



Theses and Dissertations

---

2008-06-30

## Fusion of Character Tables and Schur Rings of Dihedral Groups

Long Pham Bao Nguyen  
*Brigham Young University - Provo*

Follow this and additional works at: <https://scholarsarchive.byu.edu/etd>



Part of the [Mathematics Commons](#)

---

### BYU ScholarsArchive Citation

Nguyen, Long Pham Bao, "Fusion of Character Tables and Schur Rings of Dihedral Groups" (2008). *Theses and Dissertations*. 1429.

<https://scholarsarchive.byu.edu/etd/1429>

This Thesis is brought to you for free and open access by BYU ScholarsArchive. It has been accepted for inclusion in Theses and Dissertations by an authorized administrator of BYU ScholarsArchive. For more information, please contact [scholarsarchive@byu.edu](mailto:scholarsarchive@byu.edu), [ellen\\_amatangelo@byu.edu](mailto:ellen_amatangelo@byu.edu).

FUSIONS OF CHARACTER TABLES AND SCHUR RINGS OF DIHEDRAL GROUPS

by  
Long Bao Nguyen

A thesis submitted to the faculty of  
Brigham Young University  
in partial fulfillment of the requirements for the degree of

Master of Science

Department of Mathematics  
Brigham Young University

August 2008

Copyright © 2008 Long Bao Nguyen

All Rights Reserved

BRIGHAM YOUNG UNIVERSITY

GRADUATE COMMITTEE APPROVAL

of a thesis submitted by

Long Bao Nguyen

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

\_\_\_\_\_  
Date

\_\_\_\_\_  
Stephen Humphries, Chair

\_\_\_\_\_  
Date

\_\_\_\_\_  
Darrin Doud

\_\_\_\_\_  
Date

\_\_\_\_\_  
William E. Lang

BRIGHAM YOUNG UNIVERSITY

As chair of the candidate's graduate committee, I have read the thesis of Long Nguyen in its final form and have found that (1) its format, citations, and bibliographical style are consistent and acceptable and fulfill university and department style requirements; (2) its illustrative materials including figures, tables, and charts are in place; and (3) the final manuscript is satisfactory to the graduate committee and is ready for submission to the university library.

---

Date

---

Stephen Humphries  
Chair, Graduate Committee

Accepted for the Department

---

William E. Lang  
Graduate Coordinator

Accepted for the College

---

Thomas Sederberg, Associate Dean  
College of Physical and Mathematical Sciences

## ABSTRACT

### Fusions of Character Tables and Schur Rings of Dihedral Groups

Long Bao Nguyen

Department of Mathematics

Master of Science

A finite group  $H$  is said to *fuse to* a finite group  $G$  if the class algebra of  $G$  is isomorphic to an *S-ring* over  $H$  which is a subalgebra of the class algebra of  $H$ . We will also say that  $G$  *fuses from*  $H$ . In this case, the classes and characters of  $H$  can fuse to give the character table of  $G$ . We investigate the case where  $H$  is the dihedral group. In many cases,  $G$  can be completely determined. In general,  $G$  can be proven to have many interesting properties. The theory is developed in terms of S-ring of Schur and Wielandt.

## ACKNOWLEDGMENTS

I would like to publicly thank Dr. Humphries for his invaluable help, guidance and patience in the writing of this thesis.

# Contents

<b>1</b>	<b>Introduction</b>	<b>1</b>
<b>2</b>	<b>Main Results</b>	<b>2</b>
<b>3</b>	<b>Preliminaries</b>	<b>5</b>
3.1	The Group Algebra . . . . .	5
3.2	Magic Rectangle Condition . . . . .	7
3.3	Basic Results . . . . .	10
<b>4</b>	<b>Groups which fuse from <math>D_{2n}</math></b>	<b>14</b>
4.1	The Dihedral Group . . . . .	14
4.2	$D_{2n}$ fuses to $G$ where $n$ is odd . . . . .	16
4.3	$D_{2n}$ fuses to $G$ where $n$ is even . . . . .	20
4.4	$D_{2n}$ fuses to $G$ where $ G  = 8p$ . . . . .	29
<b>5</b>	<b>Schur Rings over Cyclic Groups</b>	<b>30</b>
5.1	Classification of S-rings over Cyclic Groups . . . . .	30
5.2	Application to Fusion . . . . .	36
5.3	$D_{2n}$ fuses to $G$ where $ G  = 8p^2$ . . . . .	42



