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Character Tables of Metacyclic Groups

Dane Christian Skabelund
Brigham Young University - Provo

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Dane Christian Skabelund

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Stephen Humphries, Chair
Darrin Doud
David Cardon

Department of Mathematics

Brigham Young University

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ABSTRACT

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Dane Christian Skabelund
Department of Mathematics, BYU
Master of Science

We show that any two split metacyclic groups with the same character tables are isomorphic. We then use this to show that among metacyclic groups that are either 2-groups or are of odd order divisible by at most two primes, that the dihedral and generalized quaternion groups of order 2^n, $n \geq 3$, are the only pairs that have the same character tables.

Keywords: finite group, metacyclic group, split metacyclic group, character table, p-group.

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Contents

Chapter 1. Introduction

In this chapter we collect definitions and results which will be used later. See [2, 6, 8] for references for all results in the first two sections of this chapter.

All calculations made in writing this thesis were accomplished using MAGMA [1].

1.1 Characters of finite groups

A *representation* of a group G over a complex vector space V is a homomorphism $G \to \mathrm{GL}(V)$. Since this is equivalent to assigning to V the structure of a $\mathbb{C}G$-module, we will sometimes refer to V as a representation. The dimension of V is called the *degree* of the representation. We say that a representation V is *irreducible* if it has no nontrivial G-invariant subspaces. If G is finite, then any irreducible representation V for G is isomorphic to a direct summand of the regular representation $\mathbb{C}G$, with multiplicity equal to its dimension. Unless stated otherwise, all groups considered will be finite. If H is a subgroup of G and V is a representation of G, then we can obtain a *restricted* representation by considering V as a $\mathbb{C}H$ module. Alternately, if V is a representation of H then $V \otimes_{\mathbb{C}H} \mathbb{C}G$ is a representation of G, called the *induced representation*.

Given a representation $\mathfrak{X}\colon G \to \mathrm{GL}(V)$, we associate to $\mathfrak{X}$ a complex valued function $\chi\colon G \to \mathbb{C}$, called the *character afforded by* $\mathfrak{X}$, which is defined by $\chi(g) = \operatorname{tr}\mathfrak{X}(g)$. We say that a character χ is *irreducible* if the corresponding representation $\mathfrak{X}$ is irreducible. The set of irreducible characters of a group G will be denoted $\mathrm{Irr}(G)$. Since characters determine their corresponding representations up to isomorphism, knowledge of all the irreducible characters of G as functions $G \to \mathbb{C}$ is equivalent to knowing the isomorphism types of all irreducible representations of G. The irreducible characters are constant on conjugacy classes of G, and form an orthonormal basis for the space of class functions $G \to \mathbb{C}$ with respect to the positive definite Hermitian inner product

$$[\chi, \psi] = \frac{1}{|G|} \sum_{g \in G} \chi(g)\overline{\psi(g)}.$$

The *character table* of a group G is the invertible matrix whose rows correspond to the irreducible representations of G and whose columns correspond to the conjugacy classes of G. Although a tremendous amount of information about a group can be extracted from its character table, the character table does not determine all groups up to isomorphism. For example, the dihedral and quaternion groups of order eight share the common character table

	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
χ_5	2	-2	0	0	0

In general, one says that two groups and G_1 and G_2 *have the same character tables* if there is a bijection $\phi\colon G_1 \to G_2$ that sends classes to classes and a bijection $\psi\colon \mathrm{Irr}(G_1) \to \mathrm{Irr}(G_2)$ such that $\chi(g) = \psi(\chi)(\phi(g))$ for all $\chi \in \mathrm{Irr}(G_1)$, $g \in G_1$. This is the same as requiring that ϕ determines an isomorphism $\phi\colon Z(\mathbb{C}G_1) \to Z(\mathbb{C}G_2)$ of centralizer algebras. It is also the same as requiring that the character table of G_1 can be obtained from the character table of G_2 by permuting its rows and columns.

1.2 Character tables

We will be interested in determining the isomorphism types of certain groups using only their character tables. Thus it will be important to specify exactly what we mean when we say that we *know the character table* of a group. When we say this, we mean that we are given the matrix of complex values itself, which we know only up to some permutation of the rows and columns. Thus we are not aware of the labeling of the columns by conjugacy classes, and we do not know the characters as functions $G \to \mathbb{C}$.

We briefly discuss information about a group which can be obtained from knowledge of its character table. Let G be a group with irreducible characters $\chi_1, \ldots, \chi_r$ and conjugacy classes $\mathcal{C}_1, \ldots, \mathcal{C}_r$. Let $x_1, \ldots, x_r$ be representatives of each of these classes. Then the size of each conjugacy class can be computed using the second orthogonality relation [6, p.21] as follows:

$$|\mathcal{C}_i| = \sum_{j=1}^{r} |\chi_j(x_i)|^2.$$

In particular, the classes comprising the center can be located, and the orders of G and $Z(G)$ can be determined.

For $X \subseteq G$, let $\overline{X} = \sum_{x \in X} x$, so that $\overline{\mathcal{C}}_i$ are the class sums in the group algebra $\mathbb{C}G$. Then the $\overline{C}_i$ form an integral basis for $Z(\mathbb{C}G)$, and there are positive integers a_{ijk} satisfying $\overline{\mathcal{C}}_i\overline{\mathcal{C}}_j = \sum_k a_{ijk}\overline{\mathcal{C}}_k$. The constants a_{ijk} can be computed using the formula [8, p. 349]

$$a_{ijk} = \frac{|\mathcal{C}_i||\mathcal{C}_j|}{|G|} \sum_{\chi \in \mathrm{Irr}(G)} \frac{\chi(x_i)\chi(x_j)\overline{\chi(x_k)}}{\chi(1)}.$$

In particular, this allows us to determine the isomorphism type of $Z(G)$, since each class in $Z(G)$ contains only one element.

The normal subgroups of G are exactly intersections of kernels of irreducible characters of G [6, p.23], and so the classes comprising each normal subgroup of G can be determined from the character table. In particular, this gives us the lattice of normal subgroups of G and the orders of each of these subgroups. The commutator subgroup G' can be obtained as the intersection of the kernels of all linear (degree 1) characters of G. The character table of a quotient G/N can be obtained as the submatrix of the character table given by deleting all characters $\chi \in \mathrm{Irr}(G)$ with $N \not\subseteq \ker \chi$ and then collapsing any repeated columns.

1.3 Metacyclic groups

The quaternion and dihedral groups mentioned above are both examples of metacyclic groups. A group G is *metacyclic* if it has a cyclic normal subgroup K such that G/K is cyclic. Equivalently, the group G has cyclic subgroups S and K, with K normal in G, such that $G = SK$. This is called a *metacyclic factorization* of G. Since we will only deal with finite groups, from now on when we say metacyclic group, we mean finite metacyclic group. Examples of metacyclic groups include dihedral groups, generalized quaternion groups, and groups all of whose Sylow subgroups are cyclic (in particular this includes all groups of squarefree order) [10, p. 366]. Subgroups and quotients of metacyclic groups are again metacyclic. It was shown by Hölder [12, Thm 7.21] that a group is metacyclic if and only if it has a presentation of the form

$$G = \langle a, b \mid a^\alpha = b^\delta, b^\beta = 1, b^a = b^\gamma \rangle,$$

where $\beta \mid (\gamma^\alpha - 1)$ and $\beta \mid \delta(\gamma - 1)$. However, in general the parameters α, β, γ, and δ are not invariants of the group G, as we shall see later.

A metacyclic factorization $G = SK$ is with $S \cap K = 1$, is called a *split metacyclic factorization.* If G has such a factorization, then we call G a *split metacyclic group.* Equivalently, G has a presentation of the form

$$G = \langle a, b \mid a^\alpha = b^\beta = 1, b^a = b^\gamma \rangle.$$

Split metacyclic groups are semidirect products of cyclic groups, and are typically easier to deal with than metacyclic groups in general.

Subgroups and quotients of split metacyclic groups need not be split. For example, in the split metacyclic group given by the presentation $\langle a, b \mid a^{10} = b^{48} = 1, b^a = b^7 \rangle$, the subgroup generated by the elements $x = a^5b^39$ and $y = b^{12}$ is isomorphic to Q_8, which is not split metacyclic. Also, the metacyclic group G given by the presentation $G = \langle a, b \mid$

$a^4 = b^4 = 1, b^a = b^{-1}\rangle$ is split, but the quotient of G by the central subgroup generated by a^2b^2 is isomorphic to Q_8. Furthermore, a metacyclic group may have one metacyclic factorization which is split and one which is not. For example, one checks that the groups given by the presentations $\langle a, b \mid a^2 = b^3, b^6 = 1, b^a = b^{-1}\rangle$ and $\langle a, b \mid a^4 = b^3 = 1, b^a = b^{-1}\rangle$ are isomorphic.

We will use the following basic lemma which describes the center and the commutator subgroup of a metacyclic group.

Lemma 1.1. *[11, Lemma 2.7] Let G be a group with a metacyclic factorization $G = SK$. Let $S = \langle a\rangle$ and $K = \langle b\rangle \trianglelefteq G$. Let γ be an integer with $b^a = b^\gamma$. Let $s = \mathrm{ord}_{|K|}(\gamma)$ and $t = |K|/\gcd(|K|, \gamma - 1)$. Then*

$$G' = \langle b^{\gamma-1}\rangle \cong C_t, \qquad Z(G) = \langle a^s, b^t\rangle.$$

There is a classical combinatorial method for constructing the irreducible representations of the symmetric groups S_n. This method makes use of Young tableau in order to construct elements of the group algebra $\mathbb{C}S_n$, called Young symmetrizers, which generate the all the minimal ideals of $\mathbb{C}G$. These ideals correspond to the irreducible representations of S_n. For more information about this, see [4, Ch 4] or [2, Ch 28]. In [9], Munkholm uses similar methods to construct the irreducible representations of metacyclic groups. Here is a statement of this result.

