2021-06-14

Developing the Definite Integral and Accumulation Function Through Adding Up Pieces: A Hypothetical Learning Trajectory

Brinley Nichole Stevens
Brigham Young University

Developing the Definite Integral and Accumulation Function Through

Adding Up Pieces: A Hypothetical Learning Trajectory

Brinley Nichole Stevens

A thesis submitted to the faculty of
Brigham Young University
in partial fulfillment of the requirements for the degree of

Master of Science

Steven R. Jones, Chair
Steve R. Williams
Douglas L. Corey

Department of Mathematics Education

Brigham Young University

ABSTRACT

Developing the Definite Integral and Accumulation Function Through
Adding Up Pieces: A Hypothetical Learning Trajectory

Brinley Nichole Stevens
Department of Mathematics Education, BYU
Master of Science

Integration is a core concept of calculus. As such, significant work has been done on understanding how students come to reason about integrals, including both the definite integral and the accumulation function. A path towards understanding the accumulation function first, then the definite integral as a single point on the accumulation function has been presented in the literature. However, there seems to be an accessible path that begins first with understanding the definite integral through an Adding Up Pieces (AUP) perspective and extending that understanding to the accumulation function.

This study provides a viable hypothetical learning trajectory (HLT) for beginning instruction with an AUP perspective of the definite integral and extending this understanding to accumulation functions. This HLT was implemented in a small-scale teaching experiment that provides empirical data for the type of student reasoning that can occur through the various learning activities. The HLT also appears to be a promising springboard into developing the Fundamental Theorem of Calculus. Additionally, this study offers a systematic framework for understanding the process- and object- level thinking that occurs at different layers of integration.

Keywords: calculus, integration, adding up pieces, hypothetical learning trajectory

ACKNOWLEDGMENTS

I would first like to thank my wonderful husband, Tristan, for his constant support. He has been my rock through it all and constantly encouraged me to reach my goals. I also want to thank Emily Adams for being the best study partner, sounding board, and friend throughout graduate school. I thank the rest of my family and friends who have helped me through this wild adventure that is writing a thesis study during a pandemic.

I thank the professors who encouraged me to pursue a master's degree and supported me during my time at BYU, particularly Dr. Dan Siebert and Professor Sharon Christensen. Finally, I especially thank my advisor, Dr. Steven Jones. His patience and guidance allowed me to find research I was passionate about and create something I am excited to share.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER ONE: RATIONALE

Cases have been made by many researchers that more classroom instruction should be focused on quantitative reasoning (Thompson, 2011; Johnson, 2016; Steen, 2001; Thompson & Carlson, 2017; Moore & Carlson, 2012). There appears to be a lack of quantitative and covariational reasoning in students across mathematics. These skills and attention to quantity are key aspects of mathematics that allow students to bridge the gap between the classroom and their real-world experience (Thompson, 2011). One subset of mathematics students who can greatly benefit from an increase in quantitative reasoning is calculus students. Bressoud et al. (2013) showed that about 78% of calculus students major in engineering, biology, physics, computer science, or business. However, research has shown that calculus students, much like other students, show a lack of quantitative reasoning in thinking about topics such as rates, limits, and integration (Thompson, 1994; Carlson et al., 2002; Oehrtman, 2009; Jones, 2015b). Quantitative reasoning might provide the richness in mathematical understanding, rather than solely procedural knowledge, that these students need in their respective fields (Smith & Thompson, 2008).

One of the most important topics in calculus is integration—its usefulness extending beyond math into chemistry, biology, business, physics, statistics, and engineering. However, when students use the typical notions of areas and antiderivatives, they have been shown to struggle to make sense of integration problems in real-world contexts (Jones, 2015a). Research has been done on ways students can reason about integration more quantitatively in these contextualized problems. Jones (2015a) showed that students attend to the meaning of an integral more successfully in contextual problems when viewing the integral as "adding up pieces" (AUP), meaning that the integral represents the summation of infinitely many small pieces of a quantity. This quantity is often created through the product of two other quantities. Jones (2015b) also found that students do not often think about these products when making sense of integrals. They more often reason about integrals as areas under curves or as antiderivatives. While not incorrect conceptualizations of integration, they proved less productive in sense-making (Jones 2015a).

Because of the importance of the “adding up pieces” way of viewing definite integrals, much work has been done to understand how students can conceptualize the definite integral quantitatively through AUP. Jones (2013; in press) has explored lessons on developing the AUP conception when introducing integrals, and Ely (2017) and Oehrtman (Chhetri & Oehrtman, 2015; Simmons & Oehrtman, 2019) have examined how to help students reason about multiple types of contexts with AUP. However, it is important to note that AUP work has focused typically on definite integrals with fixed bounds, $\int_a^b f(t)dt$. But there is an entirely other type of integral that is critical for certain contexts and the Fundamental Theorem of Calculus: accumulation function integrals with a variable upper bound, $g(x) = \int_a^x f(t)\, dt$. There has not been sufficient research done on how to extend the AUP quantitative understanding to accumulation functions.

Some work has focused on accumulation functions by having students learn them before definite integrals with fixed bounds. Yerushalmy and Swidan (2011) examined students’ semiotic meanings for the lower boundary of accumulation functions using an interactive program that allowed students to modify the bounds and argument of the integral and graphed the corresponding accumulation graph. This was done with little interaction from a teacher and only focused on initial student approaches. Also, the students only had an understanding of derivatives and had not had any previous instruction about integration. While their work has illuminated many challenges students face with accumulation functions, it does not address the potential benefit of attending to quantities as a way of making sense of the accumulation function. By contrast, Thompson and Silverman (2008) have researched how to develop the idea of accumulation quantitatively *before* applying the definite integral, with the definite integral only being the accumulation function evaluated at a specific point. The issue is that there appears to be a very high initial cognitive demand in constructing accumulation functions first. Further, Thompson and colleagues have presented a method for developing accumulation based on rate, and then imagining the definite integral as this accumulation evaluated at one point. However, since many definite integrals might not easily be conceived of as a rate (such as the density integral given above), AUP may still be an important way to develop the ideas of integrals. This leads me to believe that perhaps there is a way to first develop profound AUP understandings from contextual problems which could then be extended to accumulation functions. Because research has not yet described how AUP for definite integrals can be extended to accumulation functions, I examine in this study the possible benefits of quantitatively developing the definite integral through AUP and then extending them to accumulation functions.

The purpose of my study is twofold: a) to use existing literature to create a hypothetical learning trajectory (HLT) which builds the definite integral quantitatively and extends this quantitative understanding to the idea of accumulation and b) to answer the research question: as a student progresses through the HLT, what understandings do they develop of the definite integral and accumulation function? The first of these purposes has been achieved in preparation for my interviews, while the second is answered through my analysis of student thinking during the teaching experiments.

CHAPTER TWO: BACKGROUND

In this chapter, I review the existing literature on quantitative reasoning and integration. I start with reviewing research on quantitative reasoning and then review research on student understanding of definite integrals and accumulations functions. I then discuss my theoretical framework of Sfard's (1991) processes and objects. Lastly, I provide a conceptual breakdown of the target process-object understandings I want to elicit with the hypothetical learning trajectory.

Quantitative Reasoning

What is Quantitative Reasoning?

Smith and Thompson (2008) define quantity as an attribute of an object or system that could be measured. For example, "length" is a quantity because it measures the amount of linear space between a system of two objects. Thompson (2011) similarly points out that quantities are "mental constructions" (p. 34), meaning that quantities are not self-existent in the world but are products of our attempts at understanding the world around us.

Moore and Carlson (2012) build on Thompson and Smith's work (Smith & Thompson, 2008; Thompson, 2011) to define quantitative reasoning as "the process of analyzing a situation in terms of quantities and relationships among them" (p. 49). Therefore, quantitative reasoning includes not only identifying or conceiving of any relevant quantities in a problem, but also assessing how those quantities relate to each other. To illustrate this, I draw on an example of a student who lacked quantitative reasoning. Moore and Carlson (2012) presented students with what they called the "box problem," which involved finding the equation for the formula of a box created by cutting equal-sized squares from the corners of an 11-inch by 13-inch sheet of paper. One student, Matt, used a piece of paper to illustrate (to himself and the interviewer) how the quantities of cutout side length and the dimensions of the box change together. He made two important observations about quantitative structure: as the size of the cutout increases the box becomes less wide and at the same time becomes "deeper." When he went to create a formula for the volume of the box, he mistakenly wrote $V = 13 * 11 * x$ where x represented the side length of the squares removed from the corners. However, because he had attended to the quantities, he was able to see that his formula would not work. He used specific values for x to help himself translate the quantitative relationship between x and the dimensions of the box into a formula for the volume. Matt was able to determine that the size of x needed to be removed from both the length and width of the paper to fold the box. He refined his formula and ended with the correct equation.

Using Quantitative Reasoning

Quantitative reasoning can be utilized throughout many levels of mathematics. The literature on quantitative reasoning includes topics such as rate of change (Thompson & Carlson, 2017), trigonometry (Moore, 2012; 2014), integration (Thompson, 1994; Ely, 2017; Jones, 2015a), and function (Moore et al., 2014; Ellis, 2011; Smith & Thompson, 2008; Moore & Paoletti, 2013). Because of the vast amount of literature on this subject, I choose to focus on two applications of quantitative reasoning. The first is an example within trigonometry from Moore (2014), the second is Thompson and Carlson's (2017) work on covariation.

Moore (2014) demonstrated the utility of quantitative reasoning within trigonometry. He presented a case study of a student, Zac, who was building meaning for the sine function. After two teaching sessions of developing the concept of angle measure, Moore introduced a problem involving a bug sitting on the end of a counterclockwise-revolving fan blade. Zac's task was to create a graph of the bug's vertical distance above the 9:00 to 3:00 diameter line through the center of the fan.

Zac initially based his reasoning and graph on physical features of the scenario, such as the curvature of the fan, without attending to quantity. He created the sine graph, but his reasoning did not reflect rates of change. Moore presented Zac with an alternative graph, with the same intervals of increasing and decreasing as $sin(x)$, but with constant rates of change. Once Zac saw this graph, he began to reason about the changing rates of change as the fan revolves and was able to explain why the rates of $sin(x)$are not constant. He started to compare the changes in the bug's vertical distance from the 3:00 position for constant changes in the arc length. Attending to the size of these changes in vertical distance allowed Zac to explain why the rates of change had to change, and that for the first quarter rotation of the fan the rate of change decreased. Zac saw that as the fan approached the 12:00 position, the vertical distance changed very little, explaining why the graph should be concave down. Moore then changed the size of the fan blades so Zac began to see the radius as a unit he could use to measure distances. Through Moore's careful questioning and encouragement to focus on the quantities in context, Zac built a deeper conceptual understanding of the sine graph. Zac was able to see the covariational relationship between arc length and vertical distance because he was focused on the relative sizes of the vertical distance quantity.

Thompson and Carlson (2017) expanded the work on the application of quantitative reasoning to covariational reasoning. They developed a framework for levels of covariational reasoning which focuses on quantity and how quantities change simultaneously. For example, students may be able to envision quantities that increase together but may not yet see these as happening simultaneously. To illustrate this, consider the Bottle Problem from Carlson (1998). Carlson presented students with an image of a bottle and asked them to graph the height of the water in the bottle as a function of the volume of water in the bottle. Someone who is not attending to both quantities of height and volume simultaneously may imagine a certain volume being added, then the height increasing rather than increasing as the volume is added. The ability to coordinate quantities varying together is a key idea when developing meanings for slope and rate of change (Thompson & Carlson, 2017). Seeing rate of change not only as a slope of a line but as the relative changing of size or measure between two quantities together gives more conceptual meaning to a rate of change.

Integration

Conceptions of the Definite Integral

Jones (2013) identified different conceptualizations of integrals which students use to make sense of the meaning of integration problems. These include a conception of "perimeter and area," "function matching," and "adding up pieces" (AUP).

In function matching, the integral denotes an antiderivative of a function. Having a conception of integrals as antiderivatives means that students try to create meaning for the integral in terms of the antiderivative of the function in the integral. They might refer to velocity and position in terms of their explanations, or talk about rates of things (Jones, 2013). While using antiderivatives to solve integrals is a critical application of the Fundamental Theorem of Calculus, it is often not useful in making sense of what the answer would mean in context. Jones (2015a) found that students were less confident in making sense of problems in terms of antiderivatives. They tried to reason about the units of the function compared to the antiderivative, but without using a multiplicative comparison they struggled to find real-world meaning in the symbols.

In perimeter and area, the integral denotes the area under a curve. Relating the integral to the area between the function and the x-axis is a correct graphical interpretation but is also less helpful when explaining values in context. Jones (2015a) found that students using this method could draw a picture and attempted to explain the meaning in terms of units, but often felt frustrated or insecure about their explanations. This conceptualization of the area under a curve differs from AUP because the student only conceptualizes the integral as the undivided area of some shape bounded by the x-axis, the function, and the upper and lower bounds. They do not split this area up into pieces.

In the AUP conception, students imagine a sum of infinitesimally small quantities, which is similar in structure to the Riemann sum. Jones (2013) found that when students reasoned about the multiplication between an infinitesimally small amount of the quantity represented by the differential change in the domain and the quantity represented by the integrand function, they were able to make sense of integrals in context much more productively (Jones, 2015a). To break it down, AUP involves a student "chopping" an interval into small pieces, finding the quantity of interest within each chopped piece, and then adding the quantitative pieces up (Jones, in press).

While AUP often involves a product between a function and a differential, this is not always necessary. Ely (2017) gives an example of the arc length formula, in the form $\int_a^b \sqrt{dx^2 + dy^2}$. Each small piece here represents a tiny distance, the hypotenuse of the triangle formed by dx and dy. Adding these pieces up would give the length of a given line segment.

To illustrate the differences in these three conceptualizations of integrals, consider the integral $\int_a^b \pi[f(x)]^2 dx$, which calculates the volume created by rotating a function $f(x)$ in the interval [a,b] around the x-axis. A student who thinks of integrals as antiderivatives might try to find the antiderivative of $\pi[f(x)]^2$, but this does not have a clear connection to calculating a volume created by $f(x)$. Alternatively, thinking about the integral as the area under the curve $\pi[f(x)]^2$seems strange as well since we are interested in a volume rather than an area. Why would the area under the graph of $\pi[f(x)]^2$correspond to the volume of an object? However, an AUP perspective would begin by chopping up the interval [a,b]. In chopping up the interval, you are left with small cylindrical slivers of volume, each with a radius length of the function at that point and a height of the thickness of your "chops." The volume of each chopped piece can be represented as $\pi(x^2)^2 * dx$, as dx would necessarily be the thickness of the chops. Then adding up the pieces of volume within each section would give the total volume of the entire solid. This creates a three-dimensional volume. The integral takes every small piece of volume and adds them together, giving us the overall volume. While all three conceptualizations are valid, AUP has a much stronger connection to the context of the problem.

The often-multiplicative nature of AUP is closely tied to the structure of Riemann sums; however, explicit attention to Riemann sums in instruction will not necessarily increase students' tendency to use an AUP perspective (Jones, Lim, & Chandler, 2016). Teachers may undermine their student's abilities to reason multiplicatively by reducing the idea of Riemann sums to an approximation tool. Jones, Lim, and Chandler (2016) observed teachers who presented the integral as the area under a curve and used Riemann sums to calculate the area. However, they did not use the Riemann sum as a way to build potential conceptual meaning for the definite integral. Rather, the Riemann sum was just a calculational tool to estimate the "real" meaning of integrals: area under the curve.

While AUP is the most productive conception of the integral in contextualized problems, of the three conceptualizations discussed, students are far more likely to think of integrals as antiderivatives or as areas under a curve (Jones, 2015b). Integrals are often defined first by teachers as the area under a curve, which is why many students may gravitate towards this conceptualization (Stewart, 2016; Jones 2015b). Jones, Lim, and Chandler (2016) showed that even when teachers introduced integrals first with Riemann sums, the teachers undermined this instruction by emphasizing Riemann sums only as a calculational device that would soon be exchanged for a quicker technique as soon as the class learned the Fundamental Theorem of Calculus.

The disconnect between integration and the multiplication of quantities may also be because students are not viewing the function and differential pieces within an integral both as quantities that can be multiplied. Mathematicians themselves have a difficult time giving a definitive answer to what a differential means (McCarty & Sealey, 2019). When asked what the differential means in different contexts, they gave many different explanations, as reported in McCarty and Sealey (2019). A common response in relation to integration was the differential was simply a marker to indicate the variable to integrate with respect to. This view of differentials in integration does not attend to the multiplication of quantities and makes the differential seem more like a bookend than a critical piece of the integral. Ely (2017) provides a more useful approach to the differential, treating it as an infinitesimally small piece much like Leibniz did.

When conceptualizing the differential in an integral as an "infinitesimally small" piece of the domain, the dx begins to have a quantitative meaning. This is illustrated in Jones (2015a), as students made sense of the integral $\int_R \rho(r)\, dV$, where R is a three-dimensional object and ρ is its density at any point. A productive conceptualization of this integral included recognizing dV as a small piece of volume, multiplied by a density. The resulting product would be a small bit of mass and adding these pieces up would give the mass of the object. This attends to the quantities of volume and density and was critical for understanding the contextual meaning of the integral. Students who did not attend to the quantities were unable to explain that this integral calculated the mass of the given object, or why it calculated mass. This demonstrates the potential benefit an AUP approach would have in students' sense-making abilities.

Frameworks for the Decomposition of Integrals

Sealey (2014) created a framework (Figure 1) for an understanding of integrals which was modeled after Zandieh's (2000) framework decomposing derivatives. Sealey (2014) broke down integration into four layers: a) product, b) summation, c) limit, and d) function. She later added a preliminary step called the "orienting layer," where students make sense of the problem and its relevant quantities. Within these layers, she analyzed student thinking during three different learning activities. These learning activities were three different contextual problems that students worked through dealing with velocity, force, and pressure. Sealey added the fourth layer of "function" as a next logical step following her learning activities, but this was not something the students grappled with in this particular study. Sealey described this function layer as recognizing a function where the input is the upper bound of a definite integral and the output is the value of the integral.

Figure 1

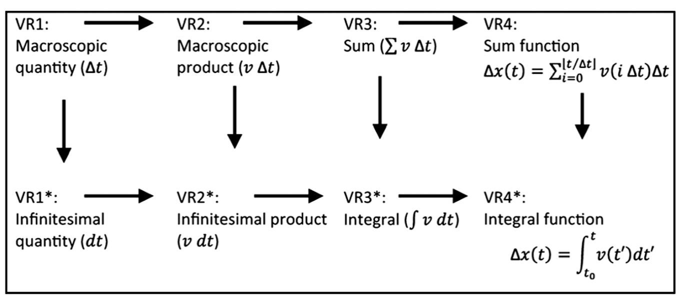
Sealey's (2014) Preliminary Framework for Integration

Layer	Symbolic representation
Pre-layer: Orienting	$\left[\frac{1}{c}\cdot f(x_i)\right]$ and/or $[c \cdot \Delta x]$
Layer 1: Product	$\left[\frac{1}{c}\cdot f(x_i)\right]\cdot[c \cdot \Delta x]$
Layer 2: Summation	$\sum_{i=1}^{n} f(x_i)\Delta x$
Layer 3: Limit	$\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$
Layer 4: Function	$f(b) = \lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\Delta x$

Von Korff and Rebello (2012) built off Sealey's (2014) work by providing a framework of process-object routes which students take in learning integration (Figure 2). Their first layer is "quantity," similar to the orienting layer that Sealey added to her framework. A significant change in Von Korff and Rebello's framework is they argue that the jump to the "limit" stage, or working with infinitesimal quantities, can occur at earlier layers of understanding integration. As such, the four stages they use in their framework corresponding to the development of the integral are: a) a quantity, b) a product, c) a sum, and d) a function. These stages are then set within the "macroscopic" and "infinitesimal" layers. In the macroscopic layer, the quantity is a portion of the domain, which is then multiplied by a function to create a product. Adding these products then leads to the Riemann sum and a function for the approximation. The infinitesimal layer is similar, but the quantities involved are infinitesimally small. This leads to an integral rather than a Riemann sum.

Figure 2

Von Korff and Rebello's (2012) Framework for Integration



Accumulation Functions

Yerushalmy and Swidan's (2012) work on students' accumulation understandings is framed with a semiotics perspective. They focused on what meanings students make for the symbols through mostly independent interactions with dynamic software. While this identified connections between the symbolic structure of integrals and graphical accumulation, what it lacks is a connection to the underlying quantitative structure of integration. On the other hand, Thompson and Silverman (2008) approach accumulation quantitatively by emphasizing the integrand function as always representing a rate of incremental bits being accumulated. While a rate is often an important feature in quantitatively assessing an integral, the literature does not support rates as always being a useful interpretation. Referring back to Ely's (2017) example of arc length, $\int_a^b \sqrt{dx^2 + dy^2} = \int_a^b \sqrt{1 + (y')^2}dx$, it is possible to interpret $\sqrt{1 + (y')^2}$ as the rate at which length is accumulated, but this feels like a forced interpretation. Further, even for more basic integrals like $\int F dx$, this approach would require thinking of force as the rate at which work accumulates, which is a somewhat unnatural interpretation. AUP appears to provide a much more flexible interpretation of quantity and allows for the definite integral to be developed first quantitatively, rather than a single value of an accumulation function. As explained earlier, this is why I am designing an HLT that *begins* with AUP and then extends to accumulation functions, rather than starting with accumulation functions as suggested by Thompson, Swidan, and colleagues.

Theoretical Framework

The framework used in this study consists of a decomposition of the integral concept into its constituent parts. This decomposition takes into account both definite integrals with fixed bounds and accumulation function integrals. This decomposition takes the quantitative "adding up pieces" meaning as the central meaning of integrals and contains how the quantitative meaning can be represented in graphical, numeric, and symbolic ways. Because the integral framework is based on the idea of process-object duality, I begin this section with a discussion of processes and objects from Sfard (1991, 1992).

