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Homomorphisms into the Fundamental Group of One-Dimensional and Planar Peano Continua

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HOMOMORPHISMS INTO THE FUNDAMENTAL GROUP OF

ONE-DIMENSIONAL AND PLANAR PEANO CONTINUA

by

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ABSTRACT

HOMOMORPHISMS INTO THE FUNDAMENTAL GROUP OF ONE-DIMENSIONAL AND PLANAR PEANO CONTINUA

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Let X be a planar or one-dimensional Peano continuum. Let $\mathbf{E}$ be a Hawaiian Earring with fundamental group $\mathbb{H}$. We show that every homomorphism $\phi : \mathbb{H} \to \pi_1(X, x_0)$ has the property that there exists a continuous function $f : \mathbf{E} \to X$ and a path $T : I \to X$ such that $\phi = \widehat{T} f_*$.

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1 Introduction and Acknowledgements

A Hawaiian Earring group, which we will denote by $\mathbb{H}$, is the fundamental group of the one-point compactification of a sequences of disjoint arcs, $\{a_i\}$. This space can be realized in the plane as the union of circles centered at $(0, \frac{1}{n})$ with radius $\frac{1}{n}$. We will use $\mathbf{E}$ to denote this subspace of the plane and a_n to denote the circle centered at $(0, \frac{1}{n})$ with radius $\frac{1}{n}$.

Cannon and Conner have shown that $\mathbb{H}$ is generated in the sense of infinite products by a countable sequence of loops corresponding to the disjoint arcs, where an infinite product is legal if each loop is transversed only finitely many times. (See [1],[2].) When there is no chance of confusion, we will refer to this infinite generating set for the fundamental group of $\mathbf{E}$ as $\{a_i\}$, i.e. a_i represents the loop which transverses counterclockwise one time the circle of radius $\frac{1}{i}$ centered at $(0, \frac{1}{i})$.

A Peano continuum is a compact, connected, locally path connected, metric space. We will prove the following two theorems.

Theorem 2.17 *Let $\phi : \mathbb{H} \to \pi_1(X, x_0)$ be a homomorphism from the Hawaiian Earring group into the fundamental group of a one-dimensional Peano continuum X. Then there exists a continuous function $f : (\boldsymbol{E}, 0) \to (X, x)$ and a path $T : (I, 0, 1) \to (X, x_0, x)$ with the property that $f_* = \widehat{T}\phi$.*

Theorem 3.9 *Let $\phi : \mathbb{H} \to \pi_1(X, x_0)$ a homomorphism into the fundamental group of a planar Peano continuum X. Then there exists a continuous function $f : (\boldsymbol{E}, 0) \to (X, x)$ and a path $\alpha : (I, 0, 1) \to (X, x_0, x)$, with the property that $f_* = \widehat{\alpha}\phi$.*

Greg Conner and Erin Summers in Summer's Masters thesis showed that homomorphisms between Hawaiian Earring groups are conjugate to homomorphisms induced by continuous maps. (See [5].) They did this by using the combinatorial word structure of the Hawaiian Earring. Katsuya Eda has shown this is still true in the case of homomorphisms for a Hawaiian Earring group into the fundamental group of a one-dimensional space in [7]. Our proof of the one-dimensional case is similarly to that of Eda.

1.1 Definitions

A path $f : I \to X$ into a one-dimensional space is *reduced* if ever nondegenerate closed subpath is essential. By definition, constant paths are *reduced*.

We will use f_r to denote the path where every nullhomotopic subpath of f is replaced by a constant path. Then f_r is a reduced representative of $[f]$. James Cannon and Greg Conner proved the existence and uniqueness (up to reparameterization) of reduced representatives of path class for one-dimensional spaces in [2].

Definition 1.1. Let X be one-dimensional space. Let $g : I \to X$ be a reduced representative for the path class $[g]$. Then we say that $a : I \to X$ is a *head* for g if there exists $b : I \to X$ such that $g = a * b$, up to reparameterization, where $a * b$ is a reduced path. We write $a \xrightarrow{h} g$. Additionally, we say that $b : I \to X$ is a *tail* for g if there exist $c : I \to X$ such that $g = c * \overline{b}$, up to reparameterization, where $c * \overline{b}$ is a reduced path and $\overline{b}$ is the path b traversed backwards. We write $b \xrightarrow{t} g$.

Since g is a reduced path; the paths a, b, and c are necessarily reduced paths.

Definition 1.2. We say that $t : I \to X$ is a *head-tail* for a reduced path $g : I \to X$ if t is a head and a tail for g and is written $t \xrightarrow{h-t} g$. We say that t is an *almost* *head-tail* for a set W of reduced paths if t is a head-tail for all but finitely many elements of W.

Definition 1.3. A path T is the *head limit* of an increasing sequence of heads $\{t_i\}$ (i.e. $t_i \xrightarrow{h} t_{i+1}$) if $t_i \xrightarrow{h} T$ for all i and if whenever $t_i \xrightarrow{h} S$ for all i, then $T \xrightarrow{h} S$. We will say that T is the *head-tail limit* of an increasing sequence of almost head-tails $\{t_i\}$ for a set W, if T is the head limit of t_i.

If both T_1 and T_2 are head limits of an single increasing sequence of heads, then $T_1 \xrightarrow{h} T_2$ and $T_2 \xrightarrow{h} T_1$ and hence $T_1 = T_2$ up to reparametrization. Thus we are justified in saying *the* head limit and *the* head-tail limit.

For a path $f : I \to X$, we will use $\widehat{f}$ to represent the standard change of base point isomorphism, $\widehat{f}([g]) = [\overline{f}] * [g] * [f]$. To prove Theorem 2.17, we will show that for a homomorphism $\phi : \mathbb{H} \to \pi_1(X, x_0)$, where X is one dimensional Peano continuum, there exists a path T (possibly trivial) such $\{\widehat{T}\phi(a_i)_r\}$ has no almost head-tail. We then use show that $\widehat{T}\phi$ is induced by a continuous function.

For Theorem 3.9, we will use an upper semicontinuous decomposition of the planar Peano continuum to get a continuous map into a one-dimensional Peano continuum which is injective on fundamental groups. If π_k is the decomposition map, we show that we can lift the path T such that $\widehat{T}\pi_{k*}\phi$ is induced be a continuous map. Then for α the lift of T, we show that $\widehat{\alpha}\phi$ is induced by a continuous function.

2 One-Dimensional Peano Continuum

For this section, we will fix X a one-dimensional Peano continuum and $\phi : \mathbb{H} \to \pi_1(X, x_0)$ a homomorphism. If $\phi(a_i)$ is eventually nullhomotopic, then Theorem

2.17 follows trivially by letting T be the constant path and sending the i-th circle of $\mathbf{E}$ to any representative of $\phi(a_i)$. Thus with no loss of generally, we will assume that $\phi(a_i)$ is not eventually trivial.

2.1 Head-tail limit

We will begin, by showing that given an increasing sequence $\{t_i\}$ of almost head-tails for $\{\phi(a_i)_r\}$, there exists a head-tail limit T which is a path in X. To do this we will define the weight of a function with respect to two sets with disjoint closures.

For a path $f : I \to X$ and U, V disjoint open subsets of X, let $r_f : f^{-1}(U \cup V) \to \{-1, 1\}$ by $r_f(b) = 1$ if $f(b) \in U$ and $r_f(b) = -1$ if $f(b) \in V$. Let $\overline{w}_U^V(f) = \sup\big(\sum_i -r_f(b_i) \cdot r_f(b_{i+1})\big)$ taken over all increasing countable subsets of $f^{-1}(U \cup V)$. For any collection consisting of 0 or 1 point, we will consider the sum to be 0.

If the image of two consecutive points in our countable subset of $f^{-1}(U \cup V)$ are contained in the same open set, then the sum would increase by deleting one. Thus the supremum is obtained by choosing an increasing sequence of points from $f^{-1}(U \cup V)$ whose image alternates between U and V. Therefore $\overline{w}$ counts the number of times that the image of f alternates between U and V. If f is continuous and U, V have disjoint closures, then its image is compact and can only alternate between sets with disjoint closures finitely many times. So the supremum is actually realized for some finite set of points. If $U' \subset U$ and $V' \subset V$, then $\overline{w}_{U'}^{V'}(f) \leq \overline{w}_U^V(f)$.

