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## Transposing Noninvertible Polynomials

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# Transposing Noninvertible Polynomials

Nathan Cordner

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## Abstract

In the class of invertible polynomials, the notion of dual polynomials  $W$  and  $W^T$ , as well as dual groups  $G$  and  $G^T$  is well-understood. In this paper we investigate finding dual pairs  $W$  and  $W^T$  for noninvertible polynomials. We find that in many instances, our intuition that stems from invertible polynomials does not extend to the noninvertible case.

## 1 Introduction

Mirror symmetry is an area of mathematical research that stems from theoretical physics, particularly from string theory. Solutions of problems in mirror symmetry yield not only interesting mathematical results, but also have important theoretical implications for high energy particle physics.

In Landau-Ginzburg mirror symmetry, there are  $\mathcal{A}$  and  $\mathcal{B}$  models which describe certain aspects of the physical world. The mirror symmetry conjecture states that each  $\mathcal{A}$ -model is isomorphic to, or the same as, a corresponding  $\mathcal{B}$ -model. Each side is built using a polynomial ( $W$ ) with its group of symmetries ( $G$ ). Constructing the isomorphism requires building a *transpose* polynomial ( $W^T$ ) and a *transpose* group ( $G^T$ ) (see Equation 1). So far this conjectured isomorphism has been proven for a small class of cases in papers such as [3] and [4], but it has not been proven in general.

$$\mathcal{A}_{W,G} \cong \mathcal{B}_{W^T,G^T}$$

**Equation 1: The Landau-Ginzburg Mirror Symmetry Conjecture**

For a certain class of polynomials called *invertible*, the transpose operation is understood well. However, we do not currently know how to define the transpose for *noninvertible* polynomials. In this paper we will explore the possibility of finding mirror partners for the noninvertible polynomials. Insight into this problem will generate interesting results for Landau-Ginzburg mirror symmetry (Equation 1).

## 2 Mathematical Preliminaries

Here we will introduce some of the concepts needed to explain the theory of this paper.

### 2.1 Polynomial Classification

**Definition.** For a polynomial  $W \in \mathbb{C}[x_1, \dots, x_n]$ , we say that  $W$  is *nondegenerate* if it has an isolated critical point at the origin.

**Definition.** Let  $W \in \mathbb{C}[x_1, \dots, x_n]$ . We say that  $W$  is *quasihomogeneous* if there exist unique positive rational numbers  $q_1, \dots, q_n$  such that for any  $c \in \mathbb{C}$ ,  $W(c^{q_1} x_1, \dots, c^{q_n} x_n) = cW(x_1, \dots, x_n)$ .

We often refer to the  $q_i$  as the *quasihomogeneous weights* of a polynomial  $W$ , or just simply the *weights* of  $W$ , and we write the weights in vector form  $J = (q_1, \dots, q_n)$ .

**Definition.**  $W \in \mathbb{C}[x_1, \dots, x_n]$  is *admissible* if  $W$  is both nondegenerate and quasihomogeneous.

We will use the following result about admissible polynomials later in the paper.

**Proposition 2.1.6 of [2].** If  $W$  is admissible, then the  $q_i$  are bounded above by  $\frac{1}{2}$ .

Because the construction of  $\mathcal{A}_{W,G}$  requires an admissible polynomial, we will only be concerned with admissible polynomials in this paper. In order for a polynomial to be admissible, it needs to have at least as many monomials as variables. Otherwise its quasihomogeneous weights cannot be uniquely determined. We now state the main subdivision of the admissible polynomials.

**Definition.** Let  $W$  be an admissible polynomial. We say that  $W$  is *invertible* if it has the same number of monomials as variables. If  $W$  has more monomials than variables, then it is *noninvertible*.

### 2.2 Dual Polynomials

We will now introduce the idea of the transpose operation for invertible polynomials.

**Definition.** Let  $W \in \mathbb{C}[x_1, \dots, x_n]$ . If we write  $W = \sum_{i=1}^m \prod_{j=1}^n x_j^{a_{ij}}$ , then the associated *exponent matrix* is defined to be  $A = (a_{ij})$ .

From this definition we notice that  $n$  is the number of variables in  $W$ , and  $m$  is the number of monomials in  $W$ .  $A$  is an  $m \times n$  matrix. Thus when  $W$  is invertible, we have that  $m = n$  which implies that  $A$  is square. When  $W$  is noninvertible,  $m > n$ .  $A$  then has more rows than columns.

**Definition.** Let  $W$  be an invertible polynomial. If  $A$  is the exponent matrix of  $W$ , then we define the *transpose polynomial* to be the polynomial  $W^T$  resulting from  $A^T$ .

