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Diagonal Entry Restrictions in Minimum Rank Matrices, and the Inverse Inertia and Eigenvalue Problems for Graphs

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ABSTRACT

Diagonal Entry Restrictions in Minimum Rank Matrices, and the Inverse Inertia and Eigenvalue Problems for Graphs

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Let F be a field, let G be an undirected graph on n vertices, and let $\mathcal{S}^F(G)$ be the set of all F-valued symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G. Let $\mathscr{MR}^F(G)$ be defined as the set of matrices in $\mathcal{S}^F(G)$ whose rank achieves the minimum of the ranks of matrices in $\mathcal{S}^F(G)$. We develop techniques involving $\widehat{Z}$, a process termed nil forcing, and induced subgraphs, that can determine when diagonal entries corresponding to specific vertices of G must be zero or nonzero for all matrices in $\mathscr{MR}^F(G)$. We call these vertices nil or nonzero vertices, respectively. If a vertex is not a nil or nonzero vertex, we call it a neutral vertex. In addition, we completely classify the vertices of trees in terms of the classifications: nil, nonzero and neutral. Next we give an example of how nil vertices can help solve the inverse inertia problem. Lastly we give results about the inverse eigenvalue problem and solve a more complex variation of the problem (the λ, μ problem) for the path on 4 vertices. We also obtain a general result for the λ, μ problem concerning the number of λ's and μ's that can be equal.

Keywords: Combinatorial Matrix Theory, Diagonal Entry Restrictions, Graph, Inverse Eigenvalue Problem, Inverse Inertia Problem, Minimum Rank, Neutral, Nil, Nonzero, Symmetric

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Chapter 1. Introduction

The *minimum rank problem* for graphs asks what is the minimum rank among all symmetric matrices whose off-diagonal zero/nonzero pattern is given by a simple graph. This question is equivalent to asking what is the maximum nullity among such matrices. It is also equivalent to asking what is the largest multiplicity of an eigenvalue of a matrix corresponding to the graph. The first paper published on the minimum rank of a graph was in 1996 by Nylen [1]. Since then much work has been done on this problem and it remains an active area of research.

The minimum rank problem is a weakening of an older and harder problem, the *inverse eigenvalue problem* for graphs. The inverse eigenvalue problem asks: Given a graph G on n vertices and n real numbers, is there a matrix, corresponding to the graph G, with the given n real numbers as eigenvalues? This problem has been of interest since at least 1960 (see [2]) and remains unsolved except in special cases.

A problem that is a simplification of the inverse eigenvalue problem and a refinement of the minimum rank problem is the *inverse inertia problem* for graphs. The inverse inertia problem asks: Given a graph G on n vertices and an ordered triple of non-negative numbers (a, b, c) such that $a + b + c = n$, is there a matrix corresponding to G with a positive eigenvalues, b negative eigenvalues, and 0 being an eigenvalue with multiplicity c?

In Chapter 1 we provide some background information and preliminary results. In Chapter 2 we focus on an interesting question related to the minimum rank problem. A natural question to ask is: Given a graph G, what structure or properties are present in matrices which achieve the minimum rank of G? Specially, we investigate the zero/nonzero pattern of the diagonal entries in matrices that achieve the minimum rank. In Chapter 3 we apply the ideas and results of Chapter 2 to the inverse inertia problem and also prove results about inertias. Chapter 4 focuses on the inverse eigenvalue problem.

To keep this thesis as self-contained as possible, we give our own proofs for corollaries of well know results, but include references when appropriate. For example, Lemma 4.6 follows from the Parter-Wiener Theorem (see [3]); however, we provide a simple alternate proof.

1.1 Preliminaries

This section consists of standard definitions, previous results, and examples we will find useful.

First, we recall that real symmetric matrices have real eigenvalues. The rank of such a matrix is equal to the number of nonzero eigenvalues, and the nullity is equal to the multiplicity of the eigenvalue 0. Also, if i and j are vertices of a graph G, we use the convention ij to denote the edge $\{i, j\}$.

Definition 1.1. Given a graph G on n vertices and a field F, let $\mathcal{S}^F(G)$ be the set of all symmetric $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij} \in F$ and $a_{ij} \neq 0$, $i \neq j$, if and only if ij is an edge of G. Then the *minimum rank* of G over F is

$$\mathrm{mr}^F(G) = \min\{\operatorname{rank} A \mid A \in \mathcal{S}^F(G)\}.$$

The *maximum nullity* of G over F is

$$\mathrm{M}^F(G) = \max\{\operatorname{nullity} A \mid A \in \mathcal{S}^F(G)\}.$$

Note that $\mathrm{mr}^F(G) + \mathrm{M}^F(G) = n$. Thus the problem of finding the minimum rank of a graph is equivalent to the problem of finding the maximum nullity of a graph.

Most of the results presented hereafter do not depend on the field F. We use the convention of including the field in the statement of the theorems, but unless a particular field affects the proof we suppress its use in the proof. Throughout the paper F will always refer to a field and G will always be a graph on n vertices.

Definition 1.2. Let $\mathcal{S}(G) = \mathcal{S}^{\mathbb{R}}(G)$. Let $S_+(G)$ be the subset of $\mathcal{S}(G)$ consisting of all positive semidefinite matrices in $\mathcal{S}(G)$. Then the *minimum positive semidefinite rank* of G over F is

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}(A) \mid A \in S_+(G)\}.$$

The *maximum positive semidefinite nullity* of G over F is

$$\mathrm{M}_+(G) = \max\{\mathrm{nullity}(A) \mid A \in S_+(G)\}.$$

Definition 1.3. Given a graph G and a field F, let

$$\mathscr{MR}^F(G) = \{A \in \mathcal{S}^F(G) \mid \mathrm{rank}\, A = \mathrm{mr}^F(G)\}.$$

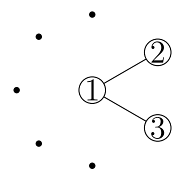
Definition 1.4.

- A *tree* is a graph in which any two vertices are connected by exactly one path.
- A *forest* is a disjoint union of trees.
- The *degree* of a vertex is the number of edges incident to the vertex.
- The *star* on n vertices, S_n, is the tree with one vertex of degree $n-1$ and $n-1$ vertices of degree 1.
- The *cycle* on n vertices, C_n, is the connected graph in which every vertex has degree 2.
- The *complete graph* on n vertices, K_n, is the graph in which every vertex is adjacent to every other vertex.
- A *pendant* vertex is a vertex of degree 1.
- A *dominating* vertex in a graph with n vertices is a vertex of degree $n-1$.
- A set of vertices in a graph G is an *independent set* if its vertices are pairwise non-adjacent.

Example 1.5. Let F be a field. In this example we show that $\mathrm{mr}^F(K_n) = 1, n \geq 2$ and $\mathrm{mr}^F(S_n) = 2, n \geq 3$.

Consider the graph K_n where $n \geq 2$. Note that the all ones matrix, J_n is in $\mathcal{S}^F(K_n)$ and has rank 1. Thus $\mathrm{mr}^F(K_n) \leq 1$. Further, every matrix in $\mathcal{S}^F(K_n)$ has a nonzero entry in the first row and second column position. The only matrix with rank 0 is the all zero matrix. It follows that the rank of every matrix in $\mathcal{S}^F(K_n)$ is at least one. Thus $\mathrm{mr}^F(K_n) = 1$.

Now consider the graph S_n, as shown below, where $n \geq 3$.

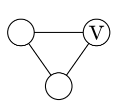
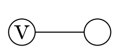
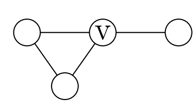


Every matrix in $\mathcal{S}^F(S_n)$ has the form $\begin{bmatrix} d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & d_2 & 0 & \cdots & 0 \\ a_{31} & 0 & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ a_{n1} & 0 & \cdots & 0 & d_n \end{bmatrix}$ (note: $a_{ij} = a_{ji}$ since matrices in $\mathcal{S}^F(S_n)$ are symmetric).

Since a_{13} is a nonzero entry, the entry directly below it is 0, and row 2 has a nonzero entry, rows 1 and 2 are linearly independent. Thus the rank of any such matrix is ≥ 2. Furthermore, letting every diagonal entry be 0 and every nonzero off diagonal entry be 1 (the adjacency matrix of S_n) results in a matrix in $\mathcal{S}^F(S_n)$ with rank 2. Therefore $\mathrm{mr}^F(S_n) = 2$.

Remark. In the previous example, we chose to label the star by labeling the dominating vertex 1. This is valid since different labelings of a graph do not affect the minimum rank, eigenvalues, or properties we are interested in. The reason for this is renumbering the vertices of a graph G corresponds to applying a permutation similarity to matrices in $\mathcal{S}^F(G)$ and a permutation similarity does not change the rank or eigenvalues of a matrix.

Definition 1.6. Let G and H be graphs with at least two vertices, each with a vertex labeled v. The *vertex-sum at v of G and H*, denoted $G \underset{v}{\oplus} H$, is the graph on $|G| + |H| - 1$ vertices obtained by identifying the vertex v in G with the vertex v in H.

Example 1.7. G :  H :  $G \underset{v}{\oplus} H$: 

Definition 1.8. Let G be a graph with a vetex labeled v. The graph $G - v$ is the graph obtained from G by removing vertex v and all edges incident to v.

Definition 1.9. Let F be a field. The *rank-spread* of a vertex v of a graph G, denoted $r_v^F(G)$, is the difference between the minimum rank of G over F and the minimum rank of $G - v$ over F. i.e.

$$r_v^F(G) = \operatorname{mr}^F(G) - \operatorname{mr}^F(G - v).$$

Remark. It is well know (see [1]) that $0 \leq r_v^F(G) \leq 2$ for any vertex v of G.

Definition 1.10. Let A be an $n \times n$ matrix and let $p \in \{1, 2, ..., n\}$. The matrix $A(p)$ is the matrix obtained by deleting the p^{th} row and column of A.

The following is a well know result in matrix theory (see p. 13 in [4]).

Proposition 1.11. *Let A and B be $m \times n$ matrices with entries in a field F. Then*

$$\operatorname{rank}(A + B) \leq \operatorname{rank} A + \operatorname{rank} B.$$

Chapter 2. Diagonal Entry Restrictions in Minimum Rank Matrices

In the definition of $\mathcal{S}^F(G)$, there are no restrictions placed on the diagonal entries of matrices in $\mathcal{S}^F(G)$. It turns out that it is often the case that certain diagonal entries must be zero or nonzero in order for a matrix in $\mathcal{S}^F(G)$ to be in $\mathscr{MR}^F(G)$. Identifying these diagonal entries is a step in determining the structure of all matrices in $\mathscr{MR}^F(G)$. Given a vertex v of a graph G, we call v a *nil* (*nonzero*) vertex if every matrix in $\mathscr{MR}^F(G)$ has a zero (nonzero) diagonal entry corresponding to v. A vertex that is neither a nil nor nonzero vertex is called a *neutral* vertex. In this section, we prove results and develop methods that can be used to find nil, nonzero, and neutral vertices. We also investigate the relationship between these vertices and common graph parameters such as rank-spread. Before doing so we provide some formal definitions, previous results, and an example.

Definition 2.1. Given a field F and a graph G, a vertex v in G is a

- *nil vertex* if its corresponding diagonal entry d_v is zero in every matrix in $\mathscr{MR}^F(G)$.

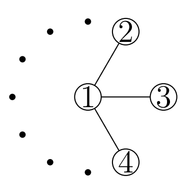
- *nonzero vertex* if its corresponding diagonal entry d_v is nonzero in every matrix in $\mathscr{MR}^F(G)$.

- *neutral vertex* if it is neither a nil vertex nor a nonzero vertex.

Example 2.2. In this example we show that over any field,

- in $S_n, n \geq 4$, the pendant vertices are nil and the dominating vertex is neutral.
- in S_3, every vertex is neutral.
- in $K_n, n \geq 2$, every vertex is nonzero.

Let F be a field. Consider a star on n vertices S_n where $n \geq 4$. The graph and the corresponding matrix are



$$A = \begin{bmatrix} d_1 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & d_2 & 0 & \cdots & 0 \\ a_{31} & 0 & \ddots & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ a_{n1} & 0 & \cdots & 0 & d_n \end{bmatrix}$$

where the a_{ij}'s are nonzero entries. By Example 1.5, $\mathrm{mr}^F(S_n) = 2$. Let $M \in \mathscr{MR}^F(S_n)$. Suppose at least one of the d_i, $i = 2, ..., n$ is nonzero. Without loss of generality, suppose $d_2 \neq 0$. Since $a_{13} \neq 0$ and $a_{23} = 0$, rows 1 and 2 are linearly independent and thus form a basis for the row space. Therefore there exists constants c_1 and c_2 such that

$$\begin{aligned} & c_1[d_1\ a_{12}\ a_{13}\ a_{14}\ \cdots] \\ + & \ c_2[a_{21}\ d_2\ 0\ 0\ \cdots] \\ = & \quad [a_{31}\ 0\ d_3\ 0 \cdots]. \end{aligned}$$

Since $c_1 a_{14} + c_2 0 = 0, c_1 = 0$. Since $c_2 d_2 = 0, c_2 = 0$, a contradiction. Hence, every pendant vertex of S_n is a nil vertex.

We also note that since $\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{bmatrix}$ are in $\mathscr{MR}^F(S_n)$, the dominating vertex of the star is a neutral vertex.

Consider a star on 3 vertices, S_3 (or equivalently a path on 3 vertices, P_3). Since $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & -1 \end{bmatrix}$ are in $\mathscr{MR}^F(S_3)$, every vertex of S_3 is a neutral vertex.

Now consider the graph K_n on $n \geq 2$ vertices. By Example 1.5, $\operatorname{mr}^F(K_n) = 1$. If any diagonal entry of a matrix $A \in \mathcal{S}^F(K_n)$ were zero, then $\operatorname{rank} A \geq 2$. Thus every vertex of K_n is a nonzero vertex.

We now introduce two graph parameters, the zero forcing number Z and the enhanced zero forcing number $\widehat{\mathrm{Z}}$, which we use to determine nil and nonzero vertices. The parameter Z first appeared in [5] and was used to put an upper bound on the maximum nullity of a graph. The zero forcing process, under the name "graph infection," has also been used by physicists to study quantum systems ([6]). The parameter $\widehat{\mathrm{Z}}$, which appears in [7] is a modification of Z and is also used to put an upper bound on the maximum nullity of a graph. The following definitions from [5] and [7] define Z and $\widehat{\mathrm{Z}}$.

Definition 2.3.

- *Color-change rule for a simple graph*: If G is a graph with each vertex colored either white or black, u is a black vertex of G, and exactly one neighbor v of u is white, then change the color of v to black.

- Given a coloring of G, the *derived coloring* is the result of applying the color-change rule for a simple graph until no more changes are possible.

• A *zero forcing set* for a graph G is a subset of vertices Z such that if initially the vertices in Z are colored black and the remaining vertices are colored white, the derived coloring of G is all black.

• The zero forcing number of a graph G, $\mathrm{Z}(G)$ is the minimum of $|\,\mathrm{Z}\,|$ over all zero forcing sets $\mathrm{Z} \subset V(G)$.