# Chapter 1

## Introduction

We investigate S-rings as developed by Schur and Wielandt and the fusion of groups. We begin Chapter Two by stating the definitions of S-rings and fusion of groups. A finite group  $H$  is said to *fuse to* a finite group  $G$  if the class algebra of  $G$  is isomorphic to an S-ring over  $H$  which is a subalgebra of the class algebra of  $H$ . We will also say that  $G$  *fuses from*  $H$ . In this case, the classes and characters of  $H$  can fuse to give the character table of  $G$ . We investigate the case where  $H$  is the dihedral group. We cover some necessary preliminaries, which include a discussion of the group algebra, the Magic Rectangle Condition and some relevant basic results in Chapter Three. In Chapter Four, we discuss the main problem of which groups fuse from  $D_{2n}$  and look at both the odd and even cases for  $n$ . We end the chapter by solving the  $2n = 8p$  case. Finally, in the Chapter Five, we discuss the classification of Schur rings over cyclic groups and apply it to our problem in the  $2n = 8p^2$  case.

# Chapter 2

## Main Results

Let  $G$  be a finite group and  $\mathbb{C}G$  be its group ring over  $\mathbb{C}$ . Suppose  $W \subset G$ , define  $\bar{W} := \sum_{w \in W} w \in \mathbb{C}G$  and  $W^{(-1)} := \{w^{-1} | w \in W\}$ . In general, for  $n \in \mathbb{Z}$ , define  $W^{(n)} := \{w^n | w \in W\}$ .

**Definition 2.0.1.** Suppose  $\{S_i\}_{i=1}^m$  is a partition of a finite group  $G$  satisfying three conditions:

(1)  $S_1 = \{1\}$ .

(2) If  $S_i = \{g_1, \dots, g_s\}$  then  $S_i^{(-1)} = \{g_1^{-1}, \dots, g_s^{-1}\} = S_j$  for some  $j$ .

(3) If  $i, j \leq m$ , then  $\bar{S}_i \bar{S}_j = \sum_k a_{ijk} \bar{S}_k$  where  $a_{ijk}$  is a non-negative integer for all  $i, j, k$ . These  $a_{ijk}$ 's are called the structure constants of the group  $G$ .

An  $S$ -ring  $S$  over  $G$  is the subalgebra of  $\mathbb{C}G$  generated by  $\bar{S}_1, \dots, \bar{S}_m$ .

We call each  $S_i$  a  $S$ -principal subset of  $G$ . The set of all  $S$ -principal subsets will be denoted by  $D(S)$ . A subgroup  $H$  of  $G$  is an  $S$ -subgroup if  $\bar{H} \in S$ .

**Example 2.0.2.** Suppose  $G$  is a finite group. Let  $S_1 = \{1\}$  and  $S_2 = G \setminus \{1\}$ . We can easily check that this forms an  $S$ -ring over  $G$ . This is called the trivial  $S$ -ring.

**Example 2.0.3.** Let the  $S$ -principal subsets be the conjugacy classes of  $G$ . It can

easily be checked that this partition also satisfies the  $S$ -ring conditions. This  $S$ -ring is the class algebra of  $G$ .

**Example 2.0.4.** If  $N$  is a normal subgroup of  $G$ , then the conjugacy classes of  $G$  contained in  $N$  form an  $S$ -ring over  $N$ . Denote this  $S$ -ring  $F_G(N)$ .

**Definition 2.0.5.** A finite group  $H$  is said to fuse to a finite group  $G$  if the class algebra of  $G$  is isomorphic to an  $S$ -ring over  $H$  which is a subalgebra of the class algebra of  $H$ . We will also say that  $G$  fuses from  $H$ . If  $G$  fuses from  $H$ , then under the isomorphism, each class  $C$  of  $G$  corresponds to a subset  $C^* \subset H$ , where  $C^*$  is a union of conjugacy classes of  $H$ . In this situation, we will say that  $C$  and  $C^*$  correspond.

If  $X$  is a set, then  $|X|$  will denote the cardinality of  $X$ . Suppose  $n = 2m$ . Let

$$Dic_{2n} = \langle a, b \mid a^m = b^2, b^4 = 1, b^{-1}ab = a^{-1} \rangle.$$

We call  $Dic_{2n}$  the *dicyclic group* of order  $2n$ .

