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Lattices and Their Applications to Rational Elliptic Surfaces

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LATTICES AND THEIR APPLICATIONS TO RATIONAL ELLIPTIC SURFACES

by

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of a thesis submitted by

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ABSTRACT

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This thesis discusses some of the invariants of rational elliptic surfaces, namely the Mordell-Weil Group, Mordell-Weil Lattice, and another lattice which will be called the *Shioda Lattice*. It will begin with a brief overview of rational elliptic surfaces, followed by a discussion of lattices, root systems and Dynkin diagrams. Known results of several authors will then be applied to determine the groups and lattices associated with a given rational elliptic surface, along with a discussion of the uses of these groups and lattices in classifying surfaces.

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1 Introduction

This paper discusses some of the invariants of rational elliptic surfaces, namely the Mordell-Weil Group, Mordell-Weil Lattice, and another lattice which we call the *Shioda Lattice*. We will begin with a brief overview of rational elliptic surfaces, discussing how they are defined, and some of their properties, including some groups that can be associated with a rational elliptic surface. We then define a lattice and discuss some lattices that can be associated to rational elliptic surfaces. We will also discuss which lattices cannot occur and as a consequence we obtain a list of configurations of singular fibres which cannot occur on a rational elliptic surface. To do this we will discuss root systems and Dynkin diagrams, and their relationships with lattices. We will then apply some known results of several authors to determine the lattices and groups associated to a given rational elliptic surface, and discuss applications of these lattices in classifying the configuration of singular fibres that may occur on the surface.

2 Rational Elliptic Surfaces

2.1 General Information on Surfaces

Let k be an algebraically closed field.

Definition 1. A *rational elliptic surface over* $\mathbb{P}^1$ consists of the following:

(i) a surface $\mathcal{E}$ over k; that is, a non-singular two dimensional projective variety which is birational to $\mathbb{P}^2$,

(ii) a morphism

$$\pi : \mathcal{E} \to \mathbb{P}^1$$

such that

(a) for all but finitely many points $t \in \mathbb{P}^1$, the fibre $\pi^{-1}(t)$ is a non-singular curve of genus 1,

(b) π is not split; that is $\pi : \mathcal{E} \to \mathbb{P}^1$ is not birational to $pr_2 : E \times \mathbb{P}^1 \to \mathbb{P}^1$ for any curve E, and

(c) π is a relatively minimal elliptic fibration; which is to say, no fibres contain exceptional curves of the first kind,

(iii) a section of π,

$$\sigma_0 : \mathbb{P}^1 \to \mathcal{E}.$$

Let E be the generic fibre of $\mathcal{E}$. Then E is an elliptic curve over K, where $K = k(\mathbb{P}^1) = k(t)$. As such, standard results about elliptic curves can be applied to E. Thus E has a Weierstrass form $f(x, y, t)$ given by

$$f(x, y, t) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6$$

where $a_i \in K$. Using a standard change of variables, the form can be rewritten so that $a_i \in k[t]$ and $\deg a_i \leq i$ if and only if the surface is rational.

Also, E has a group $E(K)$ associated with it, called the Mordell-Weil group, which is the group of all K-rational points of E. Given any $\tau \in E(K)$, there is a natural rational map $\tau = (x(t), y(t))$ with $x, y \in K$ from $\mathbb{P}^1$ to $\mathcal{E}$. Since $\mathcal{E}$

is a projective variety, this map is regular. Thus for every $\tau \in E(K)$ there is a global section of π corresponding to it. Similarly, for each global section of π, there is a τ that maps to it. Thus the global sections of π and the K-rational points of E are in one-to-one correspondence.

As E has a Weierstrass form, the vanishing of $f(x, y, t)$ defines a variety $V(f) \subset \mathbb{A}^2 \times \mathbb{A}^1$. Let $Z = \overline{V(f)} \subset \mathbb{P}^2 \times \mathbb{A}^1$. The standard change of variables, $t = \frac{1}{s}$, $x \mapsto s^{-2}x$, and $y \mapsto s^{-3}y$ gives the Weierstrass form $g(x, y, s)$ at $t = \infty$, $s = 0$. This gives another variety $W = \overline{V(g)} \subset \mathbb{P}^2 \times \mathbb{A}^1$. Clearly the change of variables map gives an isomorphism, and we see that $W|_{\mathbb{A}^1_s - 0}$ is isomorphic to $Z|_{\mathbb{A}^1_t - 0}$. Hence, W and Z can be patched together to give us a surface $\mathfrak{E}$ which is contained in a $\mathbb{P}^2$ bundle over $\mathbb{P}^1$, a surface over $\mathbb{P}^1$, which is the Weierstrass model of $\mathcal{E}$.

$\mathfrak{E}$ is in general a singular surface. The singularities of the surface can be resolved by blowing up $\mathfrak{E}$ to a smooth surface $\mathfrak{E}'$, which is relatively minimal, and which maps to $\mathfrak{E}$.

As $\mathfrak{E}$ was defined from the Weierstrass form of $\mathcal{E}$, it is birational to $\mathcal{E}$ which is by definition a relatively minimal model for $\mathfrak{E}$. From the proof of Theorem III.8.4 in Silverman [12], we know that $\mathfrak{E}'$ is also a relatively minimal model for $\mathfrak{E}$. Since the relatively minimal model of a surface is unique up to unique isomorphism, $\mathcal{E}$ is isomorphic to $\mathfrak{E}'$.

Thus we see that there are three essentially equivalent ways of describing a given elliptic surface over $\mathbb{P}^1$: as a projective variety as in the definition, as an elliptic curve over $k(t)$, which has a Weierstrass form, or as a singular Weierstrass model. As the Weierstrass form contains all of the necessary information, it is common for the surface to be described using this form. In this paper we will consider the surface in two ways, as a curve over $k(t)$, which allows us to find the Mordell-Weil group of our surface, and as the minimal resolution of a the Weierstrass model, which will allow us to more easily determine the Néron-Severi group. We will then use these two groups to look at other properties of the surface.

2.2 Singular Fibres

Consider the Weierstrass model $\mathfrak{E}$. Recall that this model is in general singular. Each singular point of this surface is found on a singular fibre. From the definition of the surface, there can only be finitely many fibres which are not elliptic curves, thus there can only be finitely many singular fibres, and finitely many singularities on our surface. It is well known that an elliptic curve has a singularity at a point if and only if the discriminant Δ of the curve vanishes at that point, where

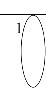
$$\Delta = -b_2^2 b_8 - 8b_4^3 - 27b_6^2 + 9b_2 b_4 b_6$$
$$b_2 = a_1^2 + 4a_2$$
$$b_4 = 2a_4 + a_1 a_3$$
$$b_6 = a_3^2 + 4a_6$$
$$b_8 = a_1^2 a_6 + 4a_2 a_6 - a_1 a_3 a_4 + a_2 a_3^2 - a_4^2.$$

Thus the surface will only have a singularity when the discriminant of the Weierstrass form is zero, and as the discriminant has degree twelve, there are at most twelve singularities, and the sum of the multiplicities of the singularities is at most twelve.

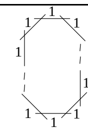
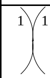
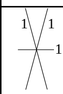
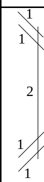
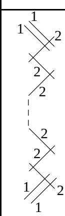
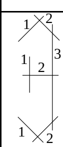
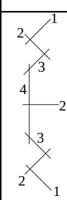
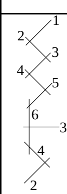
The different types of singularities on an elliptic surface were completely classified by Kodaira. They are first classified into two types according to the type of singularity of the fibre on which they are contained, and then according to the irreducible components of the resolution of the singular fibre. The two major types of singularities that occur are additive and multiplicative. The singularities are classified in this way as there are only two ways that cubic fibres in $\mathbb{P}^2$ can be singular. The fibre may contain a cusp or a node. If the fibre has a cusp, then the group that is defined on the non-singular points of the fibre is additive, thus the singularities whose fibres have cusps are called additive. Similarly, if the fibre has a node, then the group is multiplicative, and the singularities are called multiplicative. Multiplicative singularities of the surface are denoted I_n, where n is the multiplicity of the fibre, and the number of irreducible components of the resolved fibre. The additive fibres are denoted II, III, IV, I_n^* for $n = 0, \ldots, 4$, II^*, III^* and IV^*.

When the surface $\mathfrak{E}$ is blown up to $\mathcal{E}$ to resolve the singularities, each of the singular fibres becomes a possibly reducible fibre in the smooth surface. Table 1, as found in Silverman [12], gives the representation of the resolved fibre for each of the singular fibre possibilities, where the numbers indicate the multiplicity of the component. The intersection theory of the components will be used to define a lattice, and is discussed in section 3.3.

Table 1: Kodaira Symbols and Representation of Resolved Singular Fibres

Kodaira Symbol	Resolved Fibres
I_0	
	continued on next page

continued from previous page

Kodaira Symbols	Resolved Fibres
I_n	
II	
III	
IV	
I_0^*	
I_n^*	
IV^*	
III^*	
II^*	

3 Lattices

3.1 General Lattices

Following the notation used by Shioda [10], we give the formal definition of a lattice, and discuss some of the properties we will be interested in.

Definition 2. A *lattice* Λ is a finitely generated free Abelian group together with a symmetric non-degenerate bilinear pairing

$$\langle\,,\rangle : \Lambda \times \Lambda \longrightarrow \mathbb{Q},$$

where the pairing is non-degenerate if for every $x_i \in \Lambda - \{0\}$ there exists an $x_j \in \Lambda$ such that $\langle x_i, x_j \rangle \neq 0$.

We will concern ourselves only with lattices whose pairings, which we will refer to as inner products, take on integer values, as follows:

$$\langle\,,\rangle : \Lambda \times \Lambda \longrightarrow \mathbb{Z}.$$

For example, let us consider the following two lattices, called A_n and D_n. A_n is the lattice generated by $\{x_1, \ldots, x_n\}$ such that

$$\langle x_i, x_i \rangle = 2,$$
$$\langle x_i, x_j \rangle = -1, \text{ for } i = j \pm 1$$
$$\langle x_i, x_j \rangle = 0, \text{ otherwise.}$$

D_n is the lattice generated by $\{x_1, \ldots, x_n\}$ such that

$$\langle x_i, x_i \rangle = 2,$$

$$\langle x_i, x_j \rangle = -1, \text{ for } i = j \pm 1, i, j \neq n$$

$$\langle x_n, x_{n-2} \rangle = -1,$$

$$\langle x_i, x_j \rangle = 0, \text{ otherwise.}$$

Definition 3. The *rank* of a lattice Λ is defined to be the dimension of the vector space $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$.

For our purposes, we will denote the rank of the lattice in the subscript. We see in our examples that A_n and D_n have rank n.

Definition 4. The *Gram matrix* G_Λ of a given basis $\{x_1, \ldots, x_n\}$ of a lattice Λ is the matrix given by

$$G_\Lambda = [\langle x_i, x_j \rangle].$$

Thus the Gram matrix of A_n, using our choice of basis from above, is given as

$$G_{A_n} = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ & . & . & . & \\ 0 & 0 & 0 & \ldots 2 & \end{bmatrix}.$$

Similarly, we see that

$$G_{D_n} = \begin{bmatrix} 2 & -1 & \dots & 0 & 0 & 0 \\ -1 & 2 & \dots & 0 & 0 & 0 \\ . & & . & & . & \\ 0 & 0 & \dots & 2 & -1 & -1 \\ 0 & 0 & \dots & -1 & 2 & 0 \\ 0 & 0 & \dots & -1 & 0 & 2 \end{bmatrix}.$$

The inner product of our lattice can be found using the Gram matrix as follows. Let $x, y \in \Lambda$ given by $x = \Sigma \alpha_i x_i$ and $y = \Sigma \beta_i x_i$ where the x_i form a basis for Λ and $\alpha_i, \beta_i \in \mathbb{Z}$. Then we can write x and y in vector form as $\mathbf{x} = (\alpha_1, ..., \alpha_n)$ and $\mathbf{y} = (\beta_1, ..., \beta n)$. Then $\langle x, y \rangle = \mathbf{x} G_\Lambda \mathbf{y}^T$.

Definition 5. The *determinant* $\det \Lambda$ of the lattice Λ is defined to be the absolute value of the determinant of G_Λ.