Theorem 1.2. *[9, Theorem 5.1] Let G be a metacyclic group of the form*

$$G = \langle a, b \mid a^\alpha = b^\delta, b^\beta = 1, b^a = b^\gamma\rangle,$$

where $\alpha, \beta, \gamma, \delta$ are integers satisfying $(\beta, \gamma) = 1$, $\delta(\gamma - 1) \equiv \gamma^\alpha - 1 \equiv 0 \pmod{\beta}$, $\beta > 0$, $\gamma > 0$, $\alpha > 0$, $\delta \geq 0$. Let s_i be the order of γ as an element of the group of units in $Z_{\beta/(\beta,i)}$. In $\{0, 1, \ldots, \beta - 1\}$ we define an equivalence relation $\sim$ by: $i \sim i'$ if and only if there is a v with $i \equiv i'\gamma^v \pmod{\beta}$, and we let I be a set of representatives of the classes modulo $\sim$. For *each $i \in I$ we let $\{q_{i,1}, q_{i,2}, \ldots, q_{i,\alpha/s_i}\}$ be a full set of pairwise incongruent integers modulo α/s_i with $-i\delta/\beta \leq q_{i,k} < \alpha - i\delta/\beta$, and we put $j_{i,k} = i\delta/(\beta,\delta) + q_{i,k}\beta/(\beta,\delta)$. Finally, we choose a primitive βth root of unity ζ and a primitive nth root of unity η ($n = \alpha\beta/(\beta,\delta)$) such that $\zeta^{(\beta,\delta)} = \eta^{\alpha}$. Then some of the Young-elements are:*

$$e_{i,j_{k,i}} = \sum_{m=0}^{\beta-1}\sum_{n=0}^{\alpha-1} \zeta^{mi}\eta^{nj_{i,k}}a^m b^n, \quad k = 1,2,\ldots,\alpha/s_i, \quad i \in I,$$

and

$$\{(\mathbb{C}G)e_{i,j_{i,k}}; \quad k = 1,2,\ldots,\alpha/s_i, \quad i \in I\}$$

is a full set of pairwise inequivalent, irreducible left $\mathbb{C}G$-modules. The representation R_{ik} afforded by $(\mathbb{C}G)e_{i,j_{i,k}}$ is of degree s_i and it is induced from the linear representation $T_{ik}\colon G_i \to \mathbb{C}$ where G_i is the subgroup of G generated by the elements a^{s_i} and b, and where the action of T_{ik} is given by the formulae: $T_{ik}(a^{s_i}) = \eta^{s_i j_{i,k}}$, $T_{ik}(b) = \zeta^i$.

Example. We give an example of the construction of a character table using this theorem. Let $G = \langle a, b \mid a^4 = b^{10}, b^{20} = 1, b^a = b^3\rangle$. This is group $G(80,29)$, using the notation from the SmallGroups Library in MAGMA [1]. This refers to the 29th group of order 80 contained in the database. Since the linear characters are easily constructed by lifting the characters of G/G', we will only construct the nonlinear ones. The nontrivial orbits in $\mathbb{Z}/20\mathbb{Z}$ under the action of the subgroup $\langle 3\rangle \subseteq (\mathbb{Z}/20\mathbb{Z})^\times$ are:

$$\{5,15\},\ \{2,6,18,14\},\ \{4,12,16,8\},\ \{1,3,9,7\},\ \{11,13,19,17\}.$$

We choose a set of orbit representatives $I = \{5,2,4,1,11\}$. Since s_i is equal to the size of the orbit of i, we have $s_5 = 2$ and $s_2 = s_4 = s_1 = s_{11} = 4$. Our choices of $q_{i,k}$ and $j_{i,k}$ are recorded in the following table.

(i,k)	(5,0)	(5,1)	(2,0)	(4,0)	(1,0)	(11,0)
$q_{i,k}$	0	1	0	0	0	0
$j_{i,k}$	5	7	2	4	1	11

Let $\eta = e^{2\pi i/8}$ and $\zeta = e^{2\pi i/20}$. Then the character $T_{5,0}\colon \langle a^2, b\rangle \to \mathbb{C}$ sends $a^{2\sigma}b^\tau \mapsto \eta^{2\cdot 5\sigma}\zeta^{5\tau} = i^{\sigma+\tau}$. Using the transversal $1, a$ for $\langle a^2, b\rangle$ in G, we see that the character $\chi_{5,0}$ of G induced from $T_{5,0}$ is given by

$$\begin{aligned}
\chi_{5,0}(a^{2\sigma}b^\tau) &= T_{5,0}(a^{2\sigma}b^\tau) + T_{5,0}((a^{2\sigma}b^\tau)^a) \\
&= T_{5,0}(a^{2\sigma}b^\tau) + T_{5,0}(a^{2\sigma}b^{3\tau}) \\
&= i^{\sigma+\tau} + i^{\sigma+3\tau} \\
&= i^{\sigma+\tau}(1+(-1)^\tau).
\end{aligned}$$

We also have characters $T_{1,0}, T_{2,0}\colon \langle b\rangle \to \mathbb{C}$ sending $b^\tau \mapsto \zeta^i$ and $b^\tau \mapsto \zeta^{2i}$, respectively. Using the transversal $1, a, a^2, a^3$ for $\langle \beta\rangle$ in G, we compute

$$\begin{aligned}
\chi_{1,0}(b^\tau) &= \sum_{\ell=0}^{3} T_{1,0}((b^\tau)^{a^\ell}) = \sum_{\ell=0}^{3} T_{1,0}(b^{3^\ell\tau})) = \sum_{\ell=0}^{3} \zeta^{3^\ell\tau}, \\
\chi_{2,0}(b^\tau) &= \sum_{\ell=0}^{3} T_{2,0}((b^\tau)^{a^\ell}) = \sum_{\ell=0}^{3} T_{2,0}(b^{3^\ell\tau})) = \sum_{\ell=0}^{3} \zeta^{2\cdot 3^\ell\tau}.
\end{aligned}$$

The characters $\chi_{5,1}$, $\chi_{4,0}$, and $\chi_{11,0}$ are computed similarly. Most of these character values can be expressed more simply. For example, $\chi_{1,0}(b) = \zeta + \zeta^3 + \zeta^9 + \zeta^7 = i\sqrt{5}$ and $\chi_{2,0}(b) = \zeta^2 + \zeta^6 + \zeta^{18} + \zeta^{14} = 1$. The following table gives the values of the six nonlinear characters of G.

	1	b^{10}	b^5	b^2	b^4	b	b^{11}	a^2	a^2b^2	a^2b	a	ab	a^3	a^3b
$\chi_{5,0}$	2	-2	0	-2	2	0	0	$2i$	$-2i$	0	0	0	0	0
$\chi_{5,1}$	2	-2	0	-2	2	0	0	$-2i$	$2i$	0	0	0	0	0
$\chi_{2,0}$	4	4	-4	-1	-1	1	1	0	0	0	0	0	0	0
$\chi_{4,0}$	4	4	4	-1	-1	-1	-1	0	0	0	0	0	0	0
$\chi_{1,0}$	4	-4	0	1	-1	$i\sqrt{5}$	$-i\sqrt{5}$	0	0	0	0	0	0	0
$\chi_{11,0}$	4	-4	0	1	-1	$-i\sqrt{5}$	$i\sqrt{5}$	0	0	0	0	0	0	0

Chapter 2. Split metacyclic groups

In this chapter we prove the following theorem.

Theorem 2.1. *Any two split metacyclic groups with the same character tables are isomorphic.*

If G is a split metacyclic group, then it has a presentation of the form

$$G = \langle a, b \mid a^{\alpha} = b^{\beta} = 1, b^{a} = b^{\gamma} \rangle.$$

We would like to show that the parameters α and β in this presentation are invariants of G. However, in general this is not true, as the following example shows.

Example. Consider the group $G = H \times P$, where $H = \langle r, s \mid r^4 = s^2 = 1, r^s = r^{-1} \rangle \cong D_8$ and $P = \langle y \rangle \times \langle z \rangle \cong C_3 \times C_9$. Then any pairing of the generators of H and P will result in a presentation for the metacyclic group G. If we let $a = sy$ and $b = rz$, then we obtain the following presentation for G:

$$G = \langle a, b \mid a^{6} = b^{36} = 1, b^{a} = b^{19} \rangle.$$

On the other hand, choosing a different pairing of the generators and letting $a = sz$ and $b = ry$ results in the presentation

$$G = \langle a, b \mid a^{18} = b^{12} = 1, b^{a} = b^{7} \rangle.$$

In general, whenever G has a central Sylow p-subgroup $P \cong C_{p^r} \times C_{p^s}$ with $r \neq s$ then we will have this same problem. However this situation can be remedied by the following argument.

Lemma 2.2. *Let G be a group with a central Sylow p-subgroup P. Then $G = P \times H$ has P as a direct factor, and both the isomorphism type of P and the character table of the* *complement H are determined by the character table of G. Moreover, given the character table of G, the isomorphism type of G is determined by the isomorphism type of H.*

Proof. If G has a central Sylow p-subgroup P, then P is normal in G and so it is the unique normal subgroup of its order. Thus the the conjugacy classes comprising P can be located using the character table of G. Since P is central, and the multiplication table of $Z(G)$ is determined by the character table of G, from this we can determine the isomorphism type of P. Since P is a normal Hall subgroup, the group G splits over P by the Schur-Zassenhaus Theorem [7, p. 75], and we write $G = P \rtimes H$ for some H. But since P is central, this product is direct, and we have $G = P \times H$. The character table of the quotient $H = G/P$ can be obtained from the character table of G. Furthermore, since $G = P \times H$, and we know the isomorphism type of P, the isomorphism type of G is determined by the isomorphism type of H. $\square$

Suppose that G has a central Sylow p-subgroup P, and let α_p and β_p be the powers of p dividing α and β. Then $P = \langle a^{\alpha/\alpha_p} \rangle \times \langle b^{\beta/\beta_p} \rangle$, and so we have the presentation

$$G/P \cong \langle a, b \mid a^{\alpha/\alpha_p} = b^{\beta/\beta_p} = 1, b^a = b^\gamma \rangle,$$

which shows that G/P is split metacyclic. It follows from Lemma 2.2 that if we are able to determine the isomorphism type of G/P from its character table, then we will be able to determined the isomorphism type of G from its character table. From now on we assume that no Sylow p-subgroup of G is central.

In this case we will find that the parameters α and β are character table invariants. By Lemma 1.1, we know that

$$G' = \langle b^{\gamma-1} \rangle \cong C_t, \qquad Z(G) = \langle a^s, b^t \rangle,$$

where $s = \mathrm{ord}_\beta(\gamma)$ divides α, and $t = \beta/(\beta, \gamma - 1)$. Thus t can be determined from the character table. Since G is split, the subgroups generated by a^s and b^t intersect trivially, and so $Z(G) = \langle a^s \rangle \times \langle b^t \rangle \cong C_u \times C_v$, where

$$u = |a^s| = \alpha/s \qquad \text{and} \qquad v = |b^t| = \beta/t.$$

We claim that the parameters α, β, s, t, u, and v are character table invariants. So far we are able to determine

$$|G| = \alpha\beta, \quad Z(G) = uv, \quad |G'| = t, \quad \text{and} \quad s = \frac{|G|}{|Z(G)| \cdot |G'|}.$$

We note that if at any time we are able to determine one of the parameters α, β, u, or v, then we will be able to determine them all. Also, if we are able to determine the power of a prime p which divides one of these, we can determine the power of p which divides each of them.

