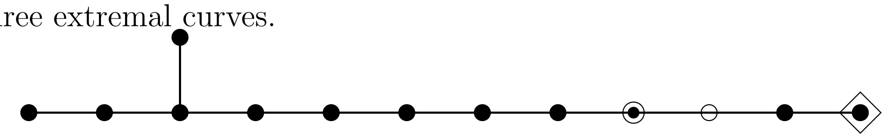
This doesn't immediately appear useful, but we observe that the invariant lattice of τ must be contained in the invariant lattices of τ^2 (having order 9) and τ^3. The conjectured invariant lattice is precisely the intersection of the two invariant lattices, so $r_\tau \neq 12$. This completes the argument. Thus $S(\tau) = U \oplus E_8 \oplus A_1(6)$.

6.2.9 Surface 16. This surface has an order 24 automorphism and the determinant of the Transcendental lattice is 2^2. From the fibration we get reducible fibers of type II^* over $t = 0$, which gives us the following configuration of curves that generate the Trivial lattice of the surface.



To see that this is the Picard lattice, we observe that we have a II^* fiber. This means that this lattice is isomorphic to the Picard lattice. (Corollary 3.16)

We can examine σ^3 and use the relations for fixed-points of order 8 to argue the action on the curves of the D_4 configuration must be order 3. If one does this, we see that it is the case that σ^3 fixes $\operatorname{Pic} X$. Thus the only action on the lattice is order 3. As the only place for an order 3 action is exchanging the extremal curves of the D_4 configuration, we conclude that the following must be $S(\sigma)$, where the diamond curve is the linear combination of the three extremal curves.



As this is clearly invariant under the action, we need only argue that $r_\sigma = 12$. This is done by examining the topological Lefschetz formula for σ^4, which is an order 6 automorphism. We see that we get 5 fixed-points of type $\frac{1}{6}(2, 5)$ and 4 of type $\frac{1}{6}(3, 4)$. There are four fibers of type II and each of these contributes an additional fixed-point of type $\frac{1}{6}(3, 4)$. This data together gives us the Lefschetz formula.

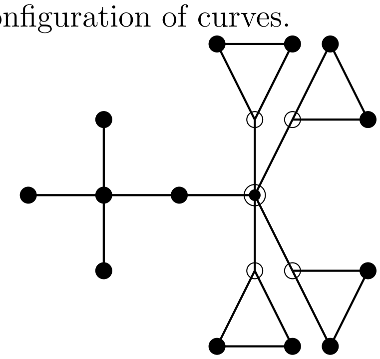
$$7 + 2\alpha = 2 + r_{\sigma^4} - l - m + 4\mu(6)$$

In this expression l is the rank of the eigenspace of -1 and m the rank of the eigenspace of a primitive 3rd root of unity. Because σ^3 fixes $\operatorname{Pic} X$, we know that $l = 0$. We can also rewrite $m = \frac{1}{2}(r_{\sigma^{12}} - r_{\sigma^4})$. Because σ^{12} is a power of σ^3, we know $r_{\sigma^{12}} = 14$. This gives us the following.

$$7 + 2\alpha = \frac{3}{2} r_{\sigma^4}$$

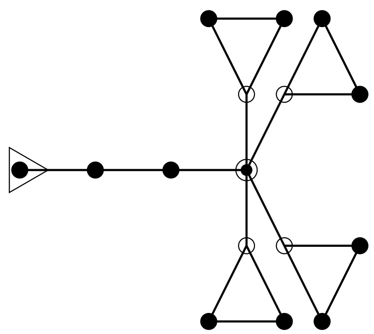
This means that $r_{\sigma^4} = 12$. Because $r_\sigma \leq 12$ and $r_\sigma \geq 12$, we know $S(\sigma)$ is generated by the proposed configuration with discriminant form $w^1_{3,1}$. Furthermore, $S(\sigma) = U \oplus E_8 \oplus A_2$.

6.2.10 Surface 18. This surface has an automorphism of order 24 and determinant of the discriminant lattice equal to $2^2 3^4$. This surface has a fiber of type I_0^* over $t = 0$ and four fibers of type IV over the 4th roots of unity. From the fibration we get the following configuration of curves.