We now have reached our fundamental problem. When a polynomial  $W$  is noninvertible, its exponent matrix  $A$  is no longer square. Taking  $A^T$  yields a polynomial with fewer monomials than variables, which is not admissible. Therefore, our approach to define what the transpose polynomial should be for noninvertibles will require a little more ingenuity.

### 2.3 Symmetry Groups and Their Duals

Though the research results do not rely heavily on the fine details of the symmetry groups, we will introduce them in this section for the interested reader.

**Definition.** Let  $W$  be an admissible polynomial, and let  $A$  be its  $m \times n$  exponent matrix. We define the *maximal symmetry group* of  $W$  to be  $G_W^{max} = \{\mathbf{g} \in (\mathbb{Q}/\mathbb{Z})^n \mid A\mathbf{g} \in \mathbb{Z}^m\}$ .

$G_W^{max}$  is a subgroup of  $(\mathbb{Q}/\mathbb{Z})^n$  with respect to component-wise addition. We represent group elements as  $n$ -tuples where each coordinate is a rational number in the interval  $[0,1)$ . The following definition of the transpose group is due to Krawitz.

**Definition** ([4]). Let  $W$  be an invertible polynomial, and let  $A$  be its associated exponent matrix. The *transpose group* of a subgroup  $G \leq G_W^{max}$  is the set  $G^T = \{g \in G_{W^T}^{max} \mid gAh^T \in \mathbb{Z} \text{ for all } h \in G\}$ .

Since this relies on knowing what  $W^T$  is, this definition currently does not extend to noninvertible polynomials. The following is a list of common results for the transpose group.

**Proposition 2 in Section 3 of [1].** Let  $W$  be an invertible polynomial with weights vector  $J$ , and let  $G \leq G_W^{max}$ .

- (1)  $(G^T)^T = G$ ,
- (2)  $\{0\}^T = G_{W^T}^{max}$  and  $(G_W^{max})^T = \{0\}$ ,
- (3)  $\langle J \rangle^T = G_{W^T}^{max} \cap \text{SL}(n, \mathbb{C})$  where  $n$  is the number of variables in  $W$ ,
- (4) if  $G_1 \leq G_2$ , then  $G_2^T \leq G_1^T$  and  $G_2/G_1 \cong G_1^T/G_2^T$ .

## 2.4 Some Notes on $\mathcal{A}$ and $\mathcal{B}$ Models

Landau-Ginzburg  $\mathcal{A}$  and  $\mathcal{B}$  models are algebraic objects that are endowed with many levels of structure. In particular we can consider these models as both rings and as graded vector spaces. In this paper, we will chiefly be concerned with their structure as graded vector spaces.

We will only develop the theory needed for the proofs in section 3. We refer the interested reader to [2] for more details on the construction of the  $\mathcal{A}$ -model. [3], [4], and [5] also contain more information on constructing  $\mathcal{A}$  and  $\mathcal{B}$  models, and related isomorphisms.

**Facts and Definitions** ( $\mathcal{B}$ -models).

- (1)  $\mathcal{Q}_W = \mathbb{C}[x_1, \dots, x_n] / (\frac{\partial W}{\partial x_1}, \dots, \frac{\partial W}{\partial x_n})$  is called the *Milnor ring* of  $W$ .
- (2)  $\mathcal{B}_{W^T, \{0\}} = \mathcal{Q}_{W^T}$ . We refer to this as the *unorbifolded  $\mathcal{B}$ -model*.
- (3)  $\dim(\mathcal{B}_{W^T, \{0\}}) = \prod_{i=1}^n \left(\frac{1}{q_i} - 1\right)$ , and the degree of its highest graded sector is  $2 \sum_{i=1}^n (1 - 2q_i)$ .

For fact (3), reference section 2.1 of [4]. We will now develop some needed ideas about  $\mathcal{A}$ -models.

**Definition.** Let  $W$  be an admissible polynomial with weights vector  $J = (q_1, \dots, q_n)$ , and let  $G \leq G_W^{max}$ . Then  $G$  is *admissible* if  $J \in G$ .

The construction of the  $\mathcal{A}$ -model requires that  $G$  be an admissible group. From parts (3) and (4) of the proposition in section 2.3, the corresponding condition for the  $\mathcal{B}$ -model is that  $G^T \leq G_{W^T}^{max} \cap \text{SL}(n, \mathbb{C})$ .

**Definition.** Let  $W \in \mathbb{C}[x_1, \dots, x_n]$  be admissible, and let  $g = (g_1, \dots, g_n) \in G_W^{max}$ . The *fixed locus* of the group element  $g$  is the set  $\text{fix}(g) = \{x_i \mid g_i = 0\}$ .