We can determine from the character table the isomorphism types of the following abelian groups in terms of the parameters given above:

$$\begin{aligned} Z(G) &\cong C_u \times C_v, \\ G/G' &\cong C_\alpha \times C_v, \\ Z/(G' \cap Z(G)) &\cong C_u \times C_{v/(t,v)}. \end{aligned}$$

Thus we know the invariant factors of each, and in particular we can determine the power of p which divides the order of each direct factor of these groups, up to the ordering of the factors. Let p be a prime. For a positive integer x, we write x_p to denote the power of p that divides x. Then

$$A = \{u_p, v_p\}, \quad B = \{\alpha_p, v_p\}, \quad \text{and} \quad C = \{u_p, v_p/(t_p, v_p)\}$$

are the sets of invariant factors for the p-primary parts of the above groups, and these sets are determined by the character table.

If $|A \cap B| = 1$, then $A \cap B = \{v_p\}$ and we are done. Thus we assume that $|A \cap B| = 2$, so that

$$u_p = \alpha_p \neq v_p.$$

Next if $|A \cap C| = 1$, then $A \cap C = \{u_p\}$ and we are done. Thus we assume that $|A \cap C| = 2$, so that $v_p = v_p/(t_p, v_p)$ and $(t_p, v_p) = 1$. If it is the case that $t_p \neq 1$, then we have $v_p = 1$ and we are done. We now assume that $t_p = 1$. But now $\beta_p = t_p v_p = v_p$, and so $(\alpha\beta)_p = \alpha_p \beta_p = u_p v_p = (uv)_p$. Therefore a Sylow p-subgroup of G is central, which contradicts our assumption above.

From now on we assume that the values of the parameters α, β, s, t, u, and v are known to us. The last piece of information needed to determine G is the multiplicative subgroup generated by γ mod β, as is shown in the following lemma. We call this subgroup H.

Lemma 2.3. *If $G_i = G(\alpha, \beta, \gamma_i)$, $i = 1, 2$, are split metacyclic groups such that $\langle \gamma_1 \rangle = \langle \gamma_2 \rangle$ as subgroups of $(Z/\beta\mathbb{Z})^\times$, then G_1 and G_2 are isomorphic.*

Proof. Suppose that $G_1 = G_1(\alpha, \beta, \gamma_1)$ and $G_2 = G_2(\alpha, \beta, \gamma_2)$ are split metacyclic groups with $\langle \gamma_1 \rangle = \langle \gamma_2 \rangle$ as subgroups of $(\mathbb{Z}/\beta\mathbb{Z})^\times$. Write $\gamma_1 = \gamma_2^k$ for some k which is invertible mod s. Define the map $\phi\colon G_1 \to G_2$ by $a^i b^j \mapsto a^{ik} b^j$. Since k is invertible mod s, it is invertible mod α, and so ϕ is bijective. The map ϕ also satisfies

$$\phi(a^i b^j \cdot a^m b^n) = \phi(a^i a^m b^{j\gamma_1^m} b^n) = \phi(a^{i+m} b^{j\gamma_1^m + n}) = a^{(i+m)k} b^{j\gamma_1^m + n}$$

and

$$\phi(a^i b^j)\phi(a^m b^n) = a^{ik} b^j a^{mk} b^n = a^{ik} a^{mk} b^{j\gamma_2^{mk}} b^n = a^{(i+m)k} b^{j\gamma_1^m + n},$$

and is therefore an isomorphism of groups. □

Let $\zeta = e^{2\pi i/\beta}$ and $\eta = e^{2\pi i/\alpha}$, and let $L = \mathbb{Q}(\zeta)$. Since $(\mathbb{Z}/\beta\mathbb{Z})^\times = \mathrm{Gal}(L/\mathbb{Q})$, there is a Galois correspondence between the subgroups of $(\mathbb{Z}/\beta\mathbb{Z})^\times$ and the subfields of L. Thus by the Lemma 2.3, we will be done if we are able to determine the fixed field L^H from the character table of G.

We restate Theorem 1.2 specialized to the situation of a split metacyclic group. Let I be a complete set of representatives of the orbits under the multiplicative action of $\langle\gamma\rangle$ on $\mathbb{Z}/\beta\mathbb{Z}$, and for each $i \in I$ let s_i denote the order of γ as an element of $\left(\mathbb{Z}\Big/\frac{\beta}{(\beta,i)}\mathbb{Z}\right)^{\times}$. Then a complete set of inequivalent irreducible complex representations of G are obtained by inducing certain linear representations T_{ik} of the subgroups $G_i = \langle a^{s_i}, b\rangle$ to G. Explicitly, these representations of G_i are given by the formulae

$$T_{ik}(a^{s_i}) = \eta^{s_i k}, \qquad T_{ik}(b) = \zeta^i,$$

where $0 \le k < \alpha/s_i$. We induce the characters T_{ik} to G to obtain characters $\chi_{i,k}$ which vanish off of the (normal) subgroup $G_i = \langle a^{s_i}, b\rangle$. We use the transversal $1, a, \ldots, a^{s_i-1}$ for G_i in G to compute the values of $\chi_{i,k}$ as follows:

$$\begin{aligned}
\chi_{i,k}(a^{s_i\sigma}b^{\tau}) &= \sum_{\ell=0}^{s_i-1} T_{ik}((a^{s_i\sigma}b^{\tau})^{a^{\ell}}) \\
&= \sum_{\ell=0}^{s_i-1} T_{ik}(a^{s_i\sigma}b^{\tau\gamma^{\ell}}) \\
&= \sum_{\ell=0}^{s_i-1} \eta^{s_i\sigma k}\zeta^{i\tau\gamma^{\ell}} \\
&= \eta^{s_i\sigma k}\sum_{\ell=0}^{s_i-1} \zeta^{i\tau\gamma^{\ell}}.
\end{aligned}$$

From now on, we will only need to consider the characters $\chi_{i,k}$ of maximal degree s, so when we refer to "the $\chi_{i,k}$" we will only mean these. These characters are those with $s_i = s$, and they take the values

$$\chi_{i,k}(a^{s\sigma}b^{\tau}) = \eta^{s\sigma k}\sum_{\ell=1}^{s} \zeta^{i\tau\gamma^{\ell}}. \tag{2.1}$$

For $i \in I$, let

$$\xi_i = \mathrm{Tr}^{L}_{L^H}(\zeta^i) = \sum_{\ell=1}^{s} \zeta^{i\gamma^{\ell}}.$$

We will call these *periods*, as they are the same as Gaussian periods when $i \in (\mathbb{Z}/\beta\mathbb{Z})^{\times}$. As traces over L^H these periods are contained in L^H. More importantly, they generate the whole fixed field L^H, as is proved in the following lemma.

Lemma 2.4. *Let L/K be a finite extension of fields with basis B, and let H be a subgroup of $\operatorname{Aut}(L/K)$. For $\lambda \in L$, let $\mathcal{C}_\lambda$ denote the orbit of λ under the action of H, and let $\xi_\lambda = \sum_{\mu\in\mathcal{C}_\lambda} \mu$ (so that the periods ξ_i above are ξ_{ζ^i}). Then the set $\{\xi_\lambda : \lambda \in B\}$ spans L^H as a K-vector space.*

Proof. For $\lambda \in L$, let $\xi'_\lambda = \sum_{\sigma\in H} \lambda^\sigma$. Then $\xi'_\lambda = |\operatorname{Aut}_K(L) : H| \cdot \xi_\lambda$, and so any set of ξ_λ span the same subspace of L as the corresponding set of ξ'_λ. Let $\lambda \in L^H$, and write $\lambda = \sum_{\mu\in B} c_{\lambda,\mu}\mu$ with $c_{\lambda,\mu} \in K$. Then for any $\sigma \in H$, we have $\lambda = \lambda^\sigma = \sum_{\mu\in B} c_{\lambda,\mu}\mu^\sigma$. It follows that

$$\lambda = \frac{1}{|H|}\sum_{\sigma\in H}\lambda^\sigma = \frac{1}{|H|}\sum_{\sigma\in H}\sum_{\mu\in B} c_{\lambda,\mu}\mu^\sigma = \frac{1}{|H|}\sum_{\mu\in B} c_{\lambda,\mu}\sum_{\sigma\in H}\mu^\sigma = \frac{1}{|H|}\sum_{\mu\in B} c_{\lambda,\mu}\xi'_\mu,$$

and so L^H is spanned by the ξ'_μ, $\mu \in B$. ☐

Let $L_{i,k}$ be the field obtained by adjoining the values of $\chi_{i,k}$ to the rational field. Then from (2.1), we have $L_{i,k} = L_i N_k$, where $N_k = \mathbb{Q}(\eta^{sk})$ and $L_i = \mathbb{Q}(\{\xi_{i\tau} : 0 \le \tau \le \beta - 1\})$. For i with $\gcd(i, \beta) = 1$, the values $\xi_{i\tau}$ run through all the periods ξ_j as τ runs through $0, 1, \dots, \beta - 1$, and so $L_i = L^H$.

By examining the values of the $\chi_{i,k}$ on $Z(G)$ we will be able to choose certain $\chi_{i,k}$ which will allow us to isolate the periods ξ_i and determine the fixed field L^H. The values of $\chi_{i,k}$ on $Z(G) = \langle a^s\rangle \times \langle b^t\rangle \cong C_u \times C_v$ are

$$\chi_{i,k}(a^{s\sigma}b^{t\tau}) = s\eta^{s\sigma k}\zeta^{t\tau i}, \qquad 0 \le \sigma \le u, \;\; 0 \le \tau \le v.$$

Let Y be the set of all $\chi_{i,k}$ such that $\chi_{i,k}(z) \in \mathbb{Q}(\zeta^t)$ for all $z \in Z(G)$. Since $\mathbb{Q}(\zeta^t)$ is the vth cyclotomic field, we know which field $\mathbb{Q}(\zeta^t)$ is. Note that the fact that a character $\chi_{i,k}$

belongs to Y puts no restraint on i, but only on k. In particular, the character $\chi_{i,0}$ for any i relatively prime to β is contained in Y. As a result, the field obtained by adjoining to $\mathbb{Q}$ the set $\{\chi_{i,k}(g) : \chi_{i,k} \in Y, g \in G\}$ is equal to $\mathbb{Q}(\zeta^t, \{\xi_{i\tau}\}_{i,\tau})$. Since $\zeta^t \in L^H$ and the $\xi_{i\tau}$ generate L^H, this field is the desired fixed field. This completes the proof. $\square$

We give two examples of the process described in Theorem 2.1 for determining the isomorphism type of a split metacyclic group.