Thus, for the choice of basis we have given, $\det A_n = n+1$, and $\det D_n = 4$.

Proposition 1. The determinant of the lattice is well defined.

Proof. If a different basis for Λ were chosen, this basis would have to preserve the inner product. Thus it must be a unimodular change of basis, which is to say the change of basis matrix M must have determinant 1. The new Gram matrix is then $G'_\Lambda = M^{tr} G_\Lambda M$ and $\det G'_\Lambda = \det G_\Lambda$. Thus the determinant of the Gram matrix is unchanged by a unimodular change of basis, and the determinant is well defined. □

We will primarily be concerned with even, positive definite integral lattices, which is to say with lattices having the property that $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in \Lambda$, and such that the Gram matrix is a positive definite matrix in the usual sense. Clearly, A_n and D_n, are both even, positive definite integral lattices.

Definition 6. The *dual lattice* Λ^* of an integral lattice Λ is defined by

$$\Lambda^* = \{x \in \Lambda \otimes \mathbb{Q} | \langle x, y \rangle \in \mathbb{Z} \text{ for all } y \in \Lambda\}$$

Definition 7. T is a *sublattice* of Λ if T is a free subgroup of Λ such that the restriction of the inner product of Λ to T is non-degenerate.

Definition 8. A sublattice T of Λ is called *primitive* if the quotient Λ/T is torsion-free. The primitive closure of a sublattice T of Λ is

$$T' = \{x \in \Lambda | nx \in T \text{ for some positive integer } n\}.$$

Definition 9. The *orthogonal complement* $T^\perp$ of a sublattice T of Λ is defined to be

$$T^\perp = \{x \in \Lambda | \langle x, y \rangle = 0 \text{ for all } y \in T\}.$$

Definition 10. For $\alpha \in \Lambda$, the *norm* of α is defined to be the inner product of α with itself, denoted $|\alpha| = \langle \alpha, \alpha \rangle$.

Note that this is the square of the usual definition of norm, but for our purposes this definition will be simpler to use.

Definition 11. An element $\alpha \in \Lambda$ is said to have *minimal norm* if $|\alpha| \leq |\beta|$ for all $\beta \in \Lambda$. Such an element is called a *minimal element.* The *minimal norm of a lattice* is the norm of a minimal element.

Both our examples have minimal norm 2, but A_n has $n(n+1)$ minimal elements while D_n has $2n(n-1)$.

Definition 12. We say that two lattices are *isomorphic* if there is a isomorphism between the two finitely generated Abelian groups that preserves the inner product.

More explicitly, let Λ_1 be a lattice, with λ_1 its finitely generated Abelian group, and $\langle x, y\rangle$ its inner product for $x, y \in \Lambda_1$, and Λ_2 be another lattice, with λ_2 its finitely generated Abelian group, and (r, s) its inner product for $r, s \in \Lambda_2$. Then Λ_1 is isomorphic to Λ_2 if there is an isomorphism $\phi : \lambda_1 \to \lambda_2$ such that $(\phi(x), \phi(y)) = \langle x, y\rangle$.

Definition 13. A lattice Λ' is said to *embed* in a lattice Λ if there is an injective group homomorphism $\varphi : \Lambda' \to \Lambda$ the preserves the inner product.

Definition 14. Given two lattices Λ_1 and Λ_2, the *direct sum* $\Lambda_1 \oplus \Lambda_2$ *of* Λ_1 *and* Λ_2 is the lattice defined on the direct sum of the groups of Λ_1 and Λ_2, together with the inner product inherent from Λ_1 and Λ_2 along with the condition that $\langle \alpha, \beta\rangle = 0$ for all $\alpha \in \Lambda_1$ and all $\beta \in \Lambda_2$.

For example, $D_4 \oplus A_2$ is the lattice given by basis elements $\{x_1, \ldots, x_4, y_1, y_2\}$ and inner product such that

$$\langle x_1, x_1\rangle = \langle x_2, x_2\rangle = \langle x_3, x_3\rangle = \langle x_4, x_4\rangle = \langle y_1, y_1\rangle = \langle y_2, y_2\rangle = 2,$$

$$\langle x_1, x_2\rangle = \langle x_2, x_3\rangle = \langle x_2, x_4\rangle = \langle y_1, y_2\rangle = -1,$$

$$\langle x_1, x_3\rangle = \langle x_1, x_4\rangle = \langle x_3, x_4\rangle = \langle x_i, y_j\rangle = 0 \qquad \text{for all } i, j.$$

The Gram matrix of $D_4 \oplus A_2$ is

$$G_{D_4 \oplus A_2} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

and

$$\det D_4 \oplus A_2 = 12$$

Since the Gram matrix of $\Lambda = \Lambda_1 \oplus \Lambda_2$ can be written in block diagonal form, with the Gram matrices of Λ_1 and Λ_2 as the blocks, we get $\det(\Lambda_1 \oplus \Lambda_2) = (\det \Lambda_1)(\det \Lambda_2)$. Also, the minimal norm of the lattice is not changed, as the elements of minimal norm in Λ_i can be written as they were with zeros in the remaining slots in Λ, and thus have the same norm in Λ as in Λ_i.

Definition 15. A lattice Λ is said to be *reducible* if it is isomorphic to a direct sum of lattices $\Lambda_1 \oplus \ldots \oplus \Lambda_n$ such that Λ_i is orthogonal to Λ_j for all i and j. Otherwise, the lattice is said to be *irreducible*.

Definition 16. A lattice which is generated by elements of norm 2 is called an *even root lattice*.

This definition is different than the usual definition of root lattice, which defines them to be those lattices which are defined from root systems. We will show that the given definition is consistent with the usual definition, but for our purposes, it is simpler to define root lattices in this way.

Note that as A_n and D_n are both generated by elements of norm 2, they are both even root lattices. We will discuss the even root lattices A_n, for $n = 1, \ldots, 8$, D_n, for $n = 4, \ldots, 8$ and E_n, for $n = 6, 7, 8$, and their direct sums, where the E_n are the lattices generated by n elements, with inner product described by their Gram matrices as follows.

$$G_{E_6} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

$$G_{E_7} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix},$$

$$
G_{E_8} = \begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.
$$

All of the irreducible lattices we will be considering, together with some of their properties, are given in Table 2, found in Section 4.3.

3.2 The Mordell-Weil Lattice

As explained above, the generic fibre E is an elliptic curve over K. Thus we can apply the Mordell-Weil Theorem for Function Fields, as found in Silverman [12], Theorem 6.1.

Theorem 1 (Mordell-Weil Theorem for Function Fields). *Let $\mathcal{E} \to C$ be an elliptic surface defined over a field k, and let E/K be the corresponding elliptic curve over the function field $K = k[C]$. If $\mathcal{E} \to C$ does not split, then $E(K)$ is a finitely generated group.*

A surface splits if it is birational to $E \times \mathbb{P}^1$, which we have excluded in our definition of rational elliptic surface.

As the Mordell-Weil group is a finitely generated free Abelian group, we can define a lattice from it, if we have an inner product. The inner product that is used is called the height pairing. Neither the height pairing nor the lattice will not be discussed in great detail in this paper, but details are discussed by Shioda in [10].

The Mordell-Weil group, along with its height pairing can more easily be determined using another group and lattice associated with the surface, as discussed in Section 5.3. The lattice that we will use is a sublattice of the Néron-Severi lattice.

3.3 The Néron-Severi Lattice and the Shioda Lattice

The Néron-Severi group $NS(\mathcal{E})$ of a rational elliptic surface $\mathcal{E}$ is the group of divisors of $\mathcal{E}$ modulo algebraic equivalence. From Shioda [10] we see that $NS(\mathcal{E})$ is finitely generated, and the intersection product is non-degenerate, thus it defines a lattice structure on $NS(\mathcal{E})$. By definition and Theorem 7.5 in Shioda [10] we see that the Néron-Severi lattice always decomposes as $NS(\mathcal{E}) = \mathcal{U} \oplus \Lambda^{-}$ where $\mathcal{U} = \langle \sigma_0, F \rangle$ is the group generated by σ_0, the zero of $E(K)$, and F, a fibre, and Λ^{-} is called the frame of the surface. Lemma 10.2 of Shioda tells us that the frame for a rational elliptic surface is always E_8. Thus all rational elliptic surfaces have $NS(\mathcal{E}) = \mathcal{U} \oplus E_8$. Although they all have the same Néron-Severi lattice, they each have a different associated sublattice of $\mathcal{U} \oplus E_8$, determined by the configuration of singular fibres on the surface. Clearly, as each elliptic surface has a section σ_0 and algebraically equivalent fibres, the associated sublattice of $NS(\mathcal{E})$, as discussed in Section 7 of [10] is of the form $\mathcal{U} \oplus \lambda$ where λ is the sublattice of E_8 generated by the irreducible components of the fibres of $\mathcal{E}$ which do not meet the zero section. We will refer to the lattice λ as the *Shioda lattice.*

As mentioned above, there is a theory of intersection associated with the divisors of $\mathcal{E}$, which we will discuss briefly. All of the fibres, their components, and the sections are divisors of $\mathcal{E}$. We have the following lemma, found in Shioda [10], that gives a relation between fibres, and thus helps us determine their intersection numbers.

Theorem 2. *Any two fibres of π, F_ν and $F_{\nu'}$ $(\nu, \nu' \in C)$, are algebraically equivalent to each other. In particular, we have*

$$F \approx e_{\nu,0} + \sum_{i=1}^{m_\nu - 1} \mu_{\nu,i} e_{\nu,i}.$$

Thus, as any two fibres are algebraically equivalent, to determine a fibre's self intersection, we need only look at its intersection with any other fibre. By definition, sections intersect individual fibres once. These ideas lead us to the following theorem, also given by Shioda [10].

Theorem 3. *For any section $P \in E(K)$ and any fibre F, we have*

$$(PF) = (OF) = 1, \ (F^2) = (F_\nu F_{\nu'}) = 0.$$

Thus we see that all fibres have self intersection number 0. As all fibres are algebraically equivalent, to determine the intersection number of an irreducible component with the fibre in which it is contained we need only consider that component's intersection with any fibre. These equivalences and the Kodaira list given in Table 1 and the fact that any components that intersect transversely have intersection number 1 show that individual components of reducible fibres all have self-intersection -2.

For example, if $e_1, \ldots e_5$ are the irreducible components of the resolution of the singular fibre of type I_5, as shown in Table 1, Section 2.2, then we note that one of these, say e_1 meets σ_0, and they all intersect two different fibres transversely. Thus $(e_ie_j) = 1$ if $i = j \pm 1$, for $i, j = 1, \ldots, 5$, $(e_1e_5) = 1$ and $(e_ie_i) = -2$ for all i.

Definition 17. For each fibre there is a subgroup of $NS(\mathcal{E})$ generated by the irreducible components of the fibre which do not meet O. Since the argument above shows that the intersection product, or its opposite, is non-degenerate, this subgroup is a sublattice of Λ and its opposite is a sublattice of Λ^-. We will call this sublattice T_ν^- of Λ^- the *Shioda lattice of the fibre* F_ν. The direct sum $T = \oplus_\nu T_\nu^-$ is called the *Shioda lattice of* $\mathcal{E}$.

We use the opposite intersection number so that our lattice will be positive definite. We also note that each generator has norm 2, thus the Shioda lattices are even root lattices.

Note that the example of the intersections given above gives us the root lattice with four generators (e_2, e_3, e_4, e_5) such that

$$\langle e_2, e_3\rangle = \langle e_3, e_4\rangle = \langle e_4, e_5\rangle = -1$$

$$\langle e_i, e_i\rangle = 2$$

$$\langle e_2, e_4\rangle = \langle e_2, e_5\rangle = \langle e_3, e_5\rangle = 0,$$

which is called A_4.

The following theorem from Shioda [10] gives the correspondence between the singular fibres, and the irreducible root lattices.

Theorem 4. *The lattice T_ν^- is a root lattice of rank $m_\nu - 1$, determined by the type of the singular fibre F_ν as follows:*

Kodaira Type of F_ν	I_m	II	I_m^*	II^*	III^*	IV^*	IV	III
T_ν^-	A_{m-1}	$\{0\}$	D_{m+4}	E_8	E_7	E_6	A_2	A_1

Using this theorem it is easy to determine the Shioda lattices associated with a given rational elliptic surface, if we know its configuration of singular fibres. For example, if we know that our surface has two singular fibres, one of type I_3 and one of type I_0^*, then it will have $D_4 \oplus A_2$ as its associated Shioda lattice.