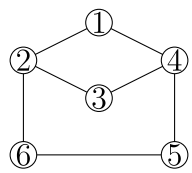
The following two results are Proposition 2.4 and Proposition 4.2 in [5].

Theorem 2.4. *For any graph G and any field F,* $\mathrm{M}^F(G) \leq \mathrm{Z}(G)$.

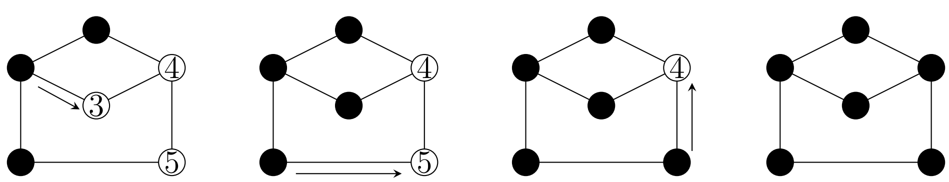
Theorem 2.5. *For any tree T and any field F,* $M^F(T) = Z(T)$.

Remark. It has been verified that $\mathrm{M}^{\mathbb{R}}(G) = \mathrm{Z}(G)$ for all graphs G on fewer than 7 vertices.

Example 2.6. The graph numbers we use correspond to those given in [8]. Consider the graph $G129$.



We show that one zero forcing set consists of vertices $1, 2$ and 6. We first color $1, 2$ and 6 black (see the illustration below). Since 2 has exactly one white neighbor, 3, it can force 3 black by the color-change rule for a simple graph. Since 6 has exactly one white neighbor, 5, it can force 5 black. Lastly since 5 has exactly one white neighbor, 4, it can force 4 black.



We note that the zero forcing set $1, 2$ and 6 is not unique nor is the order in which we forced vertices black in the above example. Also, there are no two vertices in $G129$ that if colored black can force the rest of the graph black. Thus $Z(G129) = 3$.

Definition 2.7. A *loop graph* is a graph that allows single loops at vertices, i.e., $\widehat{G} = (V_{\widehat{G}}, E_{\widehat{G}})$ where $V_{\widehat{G}}$ is the set of vertices of $\widehat{G}$ and the set of edges $E_{\widehat{G}}$ is a set of two-element multisets. Vertex u is a *neighbor* of vertex v in $\widehat{G}$ if $\{u, v\} \in E_{\widehat{G}}$; note that u is a neighbor of itself if and only if the loop $\{u, u\}$ is an edge. The *underlying simple graph* of a loop graph $\widehat{G}$ is the graph G obtained from $\widehat{G}$ by deleting all loops.

Note that if we ever write $\widehat{G}$ we think of the graph as coming with extra information, namely that the graph is a loop graph, even if there are no loops. In a loop graph, every vertex is specified as being looped or unlooped.

Definition 2.8. The *set of symmetric matrices with entries in a field F described by a loop graph $\widehat{G}$* is

$$\mathcal{S}^F(\widehat{G}) = \{A = [a_{ij}] \mid A \text{ is symmetric, } a_{ij} \in F, \text{ and } a_{ij} \neq 0 \text{ if and only if } \{i, j\} \in E_{\widehat{G}}\}.$$

and the *maximum nullity of $\widehat{G}$* over F is

$$\mathrm{M}^F(\widehat{G}) = \max\{\text{nullity } A \mid A \in \mathcal{S}^F(\widehat{G})\}$$

Definition 2.9 (Color-change rule for a loop graph)**.** Let $\widehat{G}$ be a loop graph with each vertex colored white or black. If exactly one neighbor u of v is white, then change the color of u to black.

Note that the color-change rule for a loop graph is quite similar to the color-change rule for a simple graph. The only difference is that when using a loop graph, two additional coloring forces are valid. First, a looped white vertex that has no other white neighbors may be colored black. Second if an unlooped white vertex has only one white neighbor u, u may be colored black. By $\mathrm{Z}(\widehat{G})$, we mean the same thing as in Definition 2.3, except we use the color-change rule for a loop graph. (We distinguish the two cases by whether or not the graph is a loop graph.)

The following results are from [7]. We note that the results were stated without reference to a field but the proofs hold for any field.

Theorem 2.10. *For any loop graph* $\widehat{G}$ *and any field* F, $\mathrm{M}^F(\widehat{G}) \leq \mathrm{Z}(\widehat{G})$.

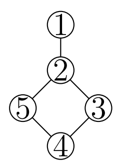
Definition 2.11. The *enhanced zero forcing number* of a graph G denoted by $\widehat{\mathrm{Z}}(G)$, is the maximum of $\mathrm{Z}(\widehat{G})$ over all loop graphs $\widehat{G}$ such that the underlying simple graph of $\widehat{G}$ is G.

Corollary 2.12. *For any graph* G *and any field* F, $\mathrm{M}^F(G) \leq \widehat{\mathrm{Z}}(G) \leq \mathrm{Z}(G)$.

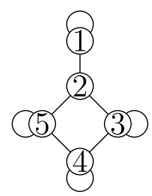
The following example illustrates the coloring rules defined above.

Example 2.13. Consider the loop graphs $\widehat{G}_1$ and $\widehat{G}_2$, each of whose underlying simple graph is $G37$.

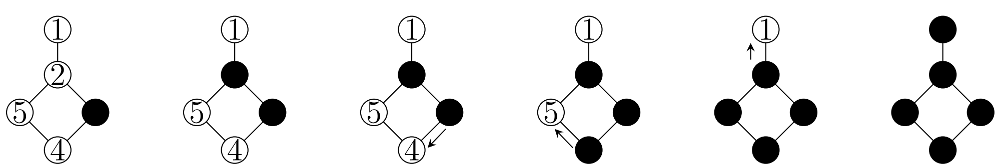
$\widehat{G}_1$:



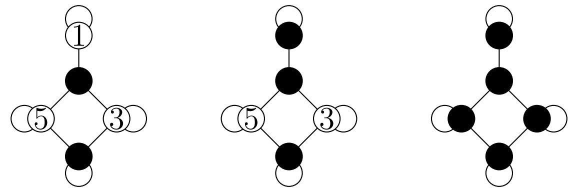
$\widehat{G}_2$:



First consider $\widehat{G}_1$. Color vertex 3 black (see illustration below). Since 1 is an unlooped vertex and only has one white neighbor 2, 2 can be colored black. Then 3 forces 4, 4 forces 5, and 2 forces 1. Thus $\mathrm{Z}(\widehat{G}_1) \leq 1$. It is straightforward to see that $\mathrm{Z}(\widehat{G}_1) \geq 1$. Thus $\mathrm{Z}(\widehat{G}_1) = 1$.



Now consider $\widehat{G}_2$. Color vertices 2 and 4 black (see illustration below). Since 1 is a looped vertex that has no white neighbors, it may be colored black. Similarly, 3 and 5 may be colored black. Thus $\mathrm{Z}(\widehat{G}_2) \leq 2$. It is straightforward to see that $\mathrm{Z}(\widehat{G}_2) \geq 2$. Thus $\mathrm{Z}(\widehat{G}_2) = 2$.



Since $\widehat{Z}(G)$ is the maximum of $\mathrm{Z}(\widehat{G})$ over all loop graphs $\widehat{G}$, $\widehat{Z}(G) \geq \mathrm{Z}(\widehat{G}_2) = 2$. It is straightforward to verify $\mathrm{Z}(G) = 2$. By Corollary 2.12, $\widehat{Z}(G) \leq \mathrm{Z}(G) = 2$ and thus $\widehat{Z}(G) = 2$.

2.1 The $\widehat{\mathrm{Z}}$ Method

The ideas and results that have been given for the zero forcing number and the enhanced zero forcing number can be combined in a way that can determine nil and nonzero vertices.

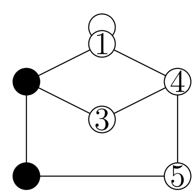
Theorem 2.14 (The $\widehat{Z}$ Method)**.** *Let G be a graph. Let $\breve{G}$ be the graph G where a vertex v is looped and no other vertices are specified looped or unlooped. Let F be a field. If there exists a set of less than $\mathrm{M}^F(G)$ vertices of $\breve{G}$ such that every vertex in $\breve{G}$ can be colored black by following the color-change rule for a simple graph (see Definition 2.3) and the additional rule that the looped vertex v may be colored black if it has no white neighbors, then v is a nil vertex of G.*

Proof. Let $\mathcal{Z}$ be a set of less than $\mathrm{M}(G)$ vertices of $\breve{G}$ such that every vertex in $\breve{G}$ can be colored black by following the color-change rule for a simple graph and the additional rule that the looped vertex v may be colored black if it has no white neighbors. Let $\widehat{G}$ be an arbitrary loop graph with underlying simple graph G such that v has a loop. Since by starting with the vertices in $\mathcal{Z}$ colored black, every vertex in $\breve{G}$ can be colored black by following the color-change rule for a simple graph and the rule that the looped vertex v may be colored black if it has no white neighbors, these same forcing moves will color every vertex of $\widehat{G}$ black. Thus $\mathrm{Z}(\widehat{G}) \leq |\mathcal{Z}| < \mathrm{M}(G)$. By Theorem 2.10, $\mathrm{M}(\widehat{G}) \leq \mathrm{Z}(\widehat{G}) < \mathrm{M}(G)$. Thus no matrix in $\mathcal{S}(\widehat{G})$ has nullity equal to $\mathrm{M}(G)$. Note that by the definition of $\mathcal{S}(\widehat{G})$, the condition that v is looped corresponds to the condition that the diagonal entry corresponding to v in every matrix in $\mathcal{S}(\widehat{G})$ is nonzero. Since $\widehat{G}$ was an arbitrary loop graph with the condition that v is looped, no matrix with a nonzero diagonal entry corresponding to v achieves $\mathrm{M}(G)$ (or equivalently $\mathrm{mr}(G)$). Therefore v is a nil vertex. □

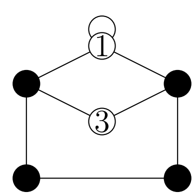
Theorem 2.15. *Let G be a graph. Let $\breve{G}$ be the graph G where a vertex v is unlooped and no other vertices are specified looped or unlooped. Let F be a field. If there exists a set of less than $\mathrm{M}^F(G)$ vertices of $\breve{G}$ such that every vertex in $\breve{G}$ can be colored black by following the color-change rule for a simple graph and the additional rule that if an v has only one white neighbor u, u may be colored black, then v is a nonzero vertex of G.*

Proof. The proof is similar to the proof of Theorem 2.14. □

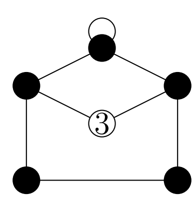
Example 2.16. We consider the graph $G129$ used in Example 2.6. We will show that vertices 1 and 3 are nil and vertices 5 and 6 are nonzero. From Example 2.6, $\mathrm{Z}(G129) = 3$. By the remark after Theorem 2.5, $\mathrm{M}^{\mathbb{R}}(G129) = 3$. We begin by placing a loop on 1 and coloring 2 and 6 black.



Using the color-change rule for a simple graph, 6 can force 5 and then 5 can force 4.

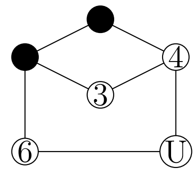


Now the looped vertex has no white neighbors and thus can be colored black.

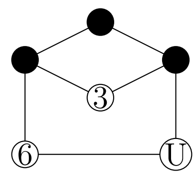


Lastly, 4 can force 3. We used a set of fewer than $M(G129) = 3$ vertices to color the entire graph black. Hence by Theorem 2.14, vertex 1 is a nil vertex. By symmetry, vertex 3 is a nil vertex.

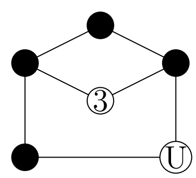
We now mark 5 as an unlooped vertex by labeling it with a U. We also color 1 and 2 black.



Using the color-change rule for a simple graph 1 can force 4.



Now the unlooped white vertex has only one white neighbor 6 and thus 6 may be colored black.



Now by the color-change rule for a simple graph, 6 can force 5 black and 2 can force 3 black. We used a set of less than 3 vertices to color the entire graph black. Hence by Theorem 2.15, vertex 5 is nonzero. By symmetry, vertex 6 is nonzero.

The above method of using the concepts of zero forcing and enhanced zero forcing can also be used to make conclusions about the rank-spread of certain vertices.

Theorem 2.17. *Let F be a field and assume that for a graph G on n vertices, $\widehat{\mathrm{Z}}(G) = \mathrm{M}^F(G)$. Let $\breve{G}$ be the graph where a vertex v is looped and no other vertices are specified looped or unlooped. If there exists a set $\mathcal{Z}$ of less than $\mathrm{M}^F(G)$ vertices of $\breve{G}$ such that every vertex in $\breve{G}$ can be colored black by following the color-change rule for a simple graph and the additional rule that the looped vertex v may by colored black if it has no white neighbors, then $r_v^F(G) = 0$.*

Proof. We claim that during the forcing process that started with $\mathcal{Z}$ and ended with all vertices black, the additional rule that v may be colored black if it has no white neighbors was used. When this additional forcing rule is used to color v black we say v "died alone". Thus we claim that v died alone in the forcing process. Suppose by way of contradiction that in the zero forcing process, the vertex v did not die alone. Thus only the color-change rule for a simple graph was used and so $\mathcal{Z}$ is a zero forcing set for G. Thus by Corollary 2.12, $\mathrm{M}(G) \le \widehat{\mathrm{Z}}(G) \le \mathrm{Z}(G) \le |\mathcal{Z}| < \mathrm{M}(G)$, a contradiction.

Since v died alone, $\mathcal{Z}$ is a zero forcing set for $G - v$. Thus

$$\mathrm{Z}(G - v) \le |\mathcal{Z}| < \widehat{\mathrm{Z}}(G).$$

By Theorem 2.4, $\mathrm{M}(G - v) \le \mathrm{Z}(G - v) < \widehat{\mathrm{Z}}(G) = \mathrm{M}(G)$. This fact along with the facts that $\mathrm{mr}(G - v) \le \mathrm{mr}(G)$ (by the remark after 1.9), $\mathrm{mr}(G) + \mathrm{M}(G) = n$ and $\mathrm{mr}(G - v) + \mathrm{M}(G - v) = n - 1$ imply $\mathrm{mr}(G) = \mathrm{mr}(G - v)$. Thus $r_v(G) = 0$. □

Example 2.18. Consider $G129$ in Example 2.16. As seen in this example, $\mathrm{M}^{\mathbb{R}}(G129) = \mathrm{Z}(G)$. By Corollary 2.12, $\mathrm{M}^{\mathbb{R}}(G129) = \widehat{\mathrm{Z}}(G129)$. Thus by Theorem 2.17, Example 2.16 shows that vertices 1 and 3 in $G129$ have rank-spread 0.

Theorem 2.17 shows that if $\widehat{\mathrm{Z}}(G) = \mathrm{M}^F(G)$ and the $\widehat{\mathrm{Z}}$ method determines a vertex v of G is nil, then $r_v^F(G) = 0$. Similarly, it will be shown in the next section that if $\mathrm{Z}(G) = \mathrm{M}^F(G)$ and the nil forcing method determines a vertex v of G is nil, then $r_v^F(G) = 0$. However it is not true that all nil vertices have rank-spread 0. An example of a nil vertex with rank-spread 1 was found by John Sinkovic and appears in [9].