**Definition.** Let  $W$  be an admissible polynomial, and  $G$  an admissible group. We define  $\mathcal{A}_{W,G} = \bigoplus_{g \in G} \left( \mathcal{Q}_{W|_{\text{fix}(g)}} \right)^G$ , where  $(\cdot)^G$  denotes all the  $G$ -invariants. This is called the  *$\mathcal{A}$ -model state space*.

We will not discuss the details of constructing the  $\mathcal{A}$ -model state space here. For further treatment of this topic, we refer the reader to section 2.4 of [5]. A brief comment on notation: we represent basis elements of  $\mathcal{A}_{W,G}$  in the form  $[m; g]$ , where  $m$  is a monomial and  $g$  is a group element. The degree of

each basis element is given by

$$\deg([m; g]) = \dim(\text{fix}(g)) + 2 \sum_{i=1}^n (g_i - q_i),$$

where  $g = (g_1, \dots, g_n)$  and  $J = (q_1, \dots, q_n)$  is the vector of quasihomogeneous weights of  $W$ . (See section 2.1 of [4])

Finally, we state one important theorem for  $\mathcal{A}$ -model isomorphisms.

**Theorem in Section 7.1 of [5]** (Group-Weights). Let  $W_1$  and  $W_2$  be admissible polynomials which have the same weights. Suppose  $G \leq G_{W_1}^{max}$  and  $G \leq G_{W_2}^{max}$ . Then  $\mathcal{A}_{W_1, G} \cong \mathcal{A}_{W_2, G}$ .

## 2.5 Properties of Invertible Polynomials

Our initial intuition tells us that some of the properties of invertible polynomials should extend to the noninvertible case. For example, we'd like to keep the results of the following proposition.

**Proposition.** Let  $W$  be an invertible polynomial. Then

- (1)  $W$  and  $W^T$  have the same number of variables.
- (2)  $(G_W^{max})^T = \{0\}$ .
- (3)  $\mathcal{A}_{W, G_W^{max}} \cong \mathcal{B}_{W^T, \{0\}}$ .

**Proof.** (1) follows from noticing that the exponent matrix of  $W$  is square. Hence its transpose is also square and of the same size, so  $W$  and  $W^T$  have the same number of variables. (2) was stated previously in section 2.3. (3) is a special case of the mirror symmetry conjecture that has been verified. Reference Theorem 4.1 in [4].  $\square$

Part (3) of the proposition is especially important, and will be what we use to look for candidate transpose polynomials. We now state our first research conjecture.

**Conjecture 1.** For any admissible polynomial  $W$  in  $n$  variables,  $W^T$  is also an admissible polynomial in  $n$  variables. Furthermore,  $(G_W^{max})^T = \{0\}$  so that  $\mathcal{A}_{W, G_W^{max}} \cong \mathcal{B}_{W^T, \{0\}}$ .

## 3 Research Results

### 3.1 Disproving Conjecture 1

**Theorem.** For any  $n \in \mathbb{N}$ ,  $n > 3$ , let  $W$  be an admissible polynomial in two variables with weight system  $J = (\frac{1}{n}, \frac{1}{n})$ , and let  $G = \langle J \rangle$ . Then there does not exist a corresponding  $W^T$  in two variables satisfying  $\mathcal{A}_{W, G} \cong \mathcal{B}_{W^T, \{0\}}$ .

Before proving this theorem, we will demonstrate the hypothesis by exhibiting a few examples of such admissible polynomials for small values of  $n$ .

$n$	$J$	Some Examples	$n$	$J$	Some Examples
4	$(\frac{1}{4}, \frac{1}{4})$	$x^4 + y^4 + x^3y$	5	$(\frac{1}{5}, \frac{1}{5})$	$x^5 + y^5 + x^4y$
		$x^4 + x^2y^2 + xy^3$			$x^4y + xy^4 + x^3y^2 + x^2y^3$
		$x^4 + xy^3$			$x^5 + x^2y^3 + xy^4$
6	$(\frac{1}{6}, \frac{1}{6})$	$x^6 + y^6 + x^5y$	7	$(\frac{1}{7}, \frac{1}{7})$	$x^7 + y^7 + x^6y$
		$x^5y + x^4y^2 + y^6$			$x^6y + x^5y^2 + y^7$
		$x^6 + x^2y^4 + xy^5 + y^6$			$x^6y + xy^6$

**Proof.** The idea of this proof is to choose an admissible polynomial with weight system  $J = (\frac{1}{n}, \frac{1}{n})$ , compute some formulas for its  $\mathcal{A}$ -model using the group  $\langle J \rangle$ , and show that there is no corresponding isomorphic unorbifolded  $\mathcal{B}$ -model. Then, under the Group-Weights isomorphism for  $\mathcal{A}$ -models, we will be able to generalize the result for any admissible polynomial with the same weights.