Definition 2.1. The *weight* of f with respect to subsets A and B of X with disjoint closures is $w_B^A(f) = \inf \overline{w}_U^V(f)$ taken over all possible separations U and V of $\overline{A}$ and $\overline{B}$. If $[f]$ is a homotopy equivalence class of functions, then $w_B^A([f]) = \inf_{f \sim f'}\{w_B^A(f')\}$.

If f is the reduced representative for $[g]$, $f = g\big|_{I-(\cup_i J_i)}$, up to reparametrization, where J_i are open subintervals of I such that $g\big|_{\overline{J_i}}$ is nullhomotopic rel endpoints.[2] Hence, $\overline{w}_V^U([f]) = \overline{w}_V^U(f_r)$.

The set $\{\overline{w}_V^U(f) \mid U, V \text{are a separation of } \overline{A}, \overline{B}\}$ is a subset of the natural numbers and hence has a minimum. Thus we may chose an open separation U, V such that $w_B^A(f) = \overline{w}_U^V(f)$. For continuous f, $f^{-1}(U \cup V)$ is a finite collection of disjoint open sets, I_i, in I with a natural ordering ($I_i \leq I_j$ if $x \leq y$ for all $x \in I_i$ and $y \in I_J$) such that $f(I_i) \subset U$ or $f(I_i) \subset V$. If for some i, $f(I_i)$ did not intersect the corresponding $\overline{A}$ or $\overline{B}$, then there would exist an open set containing the $\overline{A}$ or $\overline{B}$ which did not intersect $f(I_i)$ and thus alternate fewer times. Therefore, $f(I_i)$ must intersect $\overline{A}$ or $\overline{B}$. So points which realize the weight can be chosen in the closures of A and B. Thus there exists a finite increasing set of points $\{b_i\}$ which can be chosen to have image in the closures of A and B such that $w_B^A(f) = \overline{w}_U^V(f) = \sum\limits_i -r_f(b_i) \cdot r_f(b_{i+1})$. We will sometimes write $w_B^A(f) = \sum\limits_i -r_f(b_i) \cdot r_f(b_{i+1})$. This implicitly implies a choice of U and V to define r_f. However; if the points are chosen to have image in the closure of A and B, $r_f(b_i)$ is the same for every choice of U and V. Thus we will ignore this choice at times.

This weight function is similar to Cannon and Conner's oscillation function in [4]; except, this function is discrete and A and B are allowed to be any two sets with disjoint closures. This method of defining the weight, using open separations, allows us to maintain the weight of a function under nerve approximations. Lemma 2.2 is the precise statement of how weight is preserved.

We will now fix a sequence $\{\mathcal{O}_i\}$ of finite order one covers such that $\text{mesh}(\mathcal{O}_i) < \frac{1}{i}$ and $\mathcal{O}_{i+1}$ refines $\mathcal{O}_i$. It is well known that $X = \varprojlim \mathcal{N}(\mathcal{O}_i)$. We will give an explicit partition of unity for each $\mathcal{O}_i$. Define $\theta_j^i : X \to [0,1]$ by $\theta_j^i(x) = \dfrac{d(x, O_j^c)}{\sum\limits_{O_k \in \mathcal{O}_i} d(x, O_k^c)}$.

Let ρ_i be the map from X to $\mathcal{N}(\mathcal{O}_i)$ obtained by using $\{\theta^i_j(x)\}$ as barycentric coordinates for $\rho_i(x)$ (see [9], p.119). Then a sequence of points $(\rho_i(x_n)) \subset \mathcal{N}(\mathcal{O}_i)$ converges if and only if $(\theta^i_j(x_n))$ converges for every j.

Then for any two sets A, B in X with disjoint closures, their images in $\mathcal{N}(\mathcal{O}_i)$ under ρ_i eventually have disjoint closures by Lemma A1 and the compactness of X. The weight of a homotopy class and the weight of its image under $\rho_{i*} : \pi_1(X, x_0) \to \pi_1(\mathcal{N}(\mathcal{O}_i), \rho_i(x_0))$ should eventually be the same. Lemma 2.1 is the first step in showing the weight of a homotopy class is maintained.

Lemma 2.1. *Let A, B be two subsets of X with disjoint closures. Let A_i and B_i be the image of A and B in $\mathcal{N}(\mathcal{O}_i)$, and $f : [0,1] \to X$ continuous. Then for i sufficiently large such that $\overline{A_i}$ and $\overline{B_i}$ are disjoint, $w^{A_i}_{B_i}(\rho_i f) = w^{\rho_i^{-1}(\overline{A_i})}_{\rho_i^{-1}(\overline{B_i})}(f)$.*

Proof. Let $A_s = \rho_i^{-1}(\overline{A_i})$ and $B_s = \rho_i^{-1}(\overline{B_i})$

If $f(b) \in A_s$ (or B_s), then $\rho_i f(b) \in \overline{A_i}$ (or $\overline{B_i}$). Hence for some $\{b_i\}_i \subset A_s \cup B_s$, $w^{A_s}_{B_s}(f) = \sum_i -r_f(b_i) \cdot r_f(b_{i+1}) = \sum_i -r_{\rho_i f}(b_i) \cdot r_{\rho_i f}(b_{i+1}) \leq w^{A_i}_{B_i}(\rho_i f)$.

If $\rho_i f(b) \in \overline{A_i}$ (or $\overline{B_i}$), then $f(b) \in A_s$ (or B_s). Hence for some $\{b_i\}_i \subset \overline{A_i} \cup \overline{B_i}$, $w^{A_i}_{B_i}(\rho_i f) = \sum_i -r_{\rho_i f}(b_i) \cdot r_{\rho_i f}(b_{i+1}) = \sum_i -r_f(b_i) \cdot r_f(b_{i+1}) \leq w^{A_s}_{B_s}(f)$.

□

Lemma 2.2. *Let A, B be two subsets of X with disjoint closures. Let A_i and B_i be the image of A and B in $\mathcal{N}(\mathcal{O}_i)$, and $f : [0,1] \to X$ continuous. Then there exists a k such that, for all $i > k$, $w^A_B(f) = w^{A_i}_{B_i}(\rho_i f)$. Even more, the same points in the domain which realize $w^A_B(f)$ will also realize $w^{A_i}_{B_i}(\rho_i f)$.*

Proof. There exist disjoint open sets U, V such that $w^A_B(f) = \overline{w}^U_V(f)$. Since X is compact, $d(\overline{A}, (X - U)), d(\overline{B}, (X - V)) > 0$. Let $\epsilon = \min\{d(\overline{A}, (X - U)), d(\overline{B}, (X -$

$V))\}$. For $i > 1/\epsilon$, the mesh$(\mathcal{O}_i) < \epsilon$. Then by Lemma A1, $\rho_i^{-1}(\overline{A_i}) \subset B_{1/i}(A) \subset U$ and $\rho_i^{-1}(\overline{B_i}) \subset B_{1/i}(B) \subset V$.

By Lemma 2.1, there exists a k such that for $i > k$, $w_{B_i}^{A_i}(\rho_i f) = w_{\rho_i^{-1}(\overline{B_i})}^{\rho_i^{-1}(\overline{A_i})}(f)$.

For $i > \max\{1/\epsilon, k\}$; $w_B^A(f) = \overline{w}_V^U(f) \geq w_{\rho_i^{-1}(\overline{B_i})}^{\rho_i^{-1}(\overline{A_i})}(f) = w_{B_i}^{A_i}(\rho_i f)$, since U and V are open sets containing $\rho_i^{-1}(\overline{A_i})$ and $\rho_i^{-1}(\overline{B_i})$ respectively. The other inequality, $w_B^A(f) \leq w_{B_i}^{A_i}(\rho_i f)$, follows since if $f(a) \in \overline{A}$ (or $\overline{B}$) then $\rho_i f(a) \in \overline{A_i}$ (or $\overline{B_i}$).