To see the Trivial lattice is isomorphic to the Picard lattice, we observe that the discriminant groups of the I_2 and I_0^* are of relatively prime order. This means that $|\operatorname{MW} X| = 1$ and the Trivial lattice is isomorphic to the Picard lattice.

We turn to the problem of determining $S(\sigma)$. We consider the action of σ^8, which has order 3. Because the action on the base has order 4, the IV fibers are all fixed. Furthermore, since the action is order 3, this means all of the curves in the A_2 are fixed. This means that $r_{\sigma^8} \geq 12$ because



is clearly fixed by σ^8. We also know that the section is fixed point-wise by σ^8 so we can find an isolated fixed point at the intersection of the two central curves of the D_4 configuration. As r_{σ^8} must be even because eigenvectors of the primitive 3rd roots of unity come in pairs, this means that $r_{\sigma^8} = 12$.

An alternative way to see this is $S(\sigma^8)$ is to observe that there are 5 isolated fixed-points: one in the I_0^* fiber and four from the IV fibers. We notice that this lattice is isomorphic to $U \oplus A_2^{\oplus 5}$, which is the invariant lattice given by the $N = 5, k = 2$ case in [1].

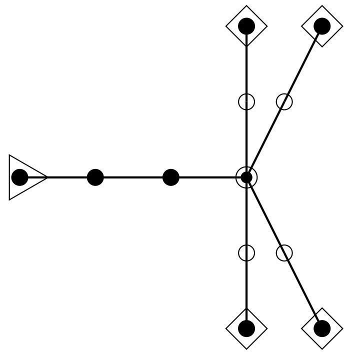
We now consider the action of σ^4, which has order 6. We see immediately that to have at least 1 fixed-point of type $\frac{1}{6}(3,4)$ and 5 of type $\frac{1}{6}(2,5)$. To fit the relations on order 6, it must be the case that we have at least one more of type $\frac{1}{6}(3,4)$. This gives the number of point-wise fixed rational curves as 1. We get the topological Lefschetz formula of σ^4 as follows.

$$7 + 2\alpha = 2 + r_{\sigma^4} - l - m + 4\mu(6)$$

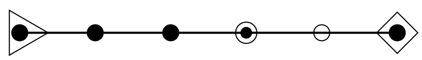
where m is the rank of the eigenspace of a primitive 3rd root of unity and l the rank associated to -1. We know that $m = 1$ and $r_{\sigma^8} = 12$, so this simplifies to the following.

$$2 + 2\alpha = 2r_{\sigma^4} - 12$$

or $r_{\sigma^4} = 8$. This gives us the following for $S(\sigma^4)$ because each of the IV fibers is invariant under the action. The triangle is the linear combination of the extremal curves of the D_4 configuration and the diamonds are the linear combinations of the two curves in each A_2.

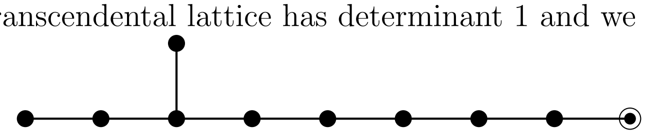


This also tells us that the action of σ on the curves of the IV fibers is order 8. We conclude then that $S(\sigma)$ is the following, where the triangle is the linear combination of the curves of the D_4 and the diamond is the sum of all 8 of the curves from the IV fibers.



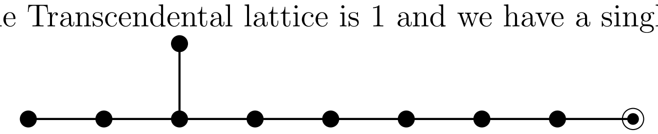
We see this is invariant under the action of σ, that no larger lattice is invariant under the action and that this lattice embeds primitively in $\operatorname{Pic} X$. Thus $S(\sigma) = U \oplus A_2 \oplus A_1(8)$ and has discriminant form $w_{3,1}^{1} \oplus w_{2,4}^{-1}$.