As the Shioda lattice is a sublattice of $NS(\mathcal{E})$, the Shioda lattice must embed in the Néron-Severi lattice. For rational elliptic surfaces, as mentioned earlier, $NS(\mathcal{E}) = \mathcal{U} \oplus E_8$. Since the Shioda lattice is generated by the irreducible components of the singular fibres which do not meet the zero section, it is clear that $\mathcal{U}$ is orthogonal to the Shioda lattice, as these components do not meet the zero section, or any other fibre. Thus the Shioda lattice must embed in E_8 for a rational elliptic surface. Dynkin [3], as cited in Oguiso and Shioda [6], classified all the lattices that embed in E_8.

Theorem 5 ([3] Chapter 2 Section 5, [6] Theorem 3.2). *Let T be a root lattice of rank s which is embedded as a sublattice of E_8, other than $\{0\}$ and E_8. Then T is isomorphic to one of the following:*

$s = 8$**:** A_8, D_8, $A_7 \oplus A_1$, $A_5 \oplus A_2 \oplus A_1$, $A_4^{\oplus 2}$,
$A_2^{\oplus 4}$, $E_6 \oplus A_2$, $E_7 \oplus A_1$, $D_6 \oplus A_1^{\oplus 2}$, $D_5 \oplus A_3$,
$D_4^{\oplus 2}$, $D_4 \oplus A_1^{\oplus 4}$, $A_3^{\oplus 2} \oplus A_1^{\oplus 2}$, $A_1^{\oplus 8}$.

$s = 7$**:** $A_6 \oplus A_1$, $A_4 \oplus A_2 \oplus A_1$, $A_5 \oplus A_2$, $A_2^{\oplus 3} \oplus A_1$, $E_6 \oplus A_1$,
E_7, D_7, $D_5 \oplus A_1^{\oplus 2}$, $D_4 \oplus A_1^{\oplus 3}$, $A_3^{\oplus 2} \oplus A_1$,

$A_1^{\oplus 7},\ D_6 \oplus A_1,\ D_5 \oplus A_2,\ A_3 \oplus A_2 \oplus A_1^{\oplus 2},\ D_4 \oplus A_3,$

$A_3 \oplus A_1^{\oplus 4},\ A_4 \oplus A_3,\ A_5 \oplus A_1^{\oplus 2},\ A_7.$

$s = 6$**:** $A_2^{\oplus 3},\ E_6,\ D_6,\ D_4 \oplus A_1^{\oplus 2},\ A_3^{\oplus 2},\ D_5 \oplus A_1,$

$A_3 \oplus A_1^{\oplus 3},\ D_4 \oplus A_2,\ A_1^{\oplus 6},\ A_2 \oplus A_1^{\oplus 4},\ A_4 \oplus A_1^{\oplus 2},$

$A_6,\ A_3 \oplus A_2 \oplus A_1,\ A_5 \oplus A_1,\ A_4 \oplus A_2,\ A_2^{\oplus 2} \oplus A_1^{\oplus 2}.$

$s = 5$**:** $D_5,\ A_3 \oplus A_1^{\oplus 2},\ A_3 \oplus A_2,\ A_5,\ A_1^{\oplus 5},$

$A_4 \oplus A_1,\ D_4 \oplus A_1,\ A_2 \oplus A_1^{\oplus 3},\ A_2^{\oplus 2} \oplus A_1.$

$s = 4$**:** $D_4,\ A_1^{\oplus 4},\ A_2 \oplus A_1^{\oplus 2},\ A_2^{\oplus 2},\ A_3 \oplus A_1,\ A_4.$

$s = 3$**:** $A_3,\ A_2 \oplus A_1,\ A_1^{\oplus 3}.$

$s = 2$**:** $A_2,\ A_1^{\oplus 2}.$

$s = 1$**:** $A_1.$

We will give a careful verification of this theorem, in which we show that there are also no additional isomorphisms between the root lattices of the form A_n, D_n, E_n or their sums except those induced by $A_1 \cong D_1$, $A_2 \cong D_2$ and $A_3 \cong D_3$. Thus we can apply the theorem to determine which configurations of singular fibres can actually exist. From a strictly lattice theoretic perspective, it is difficult to determine how lattices embed, and so which lattices do embed in E_8. Thus, we turn our attention to root systems to help us give a careful proof of this theorem.

4 Root Systems and Dynkin Diagrams

4.1 Root Systems

Following the notation of Knapp [4], we define a root system, and discuss some of its properties.

Definition 18. An *abstract root system* in a finite-dimensional real inner product space V with inner product $\langle \cdot, \cdot \rangle$ and norm $|\cdot| = \langle \alpha, \alpha \rangle$ is a finite set Δ of nonzero elements of V such that

(i) Δ spans V,

(ii) For each $\alpha \in \Delta$ the orthogonal transformations s_α, defined as

$$s_\alpha(\varphi) = \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\,\alpha$$

for $\phi \in V$, carry Δ to itself,

(iii) $\dfrac{2\langle \beta, \alpha \rangle}{|\alpha|}$ is an integer whenever α and β are in Δ.

For example, we have the root system $A_n = \{\alpha_i - \alpha_j | i \neq j,\ i, j = 1, 2, 3\}$ in the inner product space $V = \left\{ \sum_{i=1}^{n+1} a_i\alpha_i, \text{ with } \sum a_i\alpha_i = 0 \right\}$. V is a of dimension n, as the condition on $\sum a_i\alpha_i$ implies a linear equivalence among the α_i. We can also take, for example, the root system $D_n = \{\pm\alpha_i \pm \alpha_j | i \neq j,\ i, j = 1, 2, 3, 4\}$ in the inner product space $V = \left\{ \sum_{i=1}^{n} a_i\alpha_i \right\}$, where V clearly has dimension n.

The inner products of the root system are inherent from the ambient inner product space V, but we do not give them explicitly for these root systems now, as they are easier to give once we have a few more definitions. We introduce the notion of positivity, to obtain a subset of nonzero elements of V which we will refer to as positive elements. We have the following two properties of positivity to define this set.

Definition 19. A set of positive elements of V is a subset of V satisfying the following properties.

(i) for any nonzero $\varphi \in V$, exactly one of φ and $-\varphi$ is positive,

(ii) the sum of positive elements is positive, and any positive multiple of a positive element is positive.

The positivity of the inner product space can be defined using a lexicographic ordering. Taking the spanning set of V as given above $\{\alpha_1, \ldots, \alpha_n\}$, we define an element $\beta \in V$ to be positive if there exists some index k such that $\langle \beta, \alpha_i \rangle = 0$ for $1 \leq i \leq k-1$ and $\langle \beta, \alpha_k \rangle > 0$.

Definition 20. For a fixed positivity of V, a root α is said to be *simple* if α is positive and if α does not decompose as $\alpha = \beta_1 + \beta_2$ with β_1 and β_2 both positive roots.

We can define positivity on V such that the positive roots in A_n are the roots $\alpha_i - \alpha_j, i < j$. Using this same definition of positive, the simple roots in A_n are the roots

$$e_1 = \alpha_1 - \alpha_2,\, e_2 = \alpha_2 - \alpha_3, \ldots, e_n = \alpha_n - \alpha_{n-1}.$$

Similarly, we can define the positivity of V so the positive roots in D_n are the roots $\alpha_i \pm \alpha_j, i < j$, while the simple roots are the roots

$$e_1 = \alpha_1 - \alpha_2, \ldots, e_{n-2} = \alpha_{n-2} - \alpha_{n-1},\, e_{n-1} = \alpha_{n-1} - \alpha_n,\, e_n = \alpha_{n-1} + \alpha_n.$$

By Proposition 2.49 from Knapp [4], the simple roots form a basis for V. Thus, we can determine the inner product of any roots by knowing the inner product for the simple roots. In our example, the inner product of A_n is given by

$$\begin{gathered}\langle e_i, e_i\rangle = 2, \\ \langle e_i, e_j\rangle = -1, \text{ for } i = j \pm 1 \\ \langle e_i, e_j\rangle = 0, \text{ otherwise,}\end{gathered}$$

and the inner product for D_n is given by

$$\begin{gathered}\langle e_i, e_i\rangle = 2, \\ \langle e_i, e_j\rangle = -1, \text{ for } i = j \pm 1, i, j \neq n \\ \langle e_n, e_{n-2}\rangle = -1, \\ \langle e_i, e_j\rangle = 0, \text{ otherwise,}\end{gathered}$$

where the e_i are the simple roots of the root system.

Definition 21. For $\alpha \in \Delta$, the *norm* of α is defined to be $|\alpha| = \langle \alpha, \alpha\rangle$.

Just as with the lattices in Definition 10, we note that this definition of norm is the square of the usual definition, which, for our purposes, is easier to use.

Definition 22. A root system Δ is said to be an *even root system* if all the simple roots of Δ have norm 2.

Definition 23. A Π *system* is the set of simple roots of V for a given positivity.

Definition 24. Let Π be enumerated as $\Pi = \{e_1, \ldots, e_n\}$ where n =dim V. The *Cartan matrix of* Δ *relative to* Π is the matrix $C_\Delta = [C_{ij}]$ where $C_{ij} = \dfrac{2\langle e_i, e_j\rangle}{|e_i|}$.

As the Cartan matrix depends on an enumeration of the simple elements, different enumerations will give Cartan matrices that are conjugate to one another by a permutation matrix.

Using the enumeration of the simple roots from above, we get the following two Cartan matrices;

$$C_{A_n} = \begin{bmatrix} 2 & -1 & 0 & \ldots & 0 \\ -1 & 2 & -1 & \ldots & 0 \\ & \cdot & \cdot & \cdot & \\ 0 & 0 & 0 & \ldots 2 & \end{bmatrix},$$

$$C_{D_n} = \begin{bmatrix} 2 & -1 & \ldots & 0 & 0 & 0 \\ -1 & 2 & \ldots & 0 & 0 & 0 \\ \cdot & & \cdot & & \cdot & \\ 0 & 0 & \ldots & 2 & -1 & -1 \\ 0 & 0 & \ldots & -1 & 2 & 0 \\ 0 & 0 & \ldots & -1 & 0 & 2 \end{bmatrix}.$$

Since many of the previous definitions depend on the positivity of V, we need to show that they are all well defined; which is to say that the they are independent of the choice of positivity,

Definition 25. The *Weyl group* W of a root system Δ is the subgroup of the orthogonal group on V generated by the reflections s_α for $\alpha \in \Delta$.

Knapp [4] gives us the following theorems concerning the choice of positivity.

Theorem 6. *If* Π *and* Π' *are two simple systems for* Δ*, then there exists one and only one element* $s \in W$ *such that* $s\Pi = \Pi'$*.*

As the simple roots are completely determined by the positivity, this tells us that two different choices of positivity are related to each other by an element of the Weyl group. Although this is an important concept, the proof of this theorem is beyond the scope of this paper, and the reader is directed to Knapp for details.

The next theorem shows that the Cartan matrix is well defined.

Theorem 7. *The Cartan matrix is independent of the choice of positive system up to permutation of indices.*

Proof. Let Π and Π' be the two simple systems of Δ depending on two different choices of positivity. By the previous theorem, $s\Pi = \Pi'$ for some $s \in W$, which is to say we can find enumerations $\Pi = \{e_1, \ldots, e_n\}$ and $\Pi' = \{f_1, \ldots, f_n\}$ such that $se_i = f_i$. Then we have

$$\frac{2\langle e_i, e_j\rangle}{|e_i|} = \frac{2\langle se_i, se_j\rangle}{|se_i|} = \frac{2\langle f_i, f_j\rangle}{|f_i|}$$

since s is orthogonal. Thus, the Cartan matrices match, and are independent up to permutation of indices. □

This final theorem tells us that the root systems are independent of the choice of positivity.

Theorem 8. *The Cartan matrix determines the reduced root system up to isomorphism.*

Thus, as the Cartan matrix is independent of the choice of positivity, the root system is as well.

Definition 26. Given root systems Δ_1 and Δ_2, the *direct sum* $\Delta = \Delta_1 \oplus \Delta_2$ *of* Δ_1 *and* Δ_2 is the union of the roots of the systems, with their inherent inner products, together with the condition that $\langle e, f \rangle = 0$ for all $e \in \Delta_1,\ f \in \Delta_2$.