To start, we need an admissible polynomial in two variables with weight system  $J = (\frac{1}{n}, \frac{1}{n})$ . Let  $W' = x^n + y^n + x^{n-1}y$ , and let  $G' = \langle J \rangle$ . Certainly  $W'$  has weight system  $J$ , and  $G'$  fixes  $W'$ .

For the unorbifolded  $\mathcal{B}$ -model, we know that  $\dim(\mathcal{B}_{W^T, \{0\}}) = \prod_{i=1}^n (\frac{1}{q_i} - 1)$  and that the degree of its highest sector is given by  $2 \sum_{i=1}^n (1 - 2q_i)$ . In order to have  $\mathcal{A}_{W, G} \cong \mathcal{B}_{W^T, \{0\}}$ , we need the degrees of the vector spaces and the degrees of each of the graded sectors to be equal. Therefore we now need corresponding formulas for the dimension of the  $\mathcal{A}$ -model vector space and the degree of the highest sector of the  $\mathcal{A}$ -model.

**Lemma.** As a graded vector space,  $\dim(\mathcal{A}_{W', G'}) = 2n - 2$  and the degree of its highest sector is  $\frac{2(2n-4)}{n}$ . ( $n \in \mathbb{N}$ ,  $n \geq 3$ ).

**Proof of Lemma.** Recall that  $\mathcal{A}_{W, G} = \bigoplus_{g \in G} (\mathcal{Q}_{W|_{\text{fix}(g)}})^G$ . Notice that in our case,  $W = W'$ , and  $G = G' = \langle (\frac{1}{n}, \frac{1}{n}) \rangle = \{(0, 0), (\frac{1}{n}, \frac{1}{n}), \dots, (\frac{n-1}{n}, \frac{n-1}{n})\}$ . Then  $W'|_{\text{fix}(g)} = W'$  only for  $g = (0, 0)$ . Otherwise  $W'|_{\text{fix}(g)}$  is trivial.

**Case 1** When  $W'|_{\text{fix}(g)}$  is trivial, we get  $n - 1$  basis elements of the form  $[1; g]$ .

**Case 2**  $W'|_{\text{fix}(g)} = W'$ . Then  $g = (0, 0)$ . The basis elements we get in this case are of the form  $[x^a y^b; (0, 0)]$  where  $a + b \equiv n - 2 \pmod{n}$  and  $a, b \in \{0, 1, \dots, n - 2\}$ . So we have  $(a, b) = (0, n - 2), (1, n - 3), \dots, (n - 3, 1), (n - 2, 0)$ . Hence there are  $n - 1$  basis elements of this type.

The total dimension of  $\mathcal{A}_{W', G'}$  is therefore  $(n - 1) + (n - 1) = 2n - 2$ .

Now we will consider the degree of each basis element. Recall that

$$\deg([m; g]) = \dim(\text{fix}(g)) + 2 \sum_{i=1}^n (g_i - q_i),$$

where  $g = (g_1, \dots, g_n)$  and  $J = (q_1, \dots, q_n)$  is the vector of quasihomogeneous weights.

For  $g = (0, 0)$ , the degree is  $2 + (-\frac{2}{n}) + (-\frac{2}{n}) = \frac{2(n-2)}{n}$ . Also notice by the above equation that  $\deg[1; (\frac{n-1}{n}, \frac{n-1}{n})] > \deg[1; (\frac{m}{n}, \frac{m}{n})]$  for all  $m \in \{1, \dots, n-2\}$ . Compute  $\deg[1; (\frac{n-1}{n}, \frac{n-1}{n})] = \frac{2(2n-4)}{n}$ , and notice that  $\frac{2(2n-4)}{n} = 2 \left( \frac{2(n-2)}{n} \right) > \frac{2(n-2)}{n}$  for all  $n \geq 3$ . Hence the dimension of the highest sector is  $\frac{2(2n-4)}{n}$ .  $\square$

From the lemma, we now have the following system of equations

$$\begin{aligned} \left( \frac{1}{q_1} - 1 \right) \left( \frac{1}{q_2} - 1 \right) &= 2n - 2, \\ 2((1 - 2q_1) + (1 - 2q_2)) &= \frac{2(2n - 4)}{n}. \end{aligned}$$

Solutions to this system of equations yields all possible weight systems that can fix a candidate  $W^T$ .