Example 1. First we consider group $G = G(72, 27)$ in MAGMA notation. This group has 24 linear characters and 12 characters of degree 2. We quickly find the values of $\alpha\beta = |G| = 72$, $t = |G'| = 3$, and $s = \max c.d.(G) = 2$, where $c.d.(G)$ denotes the set of character degrees of G. We can also determine the isomorphism types of

$$\begin{aligned} Z(G) &\cong C_u \times C_v \cong C_{12}, \\ G/G' &\cong C_\alpha \times C_v \cong C_2 \times C_{12}, \\ Z/(G' \cap Z(G)) &\cong C_u \times C_{v/(t,v)} \cong C_{12}. \end{aligned}$$

Then $A_2 = \{u_2, v_2\} = \{1, 4\}$ and $B_2 = \{\alpha_2, v_2\} = \{2, 4\}$. Since $A_2 \cap B_2 = \{4\}$, we have $v_2 = 4$. From this we determine that $\beta_2 = t_2 v_2 = 4$, $\alpha_2 = (\alpha\beta)_2/\beta_2 = 2$, and $u_2 = \alpha_2/s_2 = 1$. We also have $A_3 = \{u_3, v_3\} = \{1, 3\}$ and $C_3 = \{u_3, v_3/(3, v_p)\} = \{1, 3\}$. Since these sets are the same, we must have $v_3 = 1$. From this we determine that $\beta_3 = t_3 v_3 = 3$, $\alpha_3 = (\alpha\beta)_3/\beta_3 = 3$, and $u_3 = \alpha_3/s_3 = 3$. Putting this information together, we have $(\alpha, \beta, s, t, u, v) = (6, 12, 2, 3, 3, 4)$.

For the remaining steps, we will refer to the partial character table for G given in Table 2.1. This table gives the values of the 12 characters of G of maximal degree 2. The central classes are labeled with asterisks across the top. The characters which take on values on $Z(G)$ which are in $\mathbb{Q}(i)$ are characters χ_1, χ_2, χ_7, and χ_8. Adjoining the values of these characters gives the field $\mathbb{Q}(i)$. Since the subgroup of $(\mathbb{Z}/12\mathbb{Z})^\times$ fixing $\mathbb{Q}(i)$ is the subgroup generated by 5, we conclude that

$$G = \langle a, b \mid a^6 = b^{12} = 1, b^a = b^5 \rangle.$$

Example 2. For the second example, let G be group $G(576, 1138)$. Then G has 48 linear characters, 60 characters of degree 2, and 18 characters of degree 4. We find that $\alpha\beta = |G| = 576$, $t = |G'| = 12$, and $s = \max c.d.(G) = 4$. We also have

$$\begin{aligned} Z(G) &\cong C_u \times C_v \cong C_{12}, \\ G/G' &\cong C_\alpha \times C_v \cong C_4 \times C_{12}, \\ Z/(G' \cap Z(G)) &\cong C_u \times C_{v/(t,v)} \cong C_3. \end{aligned}$$

Then $A_2 = \{u_2, v_2\} = \{1, 4\}$, $B_2 = \{\alpha_2, v_2\} = \{4, 4\}$, and $A_2 \cap B_2 = \{4\}$, so we have $v_2 = 4$. From this we find that $\beta_2 = t_2 v_2 = 16$, $\alpha_2 = (\alpha\beta)_2/\beta_2 = 4$, and $u_2 = \alpha_2/s_2 = 1$. We also have $A_3 = \{u_3, v_3\} = \{1, 3\}$ and $C_3 = \{u_3, v_3/(3, v_p)\} = \{1, 3\}$, so we must have $v_3 = 1$ as in the first example. From this we determine that $\beta_3 = t_3 v_3 = 3$, $\alpha_3 = (\alpha\beta)_3/\beta_3 = 3$, and $u_3 = \alpha_3/s_3 = 3$. Putting this information together, we have $(\alpha, \beta, u, v, s, t) = (12, 48, 3, 4, 4, 12)$.

Now of the 18 characters of degree 4, it will suffice to only consider the 12 characters which satisfy $Z(\chi) = Z(G)$. For the following string of equivalences

$$\begin{aligned} a^{s\sigma} b^\tau \in Z_i &\iff b^\tau \in Z_i \\ &\iff \gamma \text{ acts trivially on } \zeta^{i\tau} \\ &\iff i\tau \equiv \gamma i\tau \bmod \beta \\ &\iff i\tau(\gamma - 1) \equiv 0 \bmod \beta \\ &\iff i\tau \equiv 0 \bmod t \\ &\iff \tau \equiv 0 \bmod t/(i, t) \end{aligned}$$

shows that the characters $\chi_{i,k}$ with $Z(\chi_{i,k}) = Z(G)$ are precisely those with $\gcd(i,t) = 1$. But since $t \mid \beta$, this is satisfied by all characters $\chi_{i,k}$ with $\gcd(i,\beta) = 1$, and from just these characters we can obtain the periods ξ_i which generate the field fixed by γ.

The values of these 12 characters are shown in Table 2.2, with zero columns deleted, and where the central classes are labeled with asterisks across the top. Since $v = 4$, we look at the characters whose values on $Z(G)$ are contained in $\mathbb{Q}(i)$. These characters are χ_1, χ_4, χ_6, and χ_8. When we take the field generated by all the values of these characters, we obtain the splitting field of $x^4 + 144$ over $\mathbb{Q}$. By looking at the roots as they are expressed in terms of $\epsilon = e^{2\pi i/24}$, we see that the subgroup of $(\mathbb{Z}/48\mathbb{Z})^\times$ which fixes this field is generated by 5. Thus we have

$$G = \langle a, b \mid a^{12} = b^{48} = 1, b^a = b^5 \rangle.$$

Note that in both these examples, adjoining the values of other characters would have resulted in a field with greater degree.

	*	*			*	*				*	*			*	*								*	*	*	*										
$\boldsymbol{\chi_1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\boldsymbol{\chi_2}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
χ_3	2	2	0	0	2ω	-2ζ	-2ω	-1	ζ	2	2	0	0	-2ζ	2ω	-1	ζ	-2ω	0	0	0	0	2ω	-2ζ	2ω	-2ζ	-1	ζ	-2ω	ζ	-1	-2ω	0	0	0	0
χ_4	2	2	0	0	-2ζ	2ω	ζ	-1	-2ω	2	2	0	0	2ω	-2ζ	-1	-2ω	ζ	0	0	0	0	-2ζ	2ω	-2ζ	2ω	-1	-2ω	ζ	-2ω	-1	ζ	0	0	0	0
χ_5	2	2	0	0	-2ζ	2ω	ζ	-1	-2ω	-2	-2	0	0	2ω	-2ζ	-1	-2ω	ζ	0	0	0	0	2ζ	-2ζ	2ζ	-2ζ	1	ω	$-\zeta$	ω	1	$-\zeta$	0	0	0	0
χ_6	2	2	0	0	2ω	-2ζ	-2ω	-1	ζ	-2	-2	0	0	-2ζ	2ω	-1	ζ	-2ω	0	0	0	0	-2ζ	2ζ	-2ζ	2ζ	1	$-\zeta$	ω	$-\zeta$	1	ω	0	0	0	0
$\boldsymbol{\chi_7}$	$\mathbf{2}$	$\mathbf{-2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{-2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{2i}$	$\boldsymbol{-i}$	$\boldsymbol{-i}$	$\boldsymbol{-i}$	$\boldsymbol{i}$	$\boldsymbol{i}$	$\boldsymbol{i}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\boldsymbol{\chi_8}$	$\mathbf{2}$	$\mathbf{-2}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{-1}$	$\mathbf{-1}$	$\mathbf{-1}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\boldsymbol{2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{-2i}$	$\boldsymbol{i}$	$\boldsymbol{i}$	$\boldsymbol{i}$	$\boldsymbol{-i}$	$\boldsymbol{-i}$	$\boldsymbol{-i}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
χ_9	2	-2	0	0	-2ζ	2ω	ζ	-1	-2ω	$2i$	$-2i$	0	0	-2ζ	2ζ	1	ω	$-\zeta$	0	0	0	0	$2\eta^5$	2η	$-2\eta^5$	-2η	$-i$	η	η^5	$-\eta$	i	$-\eta^5$	0	0	0	0
χ_{10}	2	-2	0	0	2ω	-2ζ	-2ω	-1	ζ	$2i$	$-2i$	0	0	2ζ	-2ζ	1	$-\zeta$	ω	0	0	0	0	2η	$2\eta^5$	-2η	$-2\eta^5$	$-i$	η^5	η	$-\eta^5$	i	$-\eta$	0	0	0	0
χ_{11}	2	-2	0	0	-2ζ	2ω	ζ	-1	-2ω	$-2i$	$2i$	0	0	-2ζ	2ζ	1	ω	$-\zeta$	0	0	0	0	$-2\eta^5$	-2η	$2\eta^5$	2η	i	$-\eta$	$-\eta^5$	η	$-i$	η^5	0	0	0	0
χ_{12}	2	-2	0	0	2ω	-2ζ	-2ω	-1	ζ	$-2i$	$2i$	0	0	2ζ	-2ζ	1	$-\zeta$	ω	0	0	0	0	-2η	$-2\eta^5$	2η	$2\eta^5$	i	$-\eta^5$	$-\eta$	η^5	$-i$	η	0	0	0	0

Table 2.1: Nonlinear characters of group G(72,27). Here $\omega = e^{2\pi i/3}$, $i = e^{2\pi i/4}$, $\zeta = e^{2\pi i/6}$, and $\eta = e^{2\pi i/12}$.