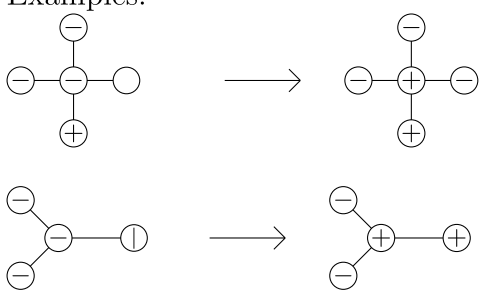
2.2 The Nil Forcing Method

In this section we describe a graph algorithm or game which we call nil forcing that can determine if a vertex is nil. One of the necessary conditions for nil forcing is that $\mathrm{M}^F(G) = \mathrm{Z}(G)$. This equality holds for a significant number of graphs. In particular, it is know that this equality hold for all graphs on less than 8 vertices. For this algorithm, we will consider 4 types of vertices which we call row, column, cross, and white vertices. When drawing graphs, we denote each of these as ⊖, ⦶, ⊕, and ○ respectively. Below are the rules of nil forcing.

Definition 2.19 (Nil forcing rules)**.** Let G be a graph with each vertex being a white, column, row, or cross vertex.

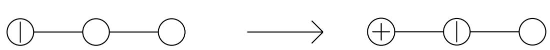
(i) Let v be a row vertex that has exactly one white or column neighbor w, then a horizontal line may be added to w (thus w becomes a row or cross vertex respectively). If this force occurs, v becomes a cross vertex.

Examples:



(ii) Let v be a column vertex that has exactly one white or row neighbor w, then a vertical line may be added to w (thus w becomes a column or cross vertex respectively). If this force occurs, v becomes a cross vertex.

Examples:



Before stating the theorem that uses nil forcing to determine a vertex is nil, we define a few more terms.

Definition 2.20. A *path* is a connected graph with exactly two degree 1 vertices with the remainder of the vertices having degree 2. A path on n vertices is denoted P_n.

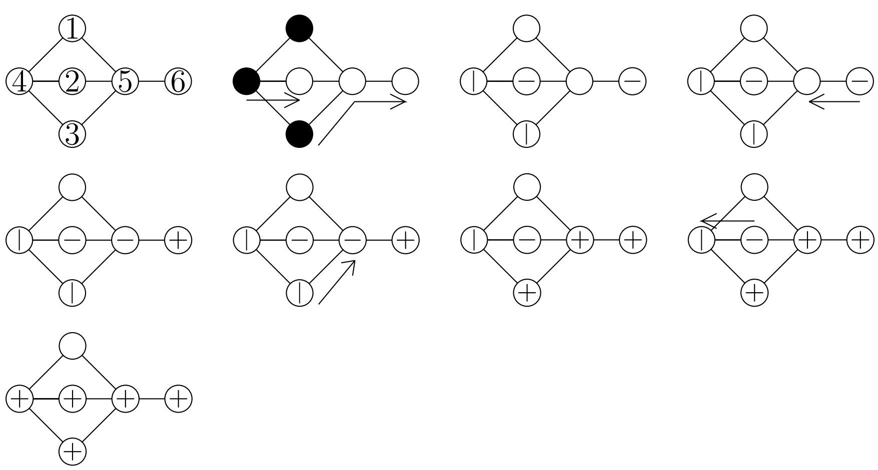
Definition 2.21. Let G be a graph. A *path cover* is a set of vertex-disjoint paths such that every vertex v of G belongs to at least one path.

Definition 2.22. Let G be a graph and let Z be a zero forcing set for G. A sequence of forces in the zero forcing process induces a path cover of G (see the proof of Proposition 2.10 in [10]). We call this a *zero forcing induced path cover.*

Theorem 2.23 (The Nil Forcing Method)**.** *Let F be a field and let G be a graph such that* $\mathrm{Z}(G) = \mathrm{M}^F(G)$, *and let* $n = |G|$. *Let v be in a path of length* 0 *in a zero forcing induced path cover with a minimal zero forcing set* Z. *Let W be the set of all vertices forced last in each zero forced path. Make each vertex of* $\mathrm{Z} - v$ *a column vertex. Make each vertex of* $W - v$ *a row vertex. If a vertex is in* $\mathrm{Z} - v$ *and* $W - v$ *then it becomes a cross vertex. Make all other vertices white. If by applying nil forcing rules to G, v can remain a white vertex and all other vertices become cross vertices, then v is a nil vertex in G.*

Before presenting the proof, it may be helpful to work through an example.

Example 2.24. For this example we let $F = \mathbb{R}$. For the following graph G, we know $\mathrm{M}(G) = \mathrm{Z}(G) = 3$. We show that vertex 1 is a nil vertex. Consider the following:



We will show directly that $d_1 = 0$, correlating our proof with the nil forcing method steps. Let G be the graph shown above. Then any matrix A in $\mathscr{MR}(G)$ is a symmetric matrix of the form:

$$\begin{bmatrix} d_1 & 0 & 0 & a_{14} & a_{15} & 0 \\ 0 & d_2 & 0 & a_{24} & a_{25} & 0 \\ 0 & 0 & d_3 & a_{34} & a_{35} & 0 \\ a_{41} & a_{42} & a_{43} & d_4 & 0 & 0 \\ a_{51} & a_{52} & a_{53} & 0 & d_5 & a_{56} \\ 0 & 0 & 0 & 0 & a_{65} & d_6 \end{bmatrix}$$

Delete the columns of A corresponding to each vertex of $\mathrm{Z} - v$ (note that these are the vertices we first made column vertices in the example). Delete the rows of A corresponding to each vertex of $W - v$ (note that these are the vertices we first made row vertices in the example). The resulting matrix is of the form:

$$B_1 = \begin{bmatrix} d_1 & 0 & a_{15} & 0 \\ 0 & 0 & a_{35} & 0 \\ a_{41} & a_{42} & 0 & 0 \\ a_{51} & a_{52} & d_5 & a_{56} \end{bmatrix}$$

Since $A \in \mathscr{MR}(G)$, $\operatorname{rank} A = \operatorname{mr}(G) = 3$. The matrix B_1 is a 4×4 submatrix of A and therefore cannot have rank 4. Thus $\det B_1 = 0$. We evaluate $\det B_1$ using the method of cofactor expansion. The order for the nil forcing gives us an algorthim for how to proceed with the cofactor expansion. The first force is the row vertex 6 forces the white vertex 5. Here 5 becomes a row vertex and 6 becomes a cross vertex. In B_1 we look at the column corresponding to 6 and see that the only nonzero entry is a_{56}. Expanding along this column we get $\det B_1 = a_{56} \det B_2$ where

$$B_2 = \begin{bmatrix} d_1 & 0 & a_{15} \\ 0 & 0 & a_{35} \\ a_{41} & a_{42} & 0 \end{bmatrix}.$$

The second force is the column vertex 3 forces the row vertex 5. Here both 5 and 3 become cross vertices. In B_2 we look at the row corresponding to 3 and see that the only nonzero entry is a_{35}. Expanding along this row we get $\det B_1 = a_{56} \det B_2 = -a_{56}a_{35} \det B_3$ where

$$B_3 = \begin{bmatrix} d_1 & 0 \\ a_{41} & a_{42} \end{bmatrix}.$$

The third force is the row vertex 2 forces the column vertex 4. Here both 4 and 2 become cross vertices. In B_3 we look at the column corresponding to 2 and see that the only nonzero entry is a_{42}. Expanding along this column we get $\det B_1 = a_{56} \det B_2 = -a_{56}a_{35} \det B_3 = -a_{56}a_{35}a_{42}d_1$. Thus $-a_{56}a_{35}a_{42}d_1 = 0 \Rightarrow d_1 = 0$.

The correlation between the nil forcing steps and the method we used to show $d_1 = 0$ is not a coincidence. The proof for Theorem 2.23 is a generalization of this technique.

Proof. The notation and symbols used here will be the same used in the hypothesis of the theorem. Let $A = [a_{i,j}] \in \mathscr{MR}(G)$. We will construct an $(\mathrm{mr}(G) + 1) \times (\mathrm{mr}(G) + 1)$ submatrix S of A. Since $\operatorname{rank} A = \mathrm{mr}(G), \operatorname{rank} S \leq \mathrm{mr}(G)$. Thus, $\det S = 0$. Evaluating the determinant by cofactor expansion will show that for $\det S$ to be 0, d_v must be 0. Create S by deleting the columns of A corresponding to the vertices of $\mathrm{Z} - v$ and deleting the rows of A corresponding to the vertices of $W - v$. Since $\mathrm{Z}(G) = \mathrm{M}(G), n - (\mathrm{Z}(G) - 1) = \mathrm{mr}(G) + 1$. Thus S is a $(\mathrm{mr}(G) + 1) \times (\mathrm{mr}(G) + 1)$ matrix.

Write

$$S = \begin{bmatrix} d_v & \cdots & a_{v,j} \\ \cdots & \cdots & \cdots \\ a_{k,v} & \cdots & a_{k,j} \end{bmatrix}.$$

We evaluate $\det S$ with cofactor expansion. The nil forcing rules will dictate what rows or columns we expand along.

Since nil forcing can force all vertices of $G - v$ to become cross vertices and v stays white, either forcing rule (i) or (ii) occurs without involving v. If (i) occurs first then there existed a row vertex, r, adjacent to all row and cross vertices except for one vertex, w. Consider the column in S corresponding to this vertex. The only nonzero entry in this column is $a_{w,r}$. Let $a_{w,r} = a_1$. We expand along this column and thus $\det S = \pm a_1 \cdot S_{w,r}$, where $S_{w,r}$ denotes the matrix obtained from S by deleting the row w and column r. Note: We know d_v is in $S_{w,r}$ since v remains a white vertex and thus neither w nor r were equal to v. Also following the graph algorithm, w has become a row vertex and r a crossed vertex.

If (ii) occurs first, a similar argument applies. Now since every vertex except v is eventually forced crossed, this same process associated with rules (i) and (ii) repeats itself. Depending on the force that occurs, a corresponding row or column will be all zeros except for one entry. We expand along this row or column. Thus once all vertices have been crossed, we have that $\det S = \pm a_1 \cdots a_m d_v = 0$. Since $a_i \neq 0, 1 \leq i \leq m, d_v = 0$. □

This nil forcing method also provides information about the rank-spread of the nil vertex.

Theorem 2.25. *Let F be a field. Let G be a graph on n vertices with $Z(G) = M^F(G)$. Let v be a vertex in G such that the nil forcing method shows v is a nil vertex. Then $r_v^F(G) = 0$.*

Proof. Since v can be determined to be a nil vertex by the nil forcing method, v is in a path of length 0 in a zero forcing induced path cover with a minimal zero forcing set Z. Thus $Z - v$ is a zero forcing set for $G - v$ of size $Z(G) - 1$. Thus

$$M(G - v) \leq Z(G - v) \leq Z(G) - 1 = M(G) - 1$$

implies $M(G - v) < M(G)$. This fact along with the facts that $\text{mr}(G - v) \leq \text{mr}(G)$ (by the remark after 1.9), $\text{mr}(G) + \text{M}(G) = n$ and $\text{mr}(G - v) + \text{M}(G - v) = n - 1$ imply $\text{mr}(G) = \text{mr}(G - v)$. Thus $r_v(G) = 0$. □

2.3 Induced Subgraph Method

We now develop another method that uses induced subgraphs to determine nil vertices.

The following is due to Nylen and is Proposition 2.2 in [1]. We note that the proposition was stated without reference to a field but the proof holds for any field.

Proposition 2.26. *For a field F and a graph G, let $B \in \mathscr{MR}^F(G)$. Then for each $p \in \{1, 2, ..., n\}$, $\operatorname{rank} B(p) \in \{\operatorname{rank} B, \operatorname{rank} B - 2\}$. In particular, $\operatorname{rank} B(p) = \operatorname{rank} B - 1$ is impossible.*

Proof. Without loss of generality, we let $p = 1$ and $\operatorname{rank} B = k$. By way of contradiction, suppose $\operatorname{rank} B(1) = k - 1$.

Denote $B = \begin{bmatrix} a & b^T \\ b & B(1) \end{bmatrix}$. Since $\operatorname{rank} B(1) = k - 1, b$ must be a linear combination of the columns of $B(1)$, that is, $B(1)x = b$ has a solution $x = c$. Form B' by replacing a with $a' = b^T c$. Thus $B' = \begin{bmatrix} b^T c & b^T \\ B(1)c & B(1) \end{bmatrix}$. Hence $B' \in S(G)$ with $\operatorname{rank} B' = \operatorname{rank} B(1) = k - 1 < k = \operatorname{mr}(G)$, a contradiction. □

Definition 2.27. Let $G = (V, E)$ be a graph. A graph $H = (V', E')$ is a *subgraph* of G if $V' \subset V$ and $E' \subset E$. A subgraph H is called an *induced subgraph* of G if H is obtained from G by deleting the vertices in $V - V'$ and the edges incident to the vertices in $V - V'$. A subgraph H is called a *proper subgraph* if $G \neq H$.

Theorem 2.28. *Let F be a field, G be a graph with $\operatorname{mr}^F(G) = r$, and H be a proper induced subgraph of G with $\operatorname{mr}^F(H) = r$. Let $v \notin V(H)$ be a vertex adjacent to exactly one vertex of each connected component of H. Then v is a nil vertex.*

Proof. Let $H_1, H_2, \cdots, H_k$ denote the disjoint components of H. Let $r_i = \operatorname{mr}(H_i), i = 1, \cdots, k$. A rank r matrix A corresponding to G can be written in the form

$$A = \begin{bmatrix} B_1 & 0 & 0 & 0 & y_1 & * \\ 0 & B_2 & 0 & 0 & y_2 & * \\ 0 & 0 & \ddots & 0 & \vdots & * \\ 0 & 0 & 0 & B_k & y_k & * \\ y_1^T & y_2^T & \cdots & y_k^T & d_v & * \\ * & * & * & * & * & * \end{bmatrix}$$

where $B_1, B_2, \cdots, B_k$ correspond to $H_1, H_2, \cdots, H_k$ respectively, the 0 entries are zero matrices of appropriate sizes, and the $*$ entries correspond to vertices of G not in $V(H) \cup \{v\}$. Further, without loss of generality and since v is adjacent to exactly one vertex of each H_i, $y_i^T = \begin{bmatrix} x_i & 0 & \cdots & 0 \end{bmatrix}, i = 1, \cdots, k$.

Let $B = \begin{bmatrix} B_1 & 0 & 0 & 0 \\ 0 & B_2 & 0 & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & B_k \end{bmatrix}$. We observe that $r = \text{mr}(H) = \text{mr}(H_1) + \cdots + \text{mr}(H_k) \leq \text{rank}\, B_1 + \cdots + \text{rank}\, B_k = \text{rank}\, B \leq \text{rank}\, A = r$, hence $\text{rank}\, B = r$. We can also conclude that $\text{rank}\, B_i = r_i, i = 1, \cdots, k$.