Simplifying this system gives us

$$\begin{aligned}(3 - 2n)q_1q_2 - q_1 - q_2 + 1 &= 0, \\ \frac{2}{n} - q_1 - q_2 &= 0.\end{aligned}$$

Solving for  $q_1$  in the second equation, we have  $q_1 = \frac{2}{n} - q_2$ . Substituting back into the first equation yields

$$\begin{aligned}(3 - 2n)\left(\frac{2}{n} - q_2\right)q_2 - q_2 - \left(\frac{2}{n} - q_2\right) + 1 &= (3 - 2n)\left(\frac{2}{n}q_2 - q_2^2\right) + 1 - \frac{2}{n} \\ &= \frac{6}{n}q_2 - 3q_2^2 - 4q_2 + 2nq_2^2 + \frac{n-2}{n} \\ &= (2n-3)q_2^2 + \left(\frac{2(3-2n)}{n}\right)q_2 + \frac{n-2}{n} = 0.\end{aligned}$$

Multiplying both sides by  $n$  yields

$$n(2n-3)q_2^2 + 2(3-2n)q_2 + n-2 = 0.$$

We now have a quadratic equation in  $q_2$ . Consider the discriminant

$$\begin{aligned}D &= (2(3-2n))^2 - 4(n(2n-3))(n-2) \\ &= 4((3-2n)^2 - (2n^2-3n)(n-2)) \\ &= 4(9-12n+4n^2 - (2n^3-7n^2+6n)) \\ &= 4(-2n^3+11n^2-18n+9).\end{aligned}$$

When  $D < 0$ , we will not have a real-valued solution for  $q_2$ . The above equation is a cubic polynomial that has roots at  $n = 1, \frac{3}{2}, 3$ . At  $n = 4$ ,  $D = -60 < 0$ . Therefore  $D < 0$  for all values of  $n > 3$ . Since  $D < 0$  for all  $n > 3$ ,  $q_2$  will not be real-valued for all  $n > 3$ . Then there are no rational-valued solutions for the quasihomogeneous weights in this case.

This shows that there is no  $W^T$  in two variables satisfying  $\mathcal{A}_{W',G'} \cong \mathcal{B}_{W^T,\{0\}}$ . Extending by the Group-Weights theorem, for any admissible polynomial  $W$  with weights  $(\frac{1}{n}, \frac{1}{n})$ , we have that  $\mathcal{A}_{W,G'} \cong \mathcal{A}_{W',G'}$ . By this isomorphism, we know that  $\dim(\mathcal{A}_{W,G'}) = 2n-2$  and the degree of its highest sector is  $\frac{2(2n-4)}{n}$ . Therefore, by what we have just shown, there cannot not exist any  $W^T$  in two variables such that  $\mathcal{A}_{W,G'} \cong \mathcal{B}_{W^T,\{0\}}$ . This proves the theorem.  $\square$

We do have the following solutions for  $n \in \{1, 2, 3\}$ .  $n = 1$  yields the solution  $\mathbf{q} = (1, 1)$ ,  $n = 2$  yields solutions  $\mathbf{q} = (1, 0)$ ,  $(0, 1)$ , and  $n = 3$  gives a solution  $\mathbf{q} = (\frac{1}{3}, \frac{1}{3})$ . However, since each coordinate must be in the interval  $(0, 1/2]$ ,  $\mathbf{q} = (\frac{1}{3}, \frac{1}{3})$  is the only valid weight system.

Our original conjecture (Conjecture 1) about the transpose of a noninvertible polynomial was that  $W$  and  $W^T$  have the same number of variables and  $(G_W^{max})^T = \{0\}$ . We will now state a corollary to demonstrate that one of these assumptions must be false.

**Corollary.** For any  $n \in \mathbb{N}$ ,  $n > 3$ , let  $W$  be a noninvertible polynomial in two variables with weight system  $J = (\frac{1}{n}, \frac{1}{n})$  and  $G_W^{max} = \langle J \rangle$ . Then there does not exist a corresponding  $W^T$  in two variables satisfying  $\mathcal{A}_{W,G_W^{max}} \cong \mathcal{B}_{W^T,\{0\}}$ .

**Proof.** Consider again the polynomial  $W' = x^n + y^n + x^{n-1}y$ .

**Lemma.** The polynomial  $W'$  has  $G_{W'}^{max} = \langle J \rangle = \langle (\frac{1}{n}, \frac{1}{n}) \rangle$  for all  $n \in \mathbb{N}$ ,  $n \geq 3$ .