6.2.11 Surface 25. For this surface we have automorphisms of order 21 and 42. The Transcendental lattice has determinant 1 and we have a single II^* fiber over $t = 0$.



The above is the Picard lattice for surface 22. In particular, because surface 22 has automorphisms of order 21 and 42, we're looking for symmetries of order 2,3 and 7. Because there are none, we see that this is also the invariant lattice of both automorphisms. So $S(\sigma) = S(\tau) = \operatorname{Pic} X = U \oplus E_8$. This is unimodular.

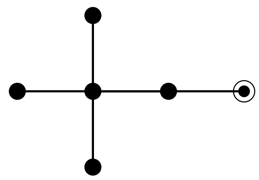
6.2.12 Surface 28. The automorphism of this surface has order 28. The determinant of the Transcendental lattice is 1 and we have a single II^* over $t = \infty$.



This is the Picard lattice for surface 25 because we have a II^* fiber. (Corollary 3.14) The configuration comes from the fiber of type II^* over $t = \infty$ and can be written $U \oplus E_8$.

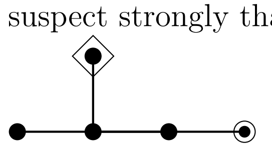
The automorphism of this surface has order 28, and since there are no symmetries of order 2 or 7, this is also the invariant lattice. Again $S(\sigma) = \operatorname{Pic} X = U \oplus E_8$.

6.2.13 Surface 31. This surface has an automorphism of order 32. The determinant of the Transcendental lattice is 2^2. We have a single reducible fiber of type I_0^* over $t = 0$.



In the figure we see the arrangement of curves derived from the fibration. To convince ourselves that this is the Picard lattice we observe that $|\operatorname{MW} X| = 1$ by Corollary 3.14. $\operatorname{Pic} X = U \oplus D_4$.

Because the automorphism of interest for surface 28 has order 32, we are interested in order 2 actions on the lattice. Because at most two of the terminal curves can be exchanged, we suspect strongly that the invariant lattice is the following:



where the diamond denotes the linear combination of the two exchanged curves.

To see that this must be the case we will make use of several different facts. We begin with the topological Lefschetz formula. Because the action has at most order 2 on this lattice, it is the case that $(\sigma^*)^2$ must fix the whole Picard lattice. This means that we have

$$N + 2\alpha = 2 + r_\sigma - (6 - r_\sigma)$$

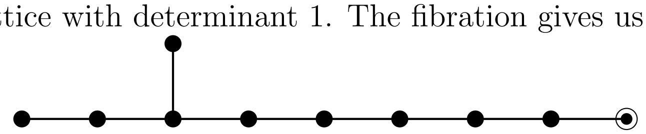
because $(6 - r_\sigma)$ is the rank of the eigenspace associated to -1. Ultimately we can rearrange this to be

$$N + 2\alpha + 4 = 2r_\sigma$$

which is part of the way to what we need. From [7] we know this automorphism has 6 isolated fixed-points. It is also the case that $(\sigma^*)^2$ fixes the central curve of the D_4 sublattice. By [24] we know that an order 16 automorphism on a surface with a rank 6 Picard lattice fixes at most one rational curve. Thus σ can't fix any other rational curve and $\alpha = 0$ for an r_σ of 5. Because the lattice embeds primitively by construction and is fixed by σ^*, we're done. Thus $S(\sigma)$ is generated by the second arrangement of divisors given above and has discriminant form $w_{2,2}^5$. By Nikulin's theorem, $S(\sigma)$ is isomorphic to $U \oplus A_3$.

6.2.14 Surface 32.

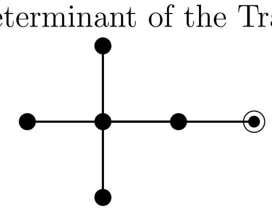
This surface has an automorphism of order 36 and transcendental lattice with determinant 1. The fibration gives us a II^* over $t = 0$.



This is the Picard lattice for surface 29 because we have a II^* fiber. (Corollary 3.14) As the order of the associated automorphism is 36, we're looking for symmetries of order 2 and 3. As none are present, we conclude this is the invariant lattice of the automorphism. Again $S(\sigma) = \operatorname{Pic} X = U \oplus E_8$. This is again unimodular.