Construct a matrix as follows. Note that by Proposition 2.26, $\text{rank}\, B_i(1) = r_i$ or $r_i - 2$. Without loss of generality assume $\text{rank}\, B_i(1) = r_i$ for $i = 1, \cdots, j$ and $\text{rank}\, B_i(1) = r_i - 2$ for $i = j+1, \cdots k$. It follows that for each $i, i = 1, \ldots j$, there exists an $r_i \times r_i$ principal submatrix of $B_i(1)$ with a nonzero determinant. Call these submatrices $C_i, i = 1, 2, ..., j$.

For each $i, i = j+1, \cdots, k$, there exists an $r_i \times r_i$ principal submatrix of B_i with a nonzero determinant. This submatrix includes the first diagonal entry of B_i because $\text{rank}\, B_i(1) = r_i - 2$. Call these submatrices $C_i, i = j+1, ..., k$. Note that since $\text{rank}\, C_i = r_i$ and $C_i(1)$ is a submatrix of $B_i(1)$ which has rank $r_i - 2$, it follows that $\text{rank}\, C_i(1) = r_i - 2, i = j+1, ..., k$.

Let $E = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{j+1} & 0 & 0 & y'_{j+1} \\ 0 & 0 & 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & 0 & C_k & y'_k \\ 0 & 0 & 0 & y'^T_{j+1} & \cdots & y'^T_k & d_v \end{bmatrix}$

where $y'^T_i = \begin{bmatrix} x_i & 0 & \cdots & 0 \end{bmatrix}, i = j+1, \cdots, k$.

The matrix E is a $(r+1)\times(r+1)$ principal submatrix of A. Since $\operatorname{rank} A = r, \det E = 0$.

$$\text{Let } D = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & C_j \end{bmatrix}, \text{ so } E = \begin{bmatrix} D & 0 & 0 & 0 & 0 \\ 0 & C_{j+1} & 0 & 0 & y'_{j+1} \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & C_k & y'_k \\ 0 & y'^T_{j+1} & \cdots & y'^T_k & d_v \end{bmatrix}.$$

Let C_i^c be the matrix C_i with the first column deleted and C_i^r be the matrix C_i with the first row deleted.

Using cofactor expansion,

$0 = \det E = d_v \det C_1 \det C_2 ... \det C_k$

$$\pm x_{j+1} \det \begin{bmatrix} D & 0 & 0 & 0 & 0 \\ 0 & C^c_{j+1} & 0 & 0 & y'_{j+1} \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & C_k & y'_k \end{bmatrix} \pm \cdots \pm x_k \times \det \begin{bmatrix} D & 0 & 0 & 0 & 0 \\ 0 & C_{j+1} & 0 & 0 & y'_{j+1} \\ 0 & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & C^c_k & y'_k \end{bmatrix}$$

$$= d_v \det C_1 \cdots \det C_k + \sum_{m=j+1}^{k} \sum_{n=j+1}^{k} (\pm x_m)(\pm x_n) \det \mathscr{C}^{\{m,n\}},$$

where

$$\mathscr{C}^{\{m,n\}} = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C^r_m & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & C^c_n & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & C_k \end{bmatrix}.$$

Recall that each C_i is a $r_i \times r_i$ matrix and $\operatorname{rank} C_i = r_i$. It follows that for the $r_i \times (r_i - 1)$ matrix C_i^c, $\operatorname{rank} C_i^c = r_i - 1$. Similarly, $\operatorname{rank} C_i^r = r_i - 1$.

$$\text{Let } \mathscr{C}^{\{k,j+1\}} = \begin{bmatrix} C_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & C_j & 0 & 0 & 0 \\ 0 & 0 & 0 & C^c_{j+1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & C^r_k \end{bmatrix}.$$

Now $\operatorname{rank} \mathscr{C}^{\{k,j+1\}} = \operatorname{rank} C_1 + ... + \operatorname{rank} C_j + \operatorname{rank} C^c_{j+1} + ... + \operatorname{rank} C^r_k = \sum_{i=1}^k r_i - 2 = r - 2$. The matrix $\mathscr{C}^{\{k,j+1\}}$ is a $(r-1) \times (r-1)$ matrix. Thus $\det \mathscr{C}^{\{k,j+1\}} = 0$. Similarly, $\det \mathscr{C}^{\{m,n\}} = 0$ when $m, n \in \{j+1, ..., k\}$ and $m \neq n$. Further, recall that $\operatorname{rank} C_i(1) = r_i - 2$ for $i = j+1, ..., m$. It follows that $\operatorname{rank} \mathscr{C}^{\{i,i\}} = r - 2$ for $i = j+1, ..., m$. Therefore $\det \mathscr{C}^{\{m,n\}} = 0$ when $m, n \in \{j+1, ..., k\}$.

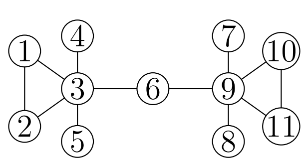
We conclude that $0 = \det E = d_v \det C_1 \det C_2 \cdots \det C_k$. Since $\det C_i \neq 0, i = 1, \cdots, k$, $d_v = 0$. □

Corollary 2.29. *Let F be a field, G be a graph with $\operatorname{mr}^F(G) = r$, and H be a connected induced subgraph of G with $\operatorname{mr}^F(H) = r$. Let v be a vertex adjacent to exactly one vertex of H. Then v is a nil-vertex.*

Corollary 2.30. *Let F be a field, G be a graph, and p be a pendant vertex of G with $r^F_v(G) = 0$, then p is a nil vertex.*

Proof. Since $\operatorname{mr}(G) = \operatorname{mr}(G-p)$ and p is adjacent to exactly one vertex of $G-p$, by Corollary 2.29, p is a nil vertex. □

Example 2.31. Let G be the following graph.



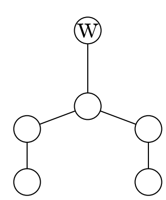
Using zero forcing and Theorem 2.4, $\operatorname{M}^{\mathbb{R}}(G) \leq \operatorname{Z}(G) = 5$ and thus $\operatorname{mr}^{\mathbb{R}}(G) \geq 6$. Also

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}^{\mathbb{R}}(G)$$

and rank $A = 6$. Thus $\mathrm{mr}^{\mathbb{R}}(G) = 6$.

Let v be vertex 6. Using zero forcing, the remark after Theorem 2.5, and the fact that $\mathrm{mr}^F(G_1 \cup G_2) = \mathrm{mr}^F(G_1) + \mathrm{mr}^F(G_2)$ for any field F and graphs G_1 and G_2, $\mathrm{M}^{\mathbb{R}}(G - v) = \mathrm{Z}(G - v) = 4$. Thus $\mathrm{mr}^{\mathbb{R}}(G - v) = 6$. Since vertex v is adjacent to exactly one vertex of each component of $G - v$, by Theorem 2.28, v is a nil vertex.

To illustrate Corollary 2.30 consider $G80$:

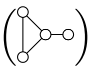
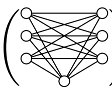


By Theorem 2.5, $\mathrm{M}^F(G80) = \mathrm{Z}(G80) = 2 \Rightarrow \mathrm{mr}^F(G80) = 4$. Similarly, since $\mathrm{M}^F(G80 - w) = \mathrm{Z}(G80 - w) = 1, \mathrm{mr}^F(G80 - w) = 4$. Hence vertex w has rank-spread 0 and thus by the corollary, w is a nil vertex.

2.4 Classification of the Vertices of Trees

A natural and significant step in determining the structure of minimum rank matrices is to classify the vertices of classes of graphs with respect to nil, nonzero and neutral vertices. The following theorem appears in [11] and classifies these vertices for all graphs whose minimum rank is 2.

Theorem 2.32. *Let G be a connected graph with $\mathrm{mr}(G) = 2$ and v be a vertex of G. Then*

- *v is a nonzero vertex if and only if v is either a non-dominating vertex of an induced paw* () *of G or else is the dominating vertex of an induced $K_{3,3,1}$* ().
- *v is a nil vertex if and only if v is in an independent set of size three or greater.*
- *v is a neutral vertex if and only if it does not meet either of the previous two conditions.*

In this section we give a classification of all nil, nonzero and neutral vertices of a tree (see Definition 1.4). Before doing so we provide some needed definitions and results.

Recall the following facts from Examples 1.5 and 2.2, which will be used extensively throughout this section.

- $\mathrm{mr}^F(K_n) = 1$ for $n \geq 2$.
- $\mathrm{mr}^F(S_n) = 2$ for $n \geq 3$.
- Pendant vertices of stars on $n \geq 4$ vertices are nil.
- The dominating vertex of a star on $n \geq 3$ vertices is neutral.
- All vertices of $S_3 = P_3$ are neutral.

Definition 2.33. A vertex v of a graph G is called a *cut-vertex* if the graph obtained by deleting v and all edges adjacent to v from G, denoted $G - v$, contains more components than G.

Definition 2.34. Given a proper subgraph H of a graph G, let $\widetilde{H}$ be the graph with vertex set $V(G)$ and edge set $E(H)$.

Definition 2.35. Let G be a graph. Then a *cover* of G is a set of subgraphs of G such that the union of the edge sets is equal to $E(G)$.

Definition 2.36. A K_2-*star cover* of G is a cover of G consisting of only K_2's and stars.

Definition 2.37. Given a graph G and a cover $\mathscr{C}$ of G, we say a vertex (edge) of G is *covered by* an element of the cover $H \in \mathscr{C}$ if the vertex (edge) is in the vertex (edge) set of H.

Definition 2.38. The *rank sum* of a cover $\mathscr{C}$ over a field F, denoted $\operatorname{rs}^F(\mathscr{C})$ is the sum of the minimum ranks over F of the graphs in $\mathscr{C}$.

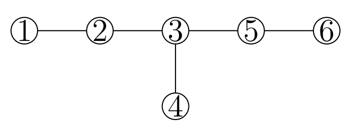
Observation 2.39. *For any edge-disjoint cover $\mathscr{C}$ of a graph G over a field F, $\operatorname{mr}^F(G) \leq \operatorname{rs}^F(\mathscr{C})$.*

Let $\mathscr{C} = \{H_1, \cdots, H_m\}$, let $\widetilde{A_i} \in \mathscr{MR}^F(\widetilde{H_i})$ for $i = 1, ..., m$ and let $A = \widetilde{A_1} + ... + \widetilde{A_m}$. Since $\mathscr{C}$ is edge-disjoint, $A \in \mathcal{S}^F(G)$, and by Proposition 1.11,

$$\operatorname{mr}^F(G) \leq \operatorname{rank} A \leq \sum_{i=1}^{m} \operatorname{rank}(\widetilde{A_i}) = \sum_{i=1}^{m} \operatorname{mr}^F(\widetilde{H_i}) = \sum_{i=1}^{m} \operatorname{mr}^F(H_i) = \operatorname{rs}^F(\mathscr{C}).$$

Definition 2.40. A *minimum rank cover* of a graph G over a field F is a cover $\mathscr{C}$ of G such that $\operatorname{rs}^F(\mathscr{C}) = \operatorname{mr}^F(G)$.

Example 2.41. Consider the graph $G80$.



As seen in Example 2.31, the minimum rank of $G80$ over any field F is 4. First consider the cover of $G80$ that consists of the five K_2's formed by vertices 1 and 2, 2 and 3, 3 and 4, 3 and 5, and 5 and 6. This cover is not a minimum rank cover since $\operatorname{mr}^F(K_2) + \operatorname{mr}^F(K_2) + \operatorname{mr}^F(K_2) + \operatorname{mr}^F(K_2) + \operatorname{mr}^F(K_2) = 5$. Now consider the cover of $G80$ consisting of the two K_2's formed by vertices 1 and 2 and vertices 5 and 6, which we denote as H_1 and H_2 respectively, and the star consisting of vertices $2, 3, 4$ and 5, which we denote as S. Since $\operatorname{rs}^F(\{H_1, H_2, S\} =$

$\mathrm{mr}^F(H_1) + \mathrm{mr}^F(H_2) + \mathrm{mr}^F(S) = 1 + 1 + 2 = \mathrm{mr}^F(G)$, $\{H_1, H_2, S\}$ is a minimum rank cover of $G80$. Furthermore, we can construct a matrix in $\mathcal{S}^F(G80)$ by taking a matrix from each of $\mathscr{MR}^F(\widetilde{H_1}), \mathscr{MR}^F(\widetilde{H_2})$, and $\mathscr{MR}^F(\widetilde{S})$ and summing them together. This is seen as follows: Let $A \in \mathscr{MR}^F(\widetilde{H_1}), B \in \mathscr{MR}^F(\widetilde{H_2})$ and $C \in \mathscr{MR}^F(\widetilde{S})$. Since $\widetilde{H_1}, \widetilde{H_2}$ and $\widetilde{S}$ are edge-disjoint, $A + B + C \in \mathcal{S}(G80)$. By Proposition 1.11, $\mathrm{mr}^F(G80) \leq \mathrm{rank}(A + B + C) \leq \mathrm{rank}\, A + \mathrm{rank}\, B + \mathrm{rank}\, C = \mathrm{mr}^F(G80)$. Therefore $A + B + C \in \mathscr{MR}^F(G80)$. For example, we can take the matrices

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathscr{MR}^F(\widetilde{H_1}), \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \in \mathscr{MR}^F(\widetilde{H_2}),$$

$$\text{and} \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathscr{MR}^F(\widetilde{S}).$$

The sum of the three matrices is in $\mathscr{MR}^F(G)$.

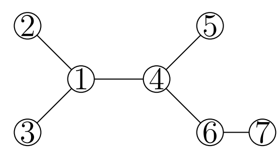
The following result appears as Corollary 3.15 in [12].

Theorem 2.42. *If T is a tree and F is any field, then there is a K_2-star cover of T whose rank sum is $\mathrm{mr}^F(T)$.*

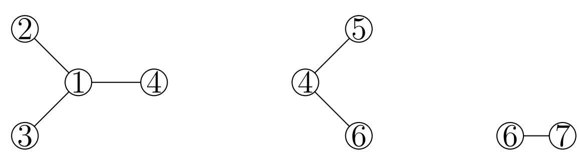
We note that the previous theorem could have stated that there is always an edge-disjoint minimum rank K_2-star cover of T. This is seen as follows. In a minimum rank K_2-star cover of T that is not edge-disjoint, the fact that the cover is a minimum rank cover implies that every overlap must occur between two stars that overlap on a single edge. An edge-disjoint cover can be obtained as follows. Whenever two stars S_j and S_k overlap on a single edge, replace S_k with a star that covers the same edges as S_k except for the edge it shared with S_j. This results in an edge-disjoint minimum rank K_2-star cover of T.

We use Theorem 2.42 extensively in the proof of the theorem that classifies the vertices of a tree (Theorem 2.51) and it is helpful to consider the following example.

Example 2.43. Consider the following tree T:



The minimum rank of T over any field F is 5. This can be seen as follows. It's straightforward that $\mathrm{Z}(T) = 2$. Thus by Theorem 2.4, $\mathrm{M}^F(T) \leq 2 \Rightarrow \mathrm{mr}^F(T) \geq 5$. Since $\mathrm{mr}^F(S_n) = 2, n > 2$ and $\mathrm{mr}^F(K_2) = 1$, the K_2-star cover of T consisting of



has rank sum 5. Thus $\mathrm{mr}^F = 5$. (Theorem 2.42 guarantees such a K_2-star cover of T whose rank-spread achieves the minimum rank.)