$\square$

We will now extend Lemma 2.2 to homotopy classes of paths, i.e. $w_B^A([f])$ is eventually equal to $w_{B_i}^{A_i}(\rho_{i*}([f]))$. We will use a well know fact that the fundamental group of a one-dimensional space embeds in the inverse limit of free groups. Curtis and Fort showed this is true for the Menger curve in [6]. Cannon and Conner later showed that this is still true in the case of compact one-dimensional metric spaces in [2].

Lemma 2.3. *Let $f : [0,1] \to X$ be a reduced path. If $(c,d) \subset I$ such that $f\big|_{(c,d)}$ is not constant, then there exists a k such that for all $i > k$, $\rho_i f\big|_{(c,d)}$ is not contained in a nullhomotopic subpath of $\rho_i f$; consequently, the image of $(\rho_i f)_r$ contains some point of $\rho_i f\big|_{(c,d)}$.*

Proof. Suppose there exists a subsequence $(i_n) \subset \mathbb{N}$ such that $\rho_{i_n} f\big|_{(c,d)}$ is contained in a nullhomotpic subpath of $\rho_{i_n} f$ for all n. Then there exists $c_n \leq c < d \leq d_n$ such that $\rho_{i_n} f\big|_{(c_n,d_n)}$ is nullhomotopic closed loop. By passing to subsequences, we may assume that $c_n \to c'$ and $d_n \to d'$. Then $d(f(c'), f(d')) \leq d(f(c'), f(c_n)) + d(f(c_n), f(d_n)) + d(f(d_n), f(d'))$. The first and last term go to zero since f is continuous and the middle term goes to zero by Lemma A1. Thus $f\big|_{[c',d']}$ is a closed subpath of f. We will now show that this loop is nullhomotopic.

By standard dimension theory, we may assume that X is embedded in $\mathbb{R}^3$ as the intersection of handlebodies, $H_1 \supset H_2 \supset \cdots \supset \cap H_i = X$. Then there exists embeddings $e_n : \mathcal{N}(\mathcal{O}_n) \to H_{k_n}$ such that $e_n \rho_n$ converge uniformly to the identity function on X. [8]

Then Cannon and Conner showed that if a path is nullhomotopic in H_k for every k, then it is nullhomotopic in X (Theorem 5.11 in [2]). Thus it is sufficient to show that $f\big|_{[c',d']}$ is nullhomotopic in H_k for any k.

We may choose, i_n sufficiently large such that the straight line homotopy from $e_{i_n}\rho_{i_n} f\big|_{[c_n,d_n]}$ to $f\big|_{[c',d']}$ is contained in H_k. Hence, $f\big|_{[c',d']}$ is freely homotopic to a nullhomotopic loop in H_k.

Then since H_k deformation retracts to a one-dimensional skeleton, $f\big|_{[c',d']}$ is nullhomotopic in H_k. (If a loop is freely nullhomotopic in a one-dimensional space, it is nullhomotopic. See [2])

$\square$

Proposition 2.4. *Let A, B be two subsets of X with disjoint closures, A_i and B_i be the image of A and B in $\mathcal{N}(\mathcal{O}_i)$, and $f : [0,1] \to X$ continuous. Then there exists a k such $w_B^A([f]) = w_{B_i}^{A_i}(\rho_{i*}([f]))$ for all $i > k$.*

Proof. By Lemma 2.2 there exists a k' such that $w_B^A([f]) = w_B^A(f_r) = w_{B_i}^{A_i}(\rho_i f_r) \geq w_{B_i}^{A_i}(\rho_{i*}([f]))$ for all $i > k'$.

There exist points $\{b_0, \cdots, b_n\}$ such that $f(b_i) \in \overline{A} \cup \overline{B}$ and $w_B^A(f_r) = \sum_{j=0}^{n} -r_f(b_j) \cdot r_f(b_{j+1})$. Then for $i > k'$, $w_{B_i}^{A_i}(\rho_i f_r) = \sum_{j=0}^{n-1} -r_{\rho_i f_r}(b_j) \cdot r_{\rho_i f_r}(b_{j+1})$.

Then we may choose U', V', U_i, and V_i such that $w_B^A(f_r) = \overline{w}_{V'}^{U'}(f_r)$ and $w_{B_i}^{A_i}(\rho_i f_r)) = \overline{w}_{V_i}^{U_i}(\rho_i f_r)$.

$U = U' \cap \rho_i^{-1}(U_i)$ and $V = V' \cap \rho_i^{-1}(V_i)$ are open sets which contain $\overline{A}$ and $\overline{B}$

and are contained inside of U' and V' respectively. Hence $w_B^A(f_r) = \overline{w}_V^U(f_r)$

Let $f_r^{-1}(U \cup V) = \bigcup I_m$ where I_m are disjoint open subintervals of $[0,1]$ with there natural ordering. Then there exists I_{m_j} such that $b_j \in I_{m_j}$. Let c_{m_j} and d_{m_j} be the right and left end points of the open interval I_{m_j}.

Then by a repeated application of Lemma 2.3 there exists a k such that for $i > k$ the image of $(\rho_i f)_r$ contains some point of $\rho_i f\big|_{(c_{m_j}, d_{m_j})}$. Thus $w_B^A([f]) = w_B^A(f_r) \leq w_{B_i}^{A_i}(\rho_{i*}([f]))$ for all $i > \max\{k, k'\}$. $\square$

Lemma 2.5. *If g is an essential closed curve, there exist sets O', O'' with disjoint closures such that, for all r, $w_{O''}^{O'}([g]^r) \geq r$.*

Proof. With no loss of generality, we may assume g is a reduced path since the weight of the reduced path is less than the weight of all paths in its homotopy class.

Since the set of head-tails for g has a natural ordering which is bounded there exists a maximal head-tail, t for g where $g = t * f * \overline{t}$ such that $f * f$ is reduced. Hence $w_B^A([g]^r) \geq w_B^A([f]^r) = r(w_B^A([f]))$ for all A and B. Since g is essential, f is essential. Hence there exists O' and O'' with disjoint closures such that $w_{O''}^{O'}([f]) \neq 0$. Then $w_{O''}^{O'}([g]^r) \geq r(w_{O''}^{O'}([f])) \geq r$, for any r. $\square$

We will call f the *core* of g.

Lemma 2.6. *If t is a head-tail for an essential reduced loop g based at x_0 and h is any loop also based at x_0, then there exists an r such that $[h * g^r]$ still has t as a tail.*

Proof. Let f be the core of g, with s the maximal head-tail of g so that $g = s * f * \overline{s}$. Since g is essential, f is essential. Then there exists O' and O'' such that $w_{O''}^{O'}([f]^{r-1}) > w_{O''}^{O'}(h * s)$. Thus $h * s$ cannot contain an inverse for $[f]^{r-1}$ and $t \xrightarrow{t} h * g^r$.

$\square$

Lemma 2.7. *If t is a head-tail for an essential reduced loop g based at x_0 and h is any loop also based at x_0, then there exists an r such that $[g^r * h]$ still has h as a head.*

The proof is the same as for Lemma 2.6.

We will denote the continuous map which retracts $\mathbf{E}$ onto the outermost i circles of $\mathbf{E}$ by P_i. We will frequently use the following theorem of Cannon and Conner.

Theorem 2.8. *For $\psi : \mathbb{H} \to \mathbb{F}$ a homomorphism where $\mathbb{F}$ is a free group, there exists an i such that ψ factors through P_{i*}.*

The main idea for this theorem is contained in Theorem 4.4 of [2] and a proof can be found in [5].

For $\{t_i\}$ an increasing sequence of almost head-tails, we will write $t_i - t_{i-1}$ for the subpath of t_i such that $t_i = t_{i-1} * (t_i - t_{i-1})$. We now have sufficient tools to be able to show that the existence of a head-tail limit of an increasing sequence of almost head-tails.

Proposition 2.9. *Let $\{t_i\}$ be an increasing sequence of almost head-tails for the set $\{\phi(a_i)_r\}$. Then $\{t_i\}$ has a head limit T, which is a path.*

Proof. By passing to a subsequence of $\{a_i\}$, we may assume that $t_i \overset{h-t}{\to} \phi(a_i)_r$. Let x_i be the terminal point of t_i. Since X is compact, we may choose a subsequence (x_{i_k}) which converges to $x \in X$.