Our main theorem is:

**Theorem 2.0.6.** Suppose  $G$  fuses from  $D_{2n}$ .

- i) If  $n$  is odd, then  $G \cong D_{2n}$ .
- ii) If  $n = 2m$ , and  $m$  is odd, then  $G \cong D_{2n}$ .
- iii) If  $2n = 8p$ , then  $G \cong D_{2n}$  or  $G \cong Dic_{2n}$ .
- iv) If  $2n = 8p^2$ ,  $p$  an odd prime, then we have either, i)  $G$  is dihedral or dicyclic; or ii)  $p=3$  and  $G$  is the Frobenius group of order 72.

In the case where  $|G| = 4m$ ,  $m$  is even with  $Z(G) = \{1\}$ , we can deduce the following important properties for  $G$ :

1.  $G$  has at least two classes of size  $m$ .

2.  $G/G' \cong \mathbb{Z}_2^2$ .
3.  $G$  has a unique non-trivial class of odd size, which is a class of involutions.
4. Each non-linear irreducible complex character of  $G$  is of even degree.
5.  $G$  has a degree 2 irreducible character.
6. Each character of  $G$  is real;
7. Each element of  $G$  is conjugate to its inverse.
8.  $G$  is 2-nilpotent, that is, the Sylow 2-subgroup of  $G$  has a normal complement in  $G$ .
9.  $G$  is solvable.
10. The Sylow 2-subgroup of  $G$  is dihedral, dicyclic or quasidihedral.
11. Let  $M$  be the normal subgroup of  $G$  which corresponds to  $\langle x \rangle \leq D_{2n}$ . Then  $(G, M, G')$  is a Camina triple, that is, conjugacy classes of  $G$  in  $G \setminus M$  are union of cosets of  $G'$ .
12.  $G$  is not a direct product.
13.  $G$  has at most  $d(n) + 3$  normal subgroups, where  $d(n)$  is the number of divisors of  $n$ . Furthermore, for each divisor  $d < n$  of  $n$ ,  $G$  has unique normal subgroup of order  $d$ .

We will hereafter refer to these properties as *Property n*,  $1 \leq n \leq 13$ .

# Chapter 3

## Preliminaries

We introduce the group algebra, the Magic Rectangle Condition and state some basic results.

### 3.1 The Group Algebra

Suppose  $G$  is a finite group with elements  $\{g_1, \dots, g_n\}$ . We define a vector space over  $\mathbb{C}$  with basis elements  $\{g_1, \dots, g_n\}$  and denote it by  $\mathbb{C}G$ . The elements of  $\mathbb{C}G$  are formal sums of the form,

$$\lambda_1 g_1 + \dots + \lambda_n g_n \quad (\lambda_i \in \mathbb{C}).$$

The rules for addition and scalar multiplication are the usual ones, that is, if

$$x = \sum_{i=1}^n \lambda_i g_i \quad \text{and} \quad y = \sum_{i=1}^n \mu_i g_i,$$

are elements of  $\mathbb{C}G$ ,  $\lambda \in \mathbb{C}$ , then,

$$x + y = \sum_{i=1}^n (\lambda_i + \mu_i) g_i \quad \text{and} \quad \lambda x = \sum_{i=1}^n (\lambda \lambda_i) g_i.$$

We now define a product on  $\mathbb{C}G$  to make it into an algebra:

$$\left(\sum_{g \in G} \lambda_g g\right) \left(\sum_{h \in H} \mu_h h\right) = \sum_{g, h \in G} \lambda_g \mu_h gh,$$

where  $\lambda_g, \mu_h \in \mathbb{C}$ . With these operations,  $\mathbb{C}G$  is an algebra over  $\mathbb{C}$  of dimension  $n$ . We call this algebra the *group algebra*. An important vector subspace of  $\mathbb{C}G$  is the *center of the group algebra*, denoted  $Z(\mathbb{C}G)$ , which is defined as,

$$Z(\mathbb{C}G) = \{z \in \mathbb{C}G : zr = rz \text{ for all } r \in \mathbb{C}G\}.$$

It is a standard result that  $Z(\mathbb{C}G)$  is generated by  $\overline{C_1}, \dots, \overline{C_s}$ , where the  $C_i$ 's are the conjugacy classes of  $G$ . We also call  $Z(\mathbb{C}G)$  the *class algebra* of  $G$ .