The following theorem was published by Hsieh in [13], and independently by Barioli, Fallat, and Hogben (see Theorem 2.3 in [14]). Van der Holst in [15] and Barrett, Grout, and Loewy in [16] proved field independent versions.

Theorem 2.44. *Let F be a field. Let G and H be graphs on at least two vertices, each with a vertex labeled v. Then*

$$\mathrm{mr}^F(G \underset{v}{\oplus} H) = \min\{\mathrm{mr}^F(G) + \mathrm{mr}^F(H), \mathrm{mr}^F(G - v) + \mathrm{mr}^F(H - v) + 2\}.$$

Equivalently,

$$r_v^F(G \underset{v}{\oplus} H) = \min\{r_v^F(G) + r_v^F(H), 2\}.$$

The next closely related result appears in [17].

Theorem 2.45. *Let F be any field, let G be the vertex-sum at v of G_1 and G_2, and let S_{k+1} be the star subgraph of G formed by the degree k vertex v and all of its neighbors.*

(i) If $r_v^F(G_1) + r_v^F(G_2) < 2$, then

$$\mathscr{MR}^F(G) = \mathscr{MR}^F(\widetilde{G_1}) + \mathscr{MR}^F(\widetilde{G_2}).$$

(ii) If $r_v^F(G_1) + r_v^F(G_2) > 2$, then

$$\mathscr{MR}^F(G) = \mathscr{MR}^F(\widetilde{G_1 - v}) + \mathscr{MR}^F(\widetilde{G_2 - v}) + \mathscr{MR}^F(\widetilde{S_{k+1}}).$$

(iii) If $r_v^F(G_1) + r_v^F(G_2) = 2$, then

$$\begin{aligned}\mathscr{MR}^F(G) = &\left(\mathscr{MR}^F(\widetilde{G_1}) + \mathscr{MR}^F(\widetilde{G_2})\right)\\ &\cup \left(\mathscr{MR}^F(\widetilde{G_1 - v}) + \mathscr{MR}^F(\widetilde{G_2 - v}) + \mathscr{MR}^F(\widetilde{S_{k+1}})\right).\end{aligned}$$

The next two results appear in [9].

Theorem 2.46. *Let p be a pendant vertex of a graph G and F a field. Then $r_p^F(G) = 0$ if and only if p is a nil vertex.*

Note: The forward implication of the above theorem is Corollary 2.30.

Lemma 2.47. *Let F be a field and v be a vertex of a graph G. If $r_v^F(G) = 2$, then v is neutral in G.*

The following lemma appears as Lemma 27 in [18]. We note that the lemma was stated without reference to a field but the proof holds for any field.

Lemma 2.48. *Let F be a field and G be the vertex-sum at v of graphs $G_1, ..., G_n$ where v is pendant in G_i for every i and $r_v^F(G) = 1$. If $w \in V(G_i), w \neq v$, then $r_w^F(G_i) = r_w^F(G)$.*

Lemma 2.49. *Let F be a field and let v be a vertex of a tree T such that $r_v^F(T) = 2$. There exists a minimum rank K_2-star cover where v is the center vertex of a star.*

Proof. Since $r_v(T) = 2$, $\mathrm{mr}(T - v) = \mathrm{mr}(T) - 2$. It follows immediately from Theorem 2.42 that given a forest, G, there is a K_2-star cover of G whose rank sum is $\mathrm{mr}(G)$. Thus there is a minimum rank K_2-star cover for $T - v$ with rank sum $\mathrm{mr}(T) - 2$. Adding a star centered at v to this cover results in a minimum rank K_2-star cover of T. □

Lemma 2.50. *Let F be a field, T be a tree, and v a vertex of T with $r_v^F(T) = 0$. Then for any vertex w adjacent to v, $r_w^F(T) = 2$.*

Proof. Let w be a vertex adjacent to v. By Theorem 2.42, there exists a K_2-star cover $\mathscr{C}$ of T with rank sum $\mathrm{mr}(T)$. In $\mathscr{C}$ the edge vw is covered by a K_2 or a star. If K_2 covers vw, then deleting v and the K_2 results in a cover of $T - v$ with rank sum at most $\operatorname{rank} T - 1$, contradicting $r_v(T) = 0$. Thus a star covers vw. This star is not centered at v since if so, deleting v and the star would result in a cover of $T - v$ with rank sum $\operatorname{rank} T - 2$. Thus w is the center of the star. Deleting w and the star centered at w gives a cover for $T - w$ with rank sum $\operatorname{rank} T - 2$. Thus $r_w(T) \geq 2$. By the remark following Definition 1.9, $r_w(T) = 2$. □

We now give the result that classifies the vertices of trees.

Theorem 2.51. *Let F be a field, and let v be a vertex of a tree T. Then*

- *v is a nil vertex if and only if $r_v^F(T) = 0$.*
- *v is a nonzero vertex if and only if v is covered by a K_2 in every minimum rank K_2-star cover of T.*
- *v is a neutral vertex if and only if it does not meet either of the previous two conditions.*

Proof. The third statement follows logically from the first two statements. Thus is suffices to prove the first two statements.

Statement 1.

We first prove the forward implication. Let v be a nil vertex. If v is a pendant vertex then by Theorem 2.46, $r_v(T) = 0$. Now suppose that v is a vertex of degree $k > 1$. Since T is a tree, v is a cut-vertex and thus T is the vertex-sum at v of $T_1, T_2, ..., T_k$ such that v is pendant in each T_i. Since v is nil, by Lemma 2.47, $r_v(T) \neq 2$. By way of contradiction, suppose that $r_v(T) = 1$. By Theorem 2.44, there is exactly one T_i with $r_v(T_i) = 1$. Without loss of generality, we let $r_v(T_1) = 1$. By Theorem 2.46, v in T_1 is not nil. Let $A_1 \in \mathscr{MR}(\widetilde{T_1})$ where the diagonal entry of corresponding to v is nonzero. Let $A_i \in \mathscr{MR}(\widetilde{T_i})$, $i = 2, ..., k$. By Theorem 2.45, $A_1 + A_2 + ... + A_k \in \mathscr{MR}(T)$. Furthermore, the diagonal entry of $A_1 + A_2 + ... + A_k$ corresponding to v is nonzero. Therefore v is not nil, a contradiction. Therefore $r_v(T) = 0$.

We now prove the reverse implication. Let $r_v(T) = 0$. Thus $\operatorname{mr}(T - v) = \operatorname{mr} T$. Since v is adjacent to exactly one vertex in each component of $T - v$, by Theorem 2.28, v is a nil vertex.

Statement 2.

We prove the contrapositive of the forward implication. Assume there exists a minimum rank K_2-star cover in which v is not covered by a K_2. Thus v is covered only by stars. Let $T_1, T_2, ..., T_k, T_{k+1}, ..., T_l$ be the elements of the cover where T_i, $i = 1, 2, ..., k$ are the elements that cover v. Since dominating vertices of stars and pendant vertices of stars are nil or neutral, there exist matrices $A_1 \in \mathscr{MR}(\widetilde{T_1}), A_2 \in \mathscr{MR}(\widetilde{T_2}), ..., A_k \in \mathscr{MR}(\widetilde{T_k})$ where in each A_i the diagonal entry corresponding to v is 0. Let $A_i \in \mathscr{MR}(\widetilde{T_i}), i = k + 1, ..., l$. Since $T_1, T_2, ..., T_k, T_{k+1}, ..., T_l$ are the elements of a minimum rank covering of T, $\operatorname{mr}(T) = \operatorname{mr}(T_1) + \operatorname{mr}(T_2) + ... + \operatorname{mr}(T_l)$. Now $A_1 + A_2 + ... + A_l \in \mathcal{S}(T)$, so by this fact and Proposition 1.11, $\operatorname{mr}(T) \leq \operatorname{rank}(A_1 + A_2 + ... + A_l) \leq \operatorname{rank} A_1 + \operatorname{rank} A_2 + ... + \operatorname{rank} A_l = \operatorname{mr}(T_1) + \operatorname{mr}(T_2) + ... + \operatorname{mr}(T_l) = \operatorname{mr}(T)$. Thus $A_1 + A_2 + ... + A_l \in \mathscr{MR}(T)$ and the diagonal entry corresponding to v is zero. Therefore v is not nonzero.

We now prove the reverse implication. Assume that v is covered by a K_2 in every minimum rank K_2-star cover of T. We first show that this implies $r_v(T) = 1$. Suppose that $r_v(T) = 2$. Taking a minimum rank cover for $T - v$ and adding a star with dominating vertex v gives a minimum rank cover for T where a K_2 is not covering v, a contradiction. Thus $r_v(G) \leq 1$. We show $r_v(T) \neq 0$. Taking a minimum rank cover for T and deleting v results in a cover of $T - v$ with rank sum at most $\mathrm{mr}(T) - 1$ (since a K_2 was used to cover v). Thus $r_v(T) = 1$.

We next show that v has no neighbors with rank-spread 0 and exactly one neighbor with rank-spread 1. Since $r_v(T) = 1$, Lemma 2.50 implies that v has no neighbors with rank-spread 0. Let $x_1, x_2, ..., x_m$ be the neighbors of v. Suppose each of these vertices has rank-spread 2. Now T is the vertex-sum at v of $T_1, T_2, ..., T_m$ where v is a pendant vertex in each T_i and where without loss of generality, T_i contains $x_i, i = 1, 2, ..., m$. By Theorem 2.44, $\sum_{i=1}^{m} r_v(T_i) = 1$ and $\mathrm{mr}(T) = \mathrm{mr}(T_1) + \mathrm{mr}(T_2) + ... + \mathrm{mr}(T_m)$. By Lemma 2.48, $r_{x_i}(T_i) = 2, i = 1, ..., m$. By Lemma 2.49, for each T_i there is a minimum rank K_2-star cover, $\mathscr{C}_i$, where x_i is the center of a star. Overlapping $\mathscr{C}_1, \mathscr{C}_2, ..., \mathscr{C}_m$ on v results in a minimum rank K_2-star cover for T where v is not covered by a K_2, a contradiction.

Therefore v has a neighbor with rank-spread 1. By a similar argument as given above, there is a minimum rank cover of T where every rank-spread 2 vertex adjacent to v is the dominating vertex of a star. Call this covering $\mathscr{C}$. Since a K_2 covers v in every minimum rank cover of T, the K_2 must cover one of the edges between v and a rank-spread 1 vertex. Let w be the vertex adjacent to v such that K_2 covers wv. Suppose that there is another vertex x adjacent to v with rank-spread 1. If a K_2 or a star centered at v covered xv in $\mathscr{C}$ then deleting v from $\mathscr{C}$ would give a cover of $T - v$ with rank sum at most $\mathrm{mr}(T) - 2$, contradicting $r_v(T) = 1$. Therefore vx is covered by a star centered at x. Deleting x gives a cover of $T - x$ with rank sum $\mathrm{mr}(T) - 2$, contradicting that $r_x(T) = 1$.

Therefore v has exactly one neighbor with rank-spread 1 and the rest of the neighbors of

v have rank-spread 2. Without loss of generality, let x_1 be the neighbor of v with rank-spread 1 and let $x_2, .., x_m$ be the neighbors of v with rank-spread 2. Now T is the vertex-sum at v of $T_1, T_2, ..., T_m$ such that v is a pendant vertex in each T_i and, without loss of generality, T_i contains $x_i, i = 1, 2, ..., m$. By Lemma 2.48, $r_{x_1}(T_1) = 1$ and $r_{x_i}(T_i) = 2, i = 2, ..., m$. We now show that $r_v(T_1) = 1$ and $r_v(T_i) = 0, i = 2, ..., m$. Take a minimum rank cover for T_1. Since $r_{x_1}(T_1) = 1$, x_1 cannot be a dominating vertex of a star in the cover. Thus vx_1 is covered by a K_2. Deleting v results in a cover of $T_1 - v$ with rank sum $\text{mr}(T_1) - 1$. Thus $r_v(T_1) \geq 1$. Since $r_v(T) = 1$, Lemma 2.44 implies that $r_v(T_1) = 1$ and $r_v(T_i) = 0, i = 2, ..., m$.

Note that T_1 is the vertex-sum at x_1 of H and $T_1 - v$ where H is the K_2 with vertex set $\{v, x_1\}$. Since $r_{x_1}(H) = 1$, Lemma 2.44 implies that $r_{x_1}(T_1 - v) = 0$. Thus by Theorem 2.45, $\mathscr{MR}(T_1) = \mathscr{MR}(\widetilde{H}) + \mathscr{MR}(\widetilde{T_1 - v})$. Since H is a K_2, v is a nonzero vertex in H. It follows that v is a nonzero vertex in T_1. By Theorem 2.46, v is a nil vertex in $T_i, i = 1, ..., m$. Since $r_v(T_1) = 1$ and $r_v(T_i) = 0, i = 2, ..., m$, by Theorem 2.45, $\mathscr{MR}(T) = \mathscr{MR}(\widetilde{T_1}) + \mathscr{MR}(\widetilde{T_2}) + ... + \mathscr{MR}(\widetilde{T_m})$. This equality along with the facts that v is nonzero in T_1 and nil in $T_i, i = 2, ..., m$ imply that v is nonzero in T. $\square$

The second part of Theorem 2.51 relies on knowing every minimum rank K_2-star cover of a tree. This theorem can be restated, without considering covers, in terms that only rely on the rank-spreads of the vertices of the tree. This is often useful since the rank-spreads of the vertices of a tree are easily computed. The theorem is as follows:

Theorem 2.52. *Let F be a field and let v be a vertex of a tree T. Then*

- *v is a nil vertex if and only if $r_v^F(T) = 0$.*
- *v is a nonzero vertex if and only if $r_v^F(T) = 1$ and a vertex adjacent to v has rank-spread 1.*
- *v is a neutral vertex if and only if $r_v^F(T) = 2$, or $r_v^F(T) = 1$ and no vertex adjacent to v has rank-spread 1.*

Proof. By Theorem 2.51, it suffices to show that $r_v(T) = 1$ and a vertex adjacent to v has rank-spread 1 if and only if v is covered by a K_2 in every minimum rank K_2-star cover of T.

Assume that $r_v(T) = 1$ and a vertex w adjacent to v has rank-spread 1. Let $\mathscr{C}$ be a minimum rank K_2-star cover of T. The edge vw is either covered by a K_2 or a star centered at v or w. If vw were covered by a star, say centered at v (without loss of generality), then deleting v would result in a cover of $T - v$ with rank sum $\mathrm{mr}(T) - 2$, contradicting that $r_v(T) = 1$. Therefore vw must be covered by a K_2. Since $\mathscr{C}$ was an arbitrary minimum rank cover, v is covered by a K_2 in every minimum rank K_2-star cover of T.