If $\{t_{i_k} - t_{i_{k-1}}\}$ is eventually contained in every open neighborhood of x, then the ray $T' = t_{k_1} * (t_{k_2} - t_{k_1}) * \cdots$ has a single limit point and can be completed to a continuous path T. For any $i \in \mathbb{N}$ there exists $k_s > i$, therefore $t_i \xrightarrow{h} t_{k_s} \xrightarrow{h} T$ for all i. Then $t_i \xrightarrow{h} T$ for all i and T is independent of the subsequence chosen.

Suppose $t_i \xrightarrow{h} S$ for all i. Then $\{x_i\} \subset S$, hence $x \in S$. Thus $T \xrightarrow{h} S$ and T is the head limit for $\{t_i\}$.

If there exists an $\epsilon > 0$ such that $\{t_i - t_{i-1}\}$ is not eventually contained in $B_\epsilon(x)$. By passing to a subsequence of $\{t_i\}$, we may assume that, for all i, $t_i - t_{i-1}$ is not contained in $B_\epsilon(x)$. Let $A = B_{\epsilon/2}(x)$ and $B = (B_\epsilon(x))^c$. There exists an N such that for all $i \geq N$, $x_i \in B_{\epsilon/2}(x)$.

Let $i_1 = N$ and $n_1 = 1$. By Proposition 2.4, we can choose an s_1 such that, for $l \geq s_1$, $w_B^A(\phi(a_{i_1})) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{i_1}))$. By Theorem 2.8, we choose r_1 such that, for all $l \geq r_1$, $\rho_{s_1*}\phi = \rho_{s_1*}\phi P_{l*}$.

Then by induction, suppose that i_{k-1}, s_{k-1}, r_{k-1}, and n_{k-1} have been chosen.

Choose $i_k > \max\{r_{k-1}, i_{k-1}\}$. Hence i_k is a strictly increasing sequence and a_m is in the kernel of $\rho_{s_{k-1}*}\phi$ for all $m \geq i_k$. By Lemma 2.6, we may choose n_k such that $t_{i_k} \xrightarrow{t} \phi(a_{i_1} \cdots a_{i_k}^{n_k})_r$. Then $w_B^A(\phi(a_{i_1} \cdots a_{i_k}^{n_k})) \geq i_k - N$ for all i_k, since $w_B^A(t_{i_k}) \geq i_k - N$ for $i_k \geq N$. There exists an $s_k > k$ such that, for $l \geq s_k$, $w_B^A(\phi(a_{i_1} \cdots a_{i_k}^{n_k})) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{i_1} \cdots a_{i_k}^{n_k}))$. Choose r_k such that $\rho_{s_k*}\phi = \rho_{s_k*}\phi P_{l*}$ for all $l \geq r_k$. Let $a = a_{i_1} a_{i_2}^{n_2} \cdots$. Then choose s such that, for $l \geq s$, $w_B^A(\phi(a)) = w_{B_l}^{A_l}(\rho_{l*}\phi(a))$.

For $k \in \mathbb{N}$ such that $s_k \geq s$,

$$
\begin{aligned}
i_k - N &\leq w_B^A(\phi(a_{i_1} \cdots a_{i_k}^{n_k})) \\
&= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k *}\phi(a_{i_1} \cdots a_{i_k}^{n_k})) \\
&= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k *}\phi(a)) \\
&= w_B^A(\phi(a)),
\end{aligned}
$$

which is a contradiction since $\phi(a)$ must have a finite weight and i_k diverges.

□

Let $S = \{t_k |\ t_k$ is an almost head-tail for $\{\phi(a_i)_r\}\}$. We will use the total oscillation function $\mathcal{T}$ defined by Cannon, Conner, and Zastrow in [4] to show that S has a countable cofinal sequence.

Theorem 2.10. *There exists a maximal head-tail limit T for the set $\{\phi(a_i)_r\}$ such that $\{(\widehat{T}\phi(a_i))_r\}$ has no non-constant almost head-tail.*

Proof. Let $S = \{t_k |\ t_k$ is an almost head-tail for $\{\phi(a_i)_r\}\}$. S is nonempty since the constant path is a head-tail for all $\phi(a_i)$. Given $t_i, t_j \in S$ there exist $\phi(a_k)$ such that $t_i \xrightarrow{h-t} \phi(a_k)$ and $t_j \xrightarrow{h-t} \phi(a_k)$. Therefore, $t_i \xrightarrow{h} t_j$ or $t_j \xrightarrow{h} t_i$. So S is totally ordered by set inclusion.

The set $\{\mathcal{T}(t_i) \mid t_i \in S\}$ is a subset of the real numbers bounded by one where $\mathcal{T}$ is the total oscillation function. Since S is totally ordered, $\mathcal{T}(t_i) \leq \mathcal{T}(t_j)$ if $t_i \xrightarrow{h} t_j$ with equality if and only if $t_i = t_j$ (see [4], Theorem 2.3). Let $C = \sup\{\mathcal{T}(t_i) \mid t_i \in S\}$. Then there exists a sequence $\{t_i\} \subset S$ such that $\mathcal{T}(t_i) > C - \frac{1}{i}$. Let T be the head limit of $\{t_i\}$. For any $t \in S$ there exists a t_i such that $\mathcal{T}(t) \leq \mathcal{T}(t_i)$. Thus $t \xrightarrow{h} t_i$. Hence $t \xrightarrow{h} T$ for any k and T is a maximal head-tail limit.

Suppose that $\{(\overline{T} * \phi(a_i) * T)_r\}$ had an almost head-tail z. Let y path $T * z$. Then y is an almost head-tail for $\{\phi(a_i)_r\}$. Hence z must be degenerate since T was maximal. □

2.2 Induced by a continuous function

Now, we will show that $\widehat{T}\phi$ is induced by a continuous map.

Definition 2.2. Let $f : I \to X$ be continuous. Then the A-*head* of f is the maximal head of f contained in the closure of A. Similarly the A-*tail* of f is the maximal tail of f contained in the closure of A.

Lemma 2.11. *Let $f, g : I \to X$ be reduced paths such that $f(1) = g(0) \in A$. If the A-head of g is not an inverse for the A-tail of f, then $w_B^A([f * g]) = w_B^A(f) + w_B^A(g)$.*

Proof. Let $f = a * t$ and $g = h * b$ with t the A-tail of g and h the A-head of g. Then the lemma follows trivially from the fact that the reduced loop for $f * g$ still contains the paths a and b and also contains some point of the path t. □

Lemma 2.12. *Let $f_i : I \to X$ be a reduced path such that the image of f_i is not contained in A and $f_i(1) = f_{i+1}(0) \in A$. If, for each i, the A-tail of f_i is not a inverse for the A-head of f_{i+1}, then $w_B^A([f_1 * \cdots * f_n]) = \sum_{i=1}^{n} w_B^A(f_i)$.*

This follows by repeated use of the argument in the proof of Lemma 2.11.

Lemma 2.13. *If $t : I \to X$ is a reduced path and $\{i \in \mathbb{N} \mid t \overset{t}{\not\to} \phi(a_i)_r$ and $t \overset{h}{\not\to} \phi(a_i)_r\}$ is finite, then there exists a head of t which is a almost head-tail for $\{\phi(a_i)_r\}$.*

Proof. Fix N such that, for all $i \geq N$, t is a head or a tail of $\phi(a_i)_r$. Let $M_t = \{i \geq N \mid t \overset{t}{\to} \phi(a_i)_r$ and $t \overset{h}{\not\to} \phi(a_i)_r\}$ and $M_h = \{i \geq N \mid t \overset{h}{\to} \phi(a_i)_r$ and $t \overset{t}{\not\to} \phi(a_i)_r\}$.

Let A be a neighborhood of the initial point of t which does not contain the image of t. Choose t' to be the maximal head of t contained in $\overline{A}$. Then $t = t' * g$ and $t' \xrightarrow{t} \phi(a_j)_r$ for $j \in M_t$. Choose B an open set in X such that $\overline{A} \cap \overline{B} = \emptyset$ and $w_B^A(g) \neq 0$.