	*	*	*	*				*	*	*	*				*	*	*	*																		
$\boldsymbol{\chi_1}$	$\mathbf{4}$	$\mathbf{-4}$	$\mathbf{4}$	$\mathbf{4}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{-2}$	$\boldsymbol{-4i}$	$\boldsymbol{4i}$	$\mathbf{-4}$	$\mathbf{-4}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\boldsymbol{4i}$	$\boldsymbol{-4i}$	$\boldsymbol{4i}$	$\boldsymbol{-4i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{z_3}$	$\boldsymbol{z_4}$	$\boldsymbol{z_2}$	$\boldsymbol{z_1}$	$\boldsymbol{z_2}$	$\boldsymbol{z_1}$	$\boldsymbol{z_3}$	$\boldsymbol{z_4}$	$\boldsymbol{z_2}$	$\boldsymbol{z_1}$	$\boldsymbol{z_4}$	$\boldsymbol{z_3}$
χ_2	4	-4	-4ζ	4ω	2ζ	-2ω	-2	$4i$	$-4i$	4ζ	-4ω	2ω	2	-2ζ	4η	-4η	$-4\eta^5$	$4\eta^5$	2η	-2η	$2\eta^5$	$-2\eta^5$	$-2i$	$2i$	y_7	y_8	y_1	y_2	z_4	z_3	y_6	y_5	y_3	y_4	z_2	z_1
χ_3	4	-4	4ω	-4ζ	-2ω	2ζ	-2	$-4i$	$4i$	-4ω	4ζ	-2ζ	2	2ω	$4\eta^5$	$-4\eta^5$	-4η	4η	$-2\eta^5$	$2\eta^5$	-2η	2η	$2i$	$-2i$	y_3	y_4	y_6	y_5	z_1	z_2	y_1	y_2	y_7	y_8	z_3	z_4
$\boldsymbol{\chi_4}$	$\mathbf{4}$	$\mathbf{-4}$	$\mathbf{4}$	$\mathbf{4}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{-2}$	$\boldsymbol{4i}$	$\boldsymbol{-4i}$	$\mathbf{-4}$	$\mathbf{-4}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\boldsymbol{-4i}$	$\boldsymbol{4i}$	$\boldsymbol{-4i}$	$\boldsymbol{4i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{z_2}$	$\boldsymbol{z_1}$	$\boldsymbol{z_3}$	$\boldsymbol{z_4}$	$\boldsymbol{z_3}$	$\boldsymbol{z_4}$	$\boldsymbol{z_2}$	$\boldsymbol{z_1}$	$\boldsymbol{z_3}$	$\boldsymbol{z_4}$	$\boldsymbol{z_1}$	$\boldsymbol{z_2}$
χ_5	4	-4	4ω	-4ζ	-2ω	2ζ	-2	$-4i$	$4i$	-4ω	4ζ	-2ζ	2	2ω	$4\eta^5$	$-4\eta^5$	-4η	4η	$-2\eta^5$	$2\eta^5$	-2η	2η	$2i$	$-2i$	y_4	y_3	y_5	y_6	z_2	z_1	y_2	y_1	y_8	y_7	z_4	z_3
$\boldsymbol{\chi_6}$	$\mathbf{4}$	$\mathbf{-4}$	$\mathbf{4}$	$\mathbf{4}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{-2}$	$\boldsymbol{4i}$	$\boldsymbol{-4i}$	$\mathbf{-4}$	$\mathbf{-4}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\boldsymbol{-4i}$	$\boldsymbol{4i}$	$\boldsymbol{-4i}$	$\boldsymbol{4i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{z_1}$	$\boldsymbol{z_2}$	$\boldsymbol{z_4}$	$\boldsymbol{z_3}$	$\boldsymbol{z_4}$	$\boldsymbol{z_3}$	$\boldsymbol{z_1}$	$\boldsymbol{z_2}$	$\boldsymbol{z_4}$	$\boldsymbol{z_3}$	$\boldsymbol{z_2}$	$\boldsymbol{z_1}$
χ_7	4	-4	-4ζ	4ω	2ζ	-2ω	-2	$-4i$	$4i$	4ζ	-4ω	2ω	2	-2ζ	-4η	4η	$4\eta^5$	$-4\eta^5$	-2η	2η	$-2\eta^5$	$2\eta^5$	$2i$	$-2i$	y_2	y_1	y_8	y_7	z_2	z_1	y_4	y_3	y_5	y_6	z_4	z_3
$\boldsymbol{\chi_8}$	$\mathbf{4}$	$\mathbf{-4}$	$\mathbf{4}$	$\mathbf{4}$	$\mathbf{-2}$	$\mathbf{-2}$	$\mathbf{-2}$	$\boldsymbol{-4i}$	$\boldsymbol{4i}$	$\mathbf{-4}$	$\mathbf{-4}$	$\mathbf{2}$	$\mathbf{2}$	$\mathbf{2}$	$\boldsymbol{4i}$	$\boldsymbol{-4i}$	$\boldsymbol{4i}$	$\boldsymbol{-4i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{2i}$	$\boldsymbol{-2i}$	$\boldsymbol{z_4}$	$\boldsymbol{z_3}$	$\boldsymbol{z_1}$	$\boldsymbol{z_2}$	$\boldsymbol{z_1}$	$\boldsymbol{z_2}$	$\boldsymbol{z_4}$	$\boldsymbol{z_3}$	$\boldsymbol{z_1}$	$\boldsymbol{z_2}$	$\boldsymbol{z_3}$	$\boldsymbol{z_4}$
χ_9	4	-4	4ω	-4ζ	-2ω	2ζ	-2	$4i$	$-4i$	-4ω	4ζ	-2ζ	2	2ω	$-4\eta^5$	$4\eta^5$	4η	-4η	$2\eta^5$	$-2\eta^5$	2η	-2η	$-2i$	$2i$	y_5	y_6	y_4	y_3	z_3	z_4	y_8	y_7	y_2	y_1	z_1	z_2
χ_{10}	4	-4	-4ζ	4ω	2ζ	-2ω	-2	$-4i$	$4i$	4ζ	-4ω	2ω	2	-2ζ	-4η	4η	$4\eta^5$	$-4\eta^5$	-2η	2η	$-2\eta^5$	$2\eta^5$	$2i$	$-2i$	y_1	y_2	y_7	y_8	z_1	z_2	y_3	y_4	y_6	y_5	z_3	z_4
χ_{11}	4	-4	-4ζ	4ω	2ζ	-2ω	-2	$4i$	$-4i$	4ζ	-4ω	2ω	2	-2ζ	4η	-4η	$-4\eta^5$	$4\eta^5$	2η	-2η	$2\eta^5$	$-2\eta^5$	$-2i$	$2i$	y_8	y_7	y_2	y_1	z_3	z_4	y_5	y_6	y_4	y_3	z_1	z_2
χ_{12}	4	-4	4ω	-4ζ	-2ω	2ζ	-2	$4i$	$-4i$	-4ω	4ζ	-2ζ	2	2ω	$-4\eta^5$	$4\eta^5$	4η	-4η	$2\eta^5$	$-2\eta^5$	2η	-2η	$-2i$	$2i$	y_6	y_5	y_3	y_4	z_4	z_3	y_7	y_8	y_1	y_2	z_2	z_1

Table 2.2: Partial character table for group G(576,1138). Here $\omega = e^{2\pi i/3}$, $i = e^{2\pi i/4}$, $\zeta = e^{2\pi i/6}$, $\eta = e^{2\pi i/12}$, and $\epsilon = e^{2\pi i/24}$. Also $z_1 = 2\epsilon^5 + 2\epsilon$, $z_2 = -2\epsilon^5 - 2\epsilon$, $z_3 = 4\epsilon^7 - 2\epsilon^3$, $z_4 = -4\epsilon^7 + 2\epsilon^3$ are the roots of $x^4 + 144$, and $y_1 = 2\epsilon^7 + 2\epsilon^3$, $y_2 = -2\epsilon^7 - 2\epsilon^3$, $y_3 = 2\epsilon^7 - 4\epsilon^3$, $y_4 = -2\epsilon^7 + 4\epsilon^3$, $y_5 = 4\epsilon^5 - 2\epsilon$, $y_6 = -4\epsilon^5 + 2\epsilon$, $y_7 = 2\epsilon^5 - 4\epsilon$, $y_8 = -2\epsilon^5 + 4\epsilon$ are the roots of $x^8 - 144x^2 + 20736$.

It is well-known that the dihedral group D_{2^n} and the generalized quaternion group Q_{2^n} of order $2^n, n \geq 3$, have the same character tables [3, p.64]. We give a proof of this fact here. Presentations for these groups are

$$\begin{aligned} D_{2^n} &= \langle a, b \mid a^2 = b^{2^{n-1}} = 1, b^a = b^{-1} \rangle, \\ Q_{2^n} &= \langle a, b \mid a^2 = b^{2^{n-2}}, b^{2^{n-1}} = 1, b^a = b^{-1} \rangle. \end{aligned}$$

Each group has $2^{n-2} + 3$ conjugacy classes, with representatives

$$\begin{aligned} &\{1\}, \quad \{b^{2^{n-2}}\}, \quad \{b^k, b^{-k}\} \text{ for } 1 \leq k \leq 2^{n-2} - 1, \\ &\{ab^k : k \text{ even}\}, \quad \{ab^k : k \text{ odd}\}. \end{aligned}$$

Each group has abelianization $G/G' \cong C_2 \times C_2$. Lifting the characters of G/G' gives the same linear characters for each group (with respect to the labeling of the generators a and b). Let $\zeta = e^{2\pi i/2^{n-2}}$. Then for $1 \leq j \leq 2^{n-2} - 1$, the maps

$$a \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix}$$

extend to irreducible 2-dimensional representations of D_{2^n}, and

$$a \mapsto \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad b \mapsto \begin{pmatrix} \zeta^j & 0 \\ 0 & \zeta^{-j} \end{pmatrix}$$

extend to irreducible representations of Q_{2^n}.

These give the common character table

	1	$b^{2^{n-2}}$	b^k	a	ab
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	$(-1)^k$	1	-1
χ_4	1	1	$(-1)^k$	-1	1
ψ_j	2	$2(-1)^j$	$\zeta^{jk}+\zeta^{-jk}$	0	0

where j and k each run through $1, 2, \ldots, 2^{n-2}-1$.

In this section, we show that this is the only example of two metacyclic p-groups with the same character table. We prove:

Theorem 3.1. *Among metacyclic p-groups, the only pairs having the same character tables are D_{2^n} and Q_{2^n}, $n \geq 3$.*

We will need the following lemma, which is Corollary 4.4 of [5].