## 3.2 Magic Rectangle Condition

Suppose  $H$  fuses to  $G$ . Let  $B_1 = \{e\}, B_2 = \{C_{2s_1}, \dots, C_{2s_2}\}, \dots, B_f = \{C_{f_1}, \dots, C_{f_{r_f}}\}$  denote the partition of the classes of  $H$  determined by the fusion. Then the character table of  $H$  satisfies the Magic Rectangle Condition[Smith], that is, there exists a partition of the irreducible characters of  $H$ ,

$$\Phi_1 = \{\chi_1\}, \Phi_2 = \{\chi_{2s_1}, \dots, \chi_{2s_2}\}, \dots, \Phi_f = \{\chi_{f_1}, \dots, \chi_{f_{r_f}}\},$$

so that for each  $\Phi_i, B_j$  the number

$$\tau_{ij} = \frac{\sum_{m=1}^{r_j} |C_{j_m}| \chi_{it}(C_{j_m})}{\left(\sum_{m=1}^{r_j} |C_{j_m}|\right) d_i} \quad (3.1)$$

is independent of  $\chi_{it} \in \Phi_i$ , for  $1 \leq t \leq s_i$ , and is equal to the value

$$\frac{\sum_{m=1}^{s_i} d_{im} \chi_{im}(C_{j_k})}{\sum_{m=1}^{s_i} d_{im}^2}, \quad (3.2)$$

which is independent of  $C_{j_k} \in B_j$  for  $1 \leq k \leq r_j$ . Thus we have

$$\tau_{ij} = \frac{\sum_{m=1}^{r_j} |C_{j_m}| \chi_{it}(C_{j_m})}{\left(\sum_{m=1}^{r_j} |C_{j_m}|\right) d_i} = \frac{\sum_{m=1}^{s_i} d_{im} \chi_{im}(C_{j_k})}{\sum_{m=1}^{s_i} d_{im}^2}. \quad (3.3)$$

Because of the well defined value of  $\tau_{ij}$  above, we obtain an  $f \times f$  fused table with the rows indexed by the  $\Phi_i$ 's and columns indexed by the  $B_j$ 's where the value of the  $ij$ th

entry is  $\eta_i \tau_{ij}$  where  $\eta_i$  is

$$\eta_i = \sqrt{\sum_{m=1}^{s_i} d_{im}^2}. \quad (3.4)$$

This  $f \times f$  fused table is the character table for the fused group  $G$ . The irreducible characters in each  $\Phi_i$  is said to *fuse* to give an irreducible character of  $G$ .

We show that  $\mathbb{Z}_6 = \langle x \rangle$  fuses to  $D_6$ . The conjugacy classes of  $D_6$  are

$$\{\{1\}, \{(12), (13), (23)\}, \{(123), (132)\}\}.$$

We can easily check that the following correspondence gives the required fusion:

$$\{1\} \mapsto \{1\},$$

$$\{x^2, x^4\} \mapsto \{(123), (132)\},$$

$$\{x, x^3, x^5\} \mapsto \{(12), (13), (23)\}.$$

The character table for the cyclic group  $C_6$  is:

	$e$	$x^2$	$x^4$	$x^3$	$x$	$x^5$
$\chi_0$	1	1	1	1	1	1
$\chi_3$	1	1	1	-1	-1	-1
$\chi_2$	1	$\rho^4$	$\rho^2$	1	$\rho^2$	$\rho^4$
$\chi_4$	1	$\rho^2$	$\rho^4$	1	$\rho^4$	$\rho^2$
$\chi_1$	1	$\rho^2$	$\rho^4$	-1	$\rho$	$\rho^{-1}$
$\chi_5$	1	$\rho^4$	$\rho^2$	-1	$\rho^{-1}$	$\rho$

where  $\rho = e^{2\pi i/6}$  and  $\chi_i(x) = \rho^i$ . If we partition the columns according to the fusion as  $\{e\}, \{x^2, x^4\}, \{x^3, x, x^5\}$  and the rows as  $\{\chi_0\}, \{\chi_3\}, \{\chi_2, \chi_4, \chi_1, \chi_5\}$ , we can easily



check that the magic rectangle condition is satisfied. The fused table is:

	$e$	$\{(1, 2, 3)\}$	$\{(1, 2)\}$
$\chi_0$	1	1	1
$\chi_1$	1	1	-1
$\chi_2$	2	-1	0

The fused table is the character table for  $D_6$ .

### 3.3 Basic Results

We now prove some basic results concerning fusions of groups.