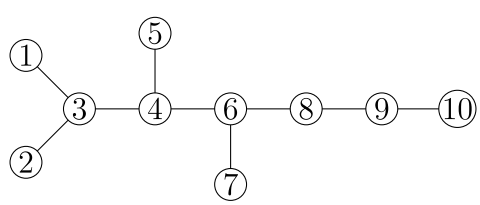
Assume that v is covered by a K_2 in every minimum rank K_2-star cover of T. From the proof of Theorem 2.51, $r_v(T) = 1$ and v has exactly one neighbor with rank-spread 1. □

Remark. The second and third statements in Theorem 2.52 could have been written as

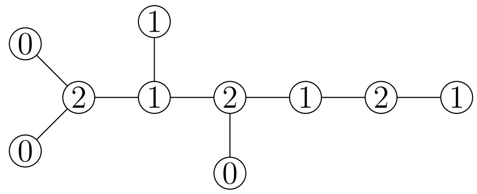
- v is a nonzero vertex if and only if $r_v^F(T) = 1$ and exactly one vertex adjacent to v has rank-spread 1.
- v is a neutral vertex if and only if $r_v^F(T) = 2$, or $r_v^F(T) = 1$ and every vertex adjacent to v has rank-spread 2.

The second statement follows from the proof given above and the third follows from Lemma 2.50 which implies that if $r_v^F(T) = 1$ then no vertex adjacent to v has rank-spread 0.

Example 2.53. Consider the following tree T.



Using zero forcing and Theorem 2.5 it is straightforward to find the rank-spreads of each vertex. They are listed below.



By Theorem 2.52, vertices $1, 2$ and 7 are nil vertices, vertices 4 and 5 are nonzero vertices, and the remaining vertices are are neutral vertices.

Chapter 3. Inverse Inertia Problem

3.1 Background and Definitions

In this chapter we consider the inverse inertia problem. In certain cases, nil, nonzero, and neutral vertices can help solve this problem. We give an example of how nil vertices help solve this problem for a particular graph. We also give an application of minimum rank and inertia to the planarity of graphs.

Definition 3.1. Given an $n \times n$ real symmetric matrix A, the *inertia* of A is the ordered triple $(\pi(A), \nu(A), \delta(A))$, where $\pi(A)$ is the number of positive eigenvalues of A, $\nu(A)$ is the number of negative eigenvalues of A, and $\delta(A)$ is the multiplicity of 0 as an eigenvalue of A.

Note that $\pi(A)+\nu(A)+\delta(A) = n$ and $\pi(A)+\nu(A) = \operatorname{rank}(A)$. Because of the first equality, if n is known then no information is lost by just considering $\pi(A)$ and $\nu(A)$. This motivates the following definition:

Definition 3.2. Given a real symmetric matrix A, the *partial inertia* of A is the ordered pair $(\pi(A), \nu(A))$, written $\operatorname{pin}(A)$.

Definition 3.3. Given a graph G, the *inertia set* $\mathcal{I}(G)$ is the set of all possible partial inertias of matrices in $\mathcal{S}(G)$. That is,

$$\mathcal{I}(G) = \{(r, s) \,|\, \operatorname{pin}(A) = (r, s) \text{ for some } A \in S(G)\}$$

Note that since $\pi(A) + \nu(A) = \operatorname{rank} A$, if $(r, s) \in \mathcal{I}(G)$, then $\operatorname{mr}(G) \le r + s \le n$. Also $\operatorname{mr}_+(G)$ is the smallest integer r such that $(r, 0)$ is in $\mathcal{I}(G)$. Lastly note if $(r, s) \in \mathcal{I}(G)$, then $(s, r) \in \mathcal{I}(G)$ since if $\operatorname{pin}(A) = (r, s)$, then $\operatorname{pin}(-A) = (s, r)$.

Definition 3.4. A *clique* in a graph is a set of vertices which are pairwise adjacent.

Definition 3.5. A *clique cover* of a graph G is a cover of G consisting of only cliques. The clique cover number of G, written $\operatorname{cc}(G)$, is the smallest number of cliques in a clique cover of G.

Definition 3.6. A graph G is *chordal* if there are no induced cycles, C_k, where $k \ge 4$.

The following is Theorem 3.6 from [19].

Theorem 3.7. *Let G be a connected chordal graph on $n \ge 2$ vertices. Then* $\operatorname{mr}_+(G) = \operatorname{cc}(G)$.

The following is a part of Proposition 1.4 from [20].

Proposition 3.8 (Subadditivity). *Let $A, B,$ and C be real symmetric $n \times n$ matrices with $A + B = C$. Then*

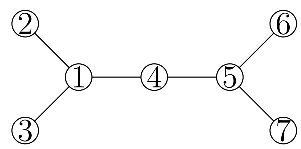
$$\pi(C) \le \pi(A) + \pi(B) \text{ and } \nu(C) \le \nu(A) + \nu(B).$$

The following observation and example illustrates how nil vertices can help solve the inverse inertia problem for minimum rank matrices.

Observation 3.9. *If G is a connected graph with a nil vertex,* $(\operatorname{mr}(G), 0) \notin \mathcal{I}(G)$.

The observation follows from the fact that if a positive semi-definite matrix has a 0 entry on the diagonal, the entire row and column corresponding to this 0 would have to be zero, contradicting that G is connected.

Example 3.10. Consider the following tree T.



Using zero forcing and Theorem 2.4, we conclude $\mathrm{mr}(T) = 4$ and $\mathrm{mr}(T-k) = 4$, $k = 2, 3, 4, 6, 7$. By Theorem 2.51, vertices $2, 3, 4, 6$, and 7 are nil vertices. Thus given $A \in \mathscr{MR}(T)$, A has the form

$$A = \begin{bmatrix} 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 \\ a & b & d_1 & c & 0 & 0 & 0 \\ 0 & 0 & c & 0 & r & 0 & 0 \\ 0 & 0 & 0 & r & d_2 & s & t \\ 0 & 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 \end{bmatrix}, abcrst \neq 0.$$

Let

$$B = \begin{bmatrix} 0 & 0 & a & 0 & 0 & 0 & 0 \\ 0 & 0 & b & 0 & 0 & 0 & 0 \\ a & b & d_1 & c & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & r & d_2 & s & t \\ 0 & 0 & 0 & 0 & s & 0 & 0 \\ 0 & 0 & 0 & 0 & t & 0 & 0 \end{bmatrix}$$

so $A = B + C$. Matrices B and C both have rank 2 and are in $\mathcal{S}(S_4 \cup 3K_1)$. Since S_4 is a chordal graph with $\mathrm{cc}(S_4) = 3$, by Theorem 3.7, $\mathrm{mr}_+(S_4) = 3$. It follows that $\mathrm{mr}_+(S_4 \cup 3K_1) = 3$ and thus $\mathrm{pin}(B) \neq (2, 0)$. Therefore $\mathrm{pin}(B) = (1, 1)$. Similarly, $\mathrm{pin}(C) = (1, 1)$. By Theorem 3.8, $\pi(A) \leq \pi(B) + \pi(C) = 2$ and $\nu(A) \leq \nu(B) + \nu(C) = 2$. Since $\pi(A) + \nu(A) = \mathrm{mr}(T) = 4$, $\pi(A) = 2$ and $\nu(A) = 2$; i.e. $\mathrm{pin}(A) = (2, 2)$. This proves that

$(4,0),(3,1),(1,3)$, and $(0,4)$ are not in $\mathcal{I}(T)$.

3.2 Application to Planarity of Graphs

Minimum rank and inertia have connections to other areas of mathematics. We illustrate that here by proving the following theorem which shows how minimum rank and inertia connects with the planarity of a graph. It is often difficult to directly prove a graph is nonplanar. This theorem gives a way to prove nonplanarity for certain graphs. We first provide a definition and a previous result we will use to prove the theorem.

Definition 3.11. Let G be a graph on n vertices. A real symmetric $n \times n$ matrix Q is a generalized Laplacian of G if $Q_{ij} < 0$ when $ij \in E(G)$ and $Q_{ij} = 0$ when $i \neq j$ and $ij \notin E(G)$. No conditions are placed on the diagonal entries of Q.

In the book, Algebraic Graph Theory [21], the following is Corollary 13.10.2.

Lemma 3.12. *Let Q be a generalized Laplacian for the graph G. If G is 3-connected and planar, then the second smallest eigenvalue $\lambda_2(G)$ of Q has multiplicity at most three.*

We are now ready to state the theorem

Theorem 3.13. *Let G be a 3-connected graph on n vertices and let $b \leq n-5$. If there exists a matrix $A \in \mathcal{S}(G)$ with $\operatorname{pin}(A) = (1,b)$ and all off-diagonal entries of A are positive, or $\operatorname{pin}(A) = (b,1)$ and all off-diagonal entries of A are negative, then G is nonplanar.*

Proof. Let $A \in \mathcal{S}(G)$ and $b \leq n-5$. If $\operatorname{pin}(A) = (1,b)$ and all off-diagonal entries of A are positive, let $B = -A$. If $\operatorname{pin}(A) = (b,1)$ and all off-diagonal entries of A are negative, let $B = A$. Note that B is a generalized Laplacian matrix of G. Since $\operatorname{rank} B \leq n-4$, 0 is an eigenvalue of B of multiplicity ≥ 4. Also, since $\operatorname{pin}(B) = (b,1)$, 0 is the second smallest eigenvalue of B. Thus by the contrapositive of Lemma 3.12, G is nonplanar. $\square$

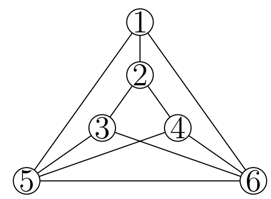
The following corollary is a special case of the theorem above.

Corollary 3.14. *Let G be a 3-connected graph on $n \geq 6$ vertices. Suppose there exists an $A \in \mathcal{S}(G)$ with*

- $\operatorname{pin}(A) = (1, 1)$.
- *all off-diagonal entries of A have the same sign*

Then G is nonplanar.

Example 3.15. Consider the following 3-connected graph G.



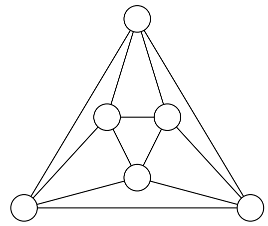
Let $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & -1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 & 1 \end{bmatrix} \in \mathcal{S}(G)$.

Clearly, $\operatorname{rank} A \geq 2$. Also, A only has 3 distinct rows and the last row is the sum of the first two rows. Therefore $\operatorname{rank} A = 2$. Neither A nor $-A$ is positive semi-definite, so $\operatorname{pin}(A) = (1, 1)$ (the eigenvalues of A are $3.854, -2.854, 0, 0, 0, 0$). By Corollary 3.14, G is nonplanar.

The contrapositive of Theorem 3.13 can also be useful in describing the structure of matrices in $\mathcal{S}(G)$ with certain inertias. This is stated in the following corollary.

Corollary 3.16. *Let G be a planar, 3-connected graph, on n vertices. Every matrix $A \in \mathcal{S}(G)$ with $\operatorname{pin}(A) = (1, b), b \leq n - 5$ has a negative off-diagonal entry. Similarly, every matrix $A \in \mathcal{S}(G)$ with $\operatorname{pin}(A) = (b, 1), b \leq n - 5$ has a positive off-diagonal entry.*

Example 3.17. Consider the following planar, 3-connected graph G.



From [22], it is know that $(1,1) \in \mathcal{I}(G)$. Thus by the above corollary, every matrix in $\mathcal{S}(G)$ with exactly one positive eigenvalue and one negative eigenvalue must have both positive and negative off-diagonal entries.

Chapter 4. Inverse Eigenvalue Problem

4.1 Background and Definitions

The following question is what is known as the inverse eigenvalue problem for graphs.

Problem 4.1 (The inverse eigenvalue problem)**.** *Given a graph G on n vertices and n real numbers*

$$\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$$

is there a matrix in $\mathcal{S}(G)$ with eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$?

We first provide an example, previous results, and lemmas that will be used in this section.

Example 4.2. Consider the complete graph on 4 vertices K_4 and the numbers $6, 2, 2, 2$. Is there a matrix in $\mathcal{S}(K_4)$ having $6, 2, 2, 2$ as eigenvalues? The answer is yes and we can construct such a matrix as follows. Consider the matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Since the row sums are all 4, 4 is an eigenvalue of A. The rank of A is 1 so the other eigenvalues are $0, 0, 0$. We shift these eigenvalues by adding a multiple of the identity. Thus

$$\begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 3 & 1 \\ 1 & 1 & 1 & 3 \end{bmatrix}$$

is in $\mathcal{S}(K_4)$ and has eigenvalues $6, 2, 2, 2$. Similarly, given any real numbers a, b, b, b, there is a matrix in $\mathcal{S}(K_4)$ having these numbers as eigenvalues.

Now consider the real numbers a, a, a, a. There is no matrix in $\mathcal{S}(K_4)$ having these as eigenvalues. If there was such a matrix A then $A - aI_4$ would be a rank 0 matrix (the 0 matrix) in $\mathcal{S}(K_4)$, which is impossible.

The next two theorems are known as the interlacing theorem (see exercise 16, p. 200, in [4]) and the Gershgorin disk theorem ([23]).

Theorem 4.3. *Let A be an $n \times n$ Hermitian matrix with eigenvalues $\lambda_1 \geq ... \geq \lambda_n$. Let B be an $n-1 \times n-1$ principal submatrix of A with eigenvalues $\mu_1 \geq ... \geq \mu_{n-1}$. Then*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \mu_{n-1} \geq \lambda_n.$$

Theorem 4.4. *If $A = [a_{ij}] \in \mathbb{C}^{n \times n}$ and*

$$r_i = \sum_{j=1, j\neq i}^{n} |a_{ij}|,$$

then every eigenvalue of A lies in at least one of the disks $\{z : |z - a_{ii}| \leq r_i\}, i = 1, 2, ..., n$ in the complex plane.

Furthermore, a set of m disks having no point in common with the remaining $n-m$ disks contains m and only m eigenvalues of A.

The following lemma is an immediate consequence of Corollary 3.9 in [24]. We note that $m_A(b)$ denotes the multiplicity of b as an eigenvalue of the matrix A.

Lemma 4.5. *Let S_n be a star on $n \geq 2$ vertices. Let $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. If $A \in \mathcal{S}(S_n)$ has eigenvalues $\lambda_1, ..., \lambda_n$ then $\lambda_1 > \lambda_2$ and $\lambda_{n-1} > \lambda_n$; i.e $m_A(\lambda_1) = 1$ and $m_A(\lambda_n) = 1$.*

Lemma 4.6. *Let S_n be a star on $n \geq 2$ vertices. Let b be an eigenvalue of a matrix $A \in \mathcal{S}(S_n)$ with $m_A(b) = n-2$. Then bI_{n-1} is a principal submatrix of A where I_{n-1} is the $n-1 \times n-1$ identity matrix.*

Proof. Let $A \in \mathcal{S}(S_n)$ have an eigenvalue b with $m_A(b) = n-2$. Then $A - bI_n \in \mathcal{S}(S_n)$ and has 0 as an eigenvalue with $m_{A-bI_n}(0) = n-2$. Thus rank $A - bI_n = 2$. Since $\text{mr}(S_n) = 2$, $A - bI_n$ is a minimum rank matrix. As shown previously every pendant vertex in S_n is a nil vertex. Thus the zero matrix of size $n-1 \times n-1$ is a principal submatrix of $A - bI_n$. Therefore bI_{n-1} is a principal submatrix of A. □

There have been relatively few results about the inverse eigenvalue problem. In 1986, Boley and Golub [25] gave a constructive proof from which it follows that given any n distinct real numbers, there is a matrix $A \in \mathcal{S}(S_n)$ with those numbers as eigenvalues. In 2002, Duarte, Johnson, and Saiago [26] completely solved the inverse eigenvalue problem for stars. The result of Boley and Golub came from their study of an important variation of the inverse eigenvalue problem. This problem, which we will call the λ, μ problem, has also been of mathematical interest and is stated below.