Lemma 3.2. *If $m \geq n \geq 1$, p is a prime, $n+p \geq 4$, and $r \geq 1$, then each of the statements*

$$(1+p^n)^{p^r} \equiv 1 \bmod p^m,$$
$$(-1+2^n)^{2^r} \equiv 1 \bmod 2^m$$

is equivalent to $r \geq m-n$.

Theorem 3.3. *Any pair of metacyclic p-groups, p odd, which have the same character tables are isomorphic.*

Proof. Let G be a noncyclic metacyclic p-group, p odd. Then in Theorem 3.5 of [11], it is shown that G has a presentation of the form

$$G = \langle a, b \mid a^{p^\alpha} = b^{p^\beta}, b^{p^{\beta+\delta}} = 1, b^a = b^{p^{1+p^\gamma}} \rangle,$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative integers satisfying $\alpha \geq \beta \geq \gamma \geq \delta$ and $\gamma \geq 1$, and that these parameters characterize the group G.

From the character table, we can determine $|G| = p^{\alpha+\beta+\delta}$ and $G/G' \cong C_{p^\alpha} \times C_{p^\gamma}$. Since $\alpha \geq \gamma$, we then know α and γ, and also $\beta + \delta$. We now determine the isomorphism type of $Z(G)$. From Lemma 1.1 we know that $Z(G) = \langle a^s, b^t \rangle$, where

$$\begin{aligned} s &= \mathrm{ord}_{|b|}(1 + p^\gamma) = \mathrm{ord}_{p^{\beta+\delta}}(1 + p^\gamma), \\ t &= |b| / \gcd(|b|, p^\gamma) = p^{\beta+\delta} / \gcd(p^{\beta+\delta}, p^\gamma) = p^{\beta+\delta-\gamma}, \end{aligned}$$

and this last equality follows since $\beta + \delta \geq \gamma$. Since $\beta + \delta \geq \gamma \geq 1$ and $\gamma + p \geq 4$, Lemma 3.2 implies that $(1 + p^\gamma)^{p^\epsilon} \equiv 1 \bmod p^{\beta+\delta}$ if and only if $\epsilon \geq \beta + \delta - \gamma$, and so

$$s = \mathrm{ord}_{p^{\beta+\delta}}(1 + p^\gamma) = p^{\beta+\delta-\gamma}.$$

Since $|a| = p^{\alpha+\delta}$ and $|b| = p^{\beta+\delta}$, we have

$$\begin{aligned} |a^s| &= |a|/s = p^{\alpha+\delta}/p^{\beta+\delta-\gamma} = p^{\alpha-\beta+\gamma}, \\ |b^t| &= |b|/t = p^{\beta+\delta}/p^{\beta+\delta-\gamma} = p^\gamma. \end{aligned}$$

Furthermore, since $\beta \leq \alpha$ and $\delta \leq \gamma$, we have

$$\begin{aligned} s &= p^{\beta+\delta-\gamma} \leq p^\alpha, \\ t &= p^{\beta+\delta-\gamma} \leq p^\beta. \end{aligned}$$

Therefore, $a^{p^\alpha} = b^{p^\beta} \in \langle a^s \rangle \cap \langle b^t \rangle$. But since a^{p^α} is the smallest power of a lying in the subgroup $\langle b \rangle$, we conclude that $\langle a^\alpha \rangle \cap \langle b^\beta \rangle$ is generated by a^{p^α}, which has order $p^{\alpha+\delta}/p^\alpha = p^\delta$. Thus

$$|Z(G)| = \frac{|\langle a^s \rangle| \cdot |\langle b^t \rangle|}{|\langle a^s \rangle \cap \langle b^t \rangle|} = \frac{|a^s| \cdot |b^t|}{|a^{p^\alpha}|} = \frac{p^{\alpha-\beta+\gamma} \cdot p^\gamma}{p^\delta} = p^{\alpha-\beta+2\gamma-\delta}.$$

Since a^s and b^t generate the abelian group $Z(G)$, we see that the exponent of $Z(G)$ is $\exp Z(G) = \max(|a^s|, |b^t|) = \max(p^{\alpha-\beta+\gamma}, p^{\gamma}) = p^{\alpha-\beta+\gamma}$. Therefore,

$$Z(G) \cong C_{\exp Z(G)} \times C_{|Z(G)|/\exp Z(G)} = C_{p^{\alpha-\beta+\gamma}} \times C_{p^{\gamma-\delta}}.$$

Since $\alpha - \beta + \gamma > \gamma - \delta$, we can then determine β and δ, and hence G. $\square$

Since nilpotent groups are direct products of their Sylow p-subgroups, the following corollary follows immediately from Theorem 3.1.

Corollary 3.4. *Any two nilpotent metacyclic groups of odd order which have the same character tables are isomorphic.*

We now deal with the case of 2-groups. We prove

Theorem 3.5. *Any two metacyclic 2-groups which are not dihedral and generalized quaternion, respectively, can be distinguished by their character tables.*

To prove this, we will make use of the following characterization of metacyclic 2-groups by Hempel [5].

Theorem 3.6. *Every metacyclic 2-group has one of the following eight types of presentation, in which the parameters r, s, t, u, v and w are nonnegative integers:*

(i) $\langle a \mid a^{2^r} = 1\rangle$ *with* $r \geq 0$,

(ii) $\langle a, b \mid a^{2^r} = b^2 = 1, b^a = b\rangle$ *with* $r \geq 1$,

(iii) $\langle a, b \mid a^2 = b^{2^r} = 1, b^a = b^{-1}\rangle$ *with* $r \geq 2$,

(iv) $\langle a, b \mid a^2 = b^{2^r}, b^{2^{r+1}} = 1, b^a = b^{-1}\rangle$ *with* $r \geq 1$,

(v) $\langle a, b \mid a^2 = b^{2^{r+1}} = 1, b^a = b^{1+2^r}\rangle$ *with* $r \geq 2$,

(vi) $\langle a, b \mid a^2 = b^{2^{r+1}} = 1, b^a = b^{-1+2^r}\rangle$ *with* $r \geq 2$,

(vii) $\langle a, b \mid a^{2^r} = b^{2^s}, b^{2^{s+t}} = 1, b^a = b^{1+2^u} \rangle$ *with* $r \geq s \geq u \geq 2$ *and* $u \geq t$,

(viii) $\langle a, b \mid a^{2^{r+s+t}} = b^{2^{r+s+u+v}}, b^{2^{r+s+u+v+w}} = 1, b^a = b^{-1+2^{r+u}} \rangle$ *where* $r \geq 2$, $v \leq r$, $w \leq 1$, $su = tu = tv = 0$, *and if* $v \geq r-1$, *then* $w = 0$.

Groups of different types or of the same type but with different parameters are not isomorphic.

Proof of Theorem 3.5. Groups of type (i) and (ii) are abelian, and so are determined by their character tables. Types (iii) and (iv) are dihedral and generalized quaternion, respectively. For the same choice of the parameters r, these groups have the same order, in which case we have already seen that they have the same character table. We will assume that $|G| > 8$.

We note that groups of types (iii), (v), (vi) are split metacyclic and so no nonisomorphic pair of these groups have the same character tables by Theorem 2.1. However, since these groups can be distinguished using only the order of G, the isomorphism type of G/G', and the number of real characters of G, we will include arguments below distinguishing between these types which are simpler than those given in the proof of Theorem 2.1.

The following table gives the orders of G and isomorphism type of G/G' for groups of different types in terms of the parameters given in the characterization above. Note that parameters with the same names but corresponding to groups of different types are unrelated.

type	$\|G\|$	G/G'
(iii), (iv)	2^{r+1}	$C_2 \times C_2$
(v)	2^{r+2}	$C_2 \times C_{2^r}$
(vi)	2^{r+2}	$C_2 \times C_2$
(vii)	2^{r+s+t}	$C_{2^r} \times C_{2^u}$
(viii)	$2^{2r+2s+t+u+v+w}$	$C_2 \times C_{2^{r+s+t}}$

The information in this table follows immediately from the presentations.

Groups of type (vii) are distinguished from all other types by G/G', since the parameters $r, u \geq 2$. Groups of type (v) have $|G'| = 2$, which distinguishes them from the remaining types (iii), (iv), (vi), and (viii). Groups of type (viii) have $G/G' \cong C_2 \times C_{2^{r+s+t}}$. Since $r \geq 2$, this distinguishes type (viii) from types (iii), (iv) and (vi). Groups of type (vi) are distinguished from groups of type (iii), (iv) by the following lemma, which is a generalization of Lemma 4.1 of [5].

Lemma 3.7. *The 2-groups given by the presentations*

$$G = \langle a, b \mid a^{2^\delta} = b^{2^{\gamma+1}} = 1, b^a = b^{-1+2^\gamma} \rangle,$$
$$H = \langle a, b \mid a^{2^\delta} = b^{2^{\gamma+1}} = 1, b^a = b^{-1} \rangle,$$

where $\gamma, \delta \geq 1$, have different numbers of real irreducible characters.

Proof. If $\delta = 1$, note that H is the dihedral group of order $2^{\gamma+2}$, and we can see from the character table above that all characters of H are real. However, in G since $b^a = b^{-1+2^\gamma} \neq b^{-1}$ and $b^{a^2} = b$, the conjugacy class b^G is not real. Therefore, not all the characters of G are real, and so G and H do not have the same character tables when $\delta = 1$.

Now assume that $\delta \geq 2$. We show first that G contains $2 + 2^\gamma$ real conjugacy classes. In G, we have $b^{a^2} = b$, and so the subgroup $M = \langle a^2, b \rangle \leq G$ is abelian of index 2. Now $(b^j)^a = (b^a)^j = b^{(-1+2^\gamma)j}$ is equal to b^j if and only if $j \equiv (-1+2^\gamma)j \bmod 2^{\gamma+1}$, if and only if $j(2-2^\gamma) \equiv 0 \bmod 2^{\gamma+1}$, if and only if $j \equiv 0 \bmod 2^\gamma$. Therefore, the G-conjugacy classes inside M are

$$(a^{2i}b^j)^G = \begin{cases} \{a^{2i}b^j\} & \text{if } j \in \{0, 2^\gamma\}, \\ \{a^{2i}b^j, a^{2i}b^{(-1+2^\gamma)j}\} & \text{otherwise.} \end{cases}$$

A central element $a^{2i}b^j \in M$ with $j \in \{0, 2^\gamma\}$ is conjugate to its inverse if and only if $a^{2i}b^j = a^{-2i}b^{-j}$, if and only if $a^{4i}b^{2j} = 1$, if and only if $4i \equiv 0 \bmod 2^\delta$ and $2j \equiv 0 \bmod 2^{\gamma+1}$. So $i \in \{0, 2^{\delta-2}, 2^{\delta-1}, 2^{\delta-2}+2^{\delta-1}\}$ and $j \in \{0, 2^\gamma\}$. Since these 4 values of i give only 2 distinct values for $2i \bmod 2^\delta$, M contains exactly 4 central elements which are real.