**Lemma 3.3.1.** *Suppose  $H$  fuses to  $G$ , where the classes of  $G$  are  $C_1, \dots, C_t$  and the corresponding subsets of  $H$  are  $H_1, \dots, H_t$ . Then  $|C_i| = |H_i|$  for all  $i \leq t$ . In particular, we have  $|H| = |G|$ .*

*Proof.* Consider  $C_i$  and  $H_i$  for some  $i$ . We have

$$\bar{C}_i \bar{G} = |C_i| \bar{G},$$

and

$$\bar{H}_i \bar{H} = |H_i| \bar{H}.$$

Since  $C_i$  and  $H_i$  correspond, we must have  $|C_i| = |H_i|$  as required.  $\square$

We then have this easy corollary.

**Corollary 3.3.2.** *If  $B$  is an abelian group and if  $A$  fuses to  $B$ , then  $A$  and  $B$  are isomorphic.*

*Proof.* By Lemma 3.3.1, we see that  $|A| = |B|$  and all the principal subsets of  $A$  are of size 1. Since  $B$  is an abelian group, the correspondence under the isomorphism of  $S$ -rings is a one-to-one map of singletons. Moreover, the multiplication tables of the two groups are the same since the structure constants are the same under the fusion.  $\square$

The following result will be useful in proving the subsequent three lemmas, of which we will make constant use.

**Lemma 3.3.3.** *[LMa] Let  $S$  be an  $S$ -ring over the group  $G$ . Suppose there is a normal subgroup  $H$  of  $G$  such that  $\bar{H} \in S$ . Let  $\rho : G \rightarrow G/H$  be the natural epimorphism.*

i) If  $D = g_1A_1 \cup \dots \cup g_kA_k$  is an  $S$ -principal subset of  $G$  where the  $A_i$ 's are nonempty subsets of  $H$  and  $g_1H, \dots, g_kH$  are distinct cosets of  $H$ , then  $|A_1| = \dots = |A_k|$ .

ii) If  $D_1, D_2$  are  $S$ -principal subsets of  $G$ , then either  $\rho D_1 \cap \rho D_2 = \emptyset$  or  $\rho D_1 = \rho D_2$ .

*Proof.* i) Since,  $A_i \subset H$  for all  $i$ , we have,

$$\begin{aligned} \bar{D}\bar{H} &= |A_1|g_1\bar{H} + \dots + |A_k|g_k\bar{H} \\ &= \lambda\bar{D} + \sum \lambda_i\bar{D}_i \\ &= \lambda(g_1\bar{A}_1 + g_2\bar{A}_2 + \dots + g_k\bar{A}_k) + \sum \lambda_i\bar{D}_i, \end{aligned}$$

for integers  $\lambda, \lambda_i$  and where the  $D_i$ 's are principal subsets. But clearly, by comparing coefficients, we must have that  $\lambda = |A_1| = \dots = |A_k|$ .

ii) Suppose  $D_1 = g_1B_1 \cup \dots \cup g_kB_k$  and  $D_2 = g_1C_1 \cup \dots \cup g_kC_k$  where  $B_i, C_i \subset H$  with  $B_i, C_i$  not necessarily nonempty and  $g_1H, \dots, g_kH$  distinct cosets of  $H$ . Assume,  $\rho D_1 \cap \rho D_2 \neq \emptyset$ . Then there is some  $j$  such that  $B_j \neq \emptyset$  and  $C_j \neq \emptyset$  and we have,

$$\bar{D}_1\bar{H} = \lambda_1\bar{D}_1 + \lambda_2\bar{D}_2 + \dots$$

where  $\lambda_1, \lambda_2$  are positive integers since  $g_j\bar{H}$  must be in the product above. Suppose  $g_iC_i \neq \emptyset$ . But this can happen only if the corresponding  $g_iB_i$  isn't empty in the decomposition of  $D_1$  else  $g_iH$  is an empty set and  $g_i\bar{C}_i$  can't appear on the right side of the equation. Similarly, if  $B_i \neq \emptyset$ , we consider  $\bar{D}_2\bar{H}$  and conclude that  $C_i \neq \emptyset$ . Thus the map either sent principal subsets to disjoint sets of cosets or to the same set.  $\square$

Now we state a useful result due to Humphries and Johnson[HumphriesJohnson].

**Definition 3.3.4.** *Let  $G, H$  be groups and let  $A$  be an  $S$ -ring on  $G$ . We say that  $H$  fuses to  $A$  if there is a subalgebra of  $Z(\mathbb{C}H)$  which is isomorphic to  $A$ .*

**Lemma 3.3.5.** [*HumphriesJohnson*] Suppose that  $H$  fuses to  $G$ . Let  $N$  be a normal subgroup of  $G$ . Then there is a normal subgroup  $M$  of  $H$  such that  $M$  fuses to  $F_G(N)$  and  $H/M$  fuses to  $G/N$ .