Process-Object Framework

I will be using Sfard's (1991, 1992) process-object theory as a framework for my study. Sfard outlined stages of development one takes to understand a concept: interiorization of some operation to become a process, then the reification of said process to an object. Note that in the history of mathematics education in recent decades, there have been many approaches that all have similar process-object ideas in them (Piaget, 1970, 1972, 1985; Davis, 1983, 1984; Greeno, 1983; Dubinsky, 1986, 1991; Grey & Tall, 1994). Sfard (1991) drew specifically from Piaget's (1970, 1972, 1985) work involving actions and operations becoming objects of thought. Davis (1983, 1984) describes sequences that become integrated and seen as a whole before becoming an entity itself. Greeno (1983) used the term "procedures," which become conceptual entities. Dubinsky (1986, 1991) discussed how actions become interiorized to processes, then those processes are encapsulated into objects. Grey and Tall (1994) examined procedures linked to algorithms, which are then conceived as a process without reliance on the algorithm, before becoming what they called a "procept," or a symbol evoking a concept or process. For a more detailed description of these approaches, refer to Tall et al. (2000).

While there are many approaches, each with its nuances and distinctions, for this study I elect to use Sfard's approach for two reasons. First, others in calculus education looking at the derivative and integral have explicitly built on Sfard's work (Zandieh, 2000; Sealey, 2014). In order to stay compatible with these other researchers, I also use Sfard's process-object approach. Secondly, I find her terminology to fit well with how I am thinking about a learning trajectory through definite integrals.

Sfard (1991) differentiated first between structural and operational conceptions of an idea. A structural conception is much like viewing a mathematical entity as an object that can be manipulated itself. In Sfard's words, it "means being able to recognize the idea "at a glance" and to manipulate it as a whole, without going into details" (Sfard, 1991, p. 4). On the other hand, an operational conception is an attention to the actions or computations that lead to that same mathematical entity. Sfard (1991) gave the example of a function being viewed as a collection of ordered pairs (structural) or as a computational process taking one system to another (operational).

However, Sfard (1991) used the word "operation" in a different way. She described a process as "operations performed on lower-level mathematical objects" (Sfard, 1991, p. 18). This use of operation is focused more on the actions a student performs rather than an understanding of the underlying process. I will be using the word operation in this way—to mean the initial actions or computations a student makes before conceiving of the process. For example, students may multiply a rate times a time, but they might not yet realize that with each new rate that occurs there would be a separate product to find the amount for that time interval. I acknowledge that this usage is influenced by Dubinsky's (1986) use of "action" as the stage preceding a person conceiving of processes.

As someone becomes comfortable with the operations, they can begin to imagine the operations being carried on without actually computing them (Sfard, 1991). Sfard (1991) called this "interiorization" of a process. Specifically, she drew on Piaget (1970) when she stated, "we would say that a process has been interiorized if it 'can be carried out through [mental] representation.' (Piaget, 1970, p. 14) and in order to be considered, analyzed, and compared it needs no longer to be actually performed" (Sfard, 1991, p. 18). Once the operations have been interiorized into a process, I will say that a student has a process-level understanding. Carrying on my previous example, a student with a process-level understanding of the products within integration would recognize that each interval of time can be multiplied by the corresponding rate without needing to carry out the steps for each one.

Sfard (1991) then describes the “condensation” of these processes as “a period of "squeezing" lengthy sequences of operations into more manageable units” (Sfard, 1991, p. 19). At this stage, students are more comfortable reasoning about the process as a whole without attending to the details (Sfard, 1991). While I would still denote this as process-level thinking, it is an important step in a student beginning to understand the entity as an object.

As the processes become condensed, they become reified (Sfard, 1991). As Sfard describes, “The new entity is soon detached from the process which produced it and begins to draw its meaning from the fact of its being a member of a certain category (Sfard, 1991, p. 20). Once the entity is reified, I will say that a student has an object-level understanding. This means they view the mathematical concept as its own entity to be manipulated, independent of the processes that built it. The student should still be able to deconstruct the object into its processes if needed, but they are able to skip this process-level reasoning. A student with an object-level understanding of the products within an integral recognizes that the products they conceived of at the process level give small amounts of some new quantity. Each layer of my integration framework has process-object levels of understanding. I break down what each of these entails following an overview of my integration framework.

Layers of Integration Framework

I will be using and extending both Sealey’s (2014) and Von Korff and Rebello’s (2012) frameworks for the layers of integration. Rather than referring to Sealey’s (2014) pre-layer of “orienting,” I feel Von Korff and Rebello’s (2012) layer of “quantity” better describes the design of my HLT. It seemed there was more involved in Sealey’s “function” layer, so we have broken that up into two new layers. These layers are “variable upper bound" and “accumulation function,” where in the variable upper bound layer, students conceive of the bound of the integral as something that can change. Then they can think of that varying bound as an input for some accumulation function.

During the interviews and in preparing for analysis, we also found two more layers of understanding that did not fit well within any of the other layers. The first of these is a “chop” layer between quantity and product. This layer involves chopping the domain into intervals of a certain size, either macroscopic or infinitesimal. The other added layer is “net amount,” where students recognize that the added sum is a net change rather than a total amount. In summary, the layers of integration I used were quantity, chop, product, sum, net amount, variable upper bound, and accumulation function. A more in-depth description of student thinking in these layers can be found in Table 1 below.

I also draw on Zandieh’s (2000) and Roundy et al.’s (2015) frameworks for derivatives in my usage of numerical, graphical, and symbolic representations. Zandieh calls these “contexts,” as she argues that they do not represent the same concepts for students. However, in my conceptual breakdown, I believe these are representations of the underlying quantitative structure at the core of integration. Thus, they are actual representations rather than contexts in which to view the structure. Within each layer of integration, a student can represent process- or object-level understanding. It is my goal in the HLT that students demonstrate understanding across all three representations.

A Note on Macroscopic and Infinitesimal Levels

Von Korff and Rebello (2012) said that the jump from macroscopic to infinitesimal thinking can occur at any layer of integration. However, in my conceptual breakdown, the infinitesimal jump would necessarily occur at the chop layer. For example, a student may think of infinitely skinny rectangles at the product layer, but that is due to the length of the interval of the domain and not the output height. This makes it seem like if a student is moving to the infinitesimal level, they have gone back to the chop layer and made their intervals smaller. Therefore, I have only included the infinitesimal level of thinking in the chop layer of my conceptual breakdown. A student could then carry this through to the other layers, reasoning about them similarly to before.

Target Process-Object Student Thinking

The table below shows the operation-, process-, and object-level thinking a student would exhibit among the different integration layers and representations. The two purposes of the table are to illustrate to the reader the type of thinking I was looking for during the interviews, as well as to provide a guide for analyzing student work following the interview.

Table 1

Target Process-Object Student Thinking

	Numerical	Graphical	Symbolic
Quantity	**Operation:** Student can interpret the meaning of a set of values, e.g., a 4 is 4 L/sec, rather than just 4. And this is associated with a particular value of time. **Process:** Student recognizes that they could do this with any value, even the ones not present, without actually needing to interpret each one.	**Operation:** Student can plot and interpret a point on the graph, e.g., the input as time and the output as the rate at that time. **Process:** Student recognizes that data points would exist between the graphed points, whether we know that data or not. **Object:** Student sees the collection of points	**Operation:** Student interprets a single output for a single input, e.g., a rate at one time, R, at time t. **Process:** Student recognizes that there are different function outputs for different inputs, the output could change (or stay constant) for any of those inputs. **Object:** Student

	Object: Student recognizes that all of these make up the function (typically, but not necessarily, a rate) at any given moment in time.	overall as being the function of one quantity as a function of the other, e.g., rate as a function of time.	understands the function notation as denoting all of the corresponding inputs and outputs, e.g., R(t) as the rate function at any given time, t.
Chop	**Operation:** Student subtracts two input quantities to find an interval of the domain. **Process:** Student recognizes they can find the difference between any two inputs as the output changes at those inputs, e.g., as the rate changes between two times, they can find the change in time. **Object:** Student sees each time interval being malleable and having a corresponding output as given by the data. The infinitesimal level makes calculations impossible, but students recognize that more data points provide them with smaller intervals of time.	**Operation:** Student chooses a discrete segment length along the horizontal axis to examine the function output. **Process:** Student recognizes they can segment the horizontal axis into any size they want for the entire domain. **Object:** Student sees the intervals along the horizontal axis as having an output value associated with them on the graph (taking the left bound, right bound, average of the outputs, etc.) At the infinitesimal level, the student makes the discrete segments as small as they reasonably can on their graph.	**Operation:** Student notates "change" using some kind of written inscription, e.g., writing Δt to represent an interval of time. **Process:** Student uses "Δ" to consistently mean change, e.g., there is a change in time, Δt, between any time values in the domain. **Object:** Student sees $\Delta[symbol]$ as inherently meaning a small change or amount of the quantity represented by the symbol corresponding to any change in time as the output changes. At the infinitesimal level, the student denotes the change in time as d[symbol], and conceptualizes this as a nonzero, infinitely small amount of the quantity represented by the symbol.

Product	**Operation**: Student multiplies two quantities (such as rate and time). **Process**: Student recognizes that each interval will have a product, without necessarily needing to enact the computation. **Object**: Student recognizes the product found represents a small amount of the resulting quantity.	**Operation**: Student uses a discrete segment along the horizontal axis and a height up to a point on the graph to draw a rectangle. **Process**: Student sees an interval as having an associated rectangle, without necessarily drawing it. **Object**: Student sees the area as a representation of the multiplication of the base and height quantities, producing a third quantity, which is the amount accumulated in that time interval.	**Operation**: Student writes $\Delta t * f(t)$(or other variables as determined by the student) to represent a specific computation. **Process**: Student recognizes that $\Delta t * f(t)$ represents a product in an arbitrary interval, and this can be done for any interval in the domain. **Object**: Student perceives that asserts $\Delta t * f(t) = \Delta A$, or a small change in the amount of the resulting quantity.
Sum	**Operation**: Student adds the products they calculated. **Process**: Student can imagine summing the products, even if not necessarily calculated. **Object**: Students identify that the summation represents a change in amount over the interval.	**Operation**: Student uses discrete segment lengths along the horizontal axis and a height up to a point on the graph to draw multiple rectangles. **Process**: Student imagines filling the graphical space with these rectangles, without necessarily drawing them in. **Object**: Student sees the summation of these rectangles as representing the amount over the entire interval.	**Operation**: Student writes Σ to mean "sum". **Process**: Students use sigma notation to denote adding up every product, since they cannot all be written. **Object**: Student sees the symbolic, sigma notation as equaling the amount over the interval.

Net Amount	**Operation:** Student adds the value of the integral to a previously existing amount of a quantity. **Process:** Student imagines that whatever value the integral produces can be added on to a previously existing amount of the quantity. **Object:** Student views the numeric integral result as being an "additional" amount of the quantity.	**Operation:** The area under a graph corresponding to the integral is literally connected to an adjacent area. **Process:** The student imagines the portion of the area corresponding to an integral as being a part of the larger area under a curve. **Object:** The area is viewed as a representation of the "additional" amount of the quantity.	**Operation:** Student notates the combination of the integral's value to an existing amount, such as "A+B." **Process:** Student sees the integral value as always potentially existing in conjunction with a previous amount, "A+∫." **Object:** Student views the integral symbols, "∫" as a "net amount."
Variable Upper Bound	**Operation**: Student calculates an additional product to add to a previously calculated net amount (thus extending the bounds of their integration). **Process**: Student envisions the process of the upper bound continually changing, even without computing the new value (i.e., as the bound extends, they will accumulate a little more of the amount). **Object**: The bound becomes an object that the student knows can change and be tracked as its own variable.	**Operation**: Student adds an additional piece of area onto the original graph. **Process**: Student can imagine adding several additional rectangles. **Object**: Student sees the bound as being able to change continuously, corresponding to increasing area.	**Operation**: Student writes definite integrals for varying bounds, choosing a new number for the bound each time. **Process**: Student sees the bound can be any of the domain values, places a variable in the integral bounds. **Object**: Student sees the integral expression with the variable bound as giving the accumulated amount at that point.

Accumulation Function	**Operation**: Student calculates the corresponding accumulated amount for a given input as the "stopping point." **Process**: Student sees this stopping point as being the input, which can become any value. They can reason about the behavior of the accumulated amount without running every calculation. **Object**: Student sees the accumulated amount as a function of the input variable.	**Operation**: Student can plot an individual bound value with its corresponding accumulated value. **Process**: Student conceptualizes each point as a new bound that can be calculated and plotted. Using ideas of concavity, students are able to plot a rough sketch of the graph. **Object**: Student sees the graph as an entity itself that contains all the information about different amounts in relation to different upper bounds.	**Operation**: Student writes A to represent the accumulated amount. **Process**: Student sees the output of A as being dependent on an input of x. **Object**: Student conceives the function A(x) equal to the variable bound integral. A(x) now means every accumulation at any upper bound.

CHAPTER THREE: THE HLT

The first purpose of this thesis is to create a hypothetical learning trajectory (HLT) for quantitatively understanding the definite integral and accumulation function using AUP. This chapter addresses this purpose by detailing the HLT that was created after reviewing the literature. I first explain what an HLT is and then describe the learning activities and the students' theoretical progression meant to develop these concepts.

According to Simon (1995), an HLT consists of "the learning goal, the learning activities, and the thinking and learning in which students might engage" (p. 133). Clements and Sarama (2004) further emphasize the importance of the interaction of these three components. They further explained the HLT as descriptions of children's thinking and learning alongside a hypothetical route of tasks that engender the desired ways of thinking to reach a mathematical goal (Clements & Sarama, 2004). However, an HLT is not unique for each mathematical idea and there are multiple hypothetical routes that could be successful (Simon, 1995; Clements & Sarama, 2004). Thus, my proposed HLT is only one *possible* route for understanding definite integrals and later accumulation functions. I do not argue it is the best or only way for students to understand, but that it is a useful route for students to take.

I will now define more specifically what I mean by each of the three components for the HLT. First, the learning goals are what I hoped students understand mathematically throughout the teaching sessions. I had overarching goals for the entire HLT, but also smaller goals within each lesson that will be described alongside the learning activities. The learning activities are the tasks and interview questions I presented to the students. Sometimes these activities involved shared contexts but viewed through another form of representation. While impossible to know exactly how students were thinking about these ideas, I attempted to capture it as best as I can by studying their verbal, written, and gestural responses throughout the interviews. The route I anticipated students taking prior to the interviews will be described in conjunction with the learning activities.

The first two lessons aim to develop the definite integral quantitatively, then the next two lessons are designed to extend this quantitative understanding to an accumulation function. The lessons on the definite integral draw heavily from Jones (2014; in press). In addition to his lesson materials, I have added a third set of problems dealing with the context of road construction. Also, through multiple pilot studies, I determined that the students focused less on the "area under a curve" conceptualization when the order of the questions was changed. As such, there are differences in the order of Jones' materials and the way I will present the lesson materials in my study. The third and fourth lessons are my own academic contributions based on the existing literature.

To begin, my overall learning goal was (a) to have students develop strong quantitative meanings for definite integrals through an AUP perspective, and (b) to extend this AUP meaning to accumulation functions and integrals through quantitative reasoning. My chart of the conceptual breakdown of integration describes how I imagined this quantitative meaning would develop. The target understanding that I aimed for students to achieve in the learning activities consisted of understanding each layer of integration at both a process and object level, making connections between that layer and the quantities in the context, and the ability to view that layer at an infinitesimal layer. The exception is the quantity layer, which does not have an infinitesimal scale in the framework.

Smaller-scale learning goals also accompany each part of the HLT. The following describes the learning activities that are to take place in this HLT in the form of the tasks, contexts, and activities that can be used in an actual lesson. As I describe the learning activities, I also describe the hypothetical route a student might take in developing an understanding of definite integrals and accumulation functions, as they engage in these activities. These routes will each be summarized with a figure following the description. These routes are not meant to suggest students would spontaneously develop these ideas themselves. They are highly dependent on the structure of the learning activities and the questions asked by the researcher. For example, the interview questions prompt students to create a graph after completing certain work. Students would likely not create this graph otherwise. Therefore, these are anticipated routes I would expect students to take as guided by the interviewer.

As this is a hypothetical trajectory planned prior to the interviews, I use the future tense in these lessons as I describe the thinking students might do. This portion of the thesis does not indicate exactly what thinking occurred, as that follows in the results. Note that the map figures used to show the trajectories do not always have the numerical, symbolic, and graphical representations in the same order across the top of the charts. This was done in order to clean up the way the arrows traced through the graph to prevent too much overlap or clutter.

Lesson One

Finding Amount Using Constant Rate

To begin, I will present students with the following context: “A fuel pipe leading to a tank has a device on it that records the fuel’s flow rate through the pipe. Over a 4-minute interval, the flow rate is 10 liters per minute.”

Key Questions:

1. What quantities are part of this context? What are their units?
2. Can we determine the amount of fuel in the tank after the 4 minutes?

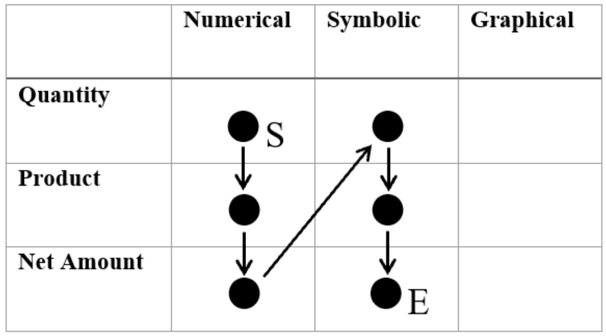
3. Were we given any information about the tank before the 4-minute interval occurred?
4. What symbols would you use to write the computation you made in this problem?

This is meant to focus attention on quantity from the start of the interview. Also, the constant rate is a much easier situation to deal with than a varying rate. It will help to have dealt with the quantities in this easier context as the situation becomes more complex. I anticipate students will have little difficulty identifying the quantities of time, flow rate, and fuel amount. I also anticipate students will say that there are 40 liters in the tank, which may or may not be accurate. In pilot studies, each student multiplied the 10 L/min by 4 minutes to get 40 L, but they did not account for any fuel that might have been in the tank prior to the 4-minute interval. Question (3) is meant to draw attention to a possible initial amount if students do not recognize that possibility. Students could then determine the 40 liters is what has been added to the tank regardless of the beginning amount. After this discussion, I will tell the students that we will assume from this point forward that the tank had 9 L before the 4-minute interval. The reason for this is to keep this initial amount in their minds throughout each problem.

Figure 3 below shows the path I anticipate students to take towards understanding. The solid dots denote both a process- and object-level understanding of the cell, which includes the ability to connect that layer back to quantities from the given context. The "S" indicates where I anticipate students beginning in the framework and the "E" indicates where I think they would end their work in the particular context. Using the given data, students would likely calculate numerically first, then translate their work to symbols when prompted by the interviewer.

Figure 3

Anticipated Student Trajectory for Initial work in the Constant Fuel Rate Context



Developing the Riemann Structure

Following the constant rate fuel context, I ask students to consider a similar situation where the fuel flow rate is not constant. Specifically, where R is a function of time, R(t). I also give students the information that at $t = 0$, the rate is $R(0) = 18\ L/min$.

Key Questions:

1. Can I multiply 18 L/min by 4 minutes to get the amount of fuel?
2. Does multiplying 18 by 4 give me any useful information?
3. What might I need to do to better approximate how much fuel was added to the tank over the 4 minutes?

I anticipate students will see the problem with using 18 multiplied by 4 since the rate is not constant. The point of question (2) is for students to see that this product does give us *some* estimate of the fuel, but it is not very precise. In pilot studies, students reasoned about how they believed the rate would either slow down or speed up based on their real-world experiences. This leads to the obvious need for more detailed information about the rate throughout the time interval, which I provided with the chart below. Specifically, the goal of my questions is to guide students toward the actions they will need to do in order to calculate an approximation.

Table 2

Table of Values for the Varying Fuel Rate Context

t (min)	0	1.25	2	2.5	3	3.75	4
R(t) (L/min)	18	12	7	6	4	3	2.5

Key Questions:

1. Can we find the total amount of fuel now, assuming it still began with the same amount as before?
2. Is your answer the exact amount? What assumptions are you making about R(t) that leads you to that answer?
3. How would you symbolically represent what you calculated with the varying rates?
4. What has changed between the procedure with the constant rate and your procedure for the varying rate function?

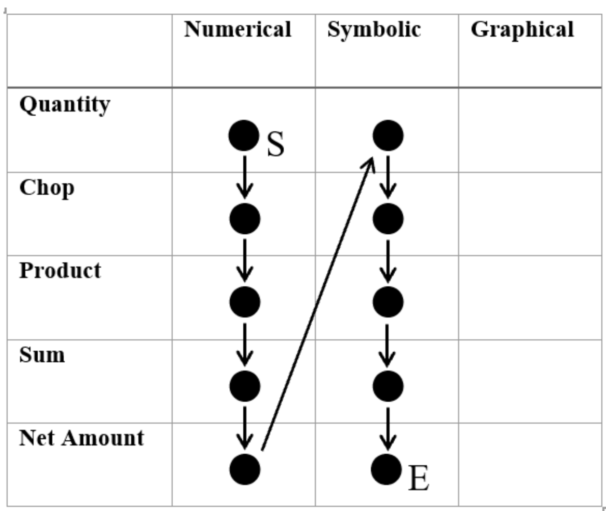
By asking them to identify differences in the procedure between constant rate and varying rate, I am trying to get them to conceptualize the steps that make up the process-level understanding. Students will not likely be at this stage yet but seeing a bigger picture may help them begin to interiorize these operations into a process. I also begin to establish part of the symbolic representation here because it naturally flowed with the numerical representation.

In order to symbolically represent the summation of the rates, students may need some short instruction about sigma notation. My goal is for them to informally write the idea of adding up products of $R(t) * \Delta t$, then I can introduce the symbol Σ to mean adding up every product we could have. Once $\Sigma\, R(t) * \Delta t$ is established, I can then demonstrate to students how to denote the indices in summation notation. Again, my focus is more on the quantitative structure of the symbols than the sigma notation, so this instruction will be kept brief.

The anticipated trajectory in Figure 4 shows a similar path as the constant fuel rate context, but with the added steps of chop and sum since there are more rate values to consider than before. I anticipate students to work numerically through the entire problem, then I will ask them to translate their work to symbols.