**Proof of Lemma.** By Lisa Bendall's theorem (see Appendix),  $G_{W'}^{max} = \langle (\frac{1}{n}, \frac{1}{n}), (0, \frac{1}{\gcd(n,1)}) \rangle = \langle (\frac{1}{n}, \frac{1}{n}) \rangle$ , since  $\gcd(n, 1) = 1$  and the generator  $(0, 1) \equiv (0, 0) \pmod{1}$  contributes nothing.  $\square$

Since  $W'$  has  $G_{W'}^{max} = \langle J \rangle$ , and since  $W'$  satisfies the hypotheses of the previous theorem, we conclude that there does not exist a corresponding  $W^T$  in two variables satisfying  $\mathcal{A}_{W', G_{W'}^{max}} \cong \mathcal{B}_{W^T, \{0\}}$ . Extending by the Group-Weights theorem shows that any noninvertible  $W$  with weights  $J$  and  $G_W^{max} = \langle J \rangle$  fails to have a  $W^T$  in two variables satisfying this mirror symmetry alignment.  $\square$

### 3.2 A Follow-up Conjecture

We will now consider finding a suitable  $W^T$  in a different number of variables. By relaxing one of the hypotheses of Conjecture 1, we formulate the following statement.

**Conjecture 2.** For any admissible  $W$ , there is a corresponding  $W^T$  satisfying  $\mathcal{A}_{W, G_W^{max}} \cong \mathcal{B}_{W^T, \{0\}}$ .

**Theorem.** For any admissible polynomial  $W$  with weight system  $J = (\frac{1}{5}, \frac{1}{5})$  and  $G = \langle J \rangle$ , there is no corresponding admissible  $W^T$  in 1, 2, or 3 variables satisfying  $\mathcal{A}_{W, G} \cong \mathcal{B}_{W^T, \{0\}}$ .

**Proof.** For  $W$  as given in the hypothesis, we have previously shown that the degree of the  $\mathcal{A}$ -model is 8, and the degree of its highest sector is  $6/5$  (after dividing by 2).

We will rule out the existence of a  $W^T$  in these three cases. In one variable, we can only have  $W^T = x^9$  to give us an unorbifolded  $\mathcal{B}$ -model of dimension 8. Then  $q_1 = \frac{1}{9}$ , but  $1 - \frac{2}{9} = \frac{7}{9} \neq \frac{6}{5}$ . The two variable case is done by the previous theorem.

Now let  $n \in \mathbb{N}$ ,  $n \geq 3$ . We have the following equations for a candidate weight system:

$$\left(\frac{1}{q_1} - 1\right) \left(\frac{1}{q_2} - 1\right) \prod_{i=3}^n \left(\frac{1}{q_i} - 1\right) = 8, \quad (1)$$

$$(1 - 2q_1) + (1 - 2q_2) + \sum_{i=3}^n (1 - 2q_i) = \frac{6}{5}. \quad (2)$$

Simplify equation (1):

$$\begin{aligned} \frac{1}{q_1 q_2} - \frac{1}{q_1} - \frac{1}{q_2} + 1 &= \frac{8}{\prod_{i=3}^n \left(\frac{1}{q_i} - 1\right)} \\ 1 - q_2 - q_1 + q_1 q_2 &= \frac{8 q_1 q_2}{\prod_{i=3}^n \left(\frac{1}{q_i} - 1\right)} \\ \left(1 - \frac{8}{\prod_{i=3}^n \left(\frac{1}{q_i} - 1\right)}\right) q_1 q_2 - q_1 - q_2 + 1 &= 0. \end{aligned}$$



Simplify equation (2):

$$\begin{aligned}
2 - 2q_1 - 2q_2 &= \frac{6}{5} - \sum_{i=3}^n (1 - 2q_i) \\
-2q_2 &= \left(\frac{6}{5} - 2\right) + 2q_1 - \sum_{i=3}^n (1 - 2q_i) \\
q_2 &= -\frac{1}{2} \left(-\frac{4}{5}\right) - q_1 + \frac{1}{2} \sum_{i=3}^n (1 - 2q_i) \\
q_2 &= \frac{2}{5} - q_1 + \frac{n-2}{2} - \sum_{i=3}^n q_i \\
q_2 &= -q_1 + \left(\frac{5n-6}{10} - \sum_{i=3}^n q_i\right).
\end{aligned}$$