Claim: *t' is an almost head-tail for M_t*

Suppose not, then there an infinite subset M' of M_t such that $t' \overset{h}{\nrightarrow} \phi(a_j)_r$ for all $j \in M'$. Let $i_1 = \min\{j \in M'\}$. There exists an s_1 such that, for $l \geq s_1$, $w_B^A(\phi(a_{i_1})) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{i_1}))$. Choose r_1 such that $\rho_{s_1*}\phi = \rho_{s_1*}\phi P_{r_1*}$.

Then by induction, suppose that i_{k-1}, s_{k-1}, and r_{k-1} have been chosen.

Choose $i_k = \min\{j \in M' \mid j > \max\{r_{k-1}, i_{k-1}\}\}$. Since $\phi(a_{i_s})_r$ has t' as a tail and $\phi(a_{i_{s+1}})_r$ does not have t' as a head for each s; Lemma 2.12 implies $w_B^A(\phi(a_{i_1} \cdots a_{i_k})) \geq k$.

There exists an $s_k > k$ such that, for $l \geq s_k$, $w_B^A(\phi(a_{i_1} \cdots a_{i_k})) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{i_1} \cdots a_{i_k}))$. Choose r_k such that $\rho_{s_k*}\phi = \rho_{s_k*}\phi P_{r_k*}$. Let $a = a_{i_1}a_{i_2}\cdots$. Then choose s such that, for $l \geq s$, $w_B^A(\phi(a)) = w_{B_l}^{A_l}(\rho_{l*}\phi(a))$.

For $k \in \mathbb{N}$ such that $s_k \geq s$,

$$
\begin{aligned}
k &\leq w_B^A(\phi(a_{i_1} \cdots a_{i_k})) \\
&= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k*}\phi(a_{i_1} \cdots a_{i_k})) \\
&= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k*}\phi(a)) \\
&= w_B^A(\phi(a)),
\end{aligned}
$$

which completes the proof of Claim 1 since $\phi(a)$ must have a finite weight.

Similarly, there exists t'' an almost head-tail for M_h. Both t' and t'' are heads of t. Thus $t' \xrightarrow{h} t''$ or $t'' \xrightarrow{h} t'$.

Hence, t' or t'' is an almost head-tail for $\{\phi(a_i)_r\}$.

$\square$

Corollary 2.14. *If a reduced path $t : I \to X$ is a head or a tail for an infinite subset of $\{\phi(a_i)_r\}$, then exists a head of t which is a almost head-tail for an infinite subset of $\{\phi(a_i)_r\}$.*

If t is not itself a head or a tail for an infinite subset of $\{\phi(a_i)_r\}$, then this is actually just the claim from the proof of Lemma 2.13.

Lemma 2.15. *Let t be a nondegenerate reduced path in X. If $\{\phi(a_n)_r\}$ has no almost head-tail, t can only be a head or a tail of finitely many of $\{\phi(a_n)_r\}$.*

Proof. Let $t : I \to X$ be a nondegenerate reduced path. Proceeding by contradiction, we will assume that t is a tail for infinitely many of $\{\phi(a_n)_r\}$.

Then by Lemma 2.14; there exists $\tilde{t}$, a head of t, such that $\tilde{t}$ is a head and a tail for an infinite subset of $\{\phi(a_n)_r\}$.

Let M be the maximal subset of $\mathbb{N}$ such that $\tilde{t}$ is a head and a tail of $\phi(a_i)_r$ for all $i \in M$.

Let A be a neighborhood of the initial point of t which does not contain the image of $\tilde{t}$. Choose t' to be the maximal head of t contained in $\overline{A}$. Then $\tilde{t} = t' * g$ and $t' \xrightarrow{t} \phi(a_j)_r$ for $j \in M$. Choose B an open set in X such that $\overline{A} \cap \overline{B} = \emptyset$ and $w_B^A(g) \neq 0$. By Lemma 2.13, there exists N an infinite subset of $\mathbb{N}$ such that, for $i \in N$, t' is neither a head nor a tail of $\phi(a_i)_r$. We will now consider two cases.

1. There exists infinitely many $i \in N$ such that $\phi(a_i)_r$ is not contained in A.

2. There exists only finitely many $i \in N$ such that $\phi(a_i)_r$ is not contained in A.

Case 1: By passing to an infinite subset of N, we may assume that $\phi(a_i)_r$ is not contained in A, for all $i \in N$.

Let $m_1 = \min\{j \in M\}$. Let $n_1 = \min\{j \in N\}$. There exists an s_1 such that, for $l \geq s_1$, $w_B^A(\phi(a_{m_1}a_{n_1}) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{m_1}a_{n_1})_r)$. Choose r_1 such that $\rho_{s_1*}\phi = \rho_{s_1*}\phi P_{r_1*}$.

Then by induction, suppose that n_{k-1}, m_{k-1}, r_{k-1} and s_{k-1} have been chosen.

Choose $m_k = \min\{j \in M \mid j > \max\{r_{k-1}, m_{k-1}\}\}$ and $n_k = \min\{j \in N \mid j > \max\{r_{k-1}, n_{k-1}\}\}$.

For all $j \leq k$, $t' \overset{h}{\not\to} \phi(a_{n_j})_r$, $t' \overset{t}{\not\to} \phi(a_{n_j})_r$, and $t' \xrightarrow{h-t} \phi(a_{m_j})_r$. Hence, Lemma 2.12 implies that

$$\begin{aligned} w_B^A(\phi(a_{m_1}a_{n_1}\cdots a_{m_k}a_{n_k})) &= \sum_{j=1}^{k} w_B^A(\phi(a_{m_j})) + w_B^A(\phi(a_{n_j})) \\ &\geq \sum_{j=1}^{k} w_B^A(\phi(a_{m_j})) \geq k \end{aligned}$$

Choose an $s_k > k$ such that $w_B^A(\phi(a_{m_1}a_{n_1}\cdots a_{m_k}a_{n_k})) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{m_1}a_{n_1}\cdots a_{m_k}a_{n_k}))$, for $l \geq s_k$. Choose r_k such that $\rho_{s_k*}\phi = \rho_{s_k*}\phi P_{r_k*}$.

Let $a = a_{m_1}a_{n_1}a_{m_2}a_{n_2}\cdots$. Then choose s such that, for $l \geq s$, $w_B^A(\phi(a)) = w_{B_l}^{A_l}(\rho_{l*}\phi(a))$.

For $k \in \mathbb{N}$ such that $s_k \geq s$,

$$\begin{aligned} k &\leq w_B^A(\phi(a_{m_1}a_{n_1}\cdots a_{m_k}a_{n_k})) \\ &= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k*}\phi(a_{m_1}a_{n_1}\cdots a_{m_k}a_{n_k})) \\ &= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k*}\phi(a)) \\ &= w_B^A(\phi(a)), \end{aligned}$$

which is a contradiction since $\phi(a)$ must have a finite weight.

Case 2: By passing to an infinite subset of N, we may assume that $\phi(a_i)_r$ is contained in A, for all $i \in N$.

Let $m_1 = \min\{j \in M\}$. Let $n_1 = \min\{j \in N\}$. Choose $p_1 > 0$ such that, for some A_1 and B_1, $w_{B_1}^{A_1}(\phi(a_{n_1}^{p_1})) > 2w_{B_1}^{A_1}(t')$.

There exists an s_1 such that, for $l \geq s_1$, $w_B^A(\phi(a_{m_1}a_{n_1}^{p_1}) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{m_1}a_{n_1}^{p_1})_r)$. Choose r_1 such that $\rho_{s_1*}\phi = \rho_{s_1*}\phi P_{r_1*}$.

Then by induction, suppose that n_{k-1}, m_{k-1}, p_{k-1}, r_{k-1} and s_{k-1} have been chosen.

Choose $m_k = \min\{j \in M \mid j > \max\{r_{k-1}, m_{k-1}\}\}$ and $n_k = \min\{j \in N \mid j > \max\{r_{k-1}, n_{k-1}\}\}$. Choose $p_k > 0$ such that, for some A_k and B_k, $w_{B_k}^{A_k}(\phi(a_{n_k}^{p_k})) > 2w_{B_k}^{A_k}(t')$.