Figure 4

Anticipated Student Trajectory for Initial Work in the Varying Fuel Rate Context



Next, I will ask students to return to the constant rate of change context and to sketch a graph of R as a function of time. It is tempting for students to instead draw the graph of fuel amount, or the change in fuel, over time. If this happens, I will redirect them to the question and ask them to first label the appropriate axes for the graph. After graphing this context correctly, students will then graph the context with a varying rate.

Key Questions:

1. How do you see $10\ L/min\ *\ 4\ min\ = 40\ L$in the graph?
    a. Where is 10 L/min on the graph?
    b. Where is the 4-minute *interval* on the graph?

2. Do we know what the rate is between the times given on the chart?
   a. What would the graph look like if we assume the rate remained constant until the next data point?
   b. Do you think it's likely that the graph would really look like this?
3. How do you see the products that you calculated earlier in the graph? (Point to a specific product, like $18\ L/min\ *\ 1.25\ min$)
4. What does each rectangle on the graph represent? What are the units?
   a. Why are the units not squared, like the units of area normally are?
5. What do all the rectangles represent together?
6. How can we get a more accurate total for the amount of fuel?

Now my goal is to introduce students to graphical representation so they can make connections between the different representations without relying on the area under the curve as their core understanding. I ask students to explain what the rectangles represent because I want to assess their level of understanding at the product level. I also ask about the summation level, which may still be developing at this stage of the lesson. My goal is that through their previous numerical work, they can draw parallels to reason about the graphical representation.

Figure 5 shows the trajectory of students moving through the integration layers in the constant fuel rate context with a graphical representation. The lighter color of the dot indicates previous understanding of the cell has been shown, whereas the darker color is the trajectory of this specific portion of the lesson. Figure 6 shows a similar trajectory with the varying fuel rate, with one notable difference—after moving through each layer from quantity through net amount graphically, students would return to the chop layer as they reason about how they could produce a more accurate estimation of the net amount of fuel. By drawing more rectangles within the interval, they can get more accurate. By prompting them to continue increasing accuracy, it is intended that students conjecture that chopping the time into infinitesimally small pieces will produce the exact net amount.

Figure 5

Anticipated Student Trajectory for Continuing Work in the Constant Fuel Rate Context

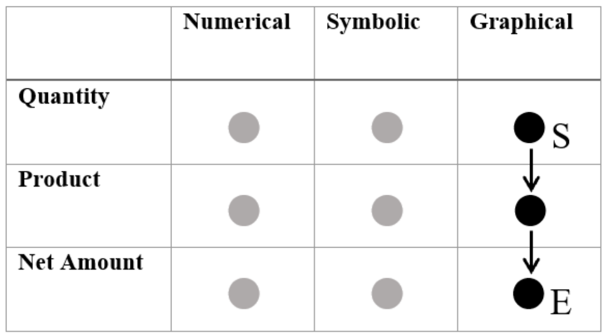
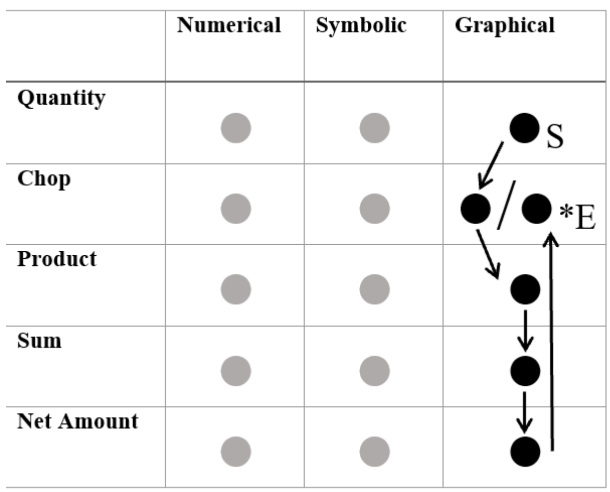


Figure 6

Anticipated Student Trajectory for Continuing Work in the Varying Fuel Rate Context



*Note: The * denotes a jump to the infinitesimal level.*

Building the Limit Idea

Table 3

First Option for Increasing the Number of Data Points

t	0	1.25	2	2.25	2.5	3	3.25	3.5	3.6	3.75	3.9	4
R(t)	18	12	7	6	6	4	4	4	3	3	3	2.5

Table 4

Second Option for Increasing the Number of Data Points

t	0	0.5	1	1.25	1.75	2	2.25	2.75	3	3.25	3.75	4
R(t)	18	21	16	12	9	7	6	5	4	4	3	2.5

Key Questions:

1. Now that we know we need more information, which of these two tables would give us a better approximation? Why?
    a. Sketch a rough graph of R(t) based on each of the two charts. Do you notice anything about the intervals?
2. How can we increase the accuracy of our calculation even more? Can you write this symbolically?
3. There will always be physical limitations, but let's assume that we can be infinitely accurate with our measurements. How can we write that summation symbolically?
4. Why can't the time interval be zero?
5. To summarize, can you explain how this notation connects to both the calculations you

made, and also to the graphs you drew?

The purpose of the two tables is for students to confront the idea of how to shrink the intervals. Only shrinking the middle intervals will not improve the accuracy of the estimate. My goal is that this line of thinking helps them conceive the process of shrinking *all* the rectangles graphically and connecting that back to the quantitative structure that the area represents. The symbolic representation will be key here because that is where students can apply the limit notation to their work. It also is the only representation where their answers can be exact, as it is not possible to find the exact answer given the numerical data.

Question (4) is meant to help avoid the "collapsing" metaphor students may use for limits (Oehrtman, 2009). This means that they imagine the width of their intervals collapsing into zero. From a quantitative perspective, this is problematic because then the resulting product will be equal to zero. As such, this will be a good point in the interview to give them a brief overview of the history and development of infinitesimals. Then I can establish a quantitative approach much like Ely's (2017) where the symbol *dx* can represent an actual size of the interval, leading nicely to the development of integral notation.

Developing Integral Notation

This will require instructor explanation since students cannot create the integral notation without this guidance. However, after introducing the idea of *dx* as an infinitesimal piece of x, I can ask students how we can represent that piece in our context. Rather than x, we have been using the variable t. So, rather than our previous symbolic notation of $\Sigma\ R(t) * \Delta t$, we now can use *dt* in place of Δt. Then, rather than "Σ ," we use the new symbol of "∫" to denote the sum of these infinitesimal products. We need to somehow identify the interval bounds, so I will show students where the bounds are placed on the integral. Therefore, our integral structure is

$\int_a^b R(t)dt$. My final task for students is for them to summarize how this integral notation aligns with the work they have done that day.

Lesson Two

Reinforcing Ideas and Identifying Thinking in Road Construction Context

The goal of the Road Construction context is to solidify the idea of AUP and as a way for the interviewer to see the students applying the ideas from the previous lesson in a new context. Students were given a simpler version first to help situate themselves in the new context and to identify the relationship of the quantities before complicating the context. This simpler situation will make the product layer of integration clearer, thus reminding them of the previous interview and setting them up for the rest of the lesson.

Figure 7

Constant Weight Road Construction Context

Engineers want to build a road connecting two cities, but while building they come across a dirt mound that needs to be removed.

Key Questions:

1. What do we need to know in order to find the weight of the dirt the engineers need to remove?
2. Do we know what the front of the mound looks like?

Impose the units of pounds/ft on the y-axis, and feet on the x-axis.

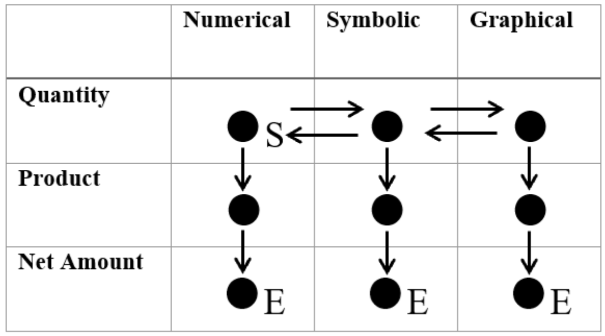
3. Why is the shape of the mound the same as the graph of pounds of dirt per foot as a function of horizontal distance?
4. Does this new graph show me how tall the mound is?
5. Why would we need a pounds per foot graph to solve this problem?
6. How is your calculation similar to other work we have done?
7. If we were to cut up the graph into rectangular slices, what would the area represent? What would be the units?

The goal of this task is to orient students to the graph we will be using in the task below. The units of pounds per foot as the dependent variable may be hard to conceive, so I introduce it within a simplified context. Then, once students have made sense of the quantities, they can work productively on the next task.

Figure 8 shows the trajectory as students may begin within any representation here and may also transition between them. Based on pilot study work, it seems most likely that the transitioning among different representations would occur at the quantity layer, but there may be more transitioning across representations than shown in the anticipated path. I placed in E in every cell of the “net amount” row to indicate students could end in any representation depending on their preferences. It is my goal that students can describe the layers in all three representations, so I may ask questions prompting them to explain their work again through the lens of a different representation.

Figure 8

Anticipated Student Trajectory in the Constant Weight Road Construction Context

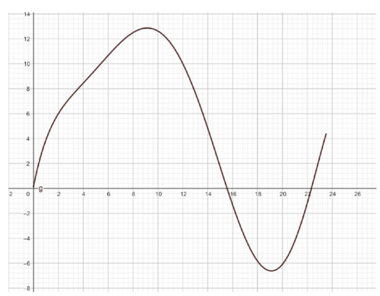


A Meaning for Negatives and Solidifying the Definite Integral

Figure 9

Varying Weight Road Construction Context

The engineers are building another road. They come across a less uniform piece of land they need to make perfectly flat to lay the road. The land has both mounds of dirt, as well as dips that need to be filled in. The graph below shows the pounds per foot of dirt as you move to the right from where the road begins. Find an approximation of the weight of the dirt the engineers need to remove.



Key Questions:

1. Will the mound be the same shape as the pound/ft graph shown, like we saw before?

   Why or why not?

2. What calculation would give us pounds?

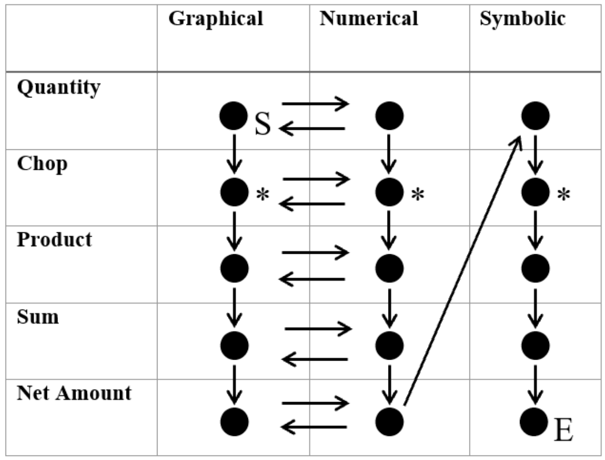
3. How do we account for the different weights of dirt at different locations?
4. How do we account for the "dip" in the road we need to fill? What would that look like symbolically?
5. Is it possible to get a completely accurate total for the weight of dirt using the tools we have? How can we make our approximation more accurate?

One key feature of the Road Construction Context is the presence of negative products. Students have to make sense of what it means for the graph to go below the x-axis and how that would or would not affect their integral. It also gives students a chance to reason with a graph to find the accumulated amount in order to have an object view within the graphical representation. This context will be built upon more as students begin exploring accumulation.

As Figure 10 shows, I anticipate a lot of movement between the numerical and graphical representations, as the data is given graphically but an estimation of the weight of dirt would require numeric calculations. I also anticipate students will use reasonably small interval sizes at the chop layer, so I classify these as infinitesimal. It would be impossible for students to draw infinitesimally small rectangles or calculate infinitesimally small products. Once they make their calculations, if they have not used symbols yet I will prompt them to translate their work into the symbolic representation.

Figure 10

Anticipated Student Trajectory in the Varying Weight Road Construction Context



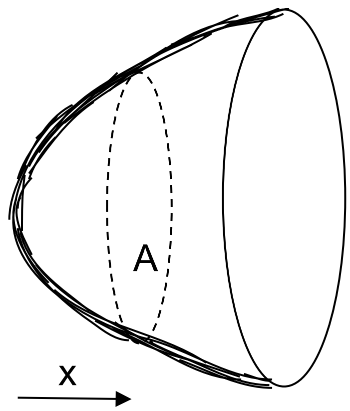
Lesson Three

Interview Questions to Check Current Understandings

Following the second interview, the researchers realized that students had only been presented integrals with rates as integrands. To expand and assess the students' views of integration, the following questions were presented.

Figure 11

Volume of a Solid Context



A is a function that gives the cross-sectional area of the shape above in terms of the length along the bottom, x (as shown in the image). *A(x)* is measured in ft^2 and x is measured in ft.

Key Questions:

1. Given the general integral $\int_a^b f(x)\,dx$, can you tell me what each piece means? What does the integral mean all together?
2. What quantity does the integral $\int_0^{83} A\,dx$ give us (using the function A(x) defined above)? What are the units of the value of this integral?
3. Given an integral $\int_{-3}^{6} F\,dy$, where F represents the force on an object in Newtons and y is measured in *ft*, what are the units of resulting quantity from this integral?

The purpose of the first question is to ascertain what students remember of integration beginning the third lesson. The second question introduces area multiplied by length, producing a volume. This is a common context for integration in second-semester calculus, and while area can be interpreted as the rate at which volume is accumulated, that is not a natural interpretation for most students. Therefore, this can help assess if students can apply the product layer of integration and make sense of the quantities despite lacking a rate. Similarly, in the third question the goal is not for students to understand the physics of work. Rather, this is just to see if they understand the integral as the sum of the products of two quantities.

Accumulating Using a Constant Rate of Change

I will then ask students to recall the constant fuel rate context, where the fuel rate was 10 L/min over a 4-minute interval and the tank began with nine liters of fuel.

Key Questions:

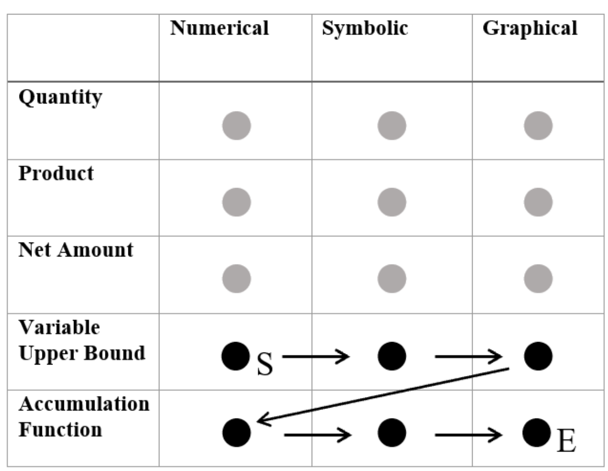
1. How much fuel will be in the tank after 1 minute? 2 minutes? 3.7 minutes?
2. How much fuel is *added* to the tank in the 1/10 second after the 4-minute interval, assuming the constant rate continues?
3. What time value represents the 1/10 second after the 4-minute interval?
4. Write an integral for each of the 4 different times (1 min., 2 min., 3.7 minutes, 1.0067 minutes/240.1 seconds).
5. If students switch to seconds: Does our multiplication still make sense? What other units need to be converted to match?
6. What symbols stay the same? What changes?
7. Sketch a graph of the amount of fuel in the tank as a function of time. How is this graph related to the fuel rate?

The graph of the fuel amount should not be too difficult for students and does not rely on understanding the accumulation layer of integration because it has a constant rate of change. However, I think that asking students to relate the graph to the fuel rate can begin to lay the groundwork for accumulating later in the lesson. The main focus of this task is in calculating the small products to be added to the rest of the amount. This is using an operation level of reasoning about the variable bound in a numerical representation. The goal of having students write integral expressions is both to solidify their prior symbolic understandings, but also to begin their operation level of thinking symbolically as well. As they write multiple integral expressions, they can notice the pattern in the bounds, leading to the interiorization of the operation to a process.

Figure 12 summarizes the anticipated path, as students will likely begin numerically based on the data given. The interview questions then prompt students to translate their reasoning into symbols, then a graph. After reasoning about the changing upper bound of the function, I anticipate students will return to the numerical data to be able to produce a graph of the amount of fuel in the tank as a function of time.

Figure 12

Anticipated Student Trajectory in Extension Work in the Constant Fuel Rate Context

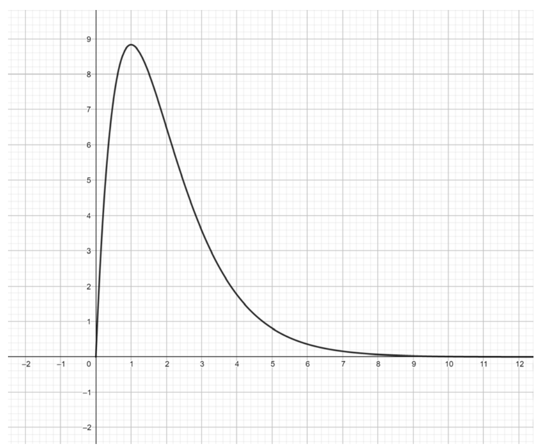


Graphical Accumulation with Varying Rate

Figure 13

Graphical Varying Fuel Rate Context

Like we saw before, the fuel rate might not be constant. Assuming the tank began with 9 L of fuel, let the graph below represent f(t).



f(t) represents the flow rate of the pipe and is modelled by $f(t) = 24te^{-t}$ for $t \geq 0$

Key Questions:

a. How do we represent the change in the amount of fuel using the graph above?
b. Approximate the amount of fuel in the tank after 4 minutes.
c. How much fuel is added to the tank 1/10 of a second after the 4 minutes? 1/100 of a second? How do we represent this amount on the graph above?
d. Write an integral expression to represent the exact amount of fuel in the tank at each of these times.
e. What do you notice about your integrals?
f. How could we represent the amount of fuel in the tank after x minutes? What would that mean?

g. Make a rough sketch of the graph of the amount of fuel in the tank as a function of time.

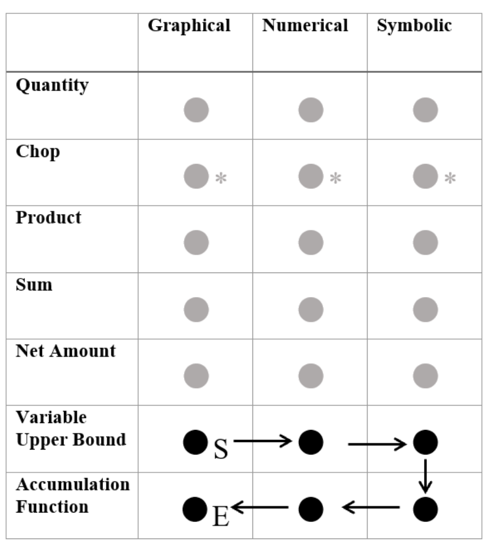
h. What shape should the graph have after the 8 minutes mark?

This task incorporates the graphical representation for the variable bound layer. The goal now is for students to identify the quantitative structure as they add small pieces of area (representing tiny changes in amount) to the previous total. This can help them begin to identify the operations necessary to graph the accumulation function, which will be a major focus in the next lesson. I also ask students to examine the integral expressions as a way to help them see the bound begin to vary. They can then reason about this bound varying without needing to carry out the calculations.

The trajectory in Figure 14 is similar to the trajectory of the constant fuel rate context. The major difference is that I anticipate students beginning graphically since that is how the data is presented. After symbolically representing the changing upper bound and the function with the bound as the input, students could then make some numerical calculations to help them in graphing.

Figure 14

Anticipated Student Trajectory in Graphical Varying Fuel Rate Context



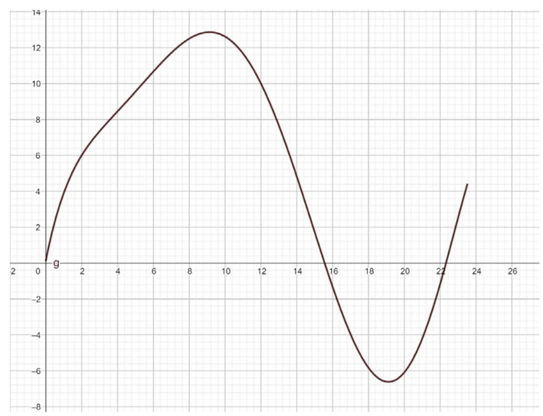
Lesson Four

Conceptualizing an Accumulation Function

Figure 15

Varying Weight Road Construction Context Revisited

Recall the Road Construction context from previous lessons. Here is the graph again.



Key Questions:

1. Imagine the engineers are making a graph of the weight of dirt they have on hand at any given time as they clear the way for the road. What factors affect the weight of the dirt the engineers have at any given time? Why might this graph be useful to the engineers?
2. Construct the graph of the weight of dirt as a function of horizontal distance travelled along the x-axis.
    a. What is the y-intercept of the graph?
    b. When does the weight of the dirt increase? Does it ever decrease?
    c. When will the engineers have the most dirt on hand? The least?
3. Let's say the engineers plot the graph, and there is a point (18, 124) on the graph.

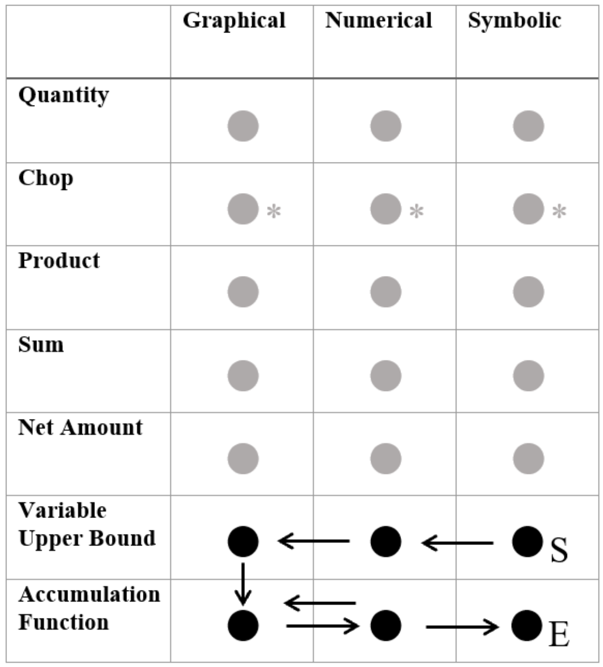
    a. What does this ordered pair tell you?
    b. What are the units of each number?
    c. What would the point (43, -10) mean?
4. Describe the input and output variables of the graph you've made.
5. Write a symbolic representation of the relationship between the input and output variables.