Problem 4.7 (The λ, μ problem)**.** *Let G be a graph on n vertices and let v be a vertex of G. Given $2n-1$ real numbers*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$$

is there a matrix M in $\mathcal{S}(G)$ with eigenvalues

$$\lambda_1, \lambda_2, ..., \lambda_n$$

such that the principal submatrix $M(v)$ has eigenvalues

$$\mu_1, \mu_2, ..., \mu_{n-1}?$$

There are physical motivations for both the inverse eigenvalue and the λ, μ problem. Variations of the inverse eigenvalue problem have been of interest in many applications, including control design, circuit theory, exploration and remote sensing, particle physics, and geophysics ([25], [27]). In particular, Hald [28] shows how the λ, μ problem aids in the study of the vibrations of a composite pendulum.

To see a mathematical motivation for the study of the λ, μ problem, as opposed to the inverse eigenvalue problem, consider the following argument: If one is concerned about solving the inverse eigenvalue problem only approximately, then there is a simple solution for any graph. The theorem below shows that given a graph on n vertices and a set of n numbers, there is a matrix corresponding to the graph with eigenvalues arbitrarily close to these numbers. This argument does not solve the λ, μ problem approximately.

Theorem 4.8. *Let $\epsilon > 0$. Given a graph G on n vertices and numbers*

$$a_1 \geq a_2 \geq ... \geq a_n$$

there is a matrix in $A \in \mathcal{S}(G)$ with eigenvalues $\lambda_1, ..., \lambda_n$ such that $|\lambda_i - a_i| < \epsilon, i = 1, 2, ..., n$.

Proof. Let G be a graph on n vertices and let $a_1 \geq a_2 \geq ... \geq a_n$ be n real numbers. Construct a matrix $A = [a_{ij}]$ in $\mathcal{S}(G)$ as follows. Let the diagonal entries of A be $a_1, a_2, ..., a_n$. Choose the off-diagonal entries so that defining $r_i = \sum_{i=1, i\neq j}^{n} |a_{ij}|$, then $A \in \mathcal{S}(G)$, $(a_i - r_i, a_i + r_i) \cap (a_j - r_j, a_j + r_j) = \emptyset$ if $a_i \neq a_j$, $r_i = r_j$ if $a_i = a_j$, and $r_i < \epsilon$ for $i, j = 1, ..., n$. Let $\lambda_1, \lambda_2, ..., \lambda_n$ be the eigenvalues of A. By the Gershgorin Disk Theorem, $\lambda_i \in (a_i - r_i, a_i + r_i), i = 1, ..., n$. Hence $|\lambda_i - a_i| < r_i < \epsilon$. $\square$

Boley and Golub [25] gave a constructive proof that given any real distinct numbers

$\lambda_1 > \mu_1 > ... > \mu_{n-1} > \lambda_n$, there is a matrix $A \in \mathcal{S}(S_n)$ with eigenvalues $\lambda_1, ..., \lambda_n$ such that $A(1)$ has eigenvalues $\mu_1, ..., \mu_{n-1}$ (where 1 corresponds to the dominating vertex of S_n). An interesting note is that in the proof, except for the sign of the off-diagonal entries, A is completely determined by the distinct eigenvalues.

We now consider the opposite extreme in that we let as many of the λ's and μ's be equal as possible. Given any real numbers $\lambda_1 > \mu_1 = \lambda_2 = ... = \lambda_{n-1} = \mu_{n-1} > \lambda_n$, we give a constructive proof that shows there is a matrix $A \in \mathcal{S}(S_n)$ with eigenvalues $\lambda_1, ..., \lambda_n$ such that $A(1)$ has eigenvalues $\mu_1, ..., \mu_{n-1}$. In contrast to the distinct eigenvalue case, the matrix is not even close to being completely determined by the eigenvalues.

Consider the matrix

$$A = \begin{bmatrix} \lambda_1 + \lambda_n - \lambda_2 & u^T \\ u & \lambda_2 I_{n-1} \end{bmatrix} \in \mathcal{S}(S_n)$$

where u is choosen so that $||u|| = \sqrt{(\lambda_2 - \lambda_n)(\lambda_1 - \lambda_2)}$. Note that $A(1)$ has eigenvalues $\mu_1, ..., \mu_{n-1}$ since $\lambda_2 = \mu_i, i = 1, ..., n-1$. By Theorem 4.3, $m_A(\lambda_2) \geq n-2$. We also know by Lemma 4.5 that the largest and smallest eigenvalues of A have multiplicity 1. Thus $m_A(\lambda_2) = n-2$.

Let a and b be the other eigenvalues of A. Since the trace of A is equal to the sum of its eigenvalues,

$$a + b + (n-2)\lambda_2 = \lambda_1 + \lambda_n - \lambda_2 + (n-1)\lambda_2$$

$$\text{so } a + b = \lambda_1 + \lambda_n.$$

Furthermore, since the sum of the squares of the entries in A is equal to the sum of the squares of the eigenvalues,

$$a^2 + b^2 + (n-2)\lambda_2^2 = (\lambda_1 + \lambda_n - \lambda_2)^2 + 2||u||^2 + (n-1)\lambda_2^2$$

$$\text{so } a^2 + b^2 + n\lambda_2^2 - 2\lambda_2^2 =$$

$$\lambda_1^2 + \lambda_n^2 + \lambda_2^2 + 2\lambda_1\lambda_n - 2\lambda_2\lambda_n - 2\lambda_1\lambda_2 + 2\lambda_1\lambda_2 - 2\lambda_2^2 - 2\lambda_1\lambda_n + 2\lambda_2\lambda_n + n\lambda_2^2 - \lambda_2^2$$

$$\text{and hence } a^2 + b^2 = \lambda_1^2 + \lambda_n^2.$$

Substituting $\lambda_1 + \lambda_n - b$ for a in the above equation, we get $b^2 + (-\lambda_1 - \lambda_n)b + \lambda_1\lambda_n = 0$. Thus $b = \lambda_1$ or λ_n. This implies $a = \lambda_n$ or λ_1. Without loss of generality, $a = \lambda_1$ and $b = \lambda_n$.

Note that Lemma 4.6 shows that any matrix in $\mathcal{S}(S_n)$ with $m_A(\lambda_2) = n - 2$, must have $\lambda_2 I_{n-1}$ as a principal submatrix. It follows that the diagonal entries of A had to be the ones chosen in the proof. The only other condition was the one placed on the length of u. We conjecture that we have seen the two extremes in the amount of structure that the eigenvalues imply upon matrices in $\mathcal{S}(S_n)$. The first extreme being that given $\lambda_1 > \mu_1 > ... > \mu_{n-1} > \lambda_n$, the matrix in $\mathcal{S}(S_n)$ corresponding to these numbers is determined up to the sign of the off-diagonal entries. The second extreme being that given $\lambda_1 > \mu_1 = ... = \mu_{n-1} > \lambda_n$, a matrix in $\mathcal{S}(S_n)$ corresponding to these numbers is only determined up to the norm of a vector.

In the 1970's both Hald [28] and Hoschstadt [29] showed that given any $2n - 1$ distinct numbers $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > ... > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, there is a matrix $A \in \mathcal{S}(P_n)$ having the λ's as eigenvalues with the μ's being the eigenvalues of $M(v)$ where v is a pendant vertex of P_n. Other previous results include the following two theorems. The first follows from a result due to Boley and Golub [25] in 1987 and the second to due to Duarte [30] in 1989.

Theorem 4.9 (Boley-Golub)**.** *Given $2n - 1$ real numbers*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > ... > \lambda_{n-1} > \mu_{n-1} > \lambda_n,$$

there exists an $M \in \mathcal{S}(\mathrm{S_n})$ with eigenvalues $\lambda_1, ..., \lambda_n$ such that $M(1)$ has eigenvalues $\mu_1, ..., \mu_{n-1}$, where 1 is the label of the dominating vertex in S_n.

Theorem 4.10 (Duarte)**.** *Let T be a tree and let v be a vertex in T. Given $2n - 1$ real* *numbers* $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > ... > \lambda_{n-1} > \mu_{n-1} > \lambda_n$, *there exists an* $M \in \mathcal{S}(T)$ *with eigenvalues* $\lambda_1, ..., \lambda_n$ *such that* $M(v)$ *has eigenvalues* $\mu_1, ..., \mu_{n-1}$.

Note that since a star is a tree, Theorem 4.9 follows from Theorem 4.10.

4.2 Some results on the λ, μ problem

The previous results we have seen pertain to specific types of graphs. The following result is nice in that it provides necessary conditions on the relationships between the λ's and the μ's for an arbitrary graph G.

Theorem 4.11. *Let* G *be a graph on* n *vertices. Let*

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n \tag{4.1}$$

be real numbers such that there exists an $A \in \mathcal{S}(G)$ *with eigenvalues* $\lambda_1, \cdots, \lambda_n$ *and such that a principal submatrix of* A *has eigenvalues* $\mu_1, \mu_2, \cdots, \mu_{n-1}$. *Then*

- *A sequence of more than* $2M(G)$ *consecutive equal signs does not occur in* (4.1).
- *A sequence of exactly* $2M(G)$ *consecutive equal signs cannot begin after a* λ_i*; i.e. the sequence cannot start with*

$$\lambda_1 = \mu_1 \quad \textit{or} \quad > \lambda_i = \mu_i = \cdots \;\; \textit{for some } i \in \{1, ..., n\}.$$

(Here exactly means the sequence of equal signs is not part of a longer sequence of equal signs).

- *A sequence of exactly* $2M(G) - 1$ *consecutive equal signs does not occur in* (4.1).

Proof. Part one: By way of contradiction suppose that a sequence of more than $2M(G)$ consecutive equal signs occur in (4.1). Then $2M(G) + 2$ terms in (4.1) are equal. Exactly half of these terms are λ's and so $M(G) + 1$ of the λ's are equal. Let α be the common value of these λ's. Let $A \in \mathcal{S}(G)$ have eigenvalues $\lambda_1, \cdots, \lambda_n$ such that a principal submatrix of A has eigenvalues $\mu_1, \cdots, \mu_{n-1}$. Then $A - \alpha I \in \mathcal{S}(G)$ and has 0 as an eigenvalue with multiplicity $M(G) + 1$. Thus

$$M(G) \geq \text{ nullity } (A - \alpha I) = M(G) + 1$$

a contradiction.

Part two: By way of contradiction suppose that a sequence of exactly $2M(G)$ equal signs begin after λ_i. Then $2M(G) + 1$ terms in (4.1) are equal. In particular, since the equal signs begin after λ_i, $M(G) + 1$ of the λ's are equal. The rest of the proof is the same as given for part one.

Part three: By way of contradiction suppose that a sequence of exactly $2M(G) - 1$ consecutive equal signs occur in (4.1). Then $2M(G)$ terms in (4.1) are equal. In particular $M(G)$ of the λ's are equal and $M(G)$ of the μ's are equal. As seen in part one, shifting the matrix A by $-\alpha I$ where α is the value of these equivalent λ's and μ's produces a matrix $B \in \mathcal{S}(G)$ with 0 as an eigenvalue of multiplicity $M(G)$ and where exactly $M(G)$ of the μ's are 0. Thus B achieves the maximum nullity (and hence minimum rank) of G. By Proposition 2.26 any principal $n - 1 \times n - 1$ submatrix of G has rank equal to the rank of B or the rank of B minus 2. However, since exactly $M(G)$ of the μ's are 0, B has an $n - 1 \times n - 1$ principal submatrix of rank $(n-1) - M(G) = \text{ rank } B - 1$, a contradiction. $\square$

4.2.1 The Graph P_n.

Theorem 4.11 is an important tool in reducing the number of strings of equalities and inequalities relating the λ's and the μ's possible in the λ, μ problem. An example of two different strings for a a graph, G, on 3 vertices is $\lambda_1 = \mu_1 > \lambda_2 > \mu_2 > \lambda_3$ and $\lambda_1 = \mu_1 = \lambda_2 > \mu_2 > \lambda_3$. Strings that are "symmetric" to each other are considered equivalent. An example of equivalent strings are $\lambda_1 = \mu_1 > \lambda_2 > \mu_2 > \lambda_3$ and $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3$. The reason these are equivalent is if $A \in \mathcal{S}(G)$ is a matrix corresponding to the first string, then $-A \in \mathcal{S}(G)$ is a matrix corresponding to the equivalent string. When the number of vertices of the graph in consideration is small, we can count the number of non-equivalent strings by listing them. For example, for graphs on 3 vertices, the number of non-equivalent strings is 10. For graphs with more than 3 vertices, Polya counting is an effective way to count the number of non-equivalent strings. Using Polya counting, we find that for graphs on 4 vertices the number of non-equivalent strings is 36 (see the proof of Theorem 4.14 for a list of these strings).

The following proposition solves the λ, μ problem for the graph P_n where the vertex being deleted is a pendant vertex. As mentioned in the paragraph preceding Theorem 4.9, Hald and Hoschstadt proved the reverse implication. The forward implication is stated without proof as Lemma 8 in [26]. It can also be deduced from Proposition 3.1 in [31] or Theorem 1 in [32]. We give a short alternate proof of the proposition using Theorem 4.10, Theorem 4.11, and the well know fact that $\mathrm{mr}(P_n) = n - 1$.

Proposition 4.12. *Let v be a pendant vertex of P_n. Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$ be real numbers. There is a matrix $A \in \mathcal{S}(P_n)$ such that the λ_i's are the eigenvalues of A and the μ_i's are the eigenvalues of $A(v)$ if and only if $\lambda > \mu_1 > \lambda_2 > \mu_2 > ... > \lambda_{n-1} > \mu_{n-1} > \lambda_n$.*

Proof. $\Leftarrow$ Since P_n is a tree, the reverse implication is given by Theorem 4.10.

$\Rightarrow$ Consider $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq ... \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$. Assume that there is a matrix $A \in \mathcal{S}(P_n)$ such that the λ_i are the eigenvalues of A and the μ_i are the eigenvalues of $A(v)$.

Case 1. Suppose there are two or more equality signs that occur consecutively. We note that Theorem 4.11 could be used here, but in this case it is simpler to give a direct argument. Since two or more equality signs occur consecutively, $\lambda_i = \lambda_{i+1}$ or $\mu_i = \mu_{i+1}$ for some i. If $\lambda_i = \lambda_{i+1}$, let $t = \lambda_i$. The matrix $A - tI$ is in $\mathcal{S}(P_n)$ and has rank $n - 2$. This contradicts the fact that $\mathrm{mr}(P_n) = n - 1$. If $\mu_i = \mu_{i+1}$, let $t = \mu_i$. The matrix $A(v) - tI \in \mathcal{S}(P_{n-1})$ and has rank $n - 3$, contradicting that $\mathrm{mr}(P_{n-1}) = n - 2$.