Definition 27. A root system Δ is said to be *reducible* if it has a nontrivial decomposition $\Delta = \Delta_1 \oplus \Delta_2$ such that e_i is orthogonal to f_j for every $e_i \in \Delta_1$ and every $f_j \in \Delta_2$. Otherwise, it is said to be *irreducible.*

Thus we could consider the root system $D_4 \oplus A_2$, which is defined by the union of the roots of the systems D_4 and A_2, with their inner products, together with the inner product condition that $\langle e, f \rangle = 0$ for $e \in D_4, f \in A_2$.

The Cartan matrix for a reducible root system can be written as a the block diagonal matrix with the Cartan matrices of the irreducible components of the root system as the blocks. For example, consider the reducible root system $D_4 \oplus A_2$;

$$C_{D_4 \oplus A_2} = \begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & -1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}.$$

Note that for an even root system $C_{ij} = \langle e_i, e_j \rangle$, thus the Cartan matrix encodes the inner product of the entire root system. Thus the inner product of two elements can be found using the Cartan matrix the same way it was found in the lattices using the Gram matrix.

Definition 28. Two root systems are said to be *isomorphic* if there is an isomorphism of their ambient inner product spaces such that the root systems are sent to each other, and the integers $\frac{2\langle \beta, \alpha \rangle}{|\alpha|}$ are preserved.

More explicitly, let Δ_1 and Δ_2 be root systems in V_1 and V_2, respectively. An isomorphism of Δ_1 and Δ_2 is a vector space isomorphism $\psi : V_1 \to V_2$ such that $\psi(\Delta_1) = \Delta_2$ and $\frac{2\langle \beta, \alpha \rangle}{|\alpha|} = \frac{2\langle \psi(\beta), \psi(\alpha) \rangle}{|\psi(\alpha)|}$ for all $\alpha, \beta \in \Delta_1$.

Definition 29. An abstract root system Δ' contained in an inner product vector space V' is said to *embed* in an abstract root system Δ contained in an inner product space V if there is an injective linear map $\psi : V' \to V$ such that $\psi(\Delta') = \Delta$ and the integers $\frac{2\langle \beta, \alpha \rangle}{|\alpha|}$ are preserved.

4.2 An Isomorphism of Categories

Theorem 9. *Let $\mathcal{L}$ be the category of even root lattices with their embeddings and $\mathcal{R}$ be the category of even root systems with their embeddings. Then there exists an isomorphism of categories $\Phi : \mathcal{L} \to \mathcal{R}$.*

Proof. To show there is an isomorphism of categories, we need to show that we can map even root lattices to even root systems and even root systems to even root lattices while preserving the embeddings of these root lattices and root systems.

It is clear that given an even root system Δ, a root lattice $\Psi(\Delta)$ can be defined, where $\Psi = \Phi^{-1}$, using the free Abelian subgroup of V generated by the simple roots of Δ, and the inner product of V restricted to $\Psi(\Delta)$.

Given a root lattice Λ, define a root system $\Phi(\Lambda)$ to be the set of elements in Λ with norm 2. Clearly $\Phi(\Lambda)$ spans the finite dimensional vector space $\Lambda \otimes \mathbb{Q}$, as the generators of the finitely generated free Abelian group are elements of norm 2, so (i) is satisfied.

Next, as we have integral lattices, and we are considering the elements of norm 2, then it clear that $\dfrac{2\langle \beta, \alpha \rangle}{|\alpha|}$ is an integer for $\alpha, \beta \in \Phi(\Lambda)$, and we see that (iii) is satisfied. Finally we show that (ii) is satisfied. First we note that as $\dfrac{2\langle \varphi, \alpha \rangle}{|\alpha|}$ is an integer for $\alpha, \varphi \in \Delta$, then $s_\alpha(\varphi) = \varphi - \dfrac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha$ is an integer linear combination of elements of Λ, thus $s_\alpha(\varphi)$ is in Λ. But we want $s_\alpha(\varphi)$ to be in $\Phi(\Lambda)$, not simply in Λ, thus we must show that $s_\alpha(\varphi)$ has norm 2 for $\alpha, \varphi \in \Lambda$. We have

$$\begin{aligned}
\langle s_\alpha(\varphi), s_\alpha(\varphi) \rangle &= \left\langle \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha, \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha \right\rangle \\
&= \left\langle \varphi, \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha \right\rangle + \left\langle -\frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha, \varphi - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha \right\rangle \\
&= \langle \varphi, \varphi \rangle + \left\langle \varphi, -\frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha \right\rangle + \left\langle -\frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha, \varphi \right\rangle \\
&\qquad + \left\langle -\frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha, -\frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\alpha \right\rangle \\
&= \langle \varphi, \varphi \rangle - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\langle \varphi, \alpha \rangle - \frac{2\langle \varphi, \alpha \rangle}{|\alpha|}\langle \alpha, \varphi \rangle + \frac{4\langle \varphi, \alpha \rangle^2}{|\alpha|^2}\langle \alpha, \alpha \rangle \\
&= \langle \varphi, \varphi \rangle - \frac{4\langle \varphi, \alpha \rangle^2}{|\alpha|} + \frac{4\langle \varphi, \alpha \rangle^2}{|\alpha|} \\
&= \langle \varphi, \varphi \rangle
\end{aligned}$$

Thus, $s_\alpha(\varphi)$ has norm 2 if and only if φ has norm 2. Hence, the orthogonal transformations $s_\alpha(\varphi)$ take Δ to itself, and condition (ii) is satisfied. Therefore, given an even root lattice, we can indeed map to an even root system.

We now need to show that an embedding $f : \Lambda \to \Lambda'$ in $\mathcal{L}$ maps to an embedding $\Phi(f) : \Phi(\Lambda) \to \Phi(\Lambda')$ in $\mathcal{R}$ under Φ, and that an embedding $g : \Delta \to \Delta'$ in $\mathcal{R}$ maps to an embedding $\Psi(g) : \Psi(\Delta) \to \Psi(\Delta')$ in $\mathcal{L}$ under Ψ, such that $\Phi \circ \Psi(g) = id_{\mathcal{R}}$ and $\Psi \circ \Phi(f) = id_{\mathcal{L}}$.

Let $f : \Lambda \to \Lambda'$ be an embedding of $\mathcal{L}$. Then we map to the even root systems $\Phi(\Lambda)$ and $\Phi(\Lambda')$. Consider the map $\Phi(f) : \Lambda \otimes \mathbb{Q} \to \Lambda' \otimes \mathbb{Q}$ which is the extension of f to $\Lambda \otimes \mathbb{Q}$. Since f is an embedding, this extension is an embedding of the ambient vector spaces of $\Phi(\Lambda)$ and $\Phi(\Lambda')$ respectively. As f preserves inner product, the inner product is preserved under $\Phi(f)$ as this inner product is simply the linear extension of the inner product of Λ. Thus the integers $\dfrac{2\langle \beta, \alpha \rangle}{|\alpha|}$ are preserved, and the elements of norm 2 are mapped to elements of norm 2, and $\Phi(f)$ maps $\Phi(\Lambda)$ into $\Phi(\Lambda')$. Thus by definition $\Phi(\Lambda)$ embeds into $\Phi(\Lambda')$ with embedding $\Phi(f)$.

Let $g : \Delta \to \Delta'$ be an embedding of $\mathcal{L}$. Then we map to the even root lattices $\Psi(\Delta)$ and $\Psi(\Delta')$. Consider the map $\Psi(g) : \Psi(\Delta) \to \Psi(\Delta')$ which is the restriction of g to $\Psi(\Delta)$. As g is an embedding of the ambient vector spaces, this restriction is an embedding of the free Abelian groups generated by the simple roots of Δ and Δ'. As g is an embedding of even root systems, it preserves the inner product of the simple roots, and so the inner product of $\Psi(\Delta)$ is preserved under $\Psi(g)$. Thus by definition $\Psi(\Delta)$ embeds into $\Psi(\Delta)$ with embedding $\Psi(g)$.

We now show that $\Phi \circ \Psi(g) = id_{\mathcal{R}}$ and that $\Psi \circ \Phi(f) = id_{\mathcal{L}}$. Clearly $\Phi \circ \Psi$ is the identity on $\mathcal{R}$, as Ψ restricts to the simple roots, and then Φ extends to the entire inner product space, while preserving the inner product. Similarly, as Φ extends to the entire vector space $\Lambda \otimes \mathbb{Q}$ with the elements of norm 2 as the root system, while preserving the inner product, followed by Φ, which contracts back to the elements in Λ with norm 2, $\Psi \circ \Phi$ is the identity on $\mathcal{L}$.

Thus we have an isomorphism of categories, and the embeddings of even root lattices are in one to one correspondence with the embeddings of even root systems. □

As Φ clearly preserves compositions, and isomorphisms are embeddings, we also know from this theorem that the even root lattices are in one to one correspondence with the even root systems, so they can be classified accordingly. Thus in order to classify the even root lattices, along with their embeddings, we need only classify the even root systems, and their embeddings. We also note, as explained by Conway and Sloane [2] that the root lattices get their names from the associated root systems, so there is no ambiguity in the names. We will use Dynkin diagrams to give the classification of our root systems up to isomorphism.

4.3 Dynkin Diagrams

From Dynkin [3] we know that root systems associated to root systems of Lie algebras have Dynkin diagrams, and following the method of Bourbaki [1] and Knapp [4], we know that associated to general root systems, there are Dynkin diagrams. A Dynkin diagram is a diagram of the simple roots in a system that encodes the rank and inner product, thus allowing the inner product to be read directly off the diagram.

4.3.1 Construction

The diagram for the root systems that we are considering is constructed in the following manner. For each simple element x_i in the root system there is one node. The nodes are then connected to one another depending on the inner products of the corresponding basis elements. If $\langle x_i, x_j \rangle = -1$, then there is a single line drawn between node i and node j. If $\langle x_i, x_j \rangle = 0$ then node i is not connected to node j. For example, we consider the root system A_2, which has for basis $\{x_1, x_2\}$ and inner product defined by

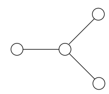
$$\langle x_1, x_2 \rangle = -1,$$
$$\langle x_1, x_1 \rangle = 2 = \langle x_2, x_2 \rangle.$$

Clearly the Dynkin diagram associated with this root system has two nodes which are connected by a single line, as shown.

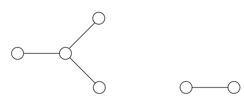
We note that as the Dynkin diagram is determined by the inner product, that it is easy to construct the diagram from the Cartan matrix of the root system. So, to determine the Dynkin diagram of D_4 we consider its Cartan matrix.

$$C_{D_4} = \begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$$

We see that this diagram must have four nodes, where node 1 in connected only to node 2, node 2 is connected to nodes 1, 3 and 4, node 3 is only connected to node 2, and node 4 is only connected to node 2, as shown.

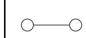
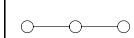
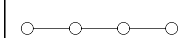


Following this same method of determining the Dynkin diagram of a root system, it is clear that the Dynkin diagram of a reducible root system is simply the disconnected union of the Dynkin diagrams of the irreducible components of the root system. This makes sense, as the inner products between the roots of the irreducible components are all 0, and so these nodes are not connected. For example, we see that the Dynkin diagram for $D_4 \oplus A_2$ is as shown.



We give a list of the irreducible root lattices Λ which we will be considering in the remainder of our paper in Table 2, along with the associated Gram matrix, the determinant of the lattice $\det \Lambda$, the minimal norm μ, the number of minimal elements τ, and the associated Dynkin diagram.