The main focus of this lesson is to take the ideas of varying the bound in lesson three, and from this conceptualize the accumulation function. Once again, I begin with quantities and contextual considerations as a way to situate students to the problem. I chose not to include the Geogebra file with this graph because I want them to reason about the shape without plotting specific calculated points. I anticipate this to be difficult for students and thus will take longer than the other tasks in previous lessons. However, this will encourage them to reason more about the process than the operations. Once they come to a consensus about the shape of the graph, I see Question (3) serves as a way to assess if students have an object understanding. They should be able to make sense of a specific data point on the graph within the context if the accumulation function has become a separate entity in their minds.

The key questions suggest the trajectory shown in Figure 16. If students do not begin with the symbolic representation with a variable upper bound, I will ask them to make connections back within that representation. For example, I might ask, how would you symbolically represent the numerical calculations you are making? I anticipate most of the work being done within the graphical and numerical representations as students sketch a graph of the amount of dirt.

Figure 16

Anticipated Student Trajectory in Extension Work in the Varying Weight Road Construction Context



Next Steps

This study does not extend beyond the concept of the accumulation function itself. However, after completing these lessons, a natural next step would be to examine the Fundamental Theorem of Calculus (FTC). As discussed in Chapter 5, all the students did see connections between their accumulation function graphs and the original graphs they were given. By comparing the accumulation function to the original, students may be able to identify features of antiderivatives and make conjectures about why those connections exist.

CHAPTER FOUR: METHODS

The HLT fulfilled the first purpose of this study. This chapter outlines the steps taken to fulfill the second purpose of the study, answering the research question: as a student progresses through the HLT, what understandings do they develop of the definite integral and accumulation function? Data was collected from a series of interviews conducted with three pairs of first-semester calculus students where students engaged in the tasks outlined in the HLT. In this chapter, I first describe how pilot studies influenced the learning activities and interview questions. I then outline the data collection and analysis processes that took place during and after the interviewing period.

Pilot Studies

While developing the HLT, multiple pilot studies were conducted with four students. These students ranged in age and mathematical experience, with two of them being undergraduates in nursing and two of them being high school students. One of the undergraduate students had taken a high school calculus class previously.

The pilot studies shaped the final format of the HLT in two significant ways. The first major influence was in the order material in the first lesson was presented. The lesson originally had graphical representations earlier; however, I found that when shown the use of area to represent the product too soon, students relied heavily on the literal area and less on the quantitative structure of the product. By first having them explore the varying rate numerically, then graphing both the constant rate of change example and the varying rate of change together, the product was the salient feature rather than the area of rectangles.

The second major influence the pilot studies had was in the exploration of the “Road Construction” context. Originally, students were presented with a literal picture of the mound, with the axes on the graph representing the height rather than lbs/ft. Then, there was a discussion about what the lbs/ft graph would look like and why it would be the same shape as the mound. This was difficult for the pilot students to conceptualize. It proved more productive to instead show them the graph of lbs/ft and question them about the similar shape of the mound. This still brought attention to quantity but felt less like “tricking” the student with a graph that is not useful.

Data Collection

The students for my teaching experiment were selected from a university, first-semester calculus class. Prior to the interviews, students had instruction on derivatives, including the product, quotient, and chain rules. During the time of the interviews, students were learning about implicit differentiation, rates of change in science contexts, related rates, and finding extrema. They had not had lessons yet on curve sketching, antiderivatives, Riemann sums, or integration. The professor of this class often utilized contexts in the problems given to students. Often, specific attention was given to the quantities involved in derivative functions. For example, students worked on problems in their lab classes where they had to describe the input and output quantities of a given function, including the units, then describe the input and output quantities of the corresponding derivative function. Due to the current COVID-19 pandemic, the classroom environment was “blended,” with both face-to-face and online instruction.

Students were recruited on a voluntary basis. The students selected had no prior calculus experience, however, nothing was known about their individual quantitative reasoning skills or their academic standing in their current calculus class. Three pairs of students were interviewed. Due to COVID-19, the interviews were conducted wearing masks and following social distancing guidelines. Students received monetary compensation for their time at the end of the fourth interview.

In the interviews, students were provided with packets containing any necessary images for the contexts, a whiteboard for sharing their work, and calculators if they did not have one available. The interviews were videotaped to capture any board work or gestures the students made and the audio was recorded for transcription. The first lesson as outlined in my HLT corresponded to the activities and questions asked in the first interview, the second lesson to the second interview, and so forth. I began each interview following the first interview by having the students briefly recap their work from the previous interviews to help them regroup. I would then direct their attention to any printed images or materials and guide them through the activities by asking them the key questions in the HLT. Occasionally, direct instruction was necessary for concepts like the integral history or notation that students could not reason through themselves. These brief episodes were planned for in the HLT. Otherwise, my role was to ask questions that helped me gain understanding of their thinking. For example, I often asked students to explain their work or document it on the whiteboard so I could better analyze their work. The planned key questions were typically enough to encourage new ways of thinking which pushed students forward in the HLT. The interviews ended once students arrived at a natural break. Most times, this was at the end of a lesson. Following the first interview, group A had completed all the tasks for the first lesson, but groups B and C were not to that stage yet. In the second interview, we picked up where each group left off and all groups were able to arrive at the same spot in the materials by the end of the second lesson. Besides the first and second lessons, the pacing was fairly uniform across all three groups. By the end of the four interviews, all three groups successfully made it through all learning activities in the HLT.

The students were interviewed in pairs. This was done for them to help each other bring up the important ideas of the lesson, and it also allowed for more student thinking to be visible for analysis. By asking them to come to consensus, I gained additional insight into how they thought and how their understanding evolved. I gave each student a pseudonym to protect their privacy. The pairings were Alice and Alan, Brian and Brad, and Calleigh and Cassie. These pairings were referred to as groups A, B, and C, respectively, when discussing each pair as a unit.

Analysis

In this section, I will first describe how I broke down the transcriptions of the interview for analysis. Then, I will describe the five categories of codes I applied to each line: integration layer, representation, process-/object-level thinking, macroscopic or infinitesimal, and quantity. Lastly, I will describe how I used these codes to create maps of the actual routes students took to understand integration throughout the HLT.

The interviews were transcribed and broken down initially by conversational turns. Each coded line in my spreadsheet reflected one person's "turn" at talking. I then began to code each line for the specific layer of integration the student was working. These codes included quantity, chop, product, sum, net amount, variable upper bound, and accumulation function. If a particular line was not mathematical in nature, it was coded as "none." If the student was talking about mathematics other than integration, the segment was coded as "other." In instances where a student spoke about multiple integration layers in their talking turn, the dialogue was split into smaller segments or "explanations." I defined each explanation to be a continuous piece of dialogue relating to a specific layer of integration as outlined by the framework. This may be an entire sentence, or a piece of a sentence, or multiple sentences. If a student was working with multiple layers that were inextricably linked, that segment was coded for both layers.

While coding for the layers of integration, I was looking for thinking that corresponded to the examples in my conceptual breakdown in Chapter 2. If a student was discussing quantities involved in the context, but not carrying out any multiplication among them, this was coded as "quantity." If students described choosing intervals to examine, taking the difference between two inputs in the domain, or finding the width of rectangles on a graph, this was coded as "chop." If students were carrying out a multiplication or discussing the area of a single rectangle, this was coded as "product." Once they added the areas of several rectangles or added several products, their work was coded as "sum." The code "net amount" was applied if students referenced a potential unknown amount from before the given interval or if they described their integral or calculated estimation as a change in the total amount rather than the total itself. For example, if students said, "we found how much fuel went into the tank in four minutes, but not how much is in the tank necessarily," this would be categorized as thinking of "net amount." If students were writing several or estimating integrals with a changing upper bound, this was coded as "variable upper bound." Lastly, if students were talking about the accumulated amount changing as they changed the upper bound, this was coded as "accumulation function."

I next coded each explanation or talking turn for the representation the student was using as either numerical, symbolic, or graphical. Numerical representation was identified by the usage of numerical data to make calculations. This was coded as "numerical" regardless of where students got the numbers from—tables, graphs, equations, etc. Symbolic representation was identified by the usage of abstracted symbols rather than numerical calculations. For example, if a student wrote an integral expression to represent the net amount rather than calculating an estimation of that amount, they were working symbolically. Graphical representation was identified by the usage of interpreting or drawing a graph of the given data. Additionally, any mention of “rectangles” to represent the products the students were taking was also coded as graphical since the area of these rectangles is a graphical representation of the products. If within a particular explanation, a student used multiple representations simultaneously, both representations were coded. For example, students often used graphical and numerical representations in tandem as they made calculations based on the graph. They interpreted their numerical calculations in terms of the areas of the rectangles they drew under the curve. The coded lines were not broken down further to separate the representations, as it was often impossible to completely isolate one representation from the other in these cases.

I then coded each explanation in terms of operation, process, or object level thinking. I used the chart in my conceptual breakdown in Chapter 2 to align these codes with the student explanations. In general, the code “operation” was used for singular actions or calculations done by the students. “Process” was a higher order of thinking, where they could imagine the operations they had been doing as continuing without having to carry out each one. For example, in thinking about a variable upper bound, students might first calculate several integrals with different upper bounds. This would be coded as “operation.” Then, as they start to say things such as, “Only the bound is changing, and we can make the bound whatever number we want,” they are exhibiting process-level thinking. Once they conceived of that ongoing process as an entity in itself, the line was coded as “object.” In the variable upper bound example, a student would recognize that the upper bound is a variable itself that can be tracked. Note: if a student’s work for a given integration layer was incorrect, it was coded as “none” for this section of the codes.

I then coded the explanations according to whether the student was using a macroscopic or infinitesimal level for their quantities (Von Korff & Rebello, 2012). If a student was talking about making the intervals as small as possible, this was coded as “infinitesimal.” It is important to note that numerical calculations would not be possible if it was truly infinitesimal, nor could infinitely narrow rectangles be drawn in on the graph. Therefore, in a numerical representation the “infinitesimal” code referred to the shrinking of intervals as far as possible, or if the student spoke of finding data for “every single point.” In a graphical representation, if a student tried to draw a rectangle as narrow as possible, this was considered infinitesimal as well. “Macroscopic” referred to any work at a larger scale than infinitesimal.

Lastly, an additional “lost quantities” code was used when students were operating at a layer of integration farther along than quantity (chop or onward) but were not attending to the quantities in their work. For example, after learning how to write integral expressions, students were clearly using integrals to represent a net amount but were trying to match their integrals to the notation from their notes rather than reasoning about the quantities. In their previous work, we had used $\int_a^b R(t)dt$ to represent the net amount of fuel going into the tank between times a and b. However, in the road construction context, students were still using $R(t)dt$, despite the function and variables being different, because they were trying to match their notes exactly. This episode was coded as “lost quantities.” Once students went back to the quantity layer and corrected their thinking, the line was coded as “returned to quantities.”

It is important to note that my analysis only illustrates the order students progressed through the HLT according to the transcript and video evidence from the interviews. It is entirely possible that a student had understanding of an integration sooner than the data shows. However, I was restricted to the work they wrote or verbalized. During the interviews I encouraged the students to voice their thinking aloud as best they could.

After coding the student explanations, the second phase of analysis was to examine the paths through the integration framework the students took as they progressed through each lesson. To illuminate these paths better, the spreadsheet columns were color-coded for each layer of integration as well as for each representation type. Therefore, if I saw a large block of blue in the representation layer, I could quickly see students were working graphically. Similarly, if I saw stripes of purple and pink in the integration layers column, I could tell that students were transitioning between two layers frequently in that segment.

The codes for both students in the groups closely matched, since they were bouncing ideas off each other. Therefore, I analyzed the paths of the pairs overall rather than individual students. So, rather than analyzing individual understandings, I looked at the shared knowledge or *distributed cognition* (Salomon, 1993), where the knowledge evolved collectively between students through their discourse when each student might not have constructed it in isolation. I began to create rough dot maps like those used in my lesson plans. I first placed a dot where the students began for a given context, then using the color blocked segments, was able to trace out the path of which layers and representations that students were working with. At first, I placed an open dot only when process-level thinking was occurring, then filled these in as object-level understanding was demonstrated. If a student referred to another layer of integration briefly (i.e., in one line of the transcript among many lines of a different layer), I did not add additional arrows to reflect that transition. This helped to maintain clarity of the overall path of the pair's understanding. If the students discussed a previous layer for more than a single line, or returned to that layer more than once, then this was included in the dot maps. These maps were then cleaned up and rendered electronically the same way the dot maps were in the lesson plans so that comparing the theorized and actual trajectories would be simpler. I created dot maps for each pair within each context and lesson, meaning there is a dot map corresponding to each of the hypothesized dot maps from Chapter 2 for each of the three pairs. Specific dot maps will be shown in the Chapter 5 but all maps can be found in Appendix B.

CHAPTER FIVE: RESULTS

In this chapter I summarize the work students did in the interviews. As previously mentioned, maps of the student thinking were created for each context within each lesson. For clarity and brevity, only specific charts have been selected for this selection. All charts for each group, lesson, and context can be found in Appendix B. Note that in order to best portray students' work switching from one type of representation to another, the figures showing their progress do not always have the numeric, symbolic, and graphical representations in the same order across the top of the charts, similarly to the charts of the hypothesized routes in Chapter 3. I also want to clarify that the charts do not imply that students spent equal time or cognitive effort in each layer and representation in their work. I will point out in my description of their work when a particular layer required more significant work for a group.

Lesson One

The first lesson began with the fuel rate context where the fuel flows at a constant rate. All three groups of students followed the hypothesized trajectory of this opening activity (Figure 17). They quickly jumped through the layers within the numerical representation to arrive at the net amount of fuel in the pipe. When asked to elaborate, the students described the quantities and products they used. For example, Brad explained, "So ten liters go in every minute, you stop the timer at four minutes. So, it would just be like four times ten, 40." Students initially called this the number of liters in the tank. I then asked them, "Were we given any information about the tank before the four-minute interval? Would any additional information affect your answer?" Both Alan and Calleigh (in separate groups) suggested that if the tank was smaller than 40 liters, then the amount of fuel in the tank would reach a maximum and the tank might overflow. This was an unexpected answer but does show they were considering the quantities in the context.

Calleigh also suggested that there may have been fuel in the tank before the four-minute interval, which Cassie seemed to agree with.

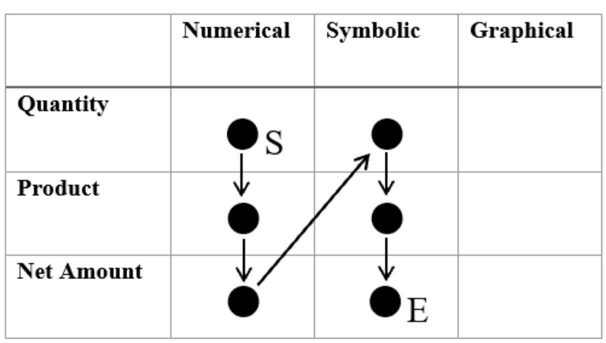
> Cassie: “That’s just the fuel that goes in [the tank]. Like during the four-minute interval it’s not the total possible fuel that’s in there.”
>
> Calleigh: “So you can’t really know exactly how much is already in there, but you know how much just went in, in the four minutes.”

This was evidence of understanding the net amount layer, and the other two groups had similar thinking to the excerpt shown.

Alan, a student in Group A, did make a graphical connection at this stage. He related the beginning amount of fuel in the tank to the y-intercept of the graph, indicating he was thinking of the graph of the amount of fuel over time as a whole. While this is graphical thinking, he was not thinking about these first three layers of integration specifically in the graphical representation. Rather, he was envisioning an early idea of the accumulation function, which is much simpler with the constant rate version of the context.

Figure 17

Paths Through the Framework for All Groups in the Constant Fuel Rate Context

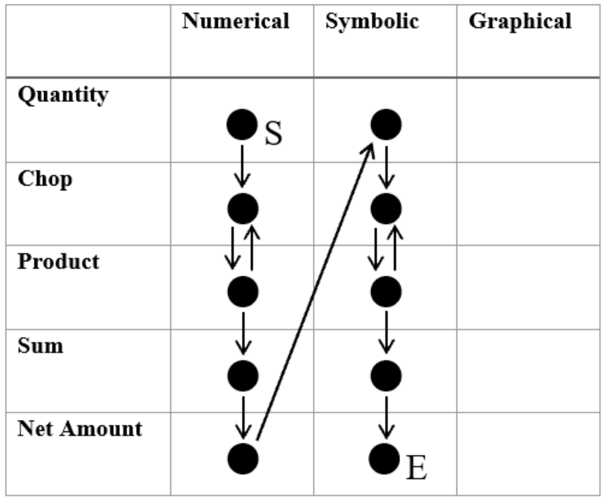


The students were then introduced to a varying rate version of the fuel rate context. The three groups had similar paths again, but they differed slightly in how they referred back to previous steps. All three groups showed substantial instances of looking back at previous steps. The “chop” layer seems to be a critical and difficult point for students, as all three groups went back to it while working in the product layer. Figures 18 below shows where these stages of going back and forth occurred. Recognizing that the product involved a change of time rather than the entire time interval was a step the constant context did not make clear, whereas with the varying rate it was a critical piece to understand. In Group B, Brian automatically began multiplying by a change in time. Brad had not yet begun multiplying and did not recognize where Brian’s numbers came from until he asked, “Where did we get that .75? Oh, you're measuring distance between seconds. I got you.” When asked how this varying context was different from the constant version, Calleigh said, “well it’s kind of more in intervals.” Once this idea was clear, students were able to move forward in the trajectory.

In moving to the symbolic representation, some students were more experienced with sigma notation. For example, Alice wrote “$\Sigma\ (\frac{L}{t})(t)$,” where she was using the variable *t* to represent time and $\frac{L}{t}$ to represent the rate of liters per minute. When asked to describe what each piece meant, Alice said that her variable *t* should be changed to Δt. Then she and Alan were able to arrive at the expression $\Sigma R(t) * \Delta t$. In Groups B and C, the students knew they were finding multiple products of $R(t) * \Delta t$ and adding them but did not know how to represent the sum of all their products. A brief introduction to sigma notation from the interviewer was provided to bridge this gap.

Figure 18

Typical Path Through the Framework for All Groups in the Varying Fuel Rate Context



Following this portion of the lesson, we returned back to the Constant Fuel Rate Context to examine it in the graphical representation. The goal here was to begin from a simpler context again and then extend that understanding to a more complicated situation, like was done previously with the numerical and symbolic representations. All three groups followed the path shown below (Figure 19). Students were asked to graph the flow rate as a function of time and find how their multiplicative work from before could be seen in their new graph. The quantity layer was important here, as most students first tried to graph the amount of fuel as a function of time, rather than the rate. They first had to recognize what quantity was important for this question. An image of Brian's work on the whiteboard is shown in Figure 20 after he recognized the graph of the rate would be a constant value throughout. He described his graph, saying, "Well because this is just constant. It never changes from ten." Students in each group were able to produce this picture. When asked about where they could see the 40 liters they had previously calculated in their picture, some students recognized the area sooner than others. Calleigh said, "The 40 would be like the area of the rectangle." In Group B, I had to ask some questions to focus their attention on the length of the time interval and the height of the rate output. However, they were also able to recognize the area in their image eventually, which can be seen in the shading done on Brian's board work in Figure 20.

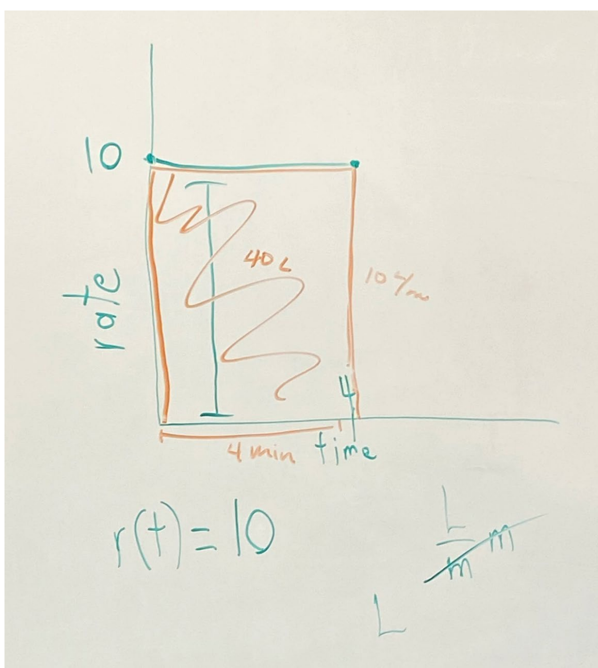
Figure 19

Path Through the Framework for All Groups in the Constant Fuel Rate Context

	Numerical	Symbolic	Graphical
Quantity	●	●	● S
Product	●	●	↓ ●
Net Amount	●	●	↓ ● E

Figure 20

Brian's Graphical Representation of the Constant Fuel Rate Context



Once it was established that the area of that rectangle could represent the product they had calculated, students were directed back to the Varying Fuel Rate Context. All three groups progressed through the integration layers similarly to their work in the Constant Fuel Rate Context, as shown in Figure 21. Alan and Alice's work on the whiteboard is shown in Figure 22. This image was typical of all the groups, except that Alan first drew in an estimated "line of best fit" as he called it, for the given data set. When asked if we knew any of the values between the points on the graph, Alan explained:

> Between zero and 1.25? We don't specifically, but if this was in a graphing calculator, we could calculate, uh, the specific point. Let's say it was at 0.75. We could find the rough value at that point, because it's got to be somewhat decreasing at a specific, constant rate if it's decreasing like this. Because it's not going to be going it's most likely not going to be going 18 liters per minute up until 1.249. And then all of a sudden that's going to drop from that to 12.

He was making assumptions that the rate would not have sudden drops or rises. When asked if there were other possible functions that could fit the data, Alan said, "It could be...a graph of some sort of like, basically something of this sort, where it's just like changing" and drew a step function over his original graph. From here, the rectangles became especially apparent to Alice and Alan, and they were able to move through the integration layers.