A noncentral element $a^{2i}b^j \in M$ with $j \neq 0, 2^\gamma$ is conjugate to its inverse if and only if $a^{2i}b^{(-1+2^\gamma)j} = a^{-2i}b^{-j}$, if and only if $a^{4i} = 1$ and $b^{2^\gamma j} = 1$, if and only if $4i \equiv 0 \bmod 2^\delta$ and

$j \equiv 0 \bmod 2$. There are 4 such i for each even $j \neq 0, 2^\gamma$, and these values for i give a total of 2 values for $2i \bmod 2^\delta$. Since the noncentral elements have class size 2, we conclude that the subgroup M contains

$$4 + 2\left(\frac{2^\gamma - 2}{2}\right) = 2 + 2^\gamma$$

real conjugacy classes.

We show that there are no real conjugacy classes outside of M. Let $g = a^{1+2i}b^j \in G \setminus M$, and write $h = a^r b^s$ where $g^{-1} = g^h$. Then $g^{-1} = a^{-1-2i}b^u$ for some u, and

$$g^h = (a^{1+2i})^{b^s} \cdot (b^j)^h = a^{1+2i} \cdot [a^{1+2i}, b^s] \cdot (b^j)^h = a^{1+2i} \cdot b^v,$$

for some v. Thus if $g^{-1} = g^h$, then we must have $1 + 2i \equiv -1 - 2i \mod 2^\delta$ i.e. $2 + 4i \equiv 0 \bmod 2^\delta$, which is a contradiction since $\delta \geq 2$. We conclude that G has $2 + 2^\gamma$ real conjugacy classes.

Now we show that the group H has $2 + 2^{\gamma+1}$ real conjugacy classes. In H, since $b^{a^2} = b^{(-1)^2} = b$, the subgroup $N = \langle a^2, b\rangle \leq H$ is abelian of index 2. Since $(a^{2i}b^j)^a = a^{2i}b^{-j}$, the conjugacy classes inside N are

$$(a^{2i}b^j)^G = \begin{cases} \{a^{2i}b^j\} & \text{if } j \in \{0, 2^\gamma\}, \\ \{a^{2i}b^j, a^{2i}b^{-j}\} & \text{otherwise.} \end{cases}$$

A central element $a^{2i}b^j \in N$ with $j \in \{0, 2^\gamma\}$ is conjugate to its inverse if $a^{2i}b^j = a^{-2i}b^{-j}$, if and only if $a^{4i}b^{2j} = 1$, if and only if $4i \equiv 0 \bmod 2^\delta$ and $2j \equiv 0 \bmod 2^{\gamma+1}$. So $i \in \{0, 2^{\delta-2}, 2^{\delta-1}, 2^{\delta-2} + 2^{\delta-1}\}$ and $j \in \{0, 2^\gamma\}$. Since these 4 values of i give only 2 distinct values for $2i \bmod 2^\delta$, N contains exactly 4 central elements which are real.

A noncentral element $a^{2i}b^j \in N$ with $j \neq 0, 2^\gamma$ is conjugate to its inverse if $a^{2i}b^{-j} = a^{-2i}b^{-j}$, if and only if $a^{4i} = 1$, if and only if $4i \equiv 0 \bmod 2^\delta$. There are 4 such i for each $j \neq 0, 2^\gamma$, and these values for i give a total of 2 values for $2i \bmod 2^\delta$. Since the noncentral elements have class size 2, the subgroup N contains

$$4 + 2\left(\frac{2^{\gamma+1} - 2}{2}\right) = 2 + 2^{\gamma+1}$$

real conjugacy classes.

The proof that there are no real conjugacy classes outside of N is the same as the proof for $M \leq G$. We conclude that H has $2 + 2^{\gamma+1}$ real conjugacy classes. $\square$

It remains to show that two groups of type (vii) or two groups of type (viii), with different parameters, cannot have the same character table.

Case (vii).

We note that the proof of this case is nearly identical to the proof of Theorem 3.3. The difference is that it appeals to the half of Lemma 3.2 which deals with $p = 2$. Suppose G is of type (vii), so that $G = \langle a, b \mid a^{2^r} = b^{2^s}, b^{2^{s+t}} = 1, b^a = b^{1+2^u}\rangle$, where $r \geq s \geq u \geq 2$ and $u \geq t$. From the character table, we can determine $G/G' \cong C_{2^r} \times C_{2^u}$. Since $r \geq u$, we then know both r and u. From Lemma 1.1, we know that $Z(G) = \langle a^\alpha, b^\beta\rangle$, where

$$\alpha = \mathrm{ord}_{|b|}(1 + 2^u) = \mathrm{ord}_{2^{s+t}}(1 + 2^u),$$
$$\beta = \frac{|b|}{\gcd(|b|, 2^u)} = \frac{2^{s+t}}{\gcd(2^{s+t}, 2^u)} = 2^{s+t-u},$$

where for the last equality we used the fact that $2 + t \geq u$. Since s, t, u satisfy $s + t \geq u \geq 1$ and $u + 2 \geq 4$, Lemma 3.2 implies that $(1 + 2^u)^{2^j} \equiv 1 \bmod 2^{s+t}$ if and only if $j \geq s + t - u$, and so

$$\alpha = \mathrm{ord}_{2^{s+t}}(1 + 2^u) = 2^{s+t-u}.$$

Since $|a| = 2^{r+t}$ and $|b| = 2^{s+t}$, we have

$$|a^\alpha| = 2^{r+t}/\alpha = 2^{r+u-s},$$
$$|b^\beta| = 2^{s+t}/\beta = 2^u.$$

Furthermore, since $r \geq s$ and $u \geq t$, we have

$$\alpha = 2^{s+t-u} \leq 2^r,$$

$$\beta = 2^{s+t-u} \leq 2^s.$$

Therefore, $a^{2^r} = b^{2^s} \in \langle a^\alpha \rangle \cap \langle b^\beta \rangle$. Since a^{2^r} is in fact the smallest power of a lying in the subgroup $\langle b \rangle$, we conclude that $\langle a^\alpha \rangle \cap \langle b^\beta \rangle$ is generated by a^{2^r}, which has order $2^{r+t}/2^r = 2^t$. Thus

$$|Z(G)| = \frac{|\langle a^\alpha \rangle| \cdot |\langle b^\beta \rangle|}{|\langle a^\alpha \rangle \cap \langle b^\beta \rangle|} = \frac{|a^\alpha| \cdot |b^\beta|}{|a^{2^r}|} = \frac{2^{r-s+u} \cdot 2^u}{2^t} = 2^{r-s-t+2u}.$$

Since $|a^\alpha| = 2^{r-s+u} > 2^u = |b^\beta|$, we conclude that the exponent of $Z(G)$ is $\exp Z(G) = 2^{r-s+u}$. Thus

$$Z(G) \cong C_{\exp Z(G)} \times C_{|Z(G)|/\exp Z(G)} = C_{2^{r-s+u}} \times C_{2^{u-t}}.$$

Since $r - s + u > u - t$ and we know r, u, we can determine s and t, and hence G.

Case (viii).

For this case we need the following lemma, which is Corollary 4.5 from [5].

Lemma 3.8. *If $G = \langle a, b \mid a^{2^k} = b^{2^l}, b^{2^m} = 1, b^a = b^{-1+2^n} \rangle$, where $m - 1 \leq l \leq m$, $2 \leq n \leq m$, and $m - n \leq k$, then*

$$Z(G) = \begin{cases} \langle a^2, b^{2^{m-1}} \rangle & \text{if } m = n, \\ \langle a^{2^{m-n}}, b^{2^{m-1}} \rangle & \text{otherwise.} \end{cases}$$

A group of type (viii) is of the form

$$G = \langle a, b \mid a^{2^{r+s+t}} = b^{2^{r+s+u+v}}, b^{2^{r+s+u+v+w}} = 1, b^a = b^{-1+2^{r+u}} \rangle,$$

where $r \geq 2$, $v \leq r$, $w \leq 1$, $su = tu = tv = 0$, and if $v \geq r - 1$, then $w = 0$.

From the above table we have:

$$|G| = 2^{2r+2s+t+u+v}, \quad G/G' \cong C_2 \times C_{2^{r+s+t}}.$$

Thus we know $2r+2s+t+u+v$ and $r+s+t$.

Lemma 3.9. *The parameter w can be determined from $Z(G)$. In particular: $w=1$ if and only if $Z(G)$ is cyclic of order greater than 2.*

Proof. Since G satisfies the hypotheses of Lemma 3.8, we have

$$Z(G) = \begin{cases} \langle a^2, b^{2^{r+u-1}} \rangle & \text{if } s+v+w=0, \\ \langle a^{2^{s+v+w}}, b^{2^{r+s+u+v+w-1}} \rangle & \text{otherwise.} \end{cases} \tag{3.1}$$

Suppose that $w=1$. Then we are in the second of the two possibilities for $Z(G)$ in (3.1). We have $b^{2^{r+s+u+v+w-1}} = b^{2^{r+s+u+v}} = a^{2^{r+s+t}}$, and so $Z(G) = \langle a^{2^{s+v+w}}, a^{2^{r+s+t}} \rangle$ is generated by $a^{2^{\min(s+v+w,r+s+t)}}$. Since $w=1$, we have $v < r-1$, and so $s+v+w < s+r \leq s+r+t$. Thus $Z(G) = \langle a^{2^{s+v+w}} \rangle$, and so $|Z(G)| = |a^{2^{s+v+w}}| = |a|/2^{s+v+w} = 2^{r+s+t+w}/2^{s+v+w} = 2^{r+t-v}$. Since $v < r-1$, we have $r+t-v \geq 2$, and so $Z(G) \cong C_{2^{r+t-v}}$ is cyclic of order $2^{r+t-v} > 2$.