Note that the weight requirement on $\phi(a_{n_k}^{p_k})$ implies that $\bar{\tilde{t}} * \phi(a_{n_k}^{p_k})_r * \tilde{t}$ cannot be homotoped off of A.

$$w_B^A(\phi(a_{m_1}a_{n_1}^{p_1} \cdots a_{m_k}a_{n_k}^{p_k})) = \sum_{j=1}^{k} w_B^A(\phi(a_{m_j})) \geq k$$

Choose an $s_k > k$ such that $w_B^A(\phi(a_{m_1}a_{n_1}^{p_1} \cdots a_{m_k}a_{n_k}^{p_k})) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{m_1}a_{n_1}^{p_1} \cdots a_{m_k}a_{n_k}^{p_k}))$, for $l \geq s_k$. Choose r_k such that $\rho_{s_k*}\phi = \rho_{s_k*}\phi P_{r_k*}$. Let $a = a_{m_1}a_{n_1}^{p_1}a_{m_2}a_{n_2}^{p_2}\cdots$. Then choose s such that, for $l \geq s$, $w_B^A(\phi(a)) = w_{B_l}^{A_l}(\rho_{l*}\phi(a))$.

For $k \in \mathbb{N}$ such that $s_k \geq s$,

$$
\begin{aligned}
k &\le w_B^A(\phi(a_{m_1}a_{n_1}^{p_1}\cdots a_{m_k}a_{n_k}^{p_k})) \\
&= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k*}\phi(a_{m_1}a_{n_1}^{p_1}\cdots a_{m_k}a_{n_k}^{p_k})) \\
&= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k*}\phi(a)) \\
&= w_B^A(\phi(a)),
\end{aligned}
$$

which is a contradiction since $\phi(a)$ must have a finite weight.

Thus t can only be a tail of finitely many of $\{\phi(a_n)_r\}$. A symmetric argument shows that t can only be a head of finitely many of $\{\phi(a_n)_r\}$, which completes the proof.

$\square$

Lemma 2.16. *Suppose that $\{\phi(a_n)\}$ has no almost head-tail, then for all $\epsilon > 0$ there exists an N such that $\{\phi(a_n)_r \mid n \ge N\} \subset i_*(\pi_1(B_\epsilon(x_0), x_0))$ where i_* is the inclusion induced homomorphism.*

Proof. Proceeding by contradiction we will assume that there exist an $\epsilon > 0$ such that $\phi(a_j)_r \not\subseteq i_*(\pi_1(B_\epsilon(x_0), x_0))$ for all $j \in J$, where i_* is the homomorphism induced by the inclusion map and J is an infinite subset of $\mathbb{N}$.

Let $A = B_{\epsilon/2}(x_0) \subset X$ and $B = (\overline{B_\epsilon(x_0)})^c \subset X$. Let be t_i the maximal tail of $\phi(a_i)_r$ contained in the closure of A.

Let $i_1 = \min\{j \mid j \in J\}$.

There exists an s_1 such that for $l \ge s_1$ $w_B^A(\phi(a_{i_1})) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{i_1}))$. Choose r_1 such that $\rho_{s_1*}\phi = \rho_{s_1*}\phi P_{r_1*}$.

Then by induction, suppose that i_{k-1}, s_{k-1}, and r_{k-1} have been chosen.

By Lemma 2.15 there exists an N_k such that $t_{i_{k-1}} \overset{h}{\not\to} \phi(a_j)_r$ for all $j > N_k$. Choose $i_k = \min\{j \in J \mid j > \max\{r_{k-1},\ i_{k-1},\ N_k\}\}$. There exists an $s_k > k$

such that for $l \geq s_k$ $w_B^A(\phi(a_{i_1}\cdots a_{i_k})) = w_{B_l}^{A_l}(\rho_{l*}\phi(a_{i_1}\cdots a_{i_k}))$. Choose r_k such that $\rho_{s_k*}\phi = \rho_{s_*}\phi P_{r_k*}$. Let $a = a_{i_1}a_{i_2}\cdots$. Then choose s such that, for $l \geq s$, $w_B^A(\phi(a)) = w_{B_l}^{A_l}(\rho_{l*}\phi(a))$.

The $w_B^A([\phi(a_{i_k})]) \geq 2$. Then by Lemma 2.12, $w_B^A([\phi(a_{i_1}\cdots a_{i_n})]) \geq 2n$.

For $k \in \mathbb{N}$ such that $s_k \geq s$,

$$\begin{aligned}
2k &\leq w_B^A(\phi(a_{i_1}\cdots a_{i_k})) \\
&= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k*}\phi(a_{i_1}\cdots a_{i_k})) \\
&= w_{B_{s_k}}^{A_{s_k}}(\rho_{s_k*}\phi(a)) \\
&= w_B^A(\phi(a)),
\end{aligned}$$

which is a contradiction since $\phi(a)$ must have a finite weight.

□

Theorem 2.17. *Let $\phi : \mathbb{H} \to \pi_1(X, x_0)$ be a homomorphism from the Hawaiian Earring group into the fundamental group of a one-dimensional Peano continuum X. Then there exists a continuous function $f : (\boldsymbol{E}, 0) \to (X, x)$ and a path $T : (I, 0, 1) \to (X, x_0, x)$ with the property that $f_* = \widehat{T}\phi$.*

Proof. By Theorem 2.10, there exists a path T such that $\{\widehat{T}\phi(a_i)\}$ has not almost head-tail.

It is sufficient to show that for any $\epsilon > 0$, there exists an N such that $\widehat{T}\phi(H_k) \subset i_*(\pi_1(B_\epsilon(x_0), x_0))$ for all $k \geq N$.

Proceeding by contradiction, suppose that there exists an $\epsilon > 0$ such that $\widehat{T}\phi(H_k) \not\subset i_*(\pi_1(B_\epsilon(x_0), x_0))$ for $k \in J$ where J is some infinite subset of $\mathbb{N}$.

By Lemma 2.16, there exists an N such that $\{\widehat{T}\phi(a_k)_r \mid k \geq N\} \subset i_*(\pi_1(B_{\epsilon/2}(x_o), x_0))$ where i_* is the inclusion induced homomorphism. Choose m such that $m \in J$ and

$m > N$. Let $b \in H_m$ such that $\widehat{T}\phi(b)_r \notin i_*(\pi_1(B_\epsilon(x_0), x_0))$. Then let $A = B_{\epsilon/2}(x_0)$ and $B = (B_\epsilon(x_0))^c$. Then $w_B^A(\widehat{T}\phi(b)_r) > 0$. Then there exists a k such that $w_{B_k}^{A_k}(\rho_{k*}\widehat{T}\phi(b)) > 0$. By Theorem 2.8, there exists an s such that $\rho_{k*}\widehat{T}\phi = \rho_{k*}\widehat{T}\phi P_{s*}$. Then $\left(\rho_{k*}\widehat{T}\phi P_{s*}(b)\right)_r$ is the a finite product of elements with image exterior to $\overline{B_k}$. Hence $w_{B_k}^{A_k}(\left(\rho_{k*}\widehat{T}\phi(b)\right)_r) = 0$, a contradiction. □

3 Planar Peano Continuum

3.1 Delineation

We will now define a decomposition map from a planar Peano continuum into a one-dimensional Peano continuum.

Definition 3.1. Let k a line in the plane and X a planar Peano continuum. Let $\pi_k : X \to X/G$ be a decomposition map where the nontrivial decomposition elements of G are the maximal line segments in X which are parallel to k.

We will use X_k to denote the decomposition space corresponding to π_k. Cannon and Conner have shown that this is actually an upper semicontinuous decomposition, that X_k is a one-dimensional Peano continuum, and that the induced homomorphism on fundamental groups is injective (Theorem 1.4 in [3]).