Figure 21

Path Through the Framework for All Groups in the Varying Fuel Rate Context

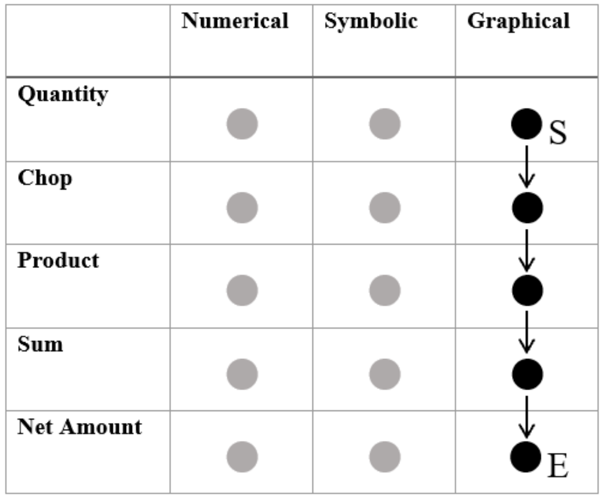
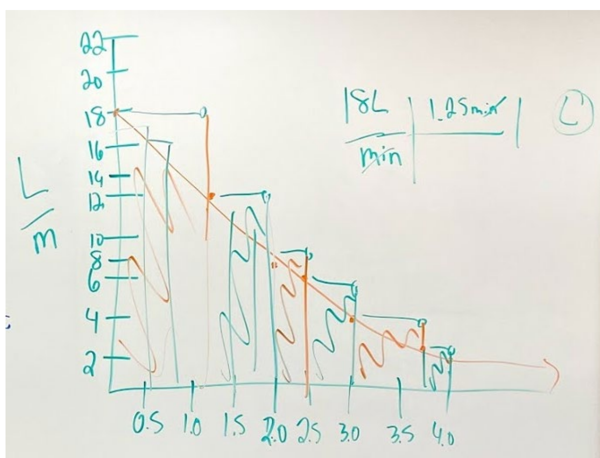


Figure 22

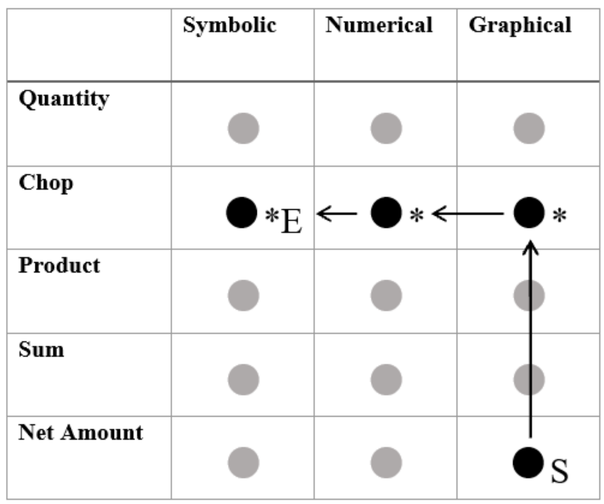
Alan and Alice's Graphical Representation of the Varying Fuel Rate Context



Group A diverged here from Groups B and C, as they finished this portion of the lesson with time to spare. Because of this, they were able to start thinking about how to make their approximations of the amount of fuel in the tank more accurate by chopping the domain into smaller intervals. Therefore, they made the jump into thinking about infinitesimal quantities in this lesson. They described how they would get a better estimate by "drawing more boxes," or having a graph of all the rate values at their disposal rather than tables of selected values. This led to a discussion of how we could symbolically represent the smallest interval possible using limits. This trajectory is summarized in Figure 23 below.

Figure 23

Path Through the Framework for Group A in the Varying Fuel Rate Context



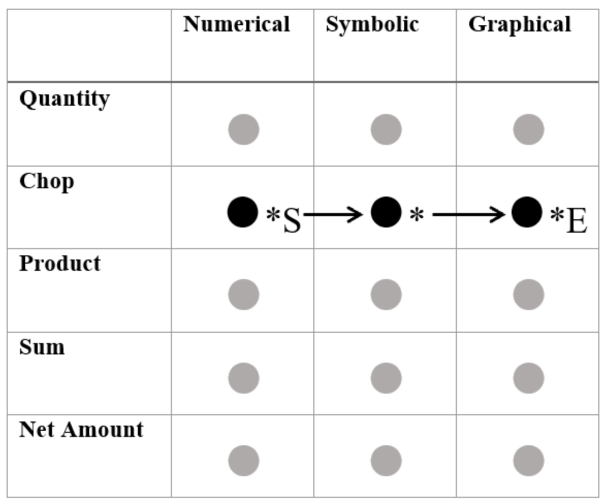
Lesson Two

Fuel Rate Context Continued

Groups B and C began with a recap of what had happened in lesson one so they could reason about how to increase the accuracy of their approximations. One interesting outcome of this is that all four students in these two groups took a different path than the pair in Group A did (Figure 24). The graphical representation was not in the forefront of their minds like it was for Group A, and they began thinking about the numerical context primarily. Rather than describing drawing more rectangles, these students said things such as, "Well, it'd be great if you had the rate for every 0.1 minute." From there, they began to write limit expressions. Cassie said, "Wouldn't we need like infinitely-small time intervals? Maybe like, I feel like it's reminding me of limits, how like the smaller it gets, the more accurate it gets." They could then think about what this would do to the rectangles they were drawing. I referred them back to their rectangles, and Cassie said, "We're not getting the actual area under the curve, we're just getting areas, like rectangles that are kind of by the curve." I asked both students how the rectangles could "fit better," and they both recognized the smaller the rectangles got, the better it would be. Calleigh said we would ideally want "infinitesimally small chunks." Cassie chimed in, calling them "baby rectangles."

Figure 24

Path Through the Framework for Groups B & C in the Varying Fuel Rate Context



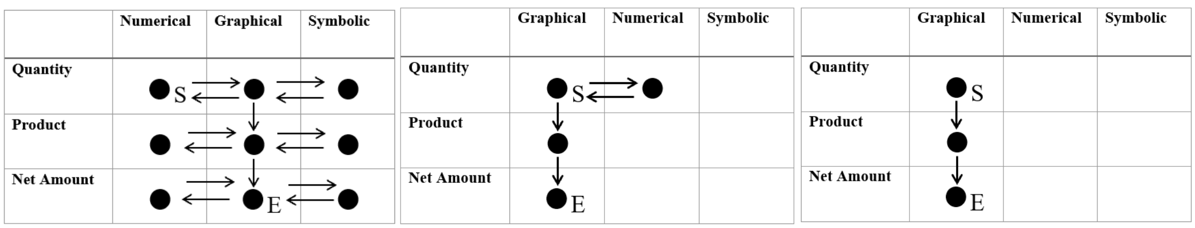
Road Construction Context

Similar to the Fuel Rate Context, the Road Construction Context began with a constant weight per foot of dirt. Since the purpose of this prompt was to get students accustomed to the quantities and units on the graph, each group did not necessarily cover every representation. However, all three groups did hit every integral layer within some representation. Group A discussed all three representations within this constant rate version of the context, whereas Group

C only used the graphical representation. Group B used a combination of numerical and graphical representations as shown in Figure 25.

Figure 25

Paths Through the Framework for All Groups in the Constant Weight Road Construction Context, with Group A on the Left, Group B in the Middle, and Group C on the Right



Following the constant context, the rate of the weight of the dirt over distance began to vary. This led to some divergence in the different groups, with Group B being very different from Groups A and C. A and C were very similar in the overall flow of their thinking throughout the map (Figure 26). Both groups worked in the numerical and graphical representations simultaneously as they calculated products they represented as rectangles on their graphs. Then after being asked how their work translated symbolically, these students were able to produce integral expressions to represent the net change in the weight of dirt. However, Group B surprisingly did not deal much within the numerical context (Figure 27). Brian latched to the symbolic integral notation very strongly, so while Brad was more focused on drawing in rectangles, Brian was writing integral expressions. Eventually, both students were able to explain both the graphical and symbolic work going on. While interesting that they did not produce any numerical calculations here, they had previously shown they were able to do it in both the first lesson and the lessons to follow. Both students felt that the integral was a better answer since it was exact.

Figure 26

Path Through the Framework for Groups A and C in the Varying Weight Road Construction Context

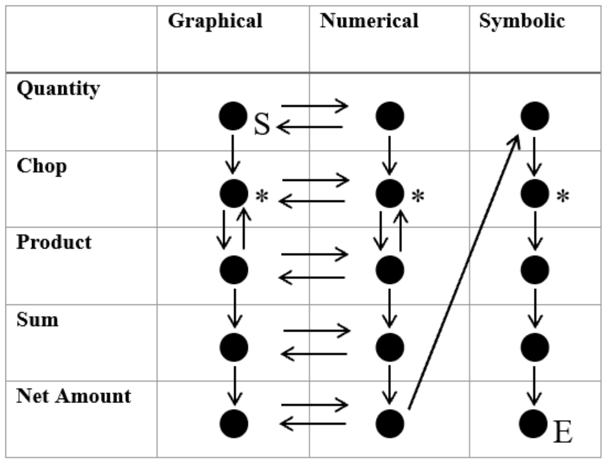
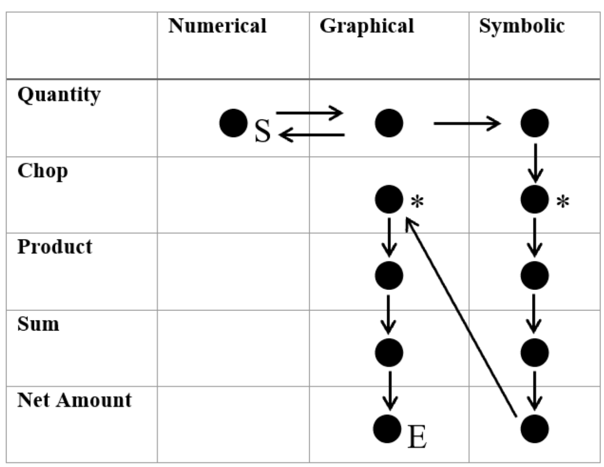


Figure 27

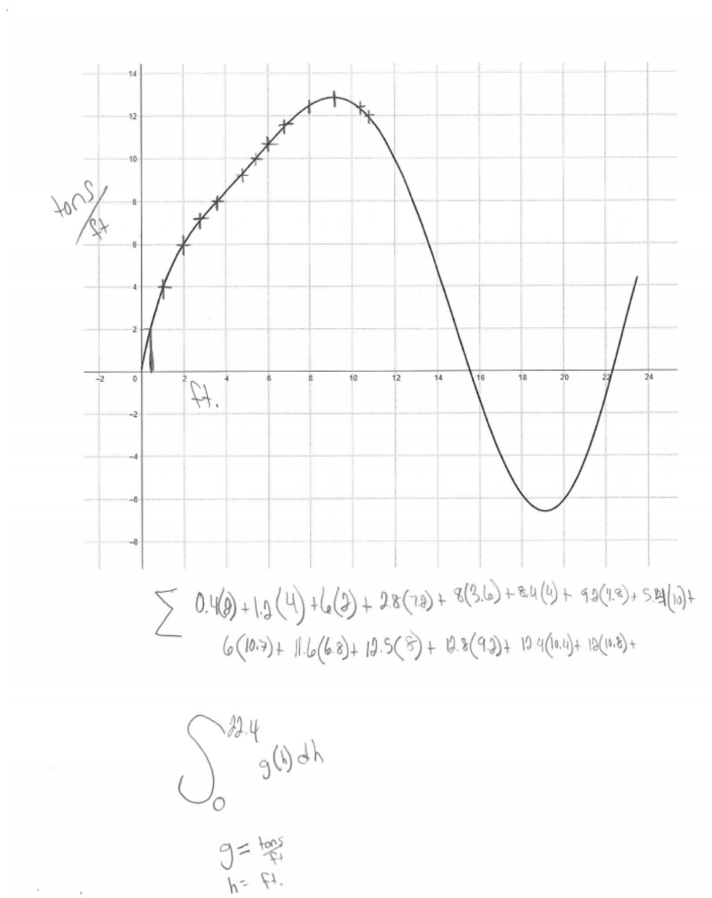
Path Through the Framework for Group B in the Varying Weight Road Construction Context



Overall, by this point all six students had shown object-level thinking of every integration layer up through the net amount layer. One example of this is Alan's work shown in Figure 28 below. His work demonstrates this thinking within all three representations. First, in his graph he began drawing in a rectangle but recognized there would be more of these without needing to fill them in. He switched to “x’s” to represent the height, and then left those off altogether. He transitioned to numerical work where he began calculating those products. It should be noted he did not complete his work here, because his partner began finding some products. Lastly, he wrote the integral expression for the entire domain. He included the units for each symbol in his product, showing he was keeping the quantities in mind even at this last representation.

Figure 28

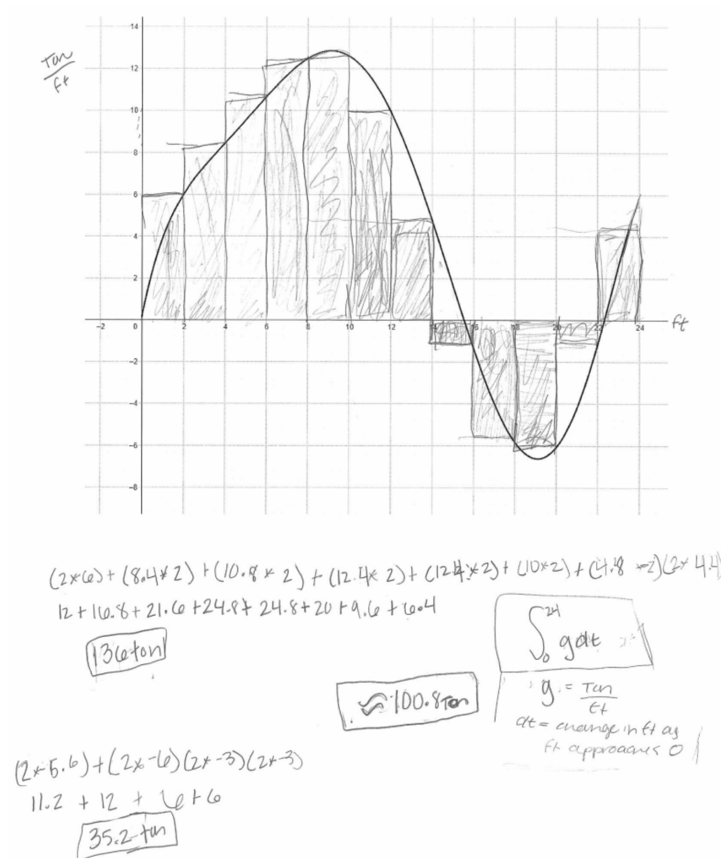
Alan's work on the Varying Weight Road Construction Context



Alan's partner Alice had similar work, but she made her rectangles evenly spaced (Figure 29). This made her numerical work easier. Alan said his goal in choosing values was to keep the rectangles small and to have values that he could accurately ascertain from the graph because they fell on one of the graph lines.

Figure 29

Alice's work on the Varying Weight Road Construction Context



Lesson Three

Expanding Meaning of the Integrand

As explained earlier in the lesson plans, at this stage the students had only been exposed to integrals where the integrand was easily interpreted as a rate. When asked what a general definite integral, $\int_a^b f(x)\,dx$, meant, the students interpreted f(x) to be the rate of something, multiplied by a small piece of the input. This is when we introduced them to two short contexts where the integrand was not generally conceived of as a rate.

When first presented with the Volume of a Solid Context, Group C struggled to make sense of what *A(x) dx* would represent. They tried to impose their ideas of rates on this context. For example, below is a dialogue from Group C.

> Cassie: "Wouldn't it give you the rate of change in uh, feet squared of the area of the cross-section as feet increase?"
>
> Calleigh: "It would give you feet I think, wouldn't it?"
>
> Interviewer: "So what, um, operation are you doing to get to feet or to get to a rate that makes sense? Like, why did you say, I think it would be feet?"
>
> Calleigh: "I was just multiplying them together. Oh, I guess not. I don't know."

The interviewer directed students back to the context to determine what *A* represented. Building from their ideas of multiplication, the students were quick to realize the resulting quantity would be a volume rather than length or area. Calleigh said, "You would get the volume? Because this is area and you're timesing [sic] it by distance. That would make it feet cubed, which is a unit of volume." Cassie agreed and added, "Yeah, the volume of the shape if the shape is 83 feet long." Groups A and B did not make the same mistake, despite describing the integrand as a rate for a general integral. It seems their conceptualization of the product layer was stronger than this assumption, so they expanded their definitions of integrals.

Returning to Constant Fuel Rate Context

Recall that the next phase of the HLT was to extend students beyond definite integrals with a fixed bound to begin thinking of variable upper bounds. All three groups followed the anticipated trajectory for extending the upper bound for the Constant Fuel Rate Context (Figure 30). They calculated and interpreted new products where the time interval's upper bound changes and were able to symbolically represent the changing bound with a variable. Brian was the exception, as he wrote a general integral first with an x as the upper bound (Figure 31). He initially wrote $10t\ dt$inside the integral. When asked what the rate would be after two minutes, he realized his mistake and erased the extra t, as his work in Figure 31 shows. His partner was working numerically from the start like the other groups did, then began working with Brian once he was ready to work symbolically.

When prompted by the interviewer, they could explain how this would appear graphically, where the fuel increased at a constant rate producing a linear graph. In fact, this confused the students in Group A. Alan said, “I didn't see the need and I didn't really make the connection and make an integral. I just recognized it's a very simple rate. It's going to be some sort of linear line that I can follow. And I'm just calculating a certain specific point.” Alice then doubted their previous work, questioning the need for an integral in the first place. The students were not incorrect here—since the rate was constant, the net amount of fuel could be calculated without integral notation. However, looking back at their equation, they recognized their set up still made sense. We discussed how you could still chop up the interval into infinitely small intervals and find the amount accumulated in that time, even if the rate is constant.

Figure 30

Path Through the Framework for All Groups in the Constant Fuel Rate Context

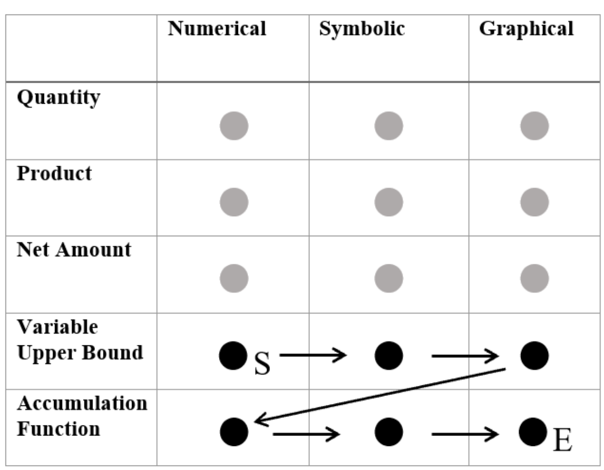
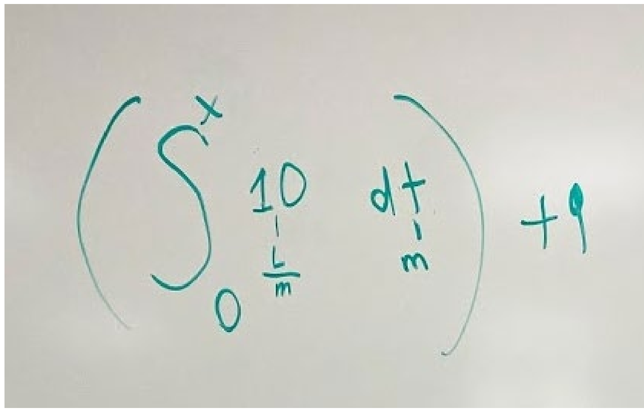


Figure 31

Brian's Whiteboard Work for Extending the Constant Fuel Rate Context



Accumulating in the Varying Fuel Rate Context

Working in the Varying Fuel Rate Context, the trajectory of Group A closely matched the anticipated path (Figure 32), whereas Groups B and C had different trajectories that were similar to each other (Figure 33). The most notable difference was that Groups B and C moved between variable upper bound and accumulation function within the symbolic representation, rather than the graphical representation like Group A. However, this did not seem to impact their understanding. Cassie's symbolic work is shown in Figure 34 and is identical to the work from Group B, as well as the work from Group A when they eventually reached that stage.

One similarity in all three groups was a difficulty with producing an accurate graph of the accumulation function. All six students knew what the graph would represent, but producing an accurate sketch was difficult for them. For this reason, rather than a solid dot in the map, I have used a dotted outline to show they did not yet have an object-level understanding. They understood the process of tracking the accumulated products. They also recognized that the amount of fuel should be increasing over time since no fuel was leaving the tank, but they could not justify the shape of the curves. Solidifying this idea became a primary goal of the fourth lesson.