Label  $A = 1 - \frac{8}{\prod_{i=3}^n \left(\frac{1}{q_i} - 1\right)}$ , and  $B = \frac{5n-6}{10} - \sum_{i=3}^n q_i$ . Substituting into (1) and (2) gives us

$$Aq_1q_2 - q_1 - q_2 + 1 = 0, \quad (3)$$

$$-q_1 + B = q_2. \quad (4)$$

Now substitute equation (4) into (3):

$$\begin{aligned}
Aq_1(-q_1 + B) - q_1 - (-q_1 + B) + 1 &= 0 \\
-Aq_1^2 + ABq_1 - q_1 + q_1 - B + 1 &= 0 \\
-Aq_1^2 + ABq_1 + (1 - B) &= 0 \\
Aq_1^2 - ABq_1 + (B - 1) &= 0.
\end{aligned}$$

We can now solve for  $q_1$  using the quadratic formula:

$$q_1 = \frac{AB \pm \sqrt{(AB)^2 - 4A(B-1)}}{2A}. \quad (5)$$

We will now analyze equations (4) and (5) to see when we obtain valid solutions for  $q_1$  and  $q_2$ . We will start by finding reasonable bounds on  $A$  and  $B$ .

For any  $q_i \in (0, 1/2]$ , we have that  $\frac{1}{q_i} - 1 \geq 1$ . By equation (1), we require that  $\prod_{i=3}^n \left(\frac{1}{q_i} - 1\right) \leq 8$ . This tells us that  $1 \leq \prod_{i=3}^n \left(\frac{1}{q_i} - 1\right) \leq 8$ . Therefore we have that  $-7 \leq A \leq 0$ . In order to use the quadratic formula in (5), we require  $A < 0$ . We will deal with the case  $A = 0$  separately.

From equation (2) we also have that

$$\sum_{i=3}^n (1 - 2q_i) \leq \frac{6}{5} \Rightarrow (n-2) - 2 \sum_{i=3}^n q_i \leq \frac{6}{5} \Rightarrow - \sum_{i=3}^n q_i \leq \frac{16-5n}{10}.$$

Substituting this into  $B$  yields

$$B = \frac{5n-6}{10} - \sum_{i=3}^n q_i \leq \frac{5n-6}{10} + \frac{16-5n}{10} = 1.$$

Consider the case  $n = 3$ . When  $A \neq 0$ , we can use equation (5) to plot the real-valued solutions of  $q_1$ . In three variables, the discriminant  $D = (AB)^2 - 4A(B-1) \geq 0$  for  $q_3 \leq 1/9$ . This yields the following:

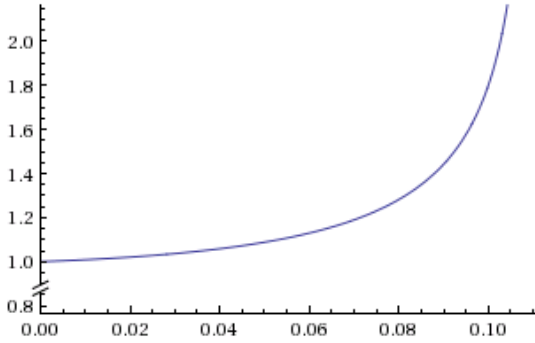


Figure 1: Positive branch of (5)

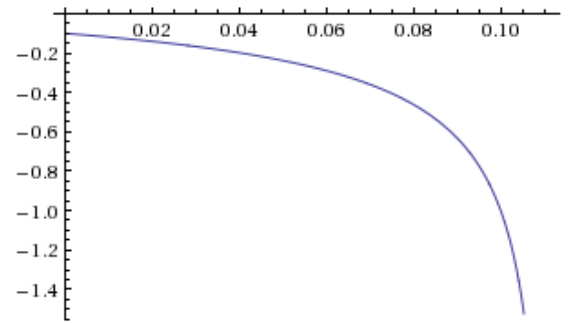


Figure 2: Negative branch of (5)

None of these values of  $q_1$  are in the interval  $(0, 1/2]$ .

Now when  $A = 0$ , we must have that  $\frac{1}{q_i} - 1 = 8$ . Therefore by equation (1) we can only have  $q_1 = q_2 = 1/2$ . But equations (1) and (2) show that if this is the case, then we could have found a satisfactory weight system in just 1 variable without considering  $q_1$  and  $q_2$ . Since we have already ruled out the case  $n = 1$ , we conclude that there are no valid weight systems for  $W^T$  in three variables.  $\square$

Using the formulas developed in the last theorem may be useful in proving the following conjecture.

**Conjecture 3.** For any admissible polynomial  $W$  with weight system  $J = (\frac{1}{5}, \frac{1}{5})$  and  $G = \langle J \rangle$ , there is no corresponding admissible  $W^T$  satisfying  $\mathcal{A}_{W,G} \cong \mathcal{B}_{W^T, \{0\}}$ .