Table 2: Irreducible Root Lattices

Λ	Gram Matrix	$\det \Lambda$	μ	τ	Dynkin Diagram
A_1	$\begin{bmatrix} 2 \end{bmatrix}$	2	2	2	
A_2	$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$	3	2	6	
A_3	$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$	4	2	12	
A_4	$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix}$	5	2	20	

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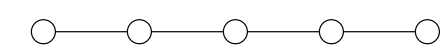
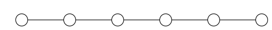
Λ	Gram Matrix	$\det \Lambda$	μ	τ	Dynkin Diagram
A_5	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	6	2	30	
A_6	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	7	2	42	

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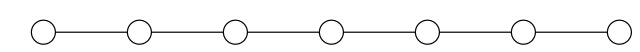
Λ	Gram Matrix	$\det\Lambda$	μ	τ	Dynkin Diagram
A_7	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	8	2	56	

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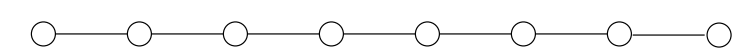
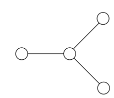
Λ	Gram Matrix	$\det\Lambda$	μ	τ	Dynkin Diagram
A_8	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	9	2	72	
D_4	$\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & -1 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}$	4	2	24	
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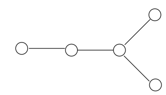
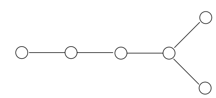
Λ	Gram Matrix	$\det\Lambda$	μ	τ	Dynkin Diagram
D_5	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & -1 & 0 & 2 \end{bmatrix}$	4	2	40	
D_6	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$	4	2	60	

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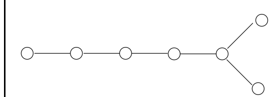
Λ	Gram Matrix	$\det\Lambda$	μ	τ	Dynkin Diagram
D_7	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$	4	2	84	

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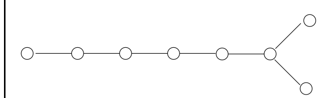
Λ	Gram Matrix	$\det\Lambda$	μ	τ	Dynkin Diagram
D_8	$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 2 \end{bmatrix}$	4	2	112	

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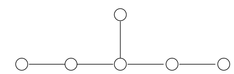
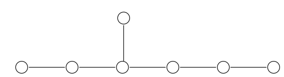
Λ	Gram Matrix	$\det \Lambda$	μ	τ	Dynkin Diagram
E_6	$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	3	2	72	
E_7	$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	2	2	126	

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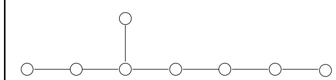
Λ	Gram Matrix	$\det\Lambda$	μ	τ	Dynkin Diagram
E_8	$\begin{bmatrix} 2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 2 \end{bmatrix}$	1	2	240	

4.3.2 Classification of Dynkin Diagrams and Root Lattices

Clearly, as the isomorphisms of root lattices preserve the rank and inner product, if two root systems are isomorphic, then they have the same Dynkin diagram. Similarly, as the rank and inner product can be determined by reading the Dynkin diagram, if two root lattices have the same Dynkin diagram, then they are isomorphic, as they have the same number of elements, and corresponding inner products. Thus, by completely classifying the Dynkin diagrams, we get a complete classification of even root lattices up to isomorphisms. Consequently, we can determine if two even root lattices are isomorphic by looking at their associated Dynkin diagrams.

We start by listing all possible lattices of rank less than or equal to 8 that are the direct sum of the root lattices A_n, for $n = 1, \ldots, 8$, D_n, for $n = 4, \ldots, 8$, and E_n, for $n = 6, 7, 8$, and their associated Dynkin diagrams.

Table 3: Root Lattices and Dynkin Diagrams

Root Lattice	Dynkin Diagram
$\{0\}$	
A_1	○
A_2	○—○
$A_1^{\oplus 2}$	○ ○
A_3	○—○—○
$A_2 \oplus A_1$	○—○ ○
$A_1^{\oplus 3}$	○ ○ ○

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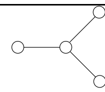
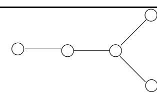
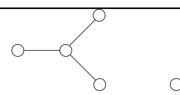
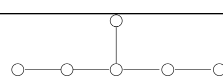
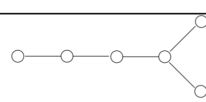
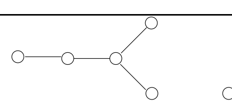
Root Lattice	Dynkin Diagram
D_4	
A_4	
$A_3 \oplus A_1$	
$A_2^{\oplus 2}$	
$A_2 \oplus A_1^{\oplus 2}$	
$A_1^{\oplus 4}$	
D_5	
$D_4 \oplus A_1$	
A_5	
$A_4 \oplus A_1$	
$A_3 \oplus A_2$	
$A_3 \oplus A_1^{\oplus 2}$	
$A_2^{\oplus 2} \oplus A_1$	
$A_2 \oplus A_1^{\oplus 3}$	
$A_1^{\oplus 5}$	
E_6	
D_6	
$D_5 \oplus A_1$	

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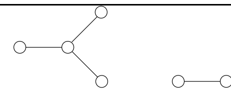
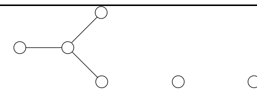
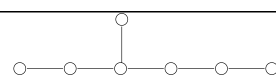
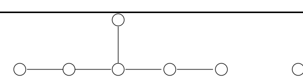
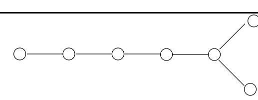
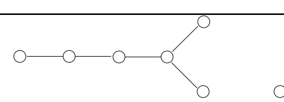
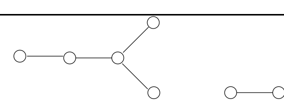
Root Lattice	Dynkin Diagram
$D_4 \oplus A_2$	
$D_4 \oplus A_1^{\oplus 2}$	
A_6	
$A_5 \oplus A_1$	
$A_4 \oplus A_2$	
$A_4 \oplus A_1^{\oplus 2}$	
$A_3^{\oplus 2}$	
$A_3 \oplus A_2 \oplus A_1$	
$A_3 \oplus A_1^{\oplus 3}$	
$A_2^{\oplus 3}$	
$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	
$A_2 \oplus A_1^{\oplus 4}$	
$A_1^{\oplus 6}$	
E_7	
$E_6 \oplus A_1$	
D_7	
$D_6 \oplus A_1$	
$D_5 \oplus A_2$	

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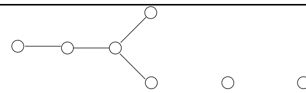
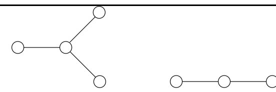
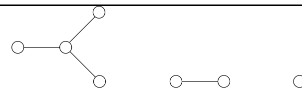
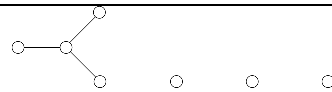
Root Lattice	Dynkin Diagram
$D_5 \oplus A_1^{\oplus 2}$	
$D_4 \oplus A_3$	
$D_4 \oplus A_2 \oplus A_1$	
$D_4 \oplus A_1^{\oplus 3}$	
A_7	
$A_6 \oplus A_1$	
$A_5 \oplus A_2$	
$A_5 \oplus A_1^{\oplus 2}$	
$A_4 \oplus A_3$	
$A_4 \oplus A_2 \oplus A_1$	
$A_4 \oplus A_1^{\oplus 3}$	
$A_3^{\oplus 2} \oplus A_1$	
$A_3 \oplus A_2^{\oplus 2}$	
$A_3 \oplus A_2 \oplus A_1^{\oplus 2}$	
$A_3 \oplus A_1^{\oplus 4}$	
$A_2^{\oplus 3} \oplus A_1$	
$A_2^{\oplus 2} \oplus A_1^{\oplus 3}$	
$A_2 \oplus A_1^{\oplus 5}$	
$A_1^{\oplus 7}$	

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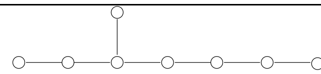
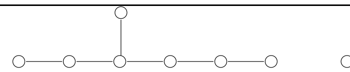
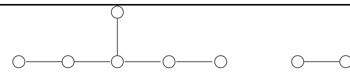
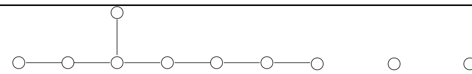
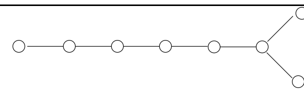
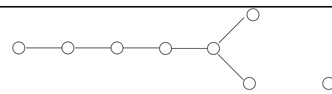
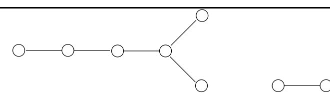
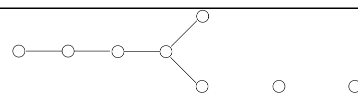
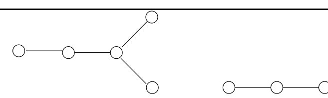
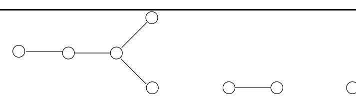
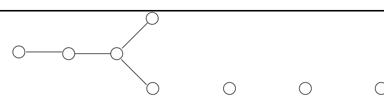
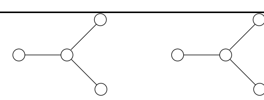
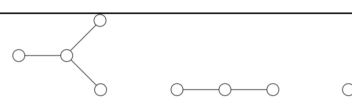
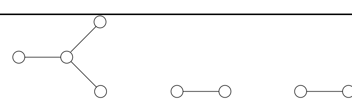
Root Lattice	Dynkin Diagram
E_8	
$E_7 \oplus A_1$	
$E_6 \oplus A_2$	
$E_6 \oplus A_1^{\oplus 2}$	
D_8	
$D_7 \oplus A_1$	
$D_6 \oplus A_2$	
$D_6 \oplus A_1^{\oplus 2}$	
$D_5 \oplus A_3$	
$D_5 \oplus A_2 \oplus A_1$	
$D_5 \oplus A_1^{\oplus 3}$	
$D_4^{\oplus 2}$	
$D_4 \oplus A_3 \oplus A_1$	
$D_4 \oplus A_2^{\oplus 2}$	

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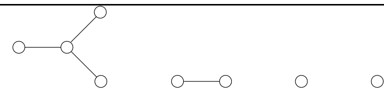
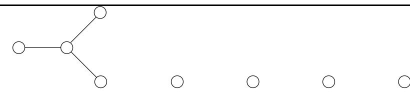
Root Lattice	Dynkin Diagram
$D_4 \oplus A_2 \oplus A_1^{\oplus 2}$	
$D_4 \oplus A_1^{\oplus 4}$	
A_8	
$A_7 \oplus A_1$	
$A_6 \oplus A_2$	
$A_6 \oplus A_1^{\oplus 2}$	
$A_5 \oplus A_3$	
$A_5 \oplus A_2 \oplus A_1$	
$A_5 \oplus A_1^{\oplus 3}$	
$A_4^{\oplus 2}$	
$A_4 \oplus A_3 \oplus A_1$	
$A_4 \oplus A_2^{\oplus 2}$	
$A_4 \oplus A_2 \oplus A_1^{\oplus 2}$	
$A_4 \oplus A_1^{\oplus 4}$	
$A_3^{\oplus 2} \oplus A_2$	
$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	
$A_3 \oplus A_2^{\oplus 2} \oplus A_1$	
$A_3 \oplus A_2 \oplus A_1^{\oplus 3}$	
$A_3 \oplus A_1^{\oplus 5}$	
$A_2^{\oplus 4}$	

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Root Lattice	Dynkin Diagram
$A_2^{\oplus 3} \oplus A_1^{\oplus 2}$	○—○ ○—○ ○—○ ○ ○
$A_2^{\oplus 2} \oplus A_1^{\oplus 4}$	○—○ ○—○ ○ ○ ○ ○
$A_2 \oplus A_1^{\oplus 6}$	○—○ ○ ○ ○ ○ ○ ○
$A_1^{\oplus 8}$	○ ○ ○ ○ ○ ○ ○ ○

By careful examination of Table 3 we see that there are no isomorphisms between any of these root lattices of rank less than or equal to 8, as none of the Dynkin diagrams are the same.

4.3.3 Derived Dynkin Diagrams

Now we will show which of these Dynkin diagrams are associated to lattices which embed in the lattice E_8 by showing which diagrams can be derived from the diagram for E_8. The following method of determining which diagrams can be derived from a given diagram comes from Dynkin [3].

Although Dynkin's argument is much broader than the one we will give, as we are only considering the Dynkin diagrams that come from root systems of type A_n, D_n and E_n, we will restrict his method to the systems which produce simply laced diagrams; that is diagrams where there is at most a single line connecting two distinct nodes.