Figure 32

Path Through the Framework for Group A in the Varying Fuel Rate Context

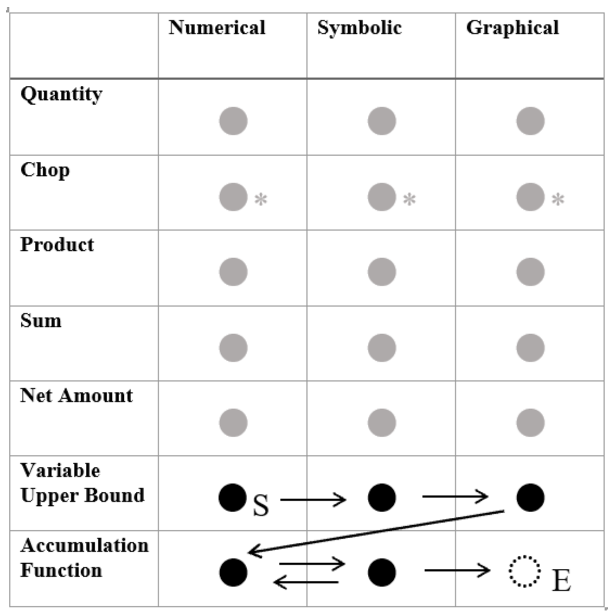


Figure 33

Path Through the Framework for Groups B and C in the Varying Fuel Rate Context

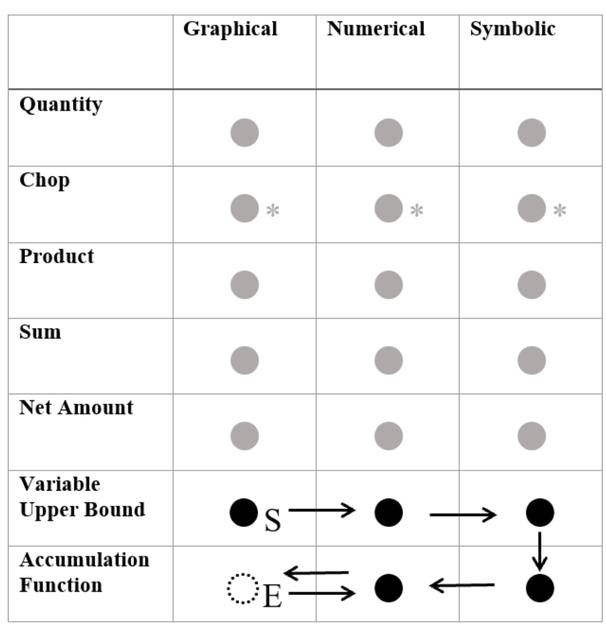
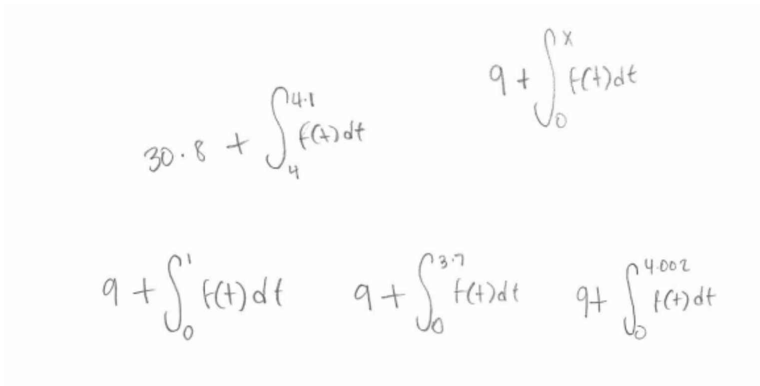


Figure 34

Cassie's Symbolic Work for the Varying Fuel Context, Typical of All Groups, Where Group A Arrived at This Stage Later



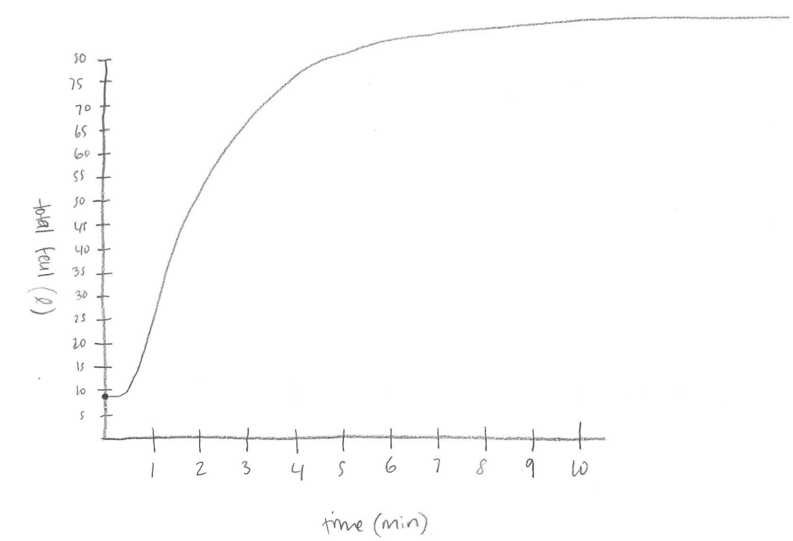
Lesson Four

Finishing the Varying Fuel Rate Accumulation Graphs

I had the students look back at the Varying Fuel Rate Context from the previous lesson. This was because students had not yet demonstrated process-level understanding of the accumulation function graphically in previous interviews. Drawing the graph of the rate up on the whiteboard, the students were then asked to reason about the amount of fuel gained from each additional product. For example, if the rate is high, that would correspond to a taller rectangle on the graph. This would be a larger value of the product of this rate with a small time interval. If the rate decreases, the amount still increases but not as much. As Brad described, "It slows down. So you're going to get less fuel. So it's going to be like a smaller rectangle." This made graphing the accumulation function much easier for students to conceptualize. Figure 35 below is a representative graph of functions students sketched.

Figure 35

Cassie's Graph of the Accumulated Fuel



Accumulation in the Road Construction Context

Students then seemed to have an understanding of the underlying process for the accumulation function graph. Moving to the Road Construction Context, students used a similar tactic of assessing the relative size of the added rectangles as the distance increased. Groups A and B had similar trajectories (Figure 36), where they first began with numerical calculations for various bounds on the integrals, representing this symbolically, then graphing the accumulated dirt.

Group C differed slightly, in a way that was also different from the anticipated trajectory. Their trajectory is summarized by Figure 37. They began and ended in the graphical representation but had lots of movement between the representations throughout both the variable upper bound layer and accumulation function layer. For example, while they began with the given graph, Cassie quickly moved to a symbolic representation, saying, "We need to write an integral right? If we want to get the exact amount." They then reasoned numerically about if their integral included the “pit” on the graph, or if they needed to subtract a separate integral to represent filling the “pit.” Calleigh says their integral already covers this subtraction, and Cassie added, “So it'll like add the negative, which will make it even out.” This thinking was similar to that of the other students, it just occurred closer to the beginning of Group C’s work on the task. The different trajectories do not seem to impact the kind of understanding students gained. All ended with fairly accurate graphs of the accumulated amount of dirt and showed evidence they understood the underlying process to create the graph.

Figure 36

Path Through the Framework for Groups A (Left) and B (Right) in the Varying Weight Road Construction Context

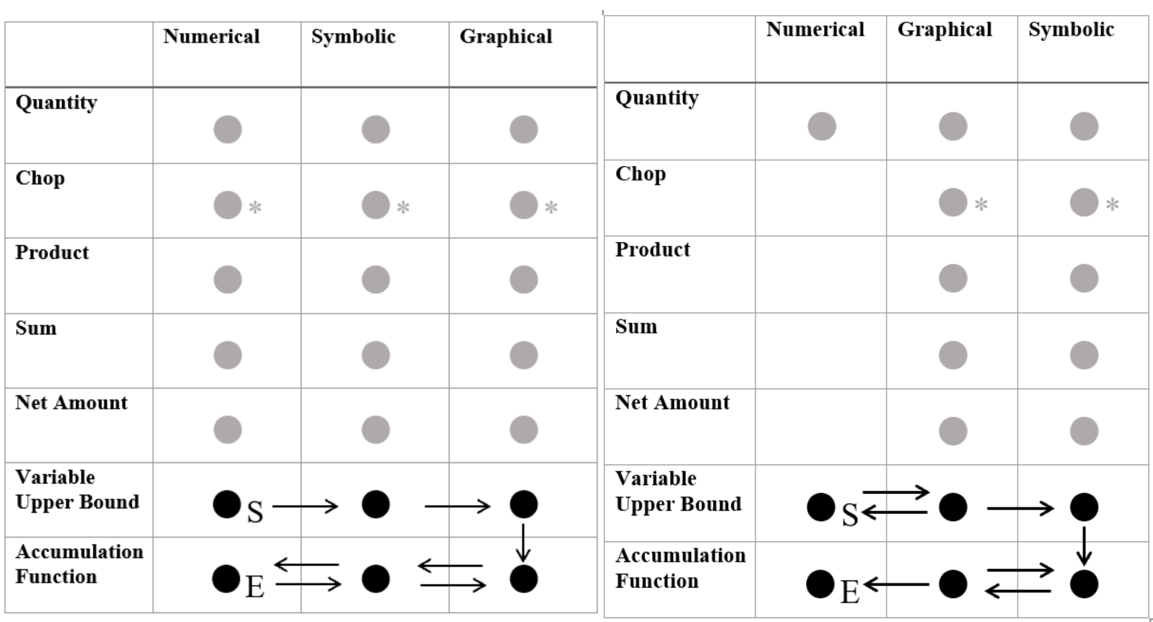
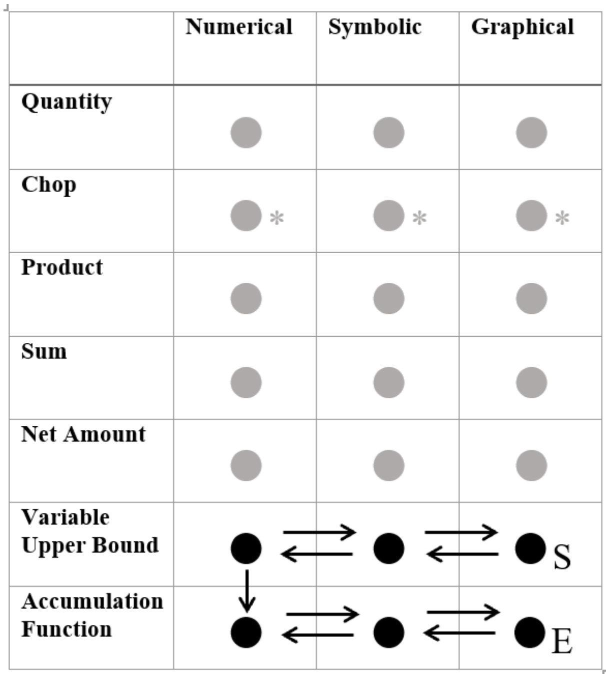


Figure 37

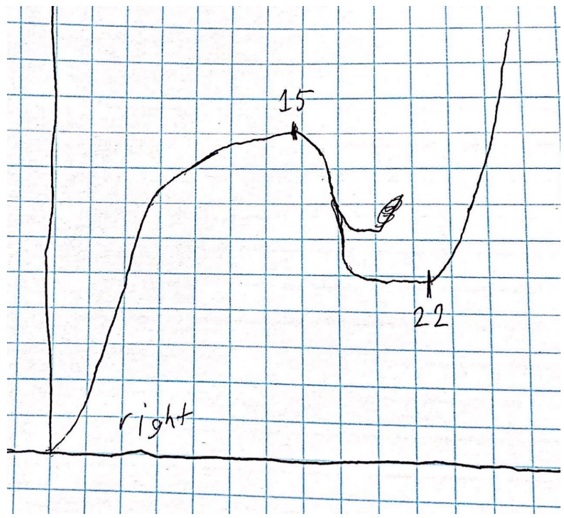
Path Through the Framework for Group C in the Varying Weight Road Construction Context



While creating the graph of the accumulated dirt, students began making connections between their new graph and the original graph of the rate. An example of Brian's work is shown in Figure 38. The marked 15 and 22 on his graph represented amounts of feet along the horizontal axis, as he explained verbally in the interview.

Figure 38

Brian's Graph of the Accumulated Dirt



Brian and Brad (along with the two pairs) recognized that when the given function had an output of zero, no dirt was gained or lost which would correspond to plateauing in their sketched graphs. These are the points at 15 and 22 on Brian's graph (Figure 38). When asked at what point on the accumulation graph would correspond to gaining the most dirt for a given stretch of distance, Cassie answered, "It'd be the highest point on it. Wait no. It would be the steepest point." Which was simultaneously confirmed by Calleigh. All three pairs made these observations.

Revisiting the Volume of a Solid Context

There was some time at the end of the fourth interview, so I referred students back to the Volume of a Solid Context to see how they interpreted accumulation functions in that context. At first glance, students recognized that if we were to graph the accumulation function $H(X) = \int_0^X A(x)dx$, the volume would increase as X increases. However, there was some uncertainty about the concavity of the function. Below is an exchange from Group C that illustrates this.

Cassie: “H(X) is the volume of the shape. The volume as a function of how far you’ve gone along.”

Interviewer: “Oh, okay. So it’s not necessarily a set point?”

Calleigh: “No because X is a variable. You don’t know what X is, it could be as long as you want it to be.”

Interviewer: “Can you draw for me the general shape of the graph of H?

Cassie: “Well, volume starts off small and then it just gets bigger. It's pretty steady. So wouldn't that just be like, it starts at zero.”

Calleigh: “Oh wait...doesn't it...just go like that [draws a linear positive slope].

Cassie: “Yeah. Wait, would it be exponential?”

I presented the students with three different increasing functions—a curve with negative concavity, a curve with positive concavity, and a line. Both students chose the curve with positive concavity. Calleigh described why that shape made the most sense, saying, “You said x is going that way [to the right], So it's going to start with a smaller sliver and then slowly the area is going to get bigger for every sliver.” Alice described it similarly:

> So as the area increases, the volume will increase at a larger rate because the—I think I agree with what [Alan] said. I think he said it really well. But we know that the volume is going to get exponen—not exponentially. It's going to get larger because there's going to be a larger area because it's going to, because each section is going to continue to get bigger.

Note that all six students in the study described this curve as being exponential, and while the function is not exponential, that was not the mathematical understanding being assessed. In Group A, this misconception was addressed due to a little extra time in the interview.

These responses were very promising and showed the students were able to translate their accumulation function work into this other context.

Toward the Fundamental Theorem of Calculus

Towards the end of the fourth interview, the students were beginning to conjecture the connection between integrals and derivatives. I began asking students about connections between the graph they were given and the accumulation graph they drew. Below is an example of this questioning with Group B, but all three groups had similar exchanges.

> Interviewer: “What about when [the given graph] is negative? What do you notice about your graph?”
>
> Brad: “The slope was negative because it was going down.”
>
> Interviewer: “What do you notice when [the given graph] is positive?”
>
> Brad: “Um, the slope was going up.”
>
> Interviewer: “What does this tell you?”
>
> Brad: “When it’s positive it goes up, when it’s negative it goes down.”
>
> Interviewer: “What does this tell you about the two graphs in relation to each other?”
>
> Brain: “It means that, it means that the value of our initial graph is kind of providing the instantaneous slopes of our graph right here.”

There had not been previous mention of derivatives or instantaneous rates of change before this point. When working with the Road Construction Context, Alice said, “So this here [the given graph] is the derivative…we found the original function...So there's a way to go, like once you have a derivative, there's a way to like go back from it.” Cassie called the process they had just done of graphing the integral “reverse differentiating.” It should be noted that the graph given to students did show a rate function, however, this was not the justification students were using to say it was the derivative of their graphs. In fact, after conjecturing the connection between the graphs, Calleigh said, “Oh wait yeah. That makes sense, duh. Wait, we literally could have figured that out.” The students later realized they could have drawn this conclusion based on the units of the graph *after* they developed their hypothesis.

The students were right at the cusp of constructing the Fundamental Theorem of Calculus on their own accord, and they showed a lot of excitement over the connections they were making. This can be summed up by Calleigh’s statement: “Oh it's like if you have the derivative, doing it the other way around, oh my gosh. So like the derivative, if you started with the original function, you have the derivative definition, but then if you start with the derivative, you use the integral to find the original function!”

CHAPTER SIX: DISCUSSION

In this chapter, I will first summarize my findings to answer my research question. Then, I will discuss how this study connects and builds upon the literature. Finally, I will address the limitations of the study and potential ideas for further research.

Answering the Research Question

My research question was: as a student progresses through the HLT, what understandings do they have of the definite integral and accumulation function? According to the data, the students had process- and object-level understanding of each layer of integration within each representation at some point in the interview process. They also regularly made connections back to quantities in the context. The actual routes throughout the layers and representations differed slightly from the original HLT, but mostly in movement between representations and less in the movement between layers. For example, see the anticipated path for the Varying Fuel Rate Context in Figure 39 as compared to the actual paths in Figure 40. The overall paths are identical to the anticipated path, with some retracing movements occurring between the layers.

Figure 39

Anticipated Path Through the Framework for the Varying Fuel Rate Context

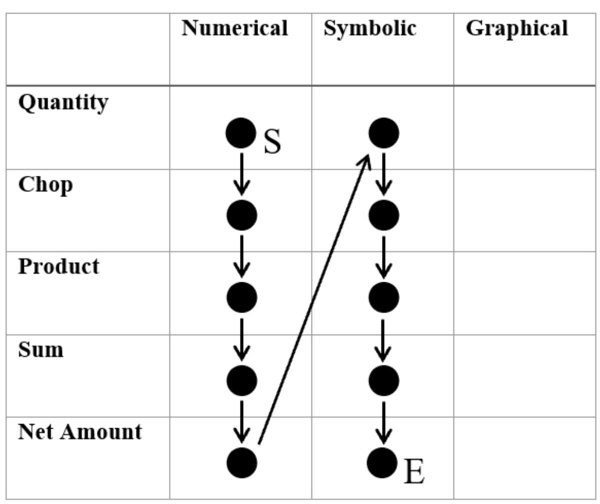
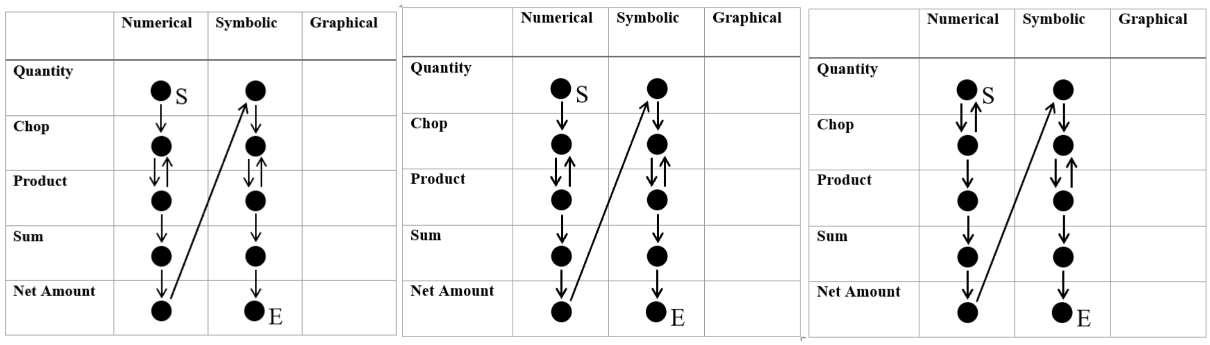


Figure 40

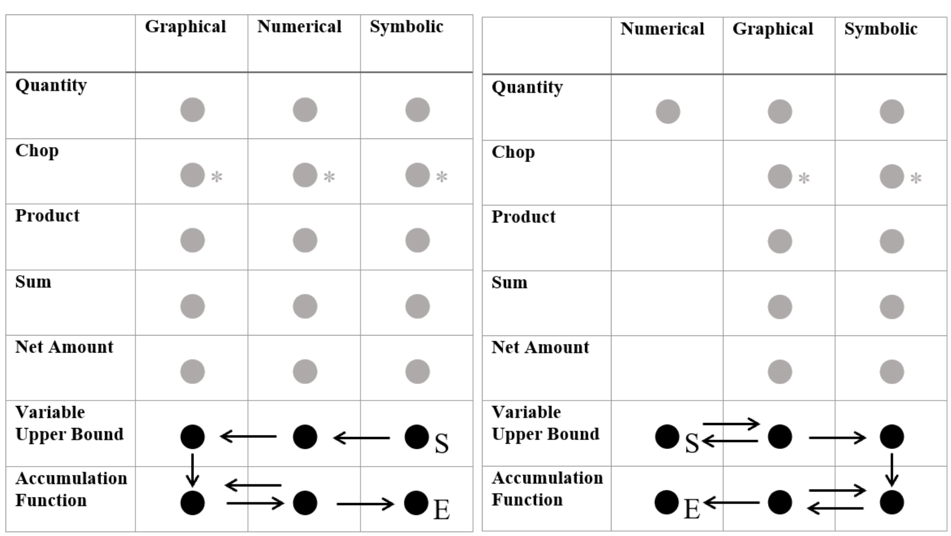
Actual Paths Through Framework for the Varying Fuel Rate Context in Order of Group A, B, Then C



The most drastic differences in the anticipated and actual paths occurred with Group B, as they did not attend to numerical calculations in their Varying Weight Road Construction Context work as thoroughly as the other groups. Due to time constraints and their previous evidence of numerical understanding, I did not push this pair away from their graphical and symbolic work because they were making great connections.

Figure 41

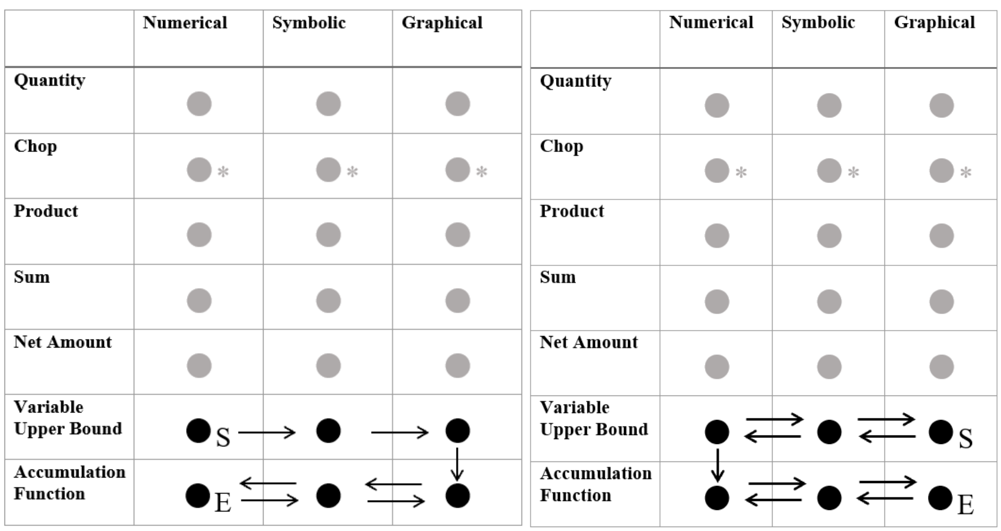
Anticipated Path Through the Framework in the Varying Weight Road Construction Context (Left) Compared to the Actual Path for Group B (Right)



Other than this instance, only small deviations from the anticipated path occurred, which is to be expected. For example, comparing the paths for Groups A and C for the same Varying Weight Road Construction Context show that each group began in different representations based on what felt most comfortable and what came naturally out of the interviewer's questions. This suggests in teaching the lessons, students may direct the lesson towards a different order of representations. However, they can make connections across representations when prompted.