Suppose that $w=0, v=r \geq 2$. Then we are again in the second case of (3.1):

$$Z(G) = \langle a^{2^{s+v+w}}, b^{2^{r+s+u+v+w-1}} \rangle = \langle a^{2^{r+s}}, b^{2^{r+s+u+v-1}} \rangle.$$

From the condition $tv=0$, we have $t=0$, and so $a^{2^{r+s}} = a^{2^{r+s+t}} = b^{2^{r+s+u+v}}$. Thus $Z(G)$ is generated by $b^{2^{r+s+u+v-1}}$, which has order $|b|/2^{r+s+u+v-1} = 2^{r+s+u+v}/2^{r+s+u+v-1} = 2$, so $Z(G) \cong C_2$.

If $w=0$ and $v \neq r$, we show that $Z(G)$ is noncyclic.

If $s+v=0$, then $Z(G) = \langle a^2, b^{2^{r+u-1}} \rangle$. Now $|a^2| = |a|/2 = 2^{r+t}/2 = 2^{r+t-1}$ and $|b^{2^{r+u-1}}| = |b|/2^{r+u-1} = 2^{r+u}/2^{r+u-1} = 2$. Furthermore, since G is split metacyclic, the intersection $\langle a^2 \rangle \cap \langle b^{2^{r+u-1}} \rangle \subseteq \langle a \rangle \cap \langle b \rangle$ is trivial, and so

$$Z(G) = \langle a^2 \rangle \times \langle b^{2^{r+u-1}} \rangle \cong C_{2^{r+t-1}} \times C_2.$$

If $s + v > 0$, then $Z(G) = \langle a^{2^{s+v}}, b^{2^{r+s+u+v-1}} \rangle$. The generators have order $|a^{s+v}| = |a|/2^{s+v} = 2^{r+s+t}/2^{s+v} = 2^{r+t-v}$ and $|b^{2^{r+s+u+v-1}}| = |b|/2^{r+s+u+v-1} = 2^{r+s+u+v}/2^{r+s+u+v-1} = 2$. Again, since G is split metacyclic, the intersection $\langle a^{2^{s+v}} \rangle \cap \langle b^{2^{r+s+u+v-1}} \rangle \subseteq \langle a \rangle \cap \langle b \rangle$ is trivial, and so

$$Z(G) = \langle a^{2^{s+v}} \rangle \times \langle b^{2^{r+s+u+v-1}} \rangle \cong C_{2^{r+t-v}} \times C_2.$$

Thus w can be determined by the isomorphism type of $Z(G)$. $\square$

If $w = 1$, then we consider the quotient

$$G/\langle b^{r+s+u+v} \rangle \cong \langle a, b \mid a^{2^{r+s+t}} = b^{2^{r+s+u+v}} = 1, b^a = b^{-1+2^{r+u}} \rangle$$

by the unique normal subgroup of G of order 2. This quotient is a group of type (viii) with the same r, s, t, u, v, but with $w = 0$. Thus we may reduce to the case where $w = 0$. Then we have

$$G = \langle a, b \mid a^{2^{r+s+t}} = b^{2^{r+s+u+v}} = 1, b^a = b^{-1+2^{r+u}} \rangle,$$

and we note that G is split metacyclic. The result now follows from Theorem 2.1. This completes the proof of Theorem 3.1. $\square$

Chapter 4. Metacyclic groups of odd order

We have the following conjecture about metacyclic groups of odd order, which has been verified using MAGMA for all groups of order up to 2023.

Conjecture. No two nonisomorphic metacyclic groups of odd order have the same character tables.

This seems to be difficult to prove. However, we have the following partial result for groups divisible by only two distinct primes, which makes use of a characterization of these groups by Sim [11].

Theorem 4.1. *Any two metacyclic $\{p, q\}$-groups, p, q odd, can be distinguished by their character tables.*

We first give a description of these groups according to the characterization given in [11]. Let p, q be odd primes with p dividing $q-1$, let μ be the largest integer such that p^μ divides $q-1$, and let $\alpha, \beta, \gamma, \delta, \kappa$ be nonnegative integers satisfying

(i) $\alpha \leq \beta, \gamma \geq \beta, \beta + \delta \geq \gamma \geq \delta$,

(ii) $\delta \geq 1$ or $\beta = 0$,

(iii) $1 \leq \kappa \leq \min(\alpha, \mu)$,

(iv) $\beta \geq \delta$ or $\alpha - \kappa < \beta$.

Let ϵ, ζ, η be nonnegative integers with $\epsilon + \eta \geq \zeta \geq \eta > 0$ and let θ be a primitive p^κth root of unity mod q^ζ. Then it is proven in [11] that the group G generated by $\{x, y, u, v\}$ and given with the relations

$$
\begin{array}{llll}
x^{p^\alpha} = y^{p^\beta}, & & & \\
y^x = y^{1+p^\delta}, & y^{p^\gamma} = 1, & & \\
u^x = u, & u^y = u, & u^{q^\epsilon} = 1, & \\
v^x = v^\theta, & v^y = v, & v^u = v^{1+q^\eta}, & v^{q^\zeta} = 1,
\end{array}
$$

is a presentation of a non-nilpotent metacyclic group of order $p^{\alpha+\gamma}q^{\epsilon+\zeta}$, and that every non-nilpotent metacyclic $\{p,q\}$-group has a presentation of this form. Moreover, it is stated in [11] that θ can be chosen in a manner depending on $\alpha, \beta, \gamma, \delta, \kappa$ so that the groups given by presentations as above form a complete and irredundant set of representatives of the isomorphism types of non-nilpotent metacyclic $\{p,q\}$-groups. Note that the elements xu and yv are generators giving a metacyclic presentation for G, and that it is not clear from the above presentation that G will be split metacyclic unless $\alpha\beta = 0$.

Proof of Theorem 4.1. Let G be a metacyclic $\{p,q\}$-group. If G is nilpotent, then the desired result follows from Corollary 3.4. Otherwise, the group G has a presentation as above, and the parameters $\alpha, \beta, \gamma, \delta, \kappa, \epsilon, \zeta, \eta$ are invariants of G which determine G up to isomorphism. Let $P = \langle x, y\rangle$ and $Q = \langle u, v\rangle$. Then Q is a normal Sylow q-subgroup, and so can be located by the character table. Since $G/Q \cong P$, we can determine the character table of P from the character table of G. Since P is a metacyclic p-group with presentation

$$P = \langle x, y \mid x^{p^\alpha} = y^{p^\beta}, y^{p^\gamma} = 1, y^x = y^{1+p^\delta}\rangle,$$

we can determine the parameters α, β, γ, and δ by Theorem 3.1.

Lemma 5.6 of [11] states that $\kappa = |P : \mathbf{O}_p(G)|$, where $\mathbf{O}_p(G)$ is the unique largest normal p-subgroup of G. Since the orders of P and of $\mathbf{O}_p(G)$ can be determined by the character table, this gives us κ.

Let $X = \langle x \rangle$ and $V = \langle v \rangle$. Note that since P centralizes u and y centralizes v, we have $G' \cap Q = [P, Q] = [X, V]$. We can write $V = C_V(X) \times [X, V]$, but since V is cyclic, one of these direct factors must be trivial. Since the action of x on v is nontrivial, we conclude that $V = [X, V]$. Now G' can be located from the character table. In particular, since G' is cyclic, it has a unique subgroup of every order dividing $|G'|$. These subgroups are normal in G, and so we can find the Sylow q-subgroup of G', which is V. Thus we can determine $\zeta = |V|$ from the character table. From the order of G, which is equal to $p^{\alpha+\gamma}q^{\epsilon+\zeta}$, we can also determine $\epsilon + \zeta$, and hence ϵ.

To find η, we examine $Q \cap Z(G)$. From the presentation of G, we see that $Q \cap Z(G) = C_{Z(Q)}(x)$. Since

$$Q = \langle u, v \mid u^{q^\epsilon} = v^{q^\zeta} = 1, v^u = v^{1+q^\eta} \rangle,$$

we know from Lemma 1.1 that $Z(Q) = \langle u^S, v^T \rangle$, where

$$S = \operatorname{ord}_{q^\zeta}(1 + q^\eta),$$
$$T = q^\zeta / \gcd(q^\zeta, q^\eta) = q^{\zeta-\eta}.$$

Since $\zeta \geq \eta \geq 1$ and $\eta + q \geq 4$, Lemma 3.2 implies that $(1 + q^\eta)^{q^n} \equiv 1 \bmod q^\zeta$ if and only if $n \geq \zeta - \eta$, and so $S = \operatorname{ord}_{q^\zeta}(1 + q^\eta) = q^{\zeta-\eta}$. The generators of $Z(Q)$ have orders

$$|u^S| = |u|/S = q^\epsilon / q^{\zeta-\eta} = q^{\epsilon-\zeta+\eta},$$
$$|v^T| = |v|/T = q^\zeta / q^{\zeta-\eta} = q^\eta.$$

Since Q is split, it then follows that

$$Z(Q) = \langle u^S, v^T \rangle \cong C_{q^{\epsilon-\zeta+\eta}} \times C_{q^\eta}.$$

Since $u^x = u$, we have

$$Q \cap Z(G) = C_{Z(Q)}(x) \cong C_{q^{\epsilon-\zeta+\eta}} \times C_{\langle v^T \rangle}(x).$$

Since this last factor is cyclic of order $\leq q^\eta \leq q^{\epsilon-\zeta+\eta}$, we conclude that the Sylow q-subgroup of $Z(G)$ has exponent $q^{\epsilon-\zeta+\eta}$. Therefore, we can determine $\epsilon - \zeta + \eta$, hence η, from the character table. This gives all the parameters for G. $\square$

Chapter 5. Questions for further research

(i) When do two metacyclic groups have the same character tables?

We note one necessary condition. Since metacyclic groups are supersolvable, they have a normal Hall π-subgroup for π the set of primes larger than any given prime [7, p.85]. It follows that if metacyclic groups G and H have the same character tables, then so do their Sylow 2-subgroups.

(ii) Can all odd order metacyclic groups be distinguished by their character tables?

(iii) Describe the automorphisms of a split metacyclic group in terms of its presentation.

(iv) Find conditions on normal subgroups of a split metacyclic group determining when the quotient is split metacyclic.

(v) The (non-metacyclic) group of order 16 given by the presentation

$$\langle a, b, c \mid a^4 = b^2 = c^2 = 1, a^b = a, c^b = c, a^c = ab \rangle$$

has the same character table as the dihedral group of order 16, which is metacyclic. Also, the two nonabelian groups of order p^3, p odd, have the same character tables, and only one of these is metacyclic. So we ask:

When can a metacyclic group have the same character table as a non-metacyclic group?

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