Definition 30. A system of roots is said to *split* if it can be written as the disjoint union of two or more subsystems that are mutually orthogonal.

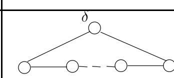
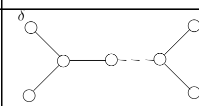
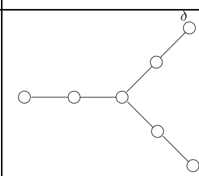
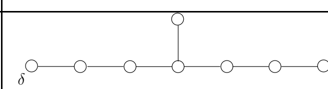
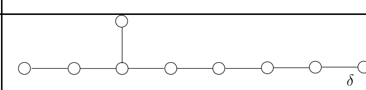
Definition 31. A root δ is said to be *expressible in terms of the roots* $\gamma_1, \ldots, \gamma_r$ if $\delta = \gamma_{i_1} + \ldots + \gamma_{i_s}$, and, for $l = 1, \ldots, s$ the sum $\gamma_{i_1} + \ldots + \gamma_{i_l} \in \Delta$, where

Δ is the entire root system.

Definition 32. Let the roots $\alpha_1, \alpha_2, \ldots, \alpha_m$ be the simple roots of a root system of Δ which does not split. As they are simple in Δ, all of these roots are positive. Adjoin to them δ, the smallest of the roots which are expressible in terms of $\pm\alpha_1, \pm\alpha_2, \ldots, \pm\alpha_m$. We obtain a system $(\alpha_1, \alpha_2, \ldots, \alpha_m, \delta)$, which we call an *extension* of the system $(\alpha_1, \alpha_2, \ldots, \alpha_m)$.

The following table lists δ, the smallest of the roots which are expressible in terms of the simple roots of the root system, and the Dynkin diagram for the extension for each of the irreducible root systems that we are considering.

Table 4: δ and the Extended Dynkin Diagrams

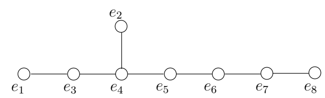
Λ	δ	Extended Dynkin Diagram
A_n	$-e_1 - \ldots - e_n$	
D_n	$-e_1 - 2e_2 - \ldots - 2e_{n-2} - e_{n-1} - e_n$	
E_6	$-e_1 - 2e_2 - 2e_3 - 3e_4 - 2e_5 - e_6$	
E_7	$-2e_1 - 2e_2 - 3e_3 - 4e_4 - 3e_5 - 2e_6 - e_7$	
E_8	$-2e_1 - 3e_2 - 4e_3 - 6e_4 - 5e_5 - 4e_6 - 3e_7 - 2e_8$	

Definition 33. Let Γ and Γ' be root systems. We say Γ' is derived from Γ by an *elementary transformation* if Γ' can be derived from Γ by the following method. Suppose that Γ reduces into s mutually orthogonal irreducible sub-

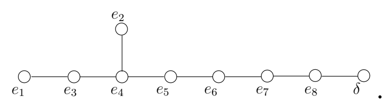
systems. By extending one of these subsystems, we get an extension of Γ. By then removing one of the elements from the extended subsystem, we obtain the new root system Γ', containing the same number of elements as Γ.

Definition 34. Diagrams which correspond to diagrams found by node removal or elementary transformations will be called *derived diagrams.*

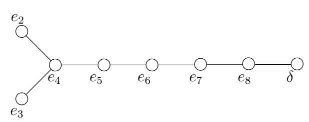
For example, starting with E_8, let us derive another root system by an elementary transformation. Begin with the diagram for E_8



and then extend with the δ given above to get

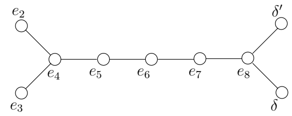


We then remove the node associated with e_1 to get the following diagram,



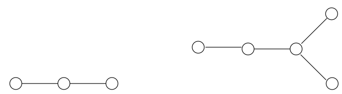
Thus we see that D_8 is derived by an elementary transformation of E_8. It is clear that this lattice D_8 embeds in the lattice E_8, with basis roots $e_2, e_3, \ldots, e_8, \delta \in E_8$, as it is defined on a simple system of elements of E_8, which respect the inner product of E_8.

We continue, to see if we can find other root lattice which embed in E_8 by finding the diagrams derived from D_8. So, we extend this D_8 by adding δ', which is the same as the δ for D_8 in the table, but with respect to D_8's new basis, and we get



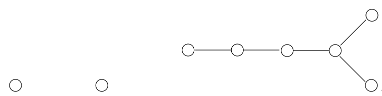
.

Clearly if we remove e_2, e_3, δ or δ', we have a D_8 again. But if we remove e_5 or e_7 we are left with



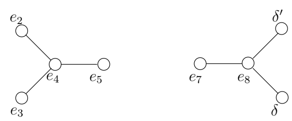
,

$D_5 \oplus A_3$. If we remove node e_4 or e_8 we are left with



,

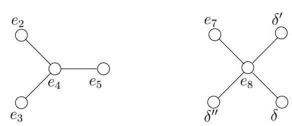
$D_6 \oplus A_1^{\oplus 2}$. And if we remove node e_6 we are left with



,

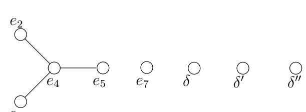
$D_4^{\oplus 2}$.

So, we see that D_8, $D_6 \oplus A_1^{\oplus 2}$, $D_5 \oplus A_3$ and $D_4^{\oplus 2}$ all embed in E_8. Continuing with $D_4^{\oplus 2}$, if we extend one of these, by adding δ'' we get



.

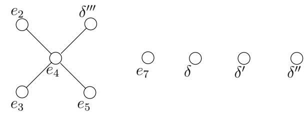
Clearly, if we remove e_7, δ, δ' or δ'', we get $D_4^{\oplus 2}$ again. But, if we remove e_8, we are left with



,

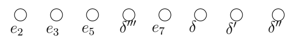
$D_4 \oplus A_1^{\oplus 4}$.

Now extending the other D_4 we get



,

and removing e_4 from this, we are left with



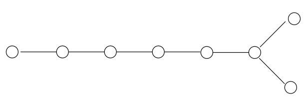
,

$A_1^{\oplus 8}$.

Thus we see that $D_4 \oplus A_1^{\oplus 4}$ and $A_1^{\oplus 8}$ also embed in E_8.

The following is a list all of the Dynkin diagrams of rank 8 that can be derived from the E_8 diagram.

1. 

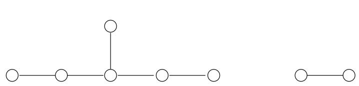
2. 

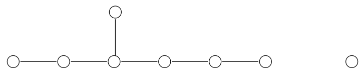
3.

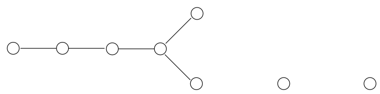
4.

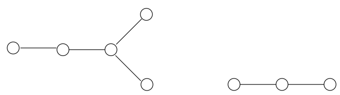
5.

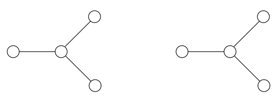
6.

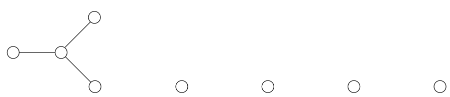
7. 

8. 

9. 

10. 

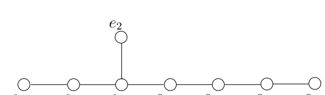
11. 

12. 

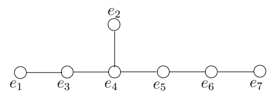
13.

14.

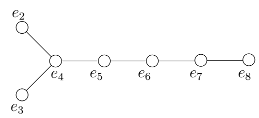
Now that we have the diagrams of rank 8 that can be derived from E_8, we next find all of the diagrams of rank less than 8 that can be derived from these diagrams. We do this by simply removing a node form the non-extended diagram, and then performing elementary transformations on lower rank diagram. For example, we start again with E_8.



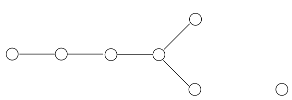
Without extending, if we simply remove e_8, then we are left with

,

E_7, which clearly embeds in E_8. Or, if we start with E_8 and remove e_1, we are left with

,

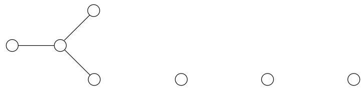
D_7. But, since these diagrams are derived from the E_8 diagram, then any diagram which can be derived from one of these by an elementary transformation will also be a diagram derived from E_8. So, for example, we use the method of elementary transformations on E_7 to see that the following diagrams of rank 7 are derived from E_7, and thus, as E_7 is derived from E_8, these can also be derived from E_8. The following is a list of all the diagrams of rank 7 that can be derived from the diagram of E_7.

1. 

2.

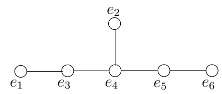
3.

4.

5. 

6.

Now, if we remove e_7 from the E_7 that we just found, we are left with



,

E_6. Thus E_6 can be derived from E_8, as well as all of the diagrams that are derived from E_6. We give the list of all the diagrams of rank 6 that can be derived from the diagram for E_6.

1.

2.

We note that we have not included the diagrams of rank n which can be derived from either D_n or A_n. This is because the only diagrams of rank n that can be derived from D_n are D_n and A_n, and the only diagram of rank n that can be derived from A_n is in fact A_n.

Although the process of determining the derived diagrams of a given diagram is not difficult, it is a bit tedious and time consuming, therefore, we do not do all of the possible elementary transformations, nor all of the node removals explicitly in this paper. But it is easy in this way to determine which of the Dynkin diagrams of rank less than or equal to 8 are derived from the diagram for E_8.

Theorem 10. *All diagrams associated with root lattices which embed in a given lattice can be found by deriving them from the given lattice's associated diagram.*

The proof of this theorem is found in Dynkin [3], Chapter II, Theorems 5.2 and 5.3, and is also not difficult to understand, but rather lengthy, and thus is not included. But from this we see that Theorem 5 is the complete list of root lattices that embed in the root lattice E_8.

5 Applications of Shioda Lattices

There are two major applications of Shioda lattices that we will consider. First, using the Shioda lattice, we can determine which configurations of singular fibres cannot exist on a rational elliptic surface. Second, the Shioda lattice can be used to determine the Mordell-Weil group associated with the surface.

5.1 Impossible Lattices and Configurations of Singular Fibres

Using the complete list of root lattices that embed in the root lattice E_8, we can deduce the root lattices of rank less than or equal to 8 which do not embed in E_8, as well as their associated configurations of singular fibres.