Figure 42

Actual Paths Through the Framework for Group A (Left) and Group C (Right) in the Varying Weight Road Construction Context



Utilizing each representation was crucial for students to have a complete view of the underlying quantitative structure of the integral. Students drew on all three representations frequently to be able to fully describe what an integral represented. When they were insecure or incorrect about their answers, referring back to the initial quantity layer helped them to make sense of their work. Overall, the HLT seems to be a viable pathway for student understanding of the definite integral and accumulation function.

Jones (2015a) found that an “adding up pieces” (AUP) perspective of integration was most useful for interpreting real-world integrals. The students in this study constructed a rich AUP meaning for definite integrals. For example, in lesson three when discussing the Volume of a Solid Volume Context, Brian described, “So we're adding up all of the little volume discs from the x values of zero to 83.” Even when using the idea of the integral as the area under a curve, when pressed the students could explain how they were adding up small bits of area to get the larger area. Rather than thinking of the entire area under the curve as one piece, they were still thinking of adding up bits to make up that area.

When students were only thinking of the integral as the area under a curve, they often lost sight of the quantities as well. In the same Solid Volume Context, Alice first described the integral as “the area of this kind of this cone from zero to 83 [feet].” However, she quickly realized when looking back at the quantities that the multiplication of feet-squared by feet should produce a volume. This led her to thinking about the discs of volume being added up, rather than her initial “area under a curve” interpretation. Alice seemed to be somewhat of an anomaly in this study. The only qualification given for participation in the study was to have no previous calculus experience, but she frequently brought up concepts from a college physical course throughout her work. She was the only student who had any preconceived notions of integrations influencing her thinking, which may be why she was more inclined to think of areas under a curve. However, she was still able to make sense of the integral through AUP with minimal questioning from her peer or the interviewer.

Students generally moved through the process and object levels of understanding quickly in the first five layers of integration (quantity, chop, product, sum, and net amount) in the first two lessons, whereas the last two layers (variable upper bound and accumulation function) took up two lessons. I believe this to be the case because the underlying mathematics in multiplying two quantities and adding the products is not new content for calculus students. The integral is based on simple mathematical ideas. The concept of “chopping up” the quantities was not likely something students had seen before, which may explain why more significant mental work was spent moving through that layer in the first two interviews. Likewise, varying the upper bound and creating an accumulation function were new concepts for students and took longer for them to develop their understanding.

Contributions of the Study

The first contribution of this thesis to the literature is the inclusion of various representations in an integration framework. I have adapted the representations or contexts from Zandieh’s (2000) and Roundy et al.’s (2015) framework for derivatives into a corresponding framework for integration built from Sealey’s (2014) and Von Korff and Rebello’s (2012) previous integration frameworks. This can allow us to more closely examine the different ways students reason about integration among numbers, graphs, and symbols. The layers in Sealey’s (2014) framework for integration could be clearly seen in the students’ verbal explanations and written work. The added layer of “quantity” by Von Korff and Rebello (2012) also played a key role in the students’ work. The data shows significant work being done in each of these layers on the way towards understanding the definite integral and accumulation function.

In addition to bringing these frameworks together, this integration framework adds two additional layers of integral understanding. Building from Jones’ (2014; in press) work on the action of “chopping " in AUP, I have incorporated a “chop” layer between quantity and product. A critical step for understanding in the interviews was for students to realize that they were working with an interval of their domain quantity rather than the quantity itself. For example, for the Fuel Rate Context, students had to multiply the rate by the change in time, or how long they were assuming the fuel flowed at a given rate, and not the actual time that rate occurred. While connected to the quantity and product layers, this felt like a different understanding. From an AUP perspective, the action of chopping is key in making sense of the integral. The domain must be broken into chunks of either macroscopic or infinitesimal size to create the pieces that will be added up.

Similarly, another crucial concept for the students to understand was exactly what quantity their integral work produced. During early pilot studies, students were often saying the amount was the total of some quantity, not taking into account any amount that accrued outside of the bounds of the integral. After confronting this issue initially in the Fuel Rate Context, the students frequently asked the interviewer if there was any fuel in the tank before the time interval given began. Or, in the Road Construction Context, they recognized the engineers could have had dirt in their truck from the day before. These observations showed they were thinking about the integral as a net change, rather than a total amount. This critical distinction did not seem to fit well within the existing framework, so we incorporated a layer of "net amount" after the sum layer. This seems to be an area of understanding that is overlooked by the current literature.

Another key contribution of this thesis is a hypothetical learning trajectory built from previous research (Jones, 2014, in press; Ely, 2017). This HLT covers the development of definite integrals and accumulation functions, most of the major topics of integration in a first-semester calculus class. In addition to this HLT, this study provides empirical documentation of how students actually progressed through the integration framework based on the activities from the HLT. The data shows students had rich understandings of the layers of integration and the HLT proved to be a viable way of describing their learning path.

Implications of the Study

This thesis implies there is another way to foreground quantitative meanings for integration by developing AUP in definite integrals first and accumulation functions after. This path to understanding is an alternative to the path already established in the literature that begins with accumulation functions (Thompson & Silverman, 2008; Yerushalmy & Swidan, 2011). Additionally, this method builds key AUP meanings early without students needing to meet the higher standard of accumulation functions. This could provide a more accessible path for students to learn integration with an AUP perspective.

Additionally, this thesis implies a way of attaching various representations (numerical, symbolic, graphical) to the quantitative meaning of AUP. The student thinking eliciting from the learning activities shows the different ways they explained the same quantitative structure among different representations. For example, students spoke about products of two numbers, two abstract symbols, and of two lengths on a graph to produce the area of a rectangle. The multiplicative structure underlies all three representations, but they appear to have different meanings to students in different representations. This expands the work of Sealey (2014) and Von Korff and Rebello (2012), as their frameworks do not consider the differences in these representations. Overall, this study implied a way of joining various perspectives of understanding integration into one systematic approach.

Limitations of the Study

There are several limitations of the study to be addressed. First, the sample size of students was small. The study represents the thinking of six students, all of whom came from the same first-semester calculus course. Therefore, their thinking of derivatives and limits was influenced by the instructor of the course. Students with different perspectives or knowledge bases of prior calculus material may be more or less primed to jump into quantitative reasoning at the start of the HLT. The students selected would only represent a subset of university majors, as well as other factors that could affect the results of the study. However, these students still provided valuable insight into the thinking involved in integration. The data shows that the proposed HLT is one promising path towards understanding.

A related limitation is the generalizability to larger groups of students. As the students were interviewed in pairs, it is not yet known how this thinking could be similarly facilitated in a full classroom setting. However, the key questions and contexts from these lesson plans did seem helpful in getting students to discuss the concepts among themselves. Group or pair work could be utilized in a class setting, but further research would be needed to see if the HLT still appears viable.

Ideas for Future Research

One promising aspect of the HLT is how naturally it led towards the Fundamental Theorem of Calculus. The students were noticing connections between the accumulated amount of some quantity and the rate function of that quantity throughout the lessons. By the third and fourth lessons, the students were frequently using phrases such as, "[The given rate graph] is the derivative, and we drew the original function." One reason this may have been so apparent to students is because the contexts used were so rate heavy. The connection between integrals and derivatives may not have been so clear if we were using the integral for the volume of a solid, for example. However, rates and the accumulation of a quantity at a given rate are logically connected to each other and the students were excited to find that connection themselves. The next step for these students would be to have them calculate the exact values of their integral expressions using this connection. Future research could see exactly how to lead students to build this idea in a meaningful way.

Another interesting finding was that students seemed to only discuss the infinitesimal level within the chop layer, saying things such as, "We want to chop into lots of small intervals." While the product of an infinitesimally small interval with the integrand would produce an infinitesimally small amount, the students did not talk about this amount explicitly in this way. They talked about having little intervals, or infinitely many intervals, showing that they were thinking at the infinitesimal level. However, it seemed that once they made the jump to infinitesimally chopping, the other layers did not require mentioning the infinitesimal nature and it was not brought up again. Von Korff and Rebello (2012) illustrated what that infinitesimal thinking might look like in other layers, but further research could examine how students differentiate their thinking at the infinitesimal level from their macroscopic reasoning.

Lastly, this study only examines initial work on integration. Students who continue beyond first-semester calculus will use integration frequently. Further research could examine how this quantitative understanding of integration can extend to other concepts of integration, such as u-substitutions, improper integrals, or calculating double integrals.

REFERENCES

Bressoud, D. M., Carlson, M. P., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insight from the Mathematical Association of America national study. *International Journal of Mathematical Education in Science and Technology, 44*(5), 685-698. https://doi.org/10.1080/0020739X.2013.798874

Carlson, M. P. (1998). A cross-sectional investigation of the development of the function concept. In J. Kaput, A. H. Schoenfeld, & E. Dubinsky (Eds.), *Research in collegiate mathematics education* (Vol. 7, pp. 114-162). Mathematical Association of America.

Chhetri, K., & Oehrtman, M. (2015). The equation has particles! How calculus students construct definite integral models. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*. Pittsburgh, PA.

Clements, D. H., & Sarama, J. (2004). Learning trajectories in mathematics education. *Mathematical Thinking and Learning, 6*(2), 81-89.

Davis, R. B. (1983). Complex mathematical cognition. In H. P. Ginsburg (Ed.), *The development of mathematical thinking* (pp. 254-290). Academic Press.

Davis, R. B. (1984). *Learning mathematics: The cognitive science approach to mathematics education*. Ablex.

Dienes, Z. P. (1960). *Building up mathematics*. London: Hutchinson Educational.

Dubinsky, E. (1986). Reflective abstraction and computer experiences: A new approach to teaching theoretical mathematics. In G. Lappan, & R. Even (Eds.), *Proceedings of the eighth annual meeting of PME-NA*. E. Lansing, MI.

Dubinsky, E. (1991). Reflective abstraction in advanced mathematical thinking. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 95-123). Kluwer Academic Publishing.

Ellis, A. B. (2011). Algebra in the middle school: Developing functional relationships through quantitative reasoning. In J. Cai & E. J. Knuth (Eds.), *Early algebraization: A global dialogue from multiple perspectives*. Springer.

Ely, R. (2017). Definite integral registers using infinitesimals. *The Journal of Mathematical Behavior, 48*, 152-167. https://doi.org/10.1016/j.jmathb.2017.10.002

Gray, E. M., & Tall, D. O. (1994). Duality, ambiguity, and flexibility: A proceptual view of simple arithmetic. *The Journal for Research in Mathematics Education, 26*, 115-141. https://www.jstor.org/stable/749505

Johnson, H. L. (2016). Quantitative reasoning in mathematics education: Directions in research and practice. In R. A. Duschl & A. Bismark (Eds.), *Reconceptualizing STEM education: The central role of practices* (pp. 149-166). Routledge.

Jones, S. R. (2013). Understanding the integral: Students' symbolic forms. *The Journal of Mathematical Behavior, 32*(2), 122-141. https://doi.org/10.1016/j.jmathb.2012.12.004

Jones, S. R. (2014). Adding it all up: Reconceiving the introduction of the integral. *Mathematics Teacher, 107*(5), 372-377. https://www.jstor.org/stable/10.5951/mathteacher.107.5.0372

Jones, S. R. (2015a). Areas, anti-derivatives, and adding up pieces: Definite integrals in pure mathematics and applied science contexts. *The Journal of Mathematical Behavior, 38*, 9-28. https://doi.org/10.1016/j.jmathb.2015.01.001

Jones, S. R. (2015b). The prevalence of area-under-a-curve and anti-derivative conceptions over Riemann sum-based conceptions in students' explanations of definite integrals.

International Journal of Mathematical Education in Science and Technology, 46(5), 721-736. https://doi.org/10.1080/0020739X.2014.1001454

Jones, S. R. (2017). An exploratory study on student understandings of derivatives in real-world, non-kinematics contexts. *The Journal of Mathematical Behavior, 45*, 95-110. https://doi.org/10.1016/j.jmathb.2016.11.002

Jones, S. R. (In press). Introducing definite integrals quantitatively through adding up pieces. In D. L. Corey & S. R. Jones (Eds.), *Building a knowledge base for teaching undergraduate mathematics: Lesson analysis*. Mathematical Association of America.

Jones, S. R., Lim, Y., & Chandler, K. R. (2016). Teaching integration: How certain instructional moves may undermine the potential conceptual value of the Riemann Sum and the Riemann Integral. *International Journal of Mathematical Education in Science and Technology, 15*(6), 1075-1095. https://doi.org/10.1007/s10763-016-9731-0

Lobato, J., & Thanheiser, E. (2002). Developing understanding of ratio and measure as a foundation for slope. In B. Litwiller & G. Bright (Eds.), *Making sense of fractions, ratios, and proportions: 2002 yearbook* (pp. 162-175). National Council of Teachers of Mathematics.

McCarty, T., & Sealey, V. (2019). What is a differential? Ask seven mathematicians, get seven different answers. In A. Weinberg, D. Moore-Russo, H. Soto, & M. Wawro (Eds.), *Proceedings of the 22nd Annual Conference on Research in Undergraduate Mathematics Education*. Oklahoma City, OK.

Moore, K. C. (2012). Coherence, quantitative reasoning, and the trigonometry of students. In R. L. Mayes & L. L. Hatfield (Eds.), *Quantitative reasoning and mathematical modeling: A* *driver for STEM integrated education and teaching in context, WISDOMe Monograph* (Vol. 2, pp. 75-92). University of Wyoming.

Moore, K. C. (2014). Quantitative reasoning and the sine function: The case of Zac. *Journal for Research in Mathematics Education, 45*(1), 102-138. https://www.jstor.org/stable/10.5951/jresematheduc.45.1.0102

Moore, K. C., & Carlson, M. P. (2012). Students' images of problem contexts when solving applied problems. *Journal of Mathematical Behavior, 31*(1), 48-59. https://doi.org/10.1016/j.jmathb.2011.09.001

Moore, K. C., Paoletti, T., & Musgrave, S. (2013). Covariational reasoning and invariance among coordinate systems. *Journal of Mathematical Behavior, 32*(3), 461-473. https://doi.org/10.1016/j.jmathb.2013.05.002

Oehrtman, M. (2009). Collapsing dimensions, physical limitation, and other student metaphors for limit concepts. *Journal for Research in Mathematics Education, 40*(4), 396-426. https://www.jstor.org/stable/40539345

Piaget, J. (1970). *Genetic Epistemology*. W. W. Norton.

Piaget, J. (1972). *The principles of genetic epistemology*. Routledge & Kegan Paul.

Piaget, J. (1985). *The equilibrium of cognitive structures*. Harvard University Press.

Roundy, D., Dray, T., Manogue, C. A., Wagner, J., & Weber, E. (2015). An extended theoretical framework for the concept of the derivative. In T. Fukawa-Connelly, N. E. Infante, K. Keene, & M. Zandieh (Eds.), *Proceedings of the 18th Annual Conference on Research in Undergraduate Mathematics Education*. Pittsburgh, PA.

Salomon, G. (Ed.). (1993). *Distributed Cognitions: Psychological and education considerations*. Cambridge University Press.

Sealey, V. (2014). A framework for characterizing student understanding of Riemann sums and definite integrals. *The Journal of Mathematical Behavior, 33*, 230-245. https://doi.org/10.1016/j.jmathb.2013.12.002

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*(1), 1-36.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification: The case of function. In E. Dubinsky & G. Harel (Eds.), The Concept of Functions: Aspects of Epistemology and Pedagogy (Vol. 25, pp. 59-84). Mathematical Association of America.

Simmons, C., & Oehrtman, M. (2019, August 6-9). *Quantitatively Based Summation: A framework for student conception of definite integrals* [Paper presentation]. Calculus in Upper Secondary and Beginning University Mathematics, Kristiansand, Norway.

Simon, M. A. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education, 26*(2), 114-145.

Smith, J. P., & Thompson, P. W. (2008). Quantitative reasoning and the development of algebraic reasoning. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 95-132). Erlbaum.

Steen, L. A. (2001). The case for quantitative literacy. In L. A. Steen (Ed.), *Mathematics and democracy: The case for quantitative literacy* (pp. 1-22). National Council on Education and the Disciplines.

Swidan, O., & Yerushalmy, M. (2014). Learning the indefinite integral in a dynamic and interactive technological environment. *ZDM Mathematics Education, 46*, 517-531. https://doi.org/10.1008/s11858-014-0583-1

Tall, D. O., Thomas, M., Davis, G., Gray, E. M., & Simpson, A. (2000). What is the object of the encapsulation of a process? *Journal of Mathematical Behavior, 18*(2), 223-241. https://doi.org/10.1016/S0732-3123(99)00029-2

Thompson, P. W. (1994). Images of rate and operational understanding of the Fundamental Theorem of Calculus. *Educational Studies in Mathematics, 26*(2-3), 229-274. https://www.jstor.org/stable/3482785

Thompson, P. W. (2011). Quantitative reasoning and mathematical modeling. In L. L. Hatfield, S. Chamberlain, & S. Belbase (Eds.), *New perspectives and directions for collaborative research in mathematics education, WISDOMe Monographs* (Vol. 1, pp. 33-57). University of Wyoming.

Thompson, P. W., & Carlson, M. P. (2017). Variation, covariation, and functions: Foundational ways of thinking mathematically. In J. Cai (Ed.), *Compendium for research in mathematics education* (pp. 421-456). NCTM.

Thompson, P. W., & Silverman, J. (2008). The concept of accumulation in calculus. In M. P. Carlson & C. L. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics* (pp. 43-52). Mathematical Association of America.

Von Korff, J., & Rebello, N. S. (2012). Teaching integration with layers and representations: A case study. *Physical Review Special Topics: Physics Education Research, 8*. https://doi.org/10.1103/PhysRevSTPER.8.010125

Yerushalmy, M., & Swidan, O. (2012). Signifying the accumulation graph in a dynamic and multi-representation environment. *Educational Studies in Mathematics, 80*, 287-306. https://doi.org/10.1007/s10649-011-9356-8

Zandieh, M. J. (2000). A theoretical framework for analyzing student understanding of the concept of derivative. In E. Dubinsky, A. H. Schoenfeld, & J. Kaput (Eds.), *Research in collegiate mathematics education, IV* (Vol. 8, pp. 103-127). American Mathematical Society.

APPENDIX A: INTERVIEW PROTOCOL

Interview One

Fuel Flow Context

Given to students:

1. A fuel pipe leading to a tank has a device on it that records the fuel's flow rate through the pipe. Over a 4-minute interval, the flow rate is 10 liters per minute.

Interview Questions:

a. What quantities are part of this context? What are their units?

b. Can we determine the amount of fuel in the tank after the 4-minute period?

c. Were we given any information about the tank before the 4-minute interval occurred?

Tell students we will set the initial amount of fuel in the tank to be 9 liters, and together assign variable names to each quantity (Ex. A for fuel amount, R for flow rate, t for time).

d. What symbols would you use to write the computation you made in this problem?

Given to students:

2. The fuel might not be passing through the pipe at a constant rate. In this case, R is a function of time, R(t). Say at t = 0 the rate is R(0) = 18 L/min.

Interview Questions:

a. Can I multiply 18 L/min by 4 minutes to get the amount of fuel?

b. Does multiplying 18 by 4 give me any useful information?

c. What might I need to do to better approximate how much fuel was added to the tank over the 4 minutes?

Given to students:

3.

t (min)	0	1.25	2	2.5	3	3.75	4
R(t) (L/min)	18	12	7	6	4	3	2.5

Interview Questions:

a. Can we find the total amount of fuel now, assuming it still began with the same amount as before?

b. Is your answer the exact amount? What assumptions are you making about R(t) that leads you to that answer?

4. It can be useful and efficient to communicate mathematically with symbols. We've done the calculations and drawn the graphs, but let's take a look at how we can write all of this work symbolically.

Interview Questions:

a. How would you symbolically represent what you calculated with the varying rates?

b. What has changed between the procedure with the constant rate and your procedure for the varying rate function?

Instruction on summation notation.

If students do not remember sigma notation from previous math classes, the instructor will quickly review it (not much is needed in order to understand the rest of the lesson and the key piece of information is that the symbol Σ represents a summation).

5. We also can communicate mathematics with a graph. Let's look at how we can sketch the graphs of these situations.

   Have students sketch a graph of R(t) when R=10 L/min, then sketch R(t) using the chart *of varying rates*

Interview Questions:

a. How do you see $10\ L/min\ *\ 4\ min\ = 40\ L$in the graph?

b. Where is 10 L/min on the graph?

c. Where is the 4-minute *interval* on the graph?

d. Do we know what the rate is between the times given on the chart?

e. What would the graph look like if we assume the rate remained constant until the next data point?

f. Do you think it's likely that the graph would really look like this?

g. How do you see the products that you calculated earlier in the graph? *(Point to a specific product, like* $18\ L/min\ *\ 1.25\ min$*)*

h. What does each rectangle on the graph represent? What are the units?

i. Why are the units not squared, like the units of area normally are?

j. How can we get a more accurate total for the amount of fuel?

Given to students:

5.

t	0	1.25	2	2.25	2.5	3	3.25	3.5	3.6	3.75	3.9	4
R(t)	18	12	7	6	6	4	4	4	3	3	3	2.5

t	0	0.5	1	1.25	1.75	2	2.25	2.75	3	3.25	3.75	4

R(t)	18	21	16	12	9	7	6	5	4	4	3	2.5

Interview Questions:

a. Now that we know we need more information, which of these two tables would give us a better approximation? Why?

b. Sketch a rough graph of R(t) based on each of the two charts. Do you notice anything about the intervals?

c. How can we increase the accuracy of our calculation even more? Can you write this symbolically?

d. There will always be physical limitations, but let's assume that we can be infinitely accurate with our measurements. How can we write that summation symbolically?

Instruction on development of infinitesimals.