6.2.15 Surface 34.

The automorphism associated with this surface has order 48. The determinant of the Transcendental lattice is 2^2. We have an I_0^* fiber over $t = \infty$.

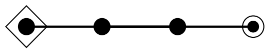


This configuration of curves is the Trivial lattice of the surface. Just as for surface 28, we use Corollary 3.14 to conclude this is the Picard lattice. Thus $\operatorname{Pic} X = U \oplus D_4$.

This means in particular that σ^{16} has order 3 and σ^3 has order 16. In particular this tells us that by [24] r_{σ^3} fixes all of $\operatorname{Pic} X$. So the only possible action of σ^* would be to exchange the three extremal curves on the left.

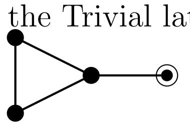
We turn to consider $r_{\sigma^{16}}$. As this is an automorphism of order 3, we know from [3] that $\alpha_{\sigma^{16}} = \frac{r_{\sigma^{16}}-8}{2}$ where $\alpha_{\sigma^{16}}$ is the α invariant computed for σ^{16}. As the central curve of the D_4 and the two curves on the right are fixed, this means that $r_{\sigma^{16}} = 4, 6$. Taki in [25] showed that there are no K3 surfaces admitting an order 3 automorphism that fixes $\operatorname{Pic} X$ and having Picard rank 6. Thus it must be the case that $r_{\sigma^{16}} = 4$.

Because the invariant lattice of σ must be contained inside the invariant lattice of σ^{16}, we conclude that $r_\sigma = 4$.



We see that $S(\sigma) = U \oplus A_2$ and has discriminant form $w^1_{3,1}$.

6.2.16 Surface 36. This surface has an automorphism of order 27 and an automorphism of order and 54. In addition to the reducible fiber IV over $t = \infty$, we have 10 fibers of type II over $t = 0$ and the 9th roots of unity. This gives us the following configuration of curves for the Trivial lattice.



This surface is discussed in [16]. If this is the Picard lattice, it is also $S(\sigma)$. To see the Trivial lattice is isomorphic to the Picard lattice, we notice that the discriminant of $\operatorname{Pic} X$ is equal to the discriminant of the IV^* fiber. This means by Corollary 3.14, the order of the Mordell-Weil group is 1 and the Trivial lattice is isomorphic to the Picard lattice. So $S(\sigma) = \operatorname{Pic} X = U \oplus A_2$ and has discriminant form $w^1_{3,1}$.

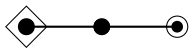
To find $S(\tau)$, we observe that τ^9 is an order 6 automorphism. We also observe that the fibration fits one of the cases in [10]. This tells us that there are 11 isolated fixed-points and 1 point-wise fixed curve. Furthermore, the automorphism τ^{18} fixes a curve of genus 5 so we know that $\alpha = 1, -4$. We also know that the action of τ^9 on the Transcendental lattice is $9\mu(6)$ (where μ denotes the Möbius function). This gives us the following topological Lefschetz formula.

$$11 + 2\alpha = 2 + r_{\tau^9} - l + 9$$

Because there's no room for an action of order 3 on this lattice, we only have to worry about r_{τ^9} and the eigenspace of -1 for τ^9 which we call l. Furthermore, because τ^{18} fixes this lattice, we know that $l = 4 - r_{\tau^9}$, so we make the substitution to get

$$2\alpha + 4 = 2r_{\tau^9}$$

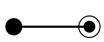
and because the rank can't be negative, this means that $r_{\tau^9} = 3$. This means that $r_\tau \leq 3$. We consider the lattice



where the diamond represents the linear combination the two curves of the IV fiber not meeting the section. It's clear that this is fixed by the action, primitively embeds into the Picard lattice and has the appropriate rank. Thus this is $S(\tau) = A_1(2) \oplus U$ with discriminant form $w_{2,2}^{-1}$.