## 4 Conclusion

Given a polynomial  $W$  fixed by a weight system  $J = (\frac{1}{n}, \frac{1}{n})$  and group  $G = \langle J \rangle$ , and  $m \in \mathbb{N}$  representing the number of variables in a candidate  $W^T$ , it is impossible to construct  $\mathcal{A}_{W,G} \cong \mathcal{B}_{W^T, \{0\}}$  in the following cases:

		$m$			
		1	2	3	...
$n$	4		X		
	5	X	X	X	
	6		X		
	$\vdots$		X		

Proving the statement of Conjecture 3 will disprove the previous conjecture that  $\mathcal{A}_{W, G_W^{max}} \cong \mathcal{B}_{W^T, \{0\}}$  extends generally to noninvertible  $W$ .

These results show that our original intuition about invertible polynomials and their transposes does not extend well to the noninvertible case. Though we have not completely ruled out the possibility of noninvertible polynomials having a transpose, we have shown that this problem is difficult and will require something cleverer than what we have already considered.

## 5 References

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## 6 Appendix

We used the following result when proving the corollary in section 3.1. The theorem and proof are due to Lisa Bendall. We will reproduce the entire proof here because it is not publicly available elsewhere.

**Theorem.** Let  $W = x^p + y^q$ . If a monomial satisfying the quasihomogeneous weights of  $W$  is added to make the new polynomial  $W' = x^p + y^q + x^r y^s$ , then  $G_{W'}^{max} = \langle (1/p, 1/q), (1/n, 0) \rangle$ , where  $n = \gcd(p, r)$ . Alternatively,  $G_{W'}^{max} = \langle (1/p, 1/q), (0, 1/m) \rangle$ , where  $m = \gcd(q, s)$ .

**Proof.** Any element  $(\theta_1, \theta_2) \in G_{W'}^{max}$  must satisfy the following matrix equation:

$$\begin{bmatrix} p & 0 \\ 0 & q \\ r & s \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} \in \mathbb{Z}^3.$$

This yields the following three equations:

$$\theta_1 = \frac{k_1}{p}, \quad \theta_2 = \frac{k_2}{q}, \quad r\theta_1 + s\theta_2 = k_3.$$

We also note that since the new monomial satisfies the weights vector of  $W$ , then  $\frac{r}{p} + \frac{s}{q} = 1$ . From this equation, it follows that  $s = \frac{pq-rq}{p}$ , which we can substitute along with the first two equations into the last equation to get the equation  $r(k_1 - k_2) = p(k_3 - k_2)$ . Dividing out by the gcd of  $r$  and  $p$ , we get the equation  $r'(k_1 - k_2) = p'(k_3 - k_2)$ , where  $r'$  and  $p'$  are relatively prime.

From this, we know that  $p' \mid (k_1 - k_2)$ , or in other words,  $k_1 - k_2 = k_4 p'$  for some  $k_4 \in \mathbb{Z}$ . Now, dividing both sides by  $p$ , we get  $\frac{k_1}{p} - \frac{k_2}{p} = \frac{k_4}{n}$ . Next, substitute  $p\theta_1$  for  $k_1$ . We find that  $\theta_1 = k_2 \left(\frac{1}{p}\right) + k_4 \left(\frac{1}{n}\right)$ . From the second equation, we already know that  $\theta_2 = \frac{k_2}{q}$ , so we have the following equation in vector form:

$$(\theta_1, \theta_2) = k_2(1/p, 1/q) + k_4(1/n, 0),$$

where  $k_2, k_4 \in \mathbb{Z}$ .

Now, to show that this generates the group, we show that anything of form  $k_2(1/p, 1/q) + k_4(1/n, 0)$  satisfies the three original equations for some three arbitrary integers. For the first equation, note that  $1/n = p'/p$ , thus  $\theta_1 = (k_2 + k_4 p')(1/p)$ , so it is satisfied for some integer. The second equation follows immediately. For the final equation, plugging in we get  $r(k_2/p + k_4/n) + s(k_2/q) = (r/p + s/q)k_2 + (r/n)k_4 = k_2 + r'k_4 \in \mathbb{Z}$ . Therefore, any element of the form  $k_2(1/p, 1/q) + k_4(1/n, 0)$  is in  $G_{W'}^{max}$ . Thus  $G_{W'}^{max} = \langle (1/p, 1/q), (1/n, 0) \rangle$ .

Note: by substituting  $r = \frac{pq-sp}{q}$  in the last equation, we get the alternate set of generators  $(1/p, 1/q)$  and  $(0, 1/m)$ .  $\square$