This idea of making intervals become smaller and smaller was a crucial part in the development of mathematics. Gottfried Liebniz was one mathematician who was particularly interested in this idea. He called these tiny intervals "infinitesimals." They represented a little quantity that was very, very small but not zero. The size of the interval is approaching zero, but it never actually becomes zero. Leibniz used a specific notation for these intervals instead of Δ--he used 'd' to represent an infinitesimal quantity (for example, a tiny interval of time would be called dt). So, dx is similar to Δx, but dx is specifically when Δx is approaching zero and becoming infinitely small.

e. Why can't the time interval be zero?

Instruction on integral notation.

This is a very useful and important concept in calculus--to be able to shrink the intervals and *find the exact sum of all these little pieces. Because of this, it has its own notation. The interviewer will then illustrate the notation for the definite integral below the sigma notation.*

f. To summarize, can you explain how this notation connects to both the calculations you made, and also to the graphs you drew?

Interview Two

Road Construction Context

Given to students:

1. Engineers want to build a road connecting two cities, but while building they come across a dirt mound that needs to be removed.

Interview Questions:

a. What do we need to know in order to find the weight of the dirt the engineers need to remove?

b. Do we know what the front of the mound looks like?

Impose the units of pounds/ft on the y-axis, and feet on the x-axis.

c. Why is the shape of the mound the same as the graph of pounds of dirt per foot as a function of horizontal distance?

d. Does this new graph show me how tall the mound is?

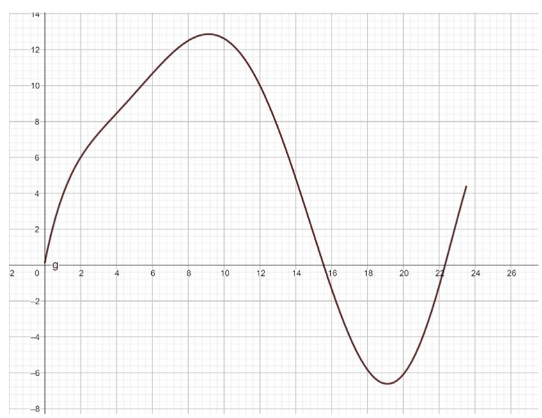
e. Why would we need a pounds per foot graph to solve this problem?

f. How is your calculation similar to other work we have done?

g. If we were to cut up the graph into rectangular slices, what would the area represent? What would be the units?

Given to students:

2. The engineers are building another road. They come across a less uniform piece of land they need to make perfectly flat to lay the road. The land has both mounds of dirt, as well as dips that need to be filled in. The graph below shows the pounds per foot of dirt as you move to the right from where the road begins. Find an approximation of the weight of the dirt the engineers need to remove.



Interview Questions:

a. Will the mound be the same shape as the pound/ft graph shown, like we saw before? Why or why not?

b. What calculation would give us pounds?

c. How do we account for the different weights of dirt at different locations?

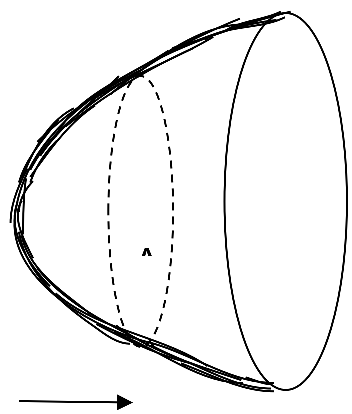
d. How do we account for the “dip” in the road we need to fill? What would that look like symbolically?

e. Is it possible to get a completely accurate total for the weight of dirt using the tools we have? How can we make our approximation more accurate?

Interview Three

Volume of a Solid and Work Contexts

Given to students:



Interview Questions:

a. Given the general integral $\int_a^b \quad f(x)\,dx$, can you tell me what each piece means? What does the integral mean all together?

b. What quantity does the integral $\int_0^{83} \quad A\,dx$ give us (using the function A(x) defined above)? What are the units of the value of this integral?

c. Given an integral $\int_{-3}^{6} \quad Fdy$, where *F* represents the force on an object in Newtons and *y* is measured in *ft*, what are the units of resulting quantity from this integral?

Fuel Flow Context Revisited

Given to students:

1. Recall the Fuel Pipe Context from before, where the fuel rate was 10 L/min over a 4-minute interval, and the tank began with 9 L of fuel.

Interview Questions:

a. How much fuel will be in the tank after 1 minute? 2 minutes? 3.7 minutes?

b. How much fuel is *added* to the tank in the 1/10 second after the 4-minute interval, assuming the constant rate continues?

c. What time value represents the 1/10 second after the 4-minute interval?

d. Write an integral for each of the 4 different times (1 min., 2 min., 3.7 minutes, 1.0067 minutes/240.1 seconds).

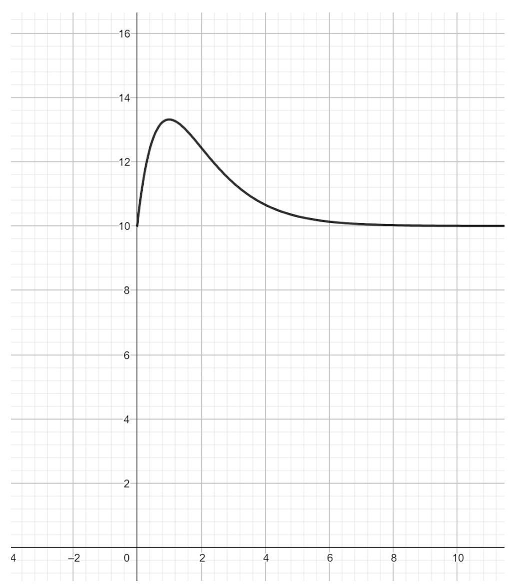
e. If students switch to seconds: Does our multiplication still make sense? What other units need to be converted to match?

f. What symbols stay the same? What changes?

g. Sketch a graph of the amount of fuel in the tank as a function of time. How is this graph related to the fuel rate?

Given to students:

2. Like we saw before, the fuel rate might not be constant. Assuming the tank began with 9 L of fuel, let the graph below represent R(t).



Interview Questions:

a. How do we represent the change in the amount of fuel using the graph above?

b. Approximate the amount of fuel in the tank after 4 minutes.

c. How much fuel is added to the tank 1/10 of a second after the 4 minutes? 1/100 of a second? How do we represent this amount on the graph above?

d. Write an integral expression to represent the exact amount of fuel in the tank at each of these times.

e. What do you notice about your integrals?

f. How could we represent the amount of fuel in the tank after x minutes? What would that mean?

g. Make a rough sketch of the graph of the amount of fuel in the tank as a function of time.

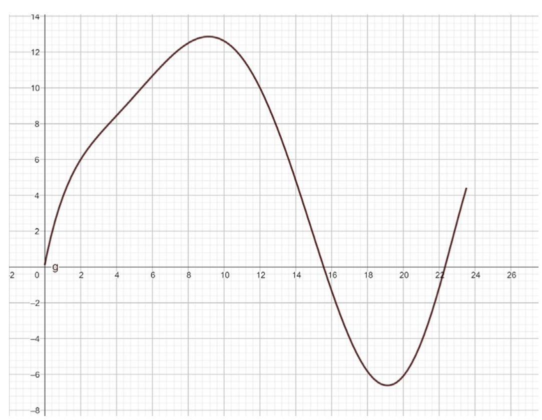
h. What shape should the graph have after the 8 minutes mark?

Interview Four

<u>Road Construction Context Revisited</u>

Given to students:

1. Recall the Road Construction context from previous lessons. Here is the graph again.



Interview Questions:

a. Imagine the engineers are making a graph of the weight of dirt they have on hand at any given time as they clear the way for the road. What factors affect the weight of the dirt the engineers have at any given time? Why might this graph be useful to the engineers?

b. Construct the graph of the weight of dirt as a function of horizontal distance travelled along the x-axis.

   i. What is the y-intercept of the graph?

   ii. When does the weight of the dirt increase? Does it ever decrease?

   iii. When will the engineers have the most dirt on hand? The least?

c. Let's say the engineers plot the graph, and there is a point (18, 124) on the graph.

   i. What does this ordered pair tell you?

   ii. What are the units of each number?

   iii. What would the point (43, -10) mean?

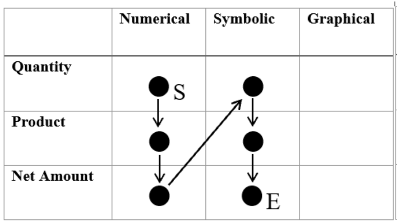
d. Describe the input and output variables of the graph you've made.

Write a symbolic representation of the relationship between the input and output variables.

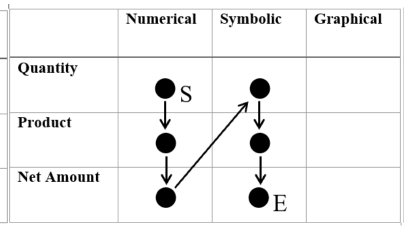
APPENDIX B: DOT MAPS OF STUDENT THINKING

Lesson One

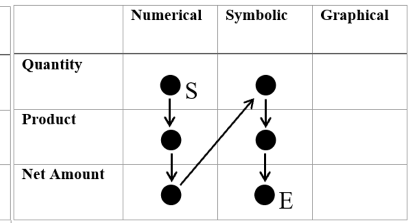
Fuel Rate Context (Constant)



Group A

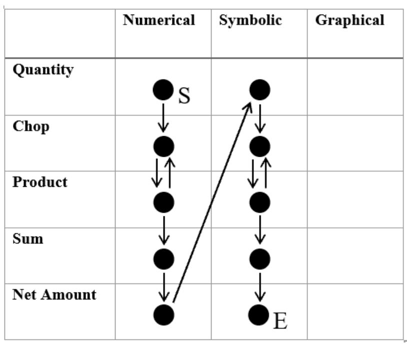


Group B

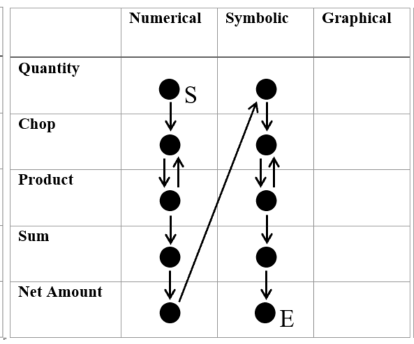


Group C

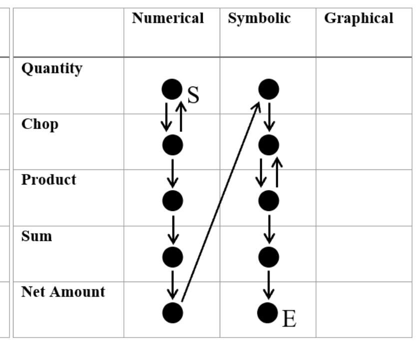
Fuel Rate Context (Varying)



Group A

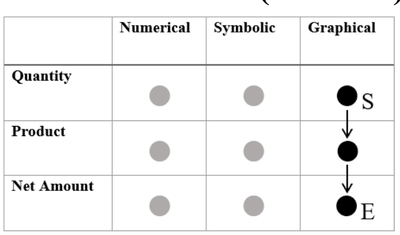


Group B

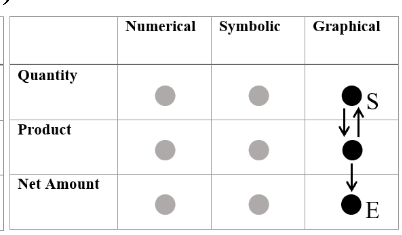


Group C

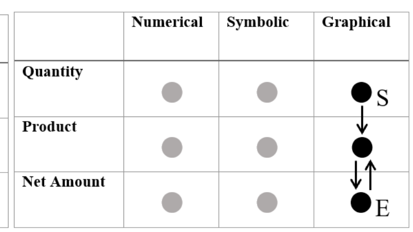
Fuel Rate Context (Constant)



Group A

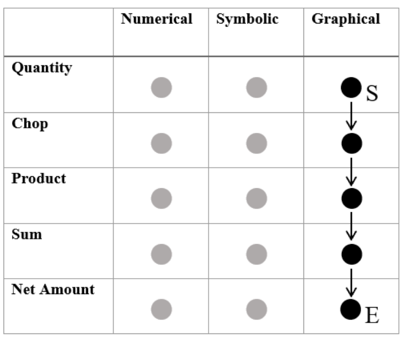


Group B

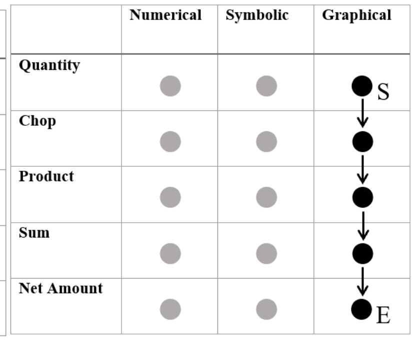


Group C

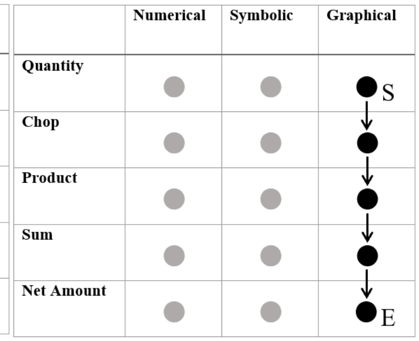
Fuel Rate Context (Varying)



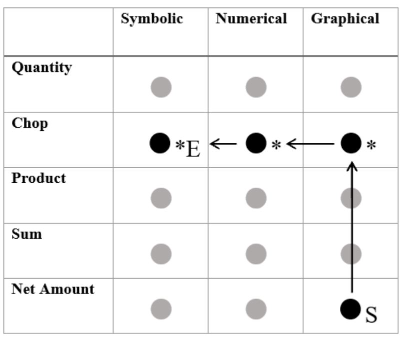
Group A



Group B



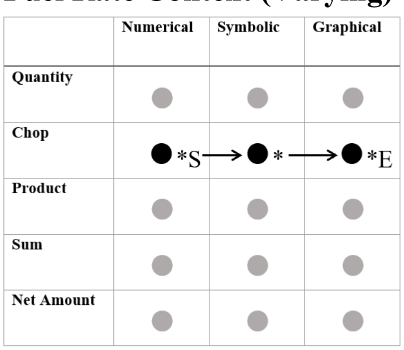
Group C



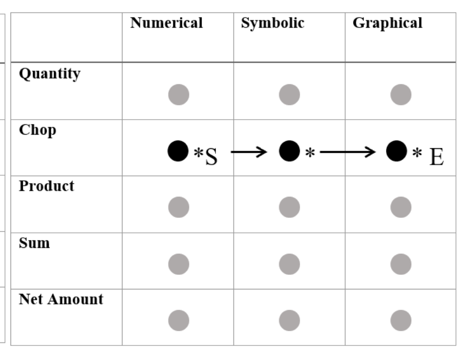
Group A

<u>Lesson Two</u>

Fuel Rate Context (Varying)

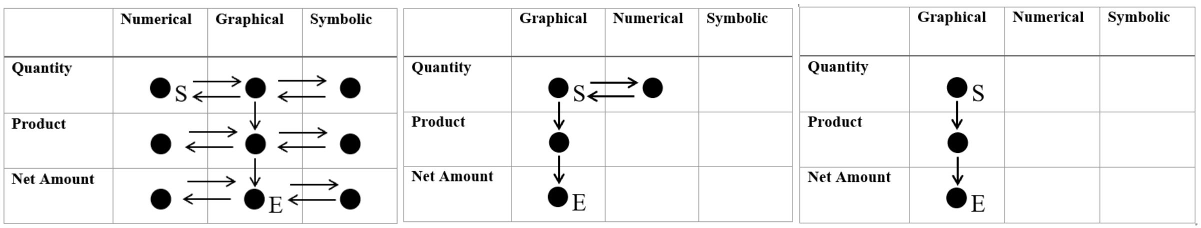


Group B



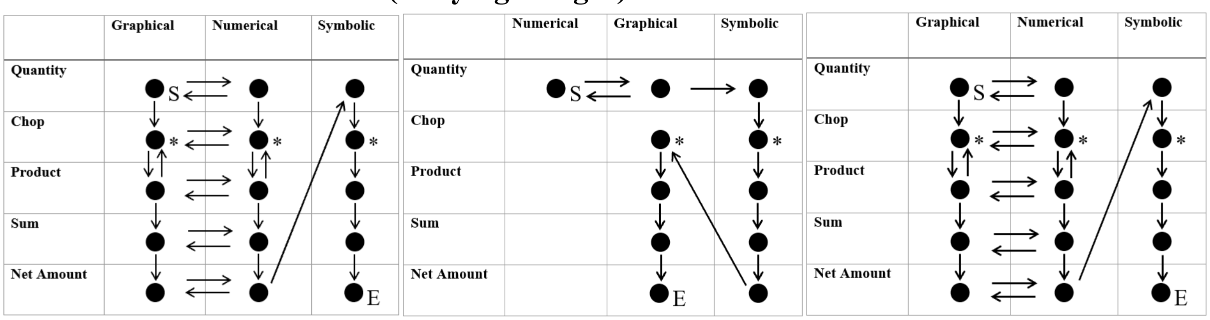
Group C

Road Construction Context (Constant Weight)



Group A Group B Group C

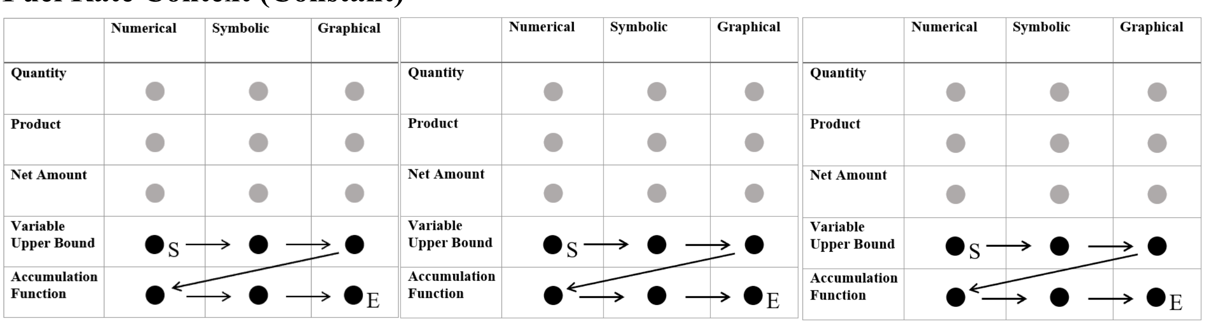
Road Construction Context (Varying Weight)



Group A Group B Group C

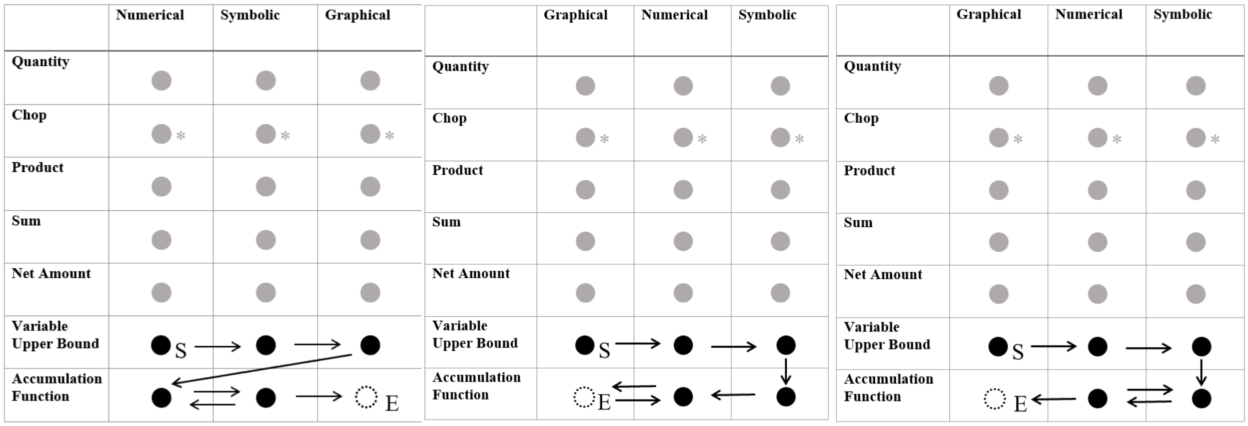
<u>Lesson Three</u>

Fuel Rate Context (Constant)



Group A Group B Group C

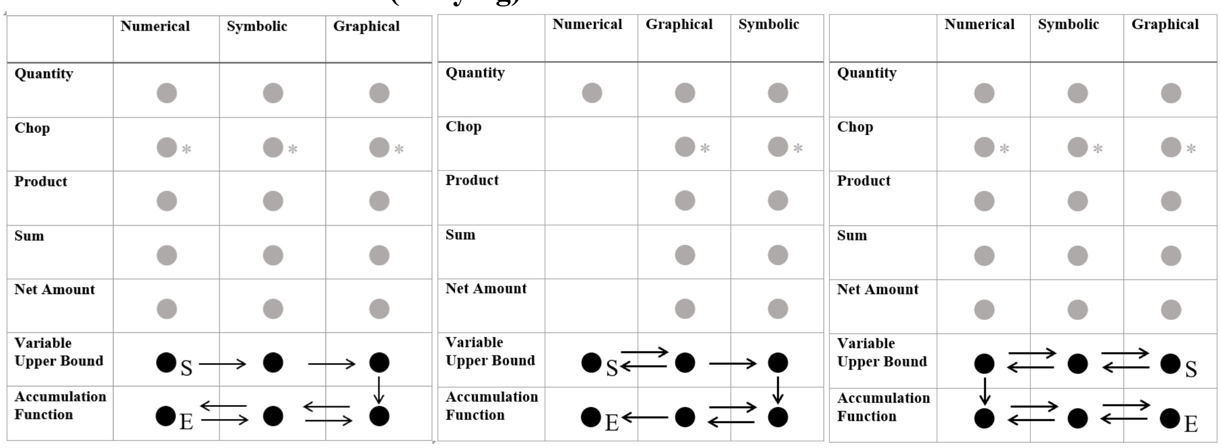
Fuel Rate Context (Varying)



Group A Group B Group C

<u>Lesson 4</u>

Road Construction Context (Varying)



Group A Group B Group C