6.2.17 Surfaces 37 and 38. These surfaces are somewhat unique. Surface 37 has II fibers over $t = 0$ and over the 11th roots of unity. Surface 38 has 22 fibers of type I_1 over the roots of the discriminant and a fiber of type II over $t = \infty$. As none of these are reducible, we don't get enough vertical divisors from the fibration to generate the Picard lattice, but we do know that the Picard lattice of these surfaces is U from [16].

This gives us the following picture of the lattice, where the self-intersection of the left curve is 0.



Surface 37 has automorphisms of order 33 and 66 and it's the order 66 automorphism that's studied in [16]. Thus this lattice is also the invariant lattice for both automorphisms. Surface 38 has only an automorphism of order 44, but we know that it fixes the Picard lattice, so this is also the invariant lattice in this case. So in all cases $S(\sigma) = S(\tau) = U = \operatorname{Pic} X$, which is unimodular.

Appendix A. Computing Singular Fibers

In the case of essentially every surface considered in this thesis, we knew from the literature what the singular fibers of the fibration were. In the event that someone in a trench coat hands you an elliptic surface, it is necessary to have a way to determine the singular fibers yourself. The normal way that this is done is by way of Tate's algorithm (see [8] [11]).

Because we are working in the setting of surfaces over an algebraically closed field of characteristic zero, it turns out this is unwieldy and unnecessary. It is the case that we can determine the fiber type by examining the vanishing order the discriminant and the coefficients $A(t), B(t)$ of the Weierstrass model. This can be done according to Table A.1 (see [19]).

The relevant information for us are the values a, b, δ, which correspond to the vanishing order of $A(t), B(t)$ and the discriminant. By examining these values, it is possible to determine the fiber over t.

Appendix B. Relations on the Number of Fixed-Points and Fixed Curves for Certain Orders

Using the Holomorphic Lefschetz Formula, it's possible to derive relations that can constrain the number of isolated fixed-points of each type and the number of point-wise fixed curves for a purely non-symplectic automorphism of a given order. We give the relations used for our computations in Table B.1 and follow the convention that N is the total number of isolated fixed-points, k the number of point-wise fixed rational curves, $\alpha = \sum_{C \in X^\sigma}(1 - g(C))$, and g the maximal genus of a point-wise fixed curve.

(IV.3.1)Table

Name	Curve	a	b	δ	J	m(J)	e	r	d	RDP	γ	Comments
I_0	smooth elliptic curve	0	0	0	$\neq 0,1,\infty$	?	0	0	1	-	0	T meets F in three points
		$a \geq 1$	0	0	0	3a						
		0	$b \geq 1$	0	1	2b						
I_1	nodal rational curve	0	0	1	∞	1	1	0	1	-	0	T is tangent to F
I_N	cycle of N smooth rational curves	0	0	N	∞	N	N	N-1	N	A_{N-1}	[N/2]	T meets F twice at the double point
I_0^*	$\tilde{D}_4$	2	3	6	$\neq 0,1,\infty$	?	6	4	4	D_4	3	T has an ordinary triple point on F
		$a \geq 3$	3	6	0	3a-6						
		2	$b \geq 4$	6	1	2b-6						
I_N^*	$\tilde{D}_{N+4}$	2	3	N+6	∞	N	N+6	N+4	4	D_{N+4}	3+[N/2]	T has a point of type (3,2) on F
II	cuspidal rational curve	$a \geq 1$	1	2	0	3a-2	2	0	1	-	0	T is flexed to F at one point
III	two tangent rational curves	1	$b \geq 2$	3	1	2b-3	3	1	2	A_1	1	T has node with F as one tangent
IV	three concurrent rational curves	$a \geq 2$	2	4	0	3a-4	4	2	3	A_2	1	T has cusp with F as tangent
IV^*	$\tilde{E}_6$	$a \geq 3$	4	8	0	3a-8	8	6	3	E_6	3	$\overline{T}$ is flexed to E
III^*	$\tilde{E}_7$	3	$b \geq 5$	9	1	2b-9	9	7	2	E_7	4	$\overline{T}$ has a node on E with one tangent E
II^*	$\tilde{E}_8$	$a \geq 4$	5	10	0	3a-10	10	8	1	E_8	4	$\overline{T}$ has a cusp on E with E as tangent