Case 2. Suppose that there is an equality sign such that the signs on either side of it are not equalities. Since $\mathrm{M}(P_n) = 1$, this contradicts Theorem 4.11.

Therefore $\lambda > \mu_1 > \lambda_2 > \mu_2 > ... > \lambda_{n-1} > \mu_{n-1} > \lambda_n$. $\square$

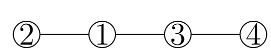
We now consider the case where the vertex being deleted from P_n is not pendant. A general solution appears to depend on n and which non-pendant vertex is being deleted. The research group, under the direction of Dr. Barrett, consisting of Anne Lazenby, Nicole Malloy, William Sexton, John Sinkovic, Robert Yang, and myself have solved the complete λ, μ problem for all n for the graph K_n and the graph S_n where the vertex being deleted in S_n is the central vertex. Thus all connected graphs with $n = 3$ have been done. Besides K_4 and S_4, there are 4 more connected graphs on 4 vertices. Some progress has been made by the research group on these graphs. I chose to focus on P_4. Because of the symmetry of P_4, deleting either non-pendant vertex is equivalent. As mentioned above, there are 36 non-equivalent strings of equalities and inequalities relating the λ's and the μ's to consider for graphs on 4 vertices. It will be seen below that Theorem 4.11 shows 34 of these strings cannot occur. Furthermore, Duarte's theorem applies to one of the two remaining cases. We consider the last case in the theorem below.

Theorem 4.13. *Let v be a non-pendant vertex of P_4. Given*

$$\lambda_1 > 0 = \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4,$$

there exists $A \in \mathcal{S}(P_4)$ such that the λ_i's are the eigenvalues of A and the μ_i's are the eigenvalues of $A(v)$.

Proof. Suppose there exists $A \in \mathcal{S}(P_4)$ such that the λ_i's are the eigenvalues of A and the μ_i's are the eigenvalues of $A(v)$. We label P_4 as shown.



Thus A is of the form $A = \begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & 0 & 0 \\ b & 0 & d_3 & c \\ 0 & 0 & c & d_4 \end{bmatrix}$ where $abc \neq 0$. By assumption, the eigenvalues of $A(1)$ are $0, 0, \mu_3$. The eigenvalues of $A(1)$ are also d_2 and the eigenvalues of $\begin{bmatrix} d_3 & c \\ c & d_4 \end{bmatrix}$. This 2×2 matrix is in $\mathcal{S}(P_2)$. An immediate consequence of Proposition 4.12 is that all matrices in $\mathcal{S}(P_2)$ have distinct eigenvalues. Thus the eigenvalues of $\begin{bmatrix} d_3 & c \\ c & d_4 \end{bmatrix}$ are $0, \mu_3$ and $d_2 = 0$.

Furthermore, using the fact that the trace of a matrix is equal to the sum of its eigenvalues, $d_3 + d_4 = \mu_3$ and $d_1 + d_3 + d_4 = \lambda_1 + \lambda_3 + \lambda_4$. Subtracting the first equation from the second equation gives $d_1 = \lambda_1 + \lambda_3 + \lambda_4 - \mu_3$. Thus

$$A = \begin{bmatrix} \lambda_1 + \lambda_3 + \lambda_4 - \mu_3 & a & b & 0 \\ a & 0 & 0 & 0 \\ b & 0 & d_3 & c \\ 0 & 0 & c & \mu_3 - d_3 \end{bmatrix}.$$

Using the fact that the determinant of a matrix is equal to the product of its eigenvalues, $\det \begin{bmatrix} d_3 & c \\ c & \mu_3 - d_3 \end{bmatrix} = 0$. Thus $c^2 = d_3(\mu_3 - d_3)$. Therefore $d_3(\mu_3 - d_3) > 0$; i.e. $0 > d_3 > \mu_3$.

Using the fact that the sum of the 2×2 principal minors of A equals $\sum_{1 \leq i < j \leq 4} \lambda_i \lambda_j$ we have

$$-a^2 + d_3(\lambda_1 + \lambda_3 + \lambda_4 - \mu_3) - b^2 + (\mu_3 - d_3)(\lambda_1 + \lambda_3 + \lambda_4 - \mu_3) + d_3(\mu_3 - d_3) - c^2 = \lambda_1\lambda_3 + \lambda_1\lambda_4 + \lambda_3\lambda_4$$

$$\text{hence } a^2 + b^2 = \mu_3(\lambda_1 + \lambda_3 + \lambda_4 - \mu_3) - \lambda_1\lambda_3 - \lambda_1\lambda_4 - \lambda_3\lambda_4$$

$$\text{so } a^2 + b^2 = (\lambda_1 - \mu_3 + \lambda_3)(\mu_3 - \lambda_4) - \lambda_1\lambda_3. \tag{4.2}$$

Since $\lambda_1 > 0 > \lambda_3 > \mu_3 > \lambda_4$, we have that $(\lambda_1 - \mu_3 + \lambda_3)(\mu_3 - \lambda_4) - \lambda_1\lambda_3 > 0$. Thus 4.2 is the equation of a circle of radius $\sqrt{(\lambda_1 - \mu_3 + \lambda_3)(\mu_3 - \lambda_4) - \lambda_1\lambda_3}$.

Similarly, the condition that results from the 3×3 principal minors of A, and using the fact $c^2 = d_3(\mu_3 - d_3)$, is

$$-d_3a^2 - a^2(\mu_3 - d_3) + d_3(\lambda_1 + \lambda_3 + \lambda_4 - \mu_3)(\mu_3 - d_3) - d_3(\mu_3 - d_3)(\lambda_1 + \lambda_3 + \lambda_4 - \mu_3) - b^2(\mu_3 - d_3) = \lambda_1\lambda_3\lambda_4$$

$$\text{thus } -\mu_3a^2 + (d_3 - \mu_3)b^2 = \lambda_1\lambda_3\lambda_4$$

$$\text{so } -\frac{\mu_3}{\lambda_1\lambda_3\lambda_4}a^2 + \frac{d_3 - \mu_3}{\lambda_1\lambda_3\lambda_4}b^2 = 1 \tag{4.3}$$

Since $\lambda_1 > 0 > \lambda_3 > \mu_3 > \lambda_4$ and $0 > d_3 > \mu_3$, we have $-\frac{\mu_3}{\lambda_1\lambda_3\lambda_4} > 0$ and $\frac{d_3-\mu_3}{\lambda_1\lambda_2\lambda_4} > 0$. Thus 4.3 is the equation of an ellipse with axes of length $2\sqrt{\frac{\lambda_1\lambda_3\lambda_4}{d_3-\mu_3}}$ and $2\sqrt{-\frac{\lambda_1\lambda_3\lambda_4}{\mu_3}}$. Since $0 > d_3 > \mu_3$ we have that $2\sqrt{\frac{\lambda_1\lambda_3\lambda_4}{d_3-\mu_3}} > 2\sqrt{-\frac{\lambda_1\lambda_3\lambda_4}{\mu_3}}$.

Because a and b satisfy 4.2 and 4.3, there is a solution of these two equations for a and b if and only the circle and the ellipse intersect; i.e. if and only if

$$\sqrt{\frac{\lambda_1\lambda_3\lambda_4}{d_3 - \mu_3}} \geq \sqrt{(\lambda_1 - \mu_3 + \lambda_3)(\mu_3 - \lambda_4) - \lambda_1\lambda_3} \geq \sqrt{-\frac{\lambda_1\lambda_3\lambda_4}{\mu_3}},$$

or equivalently, if and only if

$$\frac{\lambda_1\lambda_3\lambda_4}{d_3 - \mu_3} \geq (\lambda_1 - \mu_3 + \lambda_3)(\mu_3 - \lambda_4) - \lambda_1\lambda_3 \geq \frac{\lambda_1\lambda_3\lambda_4}{-\mu_3}.$$

Since $\lim_{d_3 \to \mu_3} \frac{\lambda_1\lambda_3\lambda_4}{d_3-\mu_3} = \infty$, there exists a number d_3 with $0 > d_3 > \mu_3$ such that the first inequality holds. Thus there is a solution for a and b if and only if

$$(\lambda_1 - \mu_3 + \lambda_3)(\mu_3 - \lambda_4) - \lambda_1\lambda_3 \geq \frac{\lambda_1\lambda_3\lambda_4}{-\mu_3},$$

or equivalently

$$-\mu_3^2\lambda_1 + \lambda_1\lambda_4\mu_3 + \mu_3^3 - \mu_3^2\lambda_4 - \mu_3^2\lambda_3 + \mu_3\lambda_3\lambda_4 + \lambda_1\lambda_3\mu_3 - \lambda_1\lambda_3\lambda_4 \geq 0.$$

This holds, if and only if

$$(\mu_3 - \lambda_4)(\lambda_3 - \mu_3)(\lambda_1 - \mu_3) \geq 0.$$

Since $\lambda_1 > 0 > \lambda_3 > \mu_3 > \lambda_4$, the above equation is always satisfied, and moreover the inequality is strict.

Therefore there is a solution to 4.2 and 4.3 simultaneously and solving for a and b gives the expressions up to a sign listed below. Thus given $\lambda_1 > 0 = \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$, let

$$\begin{aligned}
d_1 &= \lambda_1 + \lambda_3 + \lambda_4 - \mu_3 \\
d_2 &= 0 \\
d_3 &= x \\
d_4 &= \mu_3 - x \\
c &= \sqrt{x(\mu_3 - x)} \\
b &= \sqrt{\frac{\mu_3(\lambda_1 - \mu_3 + \lambda_3)(\mu_3 - \lambda_4) - \lambda_1\lambda_3\mu_3 + \lambda_1\lambda_3\lambda_4}{x}} \\
a &= \sqrt{(1 - \frac{\mu_3}{x})(\lambda_1 - \mu_3 + \lambda_3)(\mu_3 - \lambda_4) - \lambda_1\lambda_3 + \frac{\lambda_1\lambda_3\mu_3 - \lambda_1\lambda_3\lambda_4}{x}}.
\end{aligned}$$

The matrix $\begin{bmatrix} d_1 & a & b & 0 \\ a & d_2 & 0 & 0 \\ b & 0 & d_3 & c \\ 0 & 0 & c & d_4 \end{bmatrix} \in \mathcal{S}(P_4)$ has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $A(1)$ has eigenvalues μ_1, μ_2, μ_3.

The computer program Maple was used to verify that A has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ and $A(1)$ has eigenvalues μ_1, μ_2, μ_3. An alternative way to see this is the sum of the $k \times k$ principal minors of A equals $\sum_{1\leq i_1<..<i_k\leq k} \lambda_{i_1}...\lambda_{i_k}$, $k = 1, 2, 3, 4$. Thus A has eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$. Similary, the sum of the $k \times k$ principal minors of $A(1)$ equals $\sum_{1\leq i_1<...<i_k\leq k} \mu_{i_1}...\mu_{i_k}$, $k = 1, 2, 3$. Thus $A(1)$ has eigenvalues μ_1, μ_2, μ_3. $\square$

Note that since adding cI (a multiple of the identity matrix) to any matrix shifts its eigenvalues by c, it follows that the above result holds in the general case, $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$ as well. Also the result holds for the symmetric case, $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$ and the proof is similar to that of Theorem 4.13. The general case and symmetric case are stated as part of Theorem 4.14.

The following theorem gives the complete solution to the λ, μ problem when deleting a non-pendant vertex of P_4.

Theorem 4.14. *Let $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \mu_3 \geq \lambda_4$. There exists a matrix $A \in \mathcal{S}(P_4)$ with eigenvalues $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that $A(v)$ has eigenvalues μ_1, μ_2, μ_3 where v is a non-pendant vertex of P_4 if and only if one of the following sets of equalities/inequalities holds.*

- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.
- $\lambda_1 > \mu_1 = \lambda_2 = \mu_2 > \lambda_3 > \mu_3 > \lambda_4$.
- $\lambda_1 > \mu_1 > \lambda_2 > \mu_2 = \lambda_3 = \mu_3 > \lambda_4$.

Proof. The 36 different, non-equivalent strings of equalities/inequalities relating the λ's and the μ's are:

1.	$=$	$=$	$=$	$=$	$=$	$=$
2.	$>$	$=$	$=$	$=$	$=$	$=$
3.	$=$	$>$	$=$	$=$	$=$	$=$
4.	$=$	$=$	$>$	$=$	$=$	$=$
5.	$>$	$>$	$=$	$=$	$=$	$=$
6.	$>$	$=$	$>$	$=$	$=$	$=$
7.	$>$	$=$	$=$	$>$	$=$	$=$
8.	$>$	$=$	$=$	$=$	$>$	$=$
9.	$>$	$=$	$=$	$=$	$=$	$>$
10.	$=$	$>$	$>$	$=$	$=$	$=$
11.	$=$	$>$	$=$	$>$	$=$	$=$
12.	$=$	$>$	$=$	$=$	$>$	$=$
13.	$=$	$=$	$>$	$>$	$=$	$=$
14.	$>$	$>$	$>$	$=$	$=$	$=$
15.	$>$	$>$	$=$	$>$	$=$	$=$
16.	$>$	$>$	$=$	$=$	$>$	$=$
17.	$>$	$>$	$=$	$=$	$=$	$>$
18.	$>$	$=$	$>$	$>$	$=$	$=$
19.	$>$	$=$	$>$	$=$	$>$	$=$
20.	$>$	$=$	$>$	$=$	$=$	$>$
21.	$>$	$=$	$=$	$>$	$>$	$=$
22.	$=$	$>$	$>$	$>$	$=$	$=$
23.	$=$	$>$	$>$	$=$	$>$	$=$
24.	$>$	$>$	$>$	$>$	$=$	$=$
25.	$>$	$>$	$>$	$=$	$>$	$=$
26.	$>$	$>$	$=$	$>$	$>$	$=$
27.	$>$	$=$	$>$	$>$	$>$	$=$
28.	$=$	$>$	$>$	$>$	$>$	$=$
29.	$>$	$>$	$>$	$=$	$=$	$>$
30.	$>$	$>$	$=$	$>$	$=$	$>$
31.	$>$	$=$	$>$	$>$	$=$	$>$
32.	$>$	$>$	$=$	$=$	$>$	$>$
33.	$>$	$>$	$>$	$>$	$>$	$=$
34.	$>$	$>$	$>$	$>$	$=$	$>$
35.	$>$	$>$	$>$	$=$	$>$	$>$
36.	$>$	$>$	$>$	$>$	$>$	$>$

Since $\mathrm{mr}(P_4) = 4 - 1 = 3$, $\mathrm{M}(P_n) = 1$. By Theorem 4.11, it is not possible for a matrix in $\mathcal{S}(P_4)$ to satisfy the conditions of $1-28, 30-35$. By Theorem 4.10, given the condition of 36, there exists a matrix in $\mathcal{S}(P_4)$ satisfying these conditions. Lastly, it follows from Theorem 4.13, that given either of the two equivalent cases corresponding to condition 29, there exists a matrix in $\mathcal{S}(P_4)$ satisfying this condition. $\square$

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