Lemma 3.1. *If $g : I \to X$ is a path and $\pi_k g$ has reduced representative α then there exists $\tilde{g} : I \to X$ such that $\pi_k\tilde{g} = \alpha$ up to reparameterizations.*

Proof. If $\pi_k g$ is reduced, we are done. Otherwise there exists an interval $[c, d]$ such that $\pi_k g\big|_{[c,d]}$ is nullhomotopic rel endpoints. Then $\pi_k g(c) = \pi_k g(d)$ which implies that the line segment $\overline{g(c)g(d)}$ is in contained in X. Then the loop $g\big|_{[c,d]} * \overline{g(d)g(c)}$

maps to $\pi_k g\big|_{[c,d]}$ and hence must be nullhomotopic since π_{k*} is injective. Therefore g is homotopic to g' where the subpath $g\big|_{[c,d]}$ is replaced by $\overline{g(c)g(d)}$.

If f is the reduced representative for $[\pi_k g]$, then $f = \pi_k g\big|_{I-(\cup_i J_i)}$, up to reparametrization, where $J_i = (c_i, d_i)$ are disjoint open subintervals of I such that $\pi_k g\big|_{\overline{J_i}}$ is nullhomotopic rel endpoints. Let l_i be a parametrization of the line segment from $g(d_i)$ to $g(c_i)$.

Since g is uniformly continuous, diameter of $\{g\big|_{[c_i,d_i]}\}$ must converge to zero.

Claim: There exists homotopies $H_i : I \times [c_i, d_i] \to X$ with the property that $H_i\big|_{\{0\}\times[c_i,d_i]} = g\big|_{[c_i,d_i]}$, $H_i\big|_{\{1\}\times[c_i,d_i]} = \overline{l_i}$, and $H_i(I \times [c_i, d_i]) \to 0$.

Then Lemma A2 would imply that g is homotopic to $\tilde{g}$, the path where each subpath of g with nullhomotopic image is replaced by the corresponding line segment. This would then complete the proof of the lemma.

The claim is actually just a corollary of Cannon and Conner's proof that π_{k*} is injective. They show that if $h : \mathbb{S}^1 \to X \subset \mathbb{R}^2$ maps to a nullhomotopic loop under π_k, then h bounds a disk contained in the bounded component of $\mathbb{R}^2 - h(\mathbb{S}^1)$ (see [3], p. 60-65). Hence we may choose H_i such that $\mathrm{diam}(H_i(I \times [c_i, d_i])) = \mathrm{diam}(g\big|_{[c_i,d_i]} * l_i) = \mathrm{diam}(g\big|_{[c_i,d_i]})$.

$\square$

Lemma 3.2. *Let A and B be disjoint closed saturated sets and A_k,B_K their respective images under π_k. Then $w_B^A(g) = w_{B_k}^{A_k}(\pi_k g)$.*

This follows directly from the fact that the weight can be realized by a finite set of points.

Lemma 3.3. *The delineation map, π_k, preserve weights of homotopy classes on disjoint closed saturated sets, i.e. $w_B^A([g]) = w_{B_k}^{A_k}([\pi_k g])$.*

Proof. Let $\tilde{g}$ be the path homotopic to g such that $\pi_k \tilde{g}$ is reduced. Then $w_{B_k}^{A_k}([\pi_k g]) = w_{B_k}^{A_k}(\pi_k \tilde{g}) = w_B^A(\tilde{g}) \geq w_B^A([g])$. For g' homotopic to g $w_B^A(g') = w_{B_k}^{A_k}(\pi_k g') \geq w_{B_k}^{A_k}(\pi_k \tilde{g}) = w_{B_k}^{A_k}([\pi_k g])$. Then $w_B^A([g]) \geq w_{B_k}^{A_k}([\pi_k g])$.

Thus $w_B^A([g]) = w_{B_k}^{A_k}([\pi_k g])$.

□

Definition 3.2. If g maps to a reduced path under π_k, we will say that g is reduced with respect to k or g is k-reduced. For any path g, if $\tilde{g}$ is reduced and homotopic to g where $\tilde{g}$ is obtained by replacing subpaths of g by lines in X, then we will say that $\tilde{g}$ is obtained by reducing g with respect to k.

Let k_1 and k_2 be disjoint lines in $\mathbb{R}^2$ which are parallel to k. If we choose $A = A' \cap X$ where A' is the half-space with boundary k_1 which does not contain k_2 and $B = B' \cap X$ where B' is the half-space with boundary k_2 which does not contain k_1, then we can see that Lemma 3.3 implies that π_{k*} preserves oscillation with respect to all lines parallel to k.

Then it is easy to see that for g to be k reduced, a necessary condition is that it have minimal weight in its path class with respect to all disjoint half-planes A and B with boundaries parallel to k.

In fact this condition is also sufficient. Suppose g has minimal weight in its path class with respect to all subsets A and B (as above). If $\pi_k g$ is not reduced, then there exists $g(c)$ and $g(d)$ such that $\overline{g(c)g(d)}$ is in contained in X and $\pi_k g\big|_{[c,d]}$ is nullhomotopic but not constant. Then $g\big|_{[c,d]}$ must not be contained in the line segment $\overline{g(c)g(d)}$. However, then g is homotopic to $\tilde{g}$ where $g\big|_{[c,d]}$ is replace by $\overline{g(c)g(d)}$ and the weight of $\tilde{g}$ is strictly less than the weight of g for some disjoint half-planes with boundaries parallel to k.

Thus we can see that if g is k-reduced and $\tilde{g}$ is obtained by reducing g with respect to l, then $\tilde{g}$ is reduced with respect to l and k.

3.2 Induced by a continuous map

Fix a homomorphism $\phi : \mathbb{H} \to \pi_1(X, x_0)$ into the fundamental group of a planar Peano continuum. We will show that ϕ is conjugate to a homomorphism induced by a continuous map. For each line k in the plane, we will use T_k to denote the path such that $\widehat{T_k}(\pi_{k*}\phi)$ is induced by a continuous map where $\widehat{T_k}$ is the change of base isomorphism induced by T_k.

Then the real key to being able to reduced the planar case to the one-dimensional case is the following proposition.

Proposition 3.4. *For k a line in the plane, there exists α_k a path in X such that $\pi_k(\alpha_k) = T_k$.*

To prove this proposition we will construct a single word $a \in \mathbb{H}$ such that $T_k \xrightarrow{t} (\pi_k\phi(a))_r$. The main idea is contained in the following lemma.

Lemma 3.5. *Let $f : \boldsymbol{E} \to X$ be a continuous function enjoying the property $f_* = \widehat{T_k}\phi$ for some path T_k. Then there exists $a \in \mathbb{H}$ such that no nondegenerate terminal segment of T_k is a tail for $f_*(a)$; i.e. for each $s \in [0,1]$, such that $\overline{T}_k|_{[0,s]}$ is nondegenerate, $\overline{T}_k|_{[0,s]}$ is not a tail for $f_*(a)$.*

Proof. Since f is continuous, we may choose an increasing subsequence $(i_n) \subset \mathbb{N}$ such that $f_*(H_n)$ contains no inverse for $f_*(a_{i_{n-1}})$. Additionally; we may choose r_{i_n} sufficiently large such that, for some A_i and B_i, $w_{B_i}^{A_i}(T_k) < w_{B_i}^{A_i}(f_*(a_{i_n}^{r_{i_n}}))$. Let $a = \prod_{n=1}^{\infty} a_{i_n}^{r_{i_n}+1}$.

Suppose that $\overline{T}_k|_{[0,s]}$ is a nondegenerate tail for $f_*(a)$. Fix $N > 0$ such that $\operatorname{diam}(f_*(\prod_{n=N}^{\infty} a_{i_n}^{r_{i_n}+1})_r) < \frac{1}{2}\operatorname{diam}(\overline{T}_k|_{[0,s]})$. Then $f_*(\prod_{n=N}^{\infty} a_{i_n}^{r_{i_n}+1})_r \subset \overline{T}_k([0,s])$, which is a contradiction since $w_{B_i}^{A_i}(T_k) < w_{B_i}^{A_i}(f_*(a_{i_n}^{r_{i_n}})) \leq w_{B_i}^{A_i}(f_*(\prod_{n=N}^{\infty} a_{i_n}^{r_{i_n}+1}))$.