Figure A.1: Table of vanishing orders associated with singular fibers

Order	Relations	Source
3	$g = 3 + k - N$	[1]
4	$N = 2\alpha + 4$	[2]
6	$n_{3,4} + 2n_{2,5} - 6k = 6$	[10]
8	$n_{2,7} + n_{3,6} = 2 + 4\alpha$ $n_{4,5} + n_{2,7} - n_{3,6} = 2 + 2\alpha$	[23]
9	$n_{2,8} + n_{5,5} = 3\alpha + 1$ $n_{3,7} = 2\alpha + 1$ $n_{4,6} + 3n_{2,8} = 8\alpha + 4$	[18]
12	$4 = 4n_{2,11} + n_{3,10} + n_{4,9} + n_{6,7} - 14\alpha$ $6 = 6n_{2,11} + 3n_{3,10} + n_{4,9} + 2n_{5,8} + 3n_{6,7} - 24\alpha$ $2 = 2n_{2,11} - n_{3,10} + n_{4,9} - 2n_{5,8} + 2n_{6,7} - 4\alpha$ $0 = -3n_{3,10} + n_{4,9} - 4n_{5,8} + 3n_{6,7} + 6\alpha$	Computed using holomorphic Lefschetz formula and the fact that $\zeta_{12} = \frac{\sqrt{3}}{2} + i\frac{1}{2}$

Table B.1: Relations on the number of fixed-points and point-wise fixed curves by order

Appendix C. MAGMA Code for Working With Lattices

Here we present the code for computing and comparing discriminant forms of lattices.

```
disc:=function(M)

S,A,B:=SmithForm(M);

l:=[[S[i,i],i]: i in [1..NumberOfColumns(S)]| S[i,i] notin 0,1];

sA:=Matrix(Rationals(),ColumnSubmatrixRange(B,l[1][2],l[#l][2]));

for i in [1..#l] do

MultiplyColumn(sA,1/l[i][1],i);

end for;

Q:=Transpose(sA)*Matrix(Rationals(),M)*sA;

for i,j in [1..NumberOfColumns(Q)] do

if i ne j then

Q[i,j]:=Q[i,j]-Floor(Q[i,j]);

else

Q[i,j]:=Q[i,j]-Floor(Q[i,j])+ (Floor(Q[i,j]) mod 2);
```

```
end if;
end for;
return [l[i][1]: i in [1..#l]], Q;
end function;
```

The disc function takes in an even bilinear form and outputs the discriminant group and the value of the discriminant form on the generators.

```
mod2:=function(Q);
 for i,j in [1..Nrows(Q)] do
  if i ne j then Q[i,j]:=Q[i,j]-Floor(Q[i,j]);
  else Q[i,j]:=Q[i,j]-2*Floor(Q[i,j]/2);
  end if;
 end for;
 return Q;
end function;


dicompare:=function(M,Q)
v,U:=disc(M);
w,D:=disc(Q);
if v ne w then return false; end if;
A:=AbelianGroup(v);
Aut:=AutomorphismGroup(A);
f,G:=PermutationRepresentation(Aut);
h:=Inverse(f);
ll:=[Matrix(Rationals(),[Eltseq(Image(h(g),A.i)) : i in [1..Ngens(A)]]) : g in G];
dd:=[mod2(a*U*Transpose(a)) : a in ll];
return D in dd;
end function;
```

The dicompare function takes in two even bilinear forms and checks whether or not they are the same. This is useful for when we know the discriminant form of a lattice from the literature and are trying to determine if a linearly independent set of generators indeed gives the same lattice.

Bilinear forms were stored and checked as csv files and the following code was used to generate Magma inputs to run the other functions given here.

```
def read_bilinear_form_csv(file_in):
  file_reader = open(file_in)
  magma_out = 'M_1 := Matrix(['
  while True:
    line = file_reader.readline()
    if not line:
      break
    line = '['+line+'],'
    magma_out=magma_out+line
  magma_out = magma_out[:-1]
  magma_out = magma_out+']);'
  return magma_out
```

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