Table 5: Root lattices which do not embed in E_8

Root Lattice	Singular Fibres
$D_4 \oplus A_2 \oplus A_1$	$I_0^* \, IV \, III$
	$I_0^* \, IV \, I_2$
	$I_0^* \, III \, I_3$
	$I_0^* \, I_3 \, I_2$

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Root Lattice	Singular Fibres
$A_2 \oplus A_1^{\oplus 5}$	$IV\ III\ I_2\ I_2\ I_2\ I_2$
	$IV\ I_2\ I_2\ I_2\ I_2\ I_2$
	$III\ I_3\ I_2\ I_2\ I_2\ I_2$
	$III\ III\ I_3\ I_2\ I_2\ I_2$
	$I_3\ I_2\ I_2\ I_2\ I_2\ I_2$
$A_3 \oplus A_2^{\oplus 2}$	$IV\ IV\ I_4$
	$IV\ I_4\ I_3$
	$II\ I_4\ I_3\ I_3$
	$I_4\ I_3\ I_3\ I_1\ I_1$
$A_4 \oplus A_1^{\oplus 3}$	$III\ III\ I_5\ I_2$
	$III\ I_5\ I_2\ I_2$
	$I_5\ I_2\ I_2\ I_2\ I_1$
$A_2^{\oplus 2} \oplus A_1^{\oplus 3}$	$IV\ IV\ I_2\ I_2\ I_2$
	$IV\ III\ I_3\ I_2\ I_2$
	$IV\ I_3\ I_2\ I_2\ I_2$
	$III\ I_3\ I_3\ I_2\ I_2$
	$I_3\ I_3\ I_2\ I_2\ I_2$
$A_3 \oplus A_2^{\oplus 2} \oplus A_1$	$IV\ IV\ I_4\ I_2$
	$IV\ III\ I_4\ I_3$
	$IV\ I_4\ I_3\ I_2$
	$I_4\ I_3\ I_3\ I_2$
$A_4 \oplus A_3 \oplus A_1$	$III\ I_5\ I_4$

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Root Lattice	Singular Fibres
	$I_5\, I_4\, I_2\, I_1$
$A_4 \oplus A_2^{\oplus 2}$	$IV\; IV\; I_5$
	$IV\; I_5\, I_3$
	$I_5\, I_3\, I_3\, I_1$
$A_3^{\oplus 2} \oplus A_2$	$IV\; I_4\, I_4$
	$I_4\, I_4\, I_3\, I_1$
$A_6 \oplus A_2$	$IV\; I_7$
	$II\; I_7\, I_3$
	$I_7\, I_3\, I_1\, I_1$
$A_5 \oplus A_3$	$I_6\, I_4\, I_1\, I_1$
$A_6 \oplus A_1^{\oplus 2}$	$III\; III\; I_7$
	$III\; I_7\, I_2$
	$I_7\, I_2\, I_2\, I_1$
$A_4 \oplus A_2 \oplus A_1^{\oplus 2}$	$IV\; III\; I_5\, I_2$
	$III\; III\; I_5\, I_3$
	$IV\; I_5\, I_2\, I_2$
	$III\; I_5\, I_3\, I_2$
	$I_5\, I_3\, I_2\, I_2$
$A_2^{\oplus 3} \oplus A_1^{\oplus 2}$	$IV\; IV\; I_3\, I_2\, I_2$
	$IV\; III\; I_3\, I_3\, I_2$
	$III\; III\; I_3\, I_3\, I_3$
	$III\; III\; I_3\, I_3\, I_3$

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Root Lattice	Singular Fibres
	$IV\ I_3\ I_3\ I_2\ I_2$
	$III\ I_3\ I_3\ I_3\ I_2$
	$I_3\ I_3\ I_3\ I_2\ I_2$
$A_5 \oplus A_1^{\oplus 3}$	$III\ III\ I_6\ I_2$
	$III\ I_6\ I_2\ I_2$
	$I_6\ I_2\ I_2\ I_2$
$A_3 \oplus A_2 \oplus A_1^{\oplus 3}$	$IV\ III\ I_4\ I_2\ I_2$
	$III\ III\ I_4\ I_3\ I_2$
	$IV\ I_4\ I_2\ I_2\ I_2$
	$III\ I_4\ I_3\ I_2\ I_2$
	$I_4\ I_3\ I_2\ I_2\ I_2$
$A_4 \oplus A_1^{\oplus 4}$	$III\ III\ I_5\ I_2\ I_2$
	$III\ I_5\ I_2\ I_2\ I_2$
	$I_5\ I_2\ I_2\ I_2\ I_2$
$A_2^{\oplus 2} \oplus A_1^{\oplus 4}$	$IV\ IV\ I_2\ I_2\ I_2\ I_2$
	$IV\ III\ I_3\ I_2\ I_2\ I_2$
	$III\ III\ I_3\ I_3\ I_2\ I_2$
	$IV\ I_3\ I_2\ I_2\ I_2\ I_2$
	$III\ I_3\ I_3\ I_2\ I_2\ I_2$
	$I_3\ I_3\ I_2\ I_2\ I_2\ I_2$
$A_3 \oplus A_1^{\oplus 5}$	$III\ III\ I_4\ I_2\ I_2\ I_2$
	$III\ I_4\ I_2\ I_2\ I_2\ I_2$

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Root Lattice	Singular Fibres
	$I_4\, I_2\, I_2\, I_2\, I_2\, I_2$
$A_2 \oplus A_1^{\oplus 6}$	$IV\; III\; I_2\, I_2\, I_2\, I_2\, I_2$
	$IV\; I_2\, I_2\, I_2\, I_2\, I_2\, I_2$
	$III\; I_3\, I_2\, I_2\, I_2\, I_2\, I_2$
	$III\; III\; I_3\, I_2\, I_2\, I_2\, I_2$
	$I_3\, I_2\, I_2\, I_2\, I_2\, I_2\, I_2$
$D_4 \oplus A_3 \oplus A_1$	$I_0^*\; III\; I_4$
	$I_0^*\; I_4\, I_2$
$D_4 \oplus A_2 \oplus A_1^{\oplus 2}$	$I_0^*\; IV\; III\; III$
	$I_0^*\; IV\; I_2\, I_2$
	$I_0^*\; III\; I_3\, I_2$
	$I_0^*\; I_3\, I_2\, I_2$
$D_4 \oplus A_2^{\oplus 2}$	$I_0^*\; IV\; IV$
	$I_0^*\; IV\; I_3$
	$I_0^*\; I_3\, I_3$
$D_5 \oplus A_2 \oplus A_1$	$I_1^*\; IV\; III$
	$I_1^*\; IV\; I_2$
	$I_1^*\; III\; I_3$
	$I_1^*\; I_3\, I_2$
$D_5 \oplus A_1^{\oplus 3}$	$I_1^*\; III\; III\; III$
	$I_1^*\; III\; I_2\, I_2$
	$I_1^*\; I_2\, I_2\, I_2$

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Root Lattice	Singular Fibres
$D_6 \oplus A_2$	$I_2^* \, IV$
	$I_2^* \, I_3$
$D_7 \oplus A_1$	$I_3^* \, III$
	$I_3^* \, I_2$
$E_6 \oplus A_1^{\oplus 2}$	$IV^* \, III \, III$
	$IV^* \, III \, I_2$
	$IV^* \, I_2 \, I_2$

An additive singularity of type II or a multiplicative singularity of type I_1 can be added to any of these configurations, without changing the Shioda lattice as these fibres have associated root lattice $\{0\}$. Any singular fibre can be added to the above configurations, resulting in a configuration which is still not possible, as any lattice containing one of these can not embed in E_8. Also, some of the configurations listed may not be possible because the sum of the multiplicities of the singularities is greater than twelve, which, as mention previously, is not possible.

5.2 Lattices Arising in Characteristics 0, 2 and 3

Although all of the lattices of type A_n, D_n and E_n that embed in E_8 have an associated configuration of singular fibres for a rational elliptic surface, not all of these configurations actually occur, and thus not all of these lattices occur. Table 6 lists all the lattices which embed in E_8 and the appearance of at least one corresponding configuration of singular fibres in characteristics 0, 2 and

3. These lattices were determined from the existence of surfaces with a given configuration of singular fibres, given by Lang [5], Persson [7], Petrosyan and Summers [8], and Rogers [9].

Table 6: Shioda Lattices and Their Occurrences

Lattice	Char 0	Char 2	Char 3
$\{0\}$	✓	✓	✓
A_1	✓	✓	✓
A_2	✓	✓	✓
$A_1^{\oplus 2}$	✓	✓	✓
A_3	✓	✓	✓
$A_2 \oplus A_1$	✓	✓	✓
$A_1^{\oplus 3}$	✓	✓	✓
D_4	✓	✓	✓
A_4	✓	✓	✓
$A_3 \oplus A_1$	✓	✓	✓
$A_2^{\oplus 2}$	✓	✓	✓
$A_2 \oplus A_1^{\oplus 2}$	✓	✓	✓
$A_1^{\oplus 4}$	✓	✓	✓
D_5	✓	✓	✓
$D_4 \oplus A_1$	✓	✓	✓
A_5	✓	✓	✓
$A_4 \oplus A_1$	✓	✓	✓
$A_3 \oplus A_2$	✓	✓	✓

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Lattice	Char 0	Char 2	Char 3
$A_3 \oplus A_1^{\oplus 2}$	✓	✓	✓
$A_2^{\oplus 2} \oplus A_1$	✓	✓	✓
$A_2 \oplus A_1^{\oplus 3}$	✓	✓	✓
$A_1^{\oplus 5}$	✓	✓	✓
E_6	✓	✓	✓
D_6	✓	✓	✓
$D_5 \oplus A_1$	✓	✓	✓
$D_4 \oplus A_2$	✓	✓	✓
$D_4 \oplus A_1^{\oplus 2}$	✓	✓	✓
A_6	✓	✓	✓
$A_5 \oplus A_1$	✓	✓	✓
$A_4 \oplus A_2$	✓	✓	✓
$A_4 \oplus A_1^{\oplus 2}$	✓	✓	✓
$A_3^{\oplus 2}$	✓	✓	✓
$A_3 \oplus A_2 \oplus A_1$	✓	✓	✓
$A_3 \oplus A_1^{\oplus 3}$	✓	✓	✓
$A_2^{\oplus 3}$	✓	✓	✓
$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	✓	✓	✓
$A_2 \oplus A_1^{\oplus 4}$	✓	✓	✓
$A_1^{\oplus 6}$	✓		✓
E_7	✓	✓	✓
$E_6 \oplus A_1$	✓	✓	✓

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Lattice	Char 0	Char 2	Char 3
D_7	✓	✓	✓
$D_6 \oplus A_1$	✓	✓	✓
$D_5 \oplus A_2$	✓	✓	✓
$D_5 \oplus A_1^{\oplus 2}$	✓	✓	✓
$D_4 \oplus A_3$	✓	✓	✓
$D_4 \oplus A_1^{\oplus 3}$	✓		✓
A_7	✓	✓	✓
$A_6 \oplus A_1$	✓	✓	✓
$A_5 \oplus A_2$	✓	✓	✓
$A_5 \oplus A_1^{\oplus 2}$	✓		✓
$A_4 \oplus A_3$	✓	✓	✓
$A_4 \oplus A_2 \oplus A_1$	✓	✓	✓
$A_3^{\oplus 2} \oplus A_1$	✓		✓
$A_3 \oplus A_2 \oplus A_1^{\oplus 2}$	✓	✓	✓
$A_3 \oplus A_1^{\oplus 4}$	✓		✓
$A_2^{\oplus 3} \oplus A_1$	✓	✓	✓
$A_1^{\oplus 7}$			
E_8	✓	✓	✓
$E_7 \oplus A_1$	✓	✓	✓
$E_6 \oplus A_2$	✓	✓	✓
D_8	✓	✓	✓
$D_6 \oplus A_1^{\oplus 2}$	✓		✓

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Lattice	Char 0	Char 2	Char 3
$D_5 \oplus A_3$	✓	✓	
$D_4^{\oplus 2}$	✓		✓
$D_4 \oplus A_1^{\oplus 4}$			
A_8	✓	✓	✓
$A_7 \oplus A_1$	✓	✓	✓
$A_5 \oplus A_2 \oplus A_1$	✓	✓	✓
$A_4^{\oplus 2}$	✓	✓	✓
$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	✓		✓
$A_2^{\oplus 4}$	✓	✓	
$A_1^{\oplus 8}$			

We see that the lattices $A_1^{\oplus 7}$, $D_4 \oplus A_1^{\oplus 4}$ and $A_1^{\oplus 8}$ do not occur in any characteristic. This is simply because the sum of the multiplicities of the fibres is greater than 12, which is not possible for a rational elliptic surface. For this same reason $D_5 \oplus A_3$ does not occur in characteristic 3. $A_5 \oplus A_1^{\oplus 2}$ and $A_3^{\oplus 2} \oplus A_1$ are proven not to exist in characteristic two using computational methods by Lang [5], but there seem to be no lattice related reason for their failure to exist. The remaining cases that do not occur the characteristics 2 and 3 fail to do so because of the Mordell-Weil group, which we discuss in the next section.

5.3 The Mordell-Weil Group and Lattice

Once the Shioda lattice is known, the following theorem of Oguiso and Shioda [6] can be applied to determine the Mordell-Weil group and the Mordell-Weil lattice of a rational elliptic surface.

Theorem 11. *Given a rational elliptic surface $\pi : \mathcal{E} \to \mathbb{P}^1$ with section, let Λ be the Shioda lattice associated with the reducible fibres, which is a sublattice of E_8 such that the rank of Λ is n, as defined in Section 3. Let $L = \Lambda^{\perp}$ be the orthogonal complement of Λ in E_8, $\Lambda' = \Lambda \otimes \mathbb{Q} \cap E_8$ be the primitive closure of Λ in E_8, and $L^* = \{x \in \Lambda \otimes \mathbb{Q} | \langle x, y\rangle \in \mathbb{Z}$ for all $y \in \Lambda\}$ be the dual of L. Then we have*

1. $r = rank(E(K)) = 8 - n$
2. $E(K)/E(K)_{tor} \simeq L^*$
3. $E(K) \simeq L^* \oplus (T'/T)$

Here $E(K)$ the Mordell-Weil group of $\mathcal{E}$ and $E(K)/E(K)_{tor}$ the Mordell-Weil lattice of $\mathcal{E}$, and the isomorphisms in (2) and (3) take the height pairing on the left to the standard one on the right.