□

Corollary 3.6. *The path T_k is a tail for $\pi_{k*}\phi(a)$.*

Proof. Note that $(\pi_{k*}\phi(a))_r = (T_k * f_*(a) * \overline{T}_k)_r$ and by the previous lemma $(T_k * f_*(a) * \overline{T}_k)_r = (T_k * f_*(a))_r * \overline{T}_k$.

□

Then Proposition 3.4 follows from Corollary 3.6 and Lemma 3.1.

Proposition 3.7. *If k and l are non-parallel lines in the plane, then there exists a path α in X such that $\pi_k(\alpha) = T_k$ and $\pi_l(\alpha) = T_l$.*

Proof. This is actually a corollary of the proof of Proposition 3.5.

Let f by the continuous map which induces $\widehat{T}_k(\pi_{k*}\phi)$ and g the continuous map which induces $\widehat{T}_l(\pi_{l*}\phi)$.

Since f and g are continuous, we may choose an increasing subsequence $(i_n) \subset \mathbb{N}$ such that $f_*(H_n)$ contains no inverse for $f_*(a_{i_{n-1}})$ and $g_*(H_n)$ contains no inverse for $g_*(a_{i_{n-1}})$. Additionally, we may choose r_{i_n} sufficiently large such that, for some A_i, B_i, A_i', and B_i'; $w_{B_i}^{A_i}(T_k) < w_{B_i}^{A_i}(f_*(a_{i_n}^{r_{i_n}}))$ and $w_{B_i'}^{A_i'}(T_l) < w_{B_i'}^{A_i'}(g_*(a_{i_n}^{r_{i_n}}))$. Let $a = \prod_{n=1}^{\infty} a_{i_n}^{r_{i_n}+1}$.

Fix $h \in \phi(a)$, a (k,l)-reduced path. Then T_k is at tail for $\pi_k f$ and T_l is a tail for $\pi_l h$. Let α_k be the subpath of h mapping to T_k and β_l the subpath of h mapping to T_l. Notice that by our choice of a_{i_n} and r_{i_n}, $\pi_k \alpha_k$ is the maximal tail of $\pi_k h = (\pi_{k*}\phi(a))_r$ such that no terminal segment of $\pi_k \alpha_k$ is a tail for $(T_k * f_*(a))_r$.

Since no terminal segment of $\pi_k\beta_l$ is a tail for $(T_k * f_*(a))_r$, $\pi_k\beta_l$ is a subpath of $\pi_k\alpha_k$. A similiar argument applied to π_l shows that $\pi_l\alpha_k$ is a subpath of $\pi_l\beta_l$. Thus $\beta_l = \alpha_k$.

□

We will now show that α is the path such that $\widehat{\alpha}\phi$ is induced by a continuous map.

Lemma 3.8. *Let $\alpha_k : I \to X$ be a path with the property that $\pi_k\alpha_k = T_k$, up to reparametrization. Let U be a π_k-saturated neighborhood of $\pi_k^{-1}(T_k(1))$. Then for sufficiently large n, $\{\widehat{\alpha}_k\phi(H_n)\}$ is contained in $\pi_1(U, \alpha(1))$.*

Proof. If g is a loop based at a point $y \in \pi_k^{-1}(x)$ and $w_{U^c}^{\pi_k^{-1}(x)}(g) = 0$, then $[g] \in \pi_1(U, y)$.

Let U be a π_k-saturated neighborhood of $\pi_k^{-1}(T_k(1))$. Let U' by an open π_k-saturated neighborhood of $\pi_k^{-1}(T_k(1))$ with closure contained in the interior of U. We must show that, for some sufficiently large n, $w_B^A(\widehat{\alpha}\phi(b)) = 0$ for all $b \in H_n$ where $A = \pi_k^{-1}(T_k(1))$ and $B = \overline{U'}^c$.

Since $\widehat{T}_k\pi_{k*}\phi$ is induced by a continuous map, there exists an N such that, for all $n > N$, $\widehat{T}_k\pi_{k*}\phi(H_n) \subset \pi_1(\pi_k(U'), T_k(1))$. Hence $w_{B_k}^{A_k}(\widehat{T}_k\pi_{k*}\phi(b)) = 0$ for all $b \in H_n$ where $n > N$.

For $b \in H_n$, where $n > N$, let f be a k-reduced representative of $\widehat{\alpha}\phi(b)$. Then the $w_B^A(\widehat{\alpha}\phi(b)) \leq w_B^A(f) = w_{B_k}^{A_k}(\pi_k f) = w_{B_k}^{A_k}(\widehat{T}_k\pi_{k*}\phi(b)) = 0$.

□

Theorem 3.9. *Let $\phi : \mathbb{H} \to \pi_1(X, x_0)$ a homomorphism into the fundamental group of a planar Peano continuum X. Then there exists a continuous function*

$f : (\boldsymbol{E}, 0) \to (X, x)$ *and a path* $\alpha : (I, 0, 1) \to (X, x_0, x)$*, which have the property that* $f_* = \widehat{\alpha}\phi$*.*

Proof. For k and l nonparallel lines in the plane, there exists a path α in X such that $\pi_k(\alpha) = T_k$ and $\pi_l(\alpha) = T_l$, by Lemma 3.7.

It is sufficient to show that for any neighborhood U of $\alpha(1)$ there exists an N such that $\widehat{\alpha}\phi(H_n) \subset \pi_1(U, \alpha(1))$. This is done by finding U_l and U_k such that $U_k \cap U_l \subset U$ and U_l is a π_l-saturated neighborhood of $\pi_l^{-1}\pi_l(\alpha(1))$ and U_k is a π_k-saturated neighborhood of $\pi_k^{-1}\pi_k(\alpha(1))$. $\square$

4 APPENDIX

Lemma A1. *If* $A \subset X$ *and* $A_i = \rho_i(A)$*, then* $\rho_i^{-1}(\overline{A_i}) \subset B_{1/i}(A)$*.*

Proof. Let $O_j \in \mathcal{O}_i$. Then $O_j = \rho_i^{-1}(\rho_i(O_j))$, since $\theta_j^i(x) = 0$ if and only if $x \notin O_j$. If $x \in \rho_i(O_j) \cap A_i$, then $O_j \cap A \neq \emptyset$. Thus $\rho_i^{-1}(A_i) \subset B_{1/i}(A)$.

Let $x \in \rho_i^{-1}(\overline{A_i}) \cap O_j$. Then we must show that $O_j \cap A \neq \emptyset$. There exists a sequence in A_i which approaches $\rho_i(x)$. Hence there exists a sequence $(x_n) \in A$ such that $\rho_i(x_n) \to \rho_i(x)$. Since $x \in O_j$, $\theta_j^i(x) > 0$. Then $(\theta_j^i(x_n))$ is eventually greater than zero. Hence (x_n) is eventually in O_j. Thus $O_j \cap A \neq \emptyset$.

$\square$

A lemma due to Greg Conner and Mark Meilstrup.

Lemma A2. *Let* H *be a function from the first-countable space* $X \times Y$ *into* Z*. Let* $\{C_i\}$ *be a null sequence of closed sets whose union is* X*. Suppose that* $\{D_i = H(C_i \times Y)\}$ *is a null sequence of sets in* Z *and* H *is continuous on each* $C_i \times Y$*. If* *for every subsequence $C_{i_k} \to x_0$ there exists a $z_0 \in Z$ such that $D_{i_k} \to z_0$ then H is continuous on $X \times Y$.*

Proof. Consider a sequence $(x_n, y_n) \to (x_0, y_0)$. For each n, choose an i_n such that $x_n \in C_{i_N}$. If $\{C_{i_n}\}$ is finite then by restricting H to $\cup_n C_{i_n} \times Y$ we have $H(x_n, y_n) \to H(x_0, y_0)$ be a finite application of the pasting lemma. If $\{C_{i_n}\}$ is infinite, then since $\{C_i\}$ is a null sequence and $x_n \in C_{n_i}$, we have $C_{i_n} \to x_0$ and thus $H(x_n, y_n) \in D_{i_n} \to z_0 = H(x_0, y_o)$. Thus H is continuous on all of $X \times Y$. $\square$

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