Thus, if the Shioda lattice Λ can be determined for a given rational elliptic surface, then the Mordell-Weil group and lattice can also be determined. In order to do this, we must determine the orthogonal complement $T^{\perp} = L$ and primitive closure T' of the Shioda lattice, as well as the dual of the orthogonal complement, L^* .

To determine the orthogonal complement of the Shioda lattice we must know how it embeds into E_8. This is, in general, an easy thing to do. We simply need to know how we derived a given diagram from elementary transformations and node removals. This will tell us our basis elements for the Shioda lattice with respect to the basis elements of E_8. Knowing this we are then able to determine the orthogonal complement of the Shioda lattice, its dual and the primitive closure of the lattice.

For example, as mentioned earlier, in order to get an E_6, we simply remove the nodes 7 and 8 from the E_8 diagram. But, we could also remove the node 7 from the extended diagram of E_8, and this would leave us with $E_6 \oplus A_2$. Clearly this A_2 is orthogonal to E_6, and together they span E_8, thus the orthogonal complement of E_6 is A_2 as E_6 embeds into E_8 in this way. If the root lattices can be embedded into E_8 in a manner similar to this, then it is easy to determine their orthogonal complements in E_8. However, not all of the lattices embed in such a way. Take for example A_7, which can be embedded into E_8 in two different ways. First, as we see from the list of diagrams that are derived by elementary transformations of E_8, $A_7 \oplus A_1$ is such a transformation. Using this embedding it is easy to see that the orthogonal complement of A_7 is A_1. But A_7 can also be embedded into E_8 simply as a subdiagram of E_8, derived by removing node 2 from the E_8 diagram. Thus this embedding of A_7 has for a basis the elements $\{e_1, e_3, e_4, e_5, e_6, e_7, e_8\}$ of E_8, and no clear orthogonal complement. To determine the orthogonal complement of A_7 embedded in this way, we look at the set of all elements of E_8 which have inner product zero with every element of this A_7. So, $A_7^{\perp} = \{\beta \in E_8 | \langle \alpha, \beta \rangle = 0 \text{ for all } \alpha \in A_7\}$. Elements of E_8 are of the form $\beta = b_1e_1 + b_2e_2 + \ldots + b_8e_8$, and elements of this embedding of A_7 are of the form $\alpha = a_1e_1 + a_3e_3 + a_4e_4 + \ldots + a_8e_8$. We also know that the orthogonal complement must have rank 1 in this case, so we solve $\langle \alpha, \beta \rangle = 0$ where α and β are in the form described, looking for one element. We see that the element is completely determined by b_3, and is of the form $\beta = (3, 4, 6, 5, 4, 3, 2, 1)b_3$, and the smallest of such elements of $A_7^{\perp}$ has norm 8. Thus we see that the orthogonal complement of A_7 is a root with a norm of 8, which we will simply denote by $\langle 8 \rangle$. Then, from the definition of dual, it is clear that the dual lattice to this lattice is $\langle 1/8 \rangle$, which lattice is the Mordell-Weil lattice. And, since A_7 is primitive, there is no torsion in the Mordell-Weil group, so the Mordell-Weil group is the free Abelian group generated by this one element.

From Oguiso and Shioda [6] we have the following table of Mordell-Weil groups resulting from the Shioda lattices, from which groups it is easy to determine the Mordell-Weil lattices. The Mordell-Weil groups that are denoted by n-by-n matrices in the table are those groups with n generators, and the Gram matrix of the associated lattice is the given matrix.

Table 7: Shioda Lattices and Mordell-Weil Groups

Shioda Lattice	Mordell-Weil Group
$\{0\}$	E_8
A_1	E_7^*
A_2	E_6^*
$A_1^{\oplus 2}$	D_6^*
A_3	D_5^*
$A_2 \oplus A_1$	A_5^*
$A_1^{\oplus 3}$	$D_4^* \oplus A_1^*$
A_4	A_4^*
D_4	D_4^*
$A_3 \oplus A_1$	$A_3^* \oplus A_1^*$
$A_2^{\oplus 2}$	$A_2^{*\oplus 2}$

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Shioda Lattice	Mordell-Weil Group
$A_2 \oplus A_1^{\oplus 2}$	$\frac{1}{6}\begin{bmatrix} 2 & 1 & 0 & -1 \\ 1 & 5 & 3 & 1 \\ 0 & 3 & 6 & 3 \\ -1 & 1 & 3 & 5 \end{bmatrix}$
$A_1^{\oplus 4}$	$D_4^* \oplus \mathbb{Z}/2\mathbb{Z}$
$A_1^{\oplus 4}$	$A_1^{*\oplus 4}$
A_5	$A_2^* \oplus A_1^*$
D_5	A_3^*
$A_4 \oplus A_1$	$\frac{1}{10}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 7 & 3 \\ -1 & 3 & 7 \end{bmatrix}$
$D_4 \oplus A_1$	$A_1^{*\oplus 3}$
$A_3 \oplus A_2$	$\frac{1}{12}\begin{bmatrix} 7 & 1 & 2 \\ 1 & 7 & 2 \\ -1 & 3 & 7 \end{bmatrix}$
$A_2^{\oplus 2} \oplus A_1$	$A_2^* \oplus \langle 1/6 \rangle$
$A_3 \oplus A_1^{\oplus 2}$	$A_3^* \oplus \mathbb{Z}/2\mathbb{Z}$
$A_3 \oplus A_1^{\oplus 2}$	$A_1^{*\oplus 2} \oplus \langle 1/4 \rangle$
$A_2 \oplus A_1^{\oplus 3}$	$A_1^* \oplus \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$
$A_1^{\oplus 5}$	$A_1^{*\oplus 3} \oplus \mathbb{Z}/2\mathbb{Z}$

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Shioda Lattice	Mordell-Weil Group
A_6	$\frac{1}{7}\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}$
D_6	$A_1^{*\oplus 2}$
E_6	A_2^*
$A_5 \oplus A_1$	$A_2^* \oplus \mathbb{Z}/2\mathbb{Z}$
$A_5 \oplus A_1$	$A_1^* \oplus \langle 1/6 \rangle$
$D_5 \oplus A_1$	$A_1^* \oplus \langle 1/4 \rangle$
$A_4 \oplus A_2$	$\frac{1}{15}\begin{bmatrix} 2 & 1 \\ 1 & 8 \end{bmatrix}$
$D_4 \oplus A_2$	$A_1^* \oplus \langle 1/6 \rangle$
$A_4 \oplus A_1^{\oplus 2}$	$\frac{1}{10}\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$
$D_4 \oplus A_1^{\oplus 2}$	$A_1^{*\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$
$A_3^{\oplus 2}$	$A_1^{*\oplus 2} \oplus \mathbb{Z}/2\mathbb{Z}$
$A_3^{\oplus 2}$	$\langle 1/4 \rangle^{\oplus 2}$
$A_3 \oplus A_2 \oplus A_1$	$A_1^* \oplus \langle 1/12 \rangle$
$A_3 \oplus A_1^{\oplus 3}$	$A_1^* \oplus \langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$
$A_2^{\oplus 3}$	$A_2^* \oplus \mathbb{Z}/3\mathbb{Z}$
$A_2^{\oplus 2} \oplus A_1^{\oplus 2}$	$\langle 1/6 \rangle^{\oplus 2}$
$A_2 \oplus A_1^{\oplus 4}$	$\frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \oplus \mathbb{Z}/2\mathbb{Z}$

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Shioda Lattice	Mordell-Weil Group
$A_1^{\oplus 6}$	$A_1^{*\oplus 2} \oplus (\mathbb{Z}/2\mathbb{Z})^2$
E_7	A_1^*
A_7	$A_1^* \oplus \mathbb{Z}/2\mathbb{Z}$
A_7	$\langle 1/8 \rangle$
D_7	$\langle 1/4 \rangle$
$A_6 \oplus A_1$	$\langle 1/14 \rangle$
$D_6 \oplus A_1$	$A_1^* \oplus \mathbb{Z}/2\mathbb{Z}$
$E_6 \oplus A_1$	$\langle 1/6 \rangle$
$D_5 \oplus A_2$	$\langle 1/12 \rangle$
$A_5 \oplus A_2$	$A_1^* \oplus \mathbb{Z}/3\mathbb{Z}$
$D_5 \oplus A_1^{\oplus 2}$	$\langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$
$A_5 \oplus A_1^{\oplus 2}$	$\langle 1/6 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$
$D_4 \oplus A_3$	$\langle 1/4 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$
$A_4 \oplus A_3$	$\langle 1/20 \rangle$
$A_4 \oplus A_2 \oplus A_1$	$\langle 1/30 \rangle$
$D_4 \oplus A_1^{\oplus 3}$	$A_1^* \oplus (\mathbb{Z}/2\mathbb{Z})^2$
$A_3^{\oplus 2} \oplus A_1$	$A_1^* \oplus \mathbb{Z}/4\mathbb{Z}$
$A_3 \oplus A_2 \oplus A_1^{\oplus 2}$	$\langle 1/12 \rangle \oplus \mathbb{Z}/2\mathbb{Z}$
$A_3 \oplus A_1^{\oplus 4}$	$\langle 1/4 \rangle \oplus (\mathbb{Z}/2\mathbb{Z})^2$
$A_2^{\oplus 3} \oplus A_1$	$\langle 1/6 \rangle \oplus \mathbb{Z}/3\mathbb{Z}$
E_8	$\{0\}$
A_8	$\mathbb{Z}/3\mathbb{Z}$

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Shioda Lattice	Mordell-Weil Group
D_8	$\mathbb{Z}/2\mathbb{Z}$
$E_7 \oplus A_1$	$\mathbb{Z}/2\mathbb{Z}$
$A_5 \oplus A_2 \oplus A_1$	$\mathbb{Z}/6\mathbb{Z}$
$A_4^{\oplus 2}$	$\mathbb{Z}/5\mathbb{Z}$
$A_2^{\oplus 4}$	$(\mathbb{Z}/3\mathbb{Z})^2$
$E_6 \oplus A_2$	$\mathbb{Z}/3\mathbb{Z}$
$A_7 \oplus A_1$	$(\mathbb{Z}/2\mathbb{Z})^2$
$D_6 \oplus A_1^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^2$
$D_5 \oplus A_3$	$\mathbb{Z}/4\mathbb{Z}$
$D_4^{\oplus 2}$	$(\mathbb{Z}/2\mathbb{Z})^2$
$A_3^{\oplus 2} \oplus A_1^{\oplus 2}$	$\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$

We note that the lattices $A_1^{\oplus 7}$, $D_4 \oplus A_1^{\oplus 4}$ and $A_1^{\oplus 8}$ are not included in the table as the sum of the multiplicities of the singular fibres exceeds 12, which, as we mentioned, is not possible.

The remaining configurations from Table 6 that fail to exist do so because of the Mordell-Weil group of the surface for the given configurations. From Silverman [11], Corollary 6.4, we see that in characteristic 2, the Mordell-Weil group cannot contain a subgroup isomorphic to $\mathbb{Z}/2^k\mathbb{Z} \oplus \mathbb{Z}/2^l\mathbb{Z}$ for $k, l \in \mathbb{N}^+$. Similarly, in characteristic 3, the Mordell-Weil group cannot contain a subgroup isomorphic to $\mathbb{Z}/3^k\mathbb{Z} \oplus \mathbb{Z}/3^l\mathbb{Z}$ for $k, l \in \mathbb{N}^+$. Thus the lattices $D_4^{\oplus 2}$, $A_1^{\oplus 6}$, $D_4 \oplus A_1^{\oplus 3}$, $A_3 \oplus A_1^{\oplus 4}$, $A_7 \oplus A_1$, $D_6 \oplus A_1^{\oplus 2}$ and $A_3^{\oplus 2} \oplus A_1^{\oplus 2}$ do not exist in characteristic two, and $A_2^{\oplus 4}$ does not exist in characteristic three.

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