*Proof.* Since  $N$  is normal in  $G$ ,  $N = C_1 \cup \cdots \cup C_t$ , where the  $C_i$ 's are conjugacy classes of  $G$ . Let  $H_1, \dots, H_t$  be the corresponding subsets of  $H$  under the fusion and  $M = H_1 \cup \cdots \cup H_t$ . We first prove that  $M$  is a subgroup. Since the structure constants of products of the  $\bar{H}_i$ 's and  $\bar{C}_i$ 's are the same, we see that  $M$  is closed under multiplication and thus is a subgroup. It is normal, being a union of conjugacy classes.

To prove the second part, let  $\pi : G \rightarrow G/N$  be the quotient map and  $G/N = \{w_1, \dots, w_u\}$ . Let  $D_i = \{w_{i_1}, \dots, w_{i_{k_i}}\}, i \leq s$ , be the conjugacy classes of  $G/N$ . Let  $E_i = \pi^{-1}(D_i), i \leq s$ , which is a union of cosets of  $N$ . If  $\pi(v_i) = w_i$ , for  $v_i \in G$ , then  $E_i = \{v_{i_1}N, \dots, v_{i_{k_i}}N\}$ . If  $w_i w_j = w_k$ , then we have  $(v_i N)(v_j N) = v_k N$  so that  $\overline{v_i N v_j N} = |N| \overline{v_k N}$ . Thus if  $\bar{D}_i \bar{D}_j = \sum_k \lambda_{ijk} \bar{D}_k$ , then  $\bar{E}_i \bar{E}_j = |N| \sum_k \lambda_{ijk} \bar{E}_k$  and the structural constants for the  $D_i$ 's and  $E_i$ 's differ by a factor of  $|N|$ .

Now to simplify notation, we write the conjugacy class  $D_i$ , a set of cosets of  $N$ , as  $D_i = \{N_1, \dots, N_t\}$ . Then by Lemma 3.3.3,  $E_i = \pi^{-1}(D_i)$  is a union of conjugacy classes of  $G$ , say  $J_1, \dots, J_n$  where each  $J_j$  is evenly spread out over the  $N_k$ 's as defined by that lemma. More precisely,  $J_j = g_1 A_1 \cup \cdots \cup g_t A_t$ , where the  $A_i$ 's are nonempty subsets of  $N$  of equal size and  $g_i A_i \subset N_i, 1 \leq i \leq t$ . In other words, the  $J_j$ 's partition  $E_i$ 's. Consider  $J_1 = g_1 A_1 \cup \cdots \cup g_t A_t$ . Since  $H$  fuses to  $G$ , let  $K_1$  denotes the corresponding set in  $H$ . We note that  $K_1$  is a union of classes of  $H$ . By the same lemma,  $K_1$  is evenly spread out over the  $t$  cosets of  $M$ , say,  $M_1, \dots, M_t$ . To see this, consider  $\overline{J_1 N} = |A_1| \sum_{i=1}^t \overline{N_i}$ . Since  $H$  fuses to  $G$ , this implies  $\overline{K_1 M} = |A_1| \sum_{i=1}^t \overline{M_i}$ , where the  $M_i$ 's are cosets of  $M$ . Let  $B_i$  denote this set of cosets.

We claim that the correspondence between such  $B_i$ 's and  $D_i$ 's will be our required

fusion of  $H/M$  to  $G/N$ . Now,  $F_i = (\pi')^{-1}(B_i)$  is a union of conjugacy classes of  $H$ . We now show that just as  $E_i$  was partitioned into  $J_j$ 's above,  $F_i$  can be partitioned into  $K_1, \dots, K_n$  such that  $K_j$  correspond to  $J_j$  under the fusion of  $H$  to  $G$ . Consider  $J_k$ , for some  $k$ . Suppose  $J_k$  corresponds to  $J'_k$  in  $H$ . Then  $\bar{J}_k \bar{N}$  is a linear combination of  $N_1, \dots, N_s$ , a set of cosets which contain  $J_1$ . By looking at structure constants, we see that  $\bar{J}'_k \bar{M}$  must be a linear combination of cosets which contain  $K_1$ , i.e., the set  $M_1, \dots, M_t$ . Thus  $J'_k$  must be one of  $K_1, \dots, K_n$ . Similarly we can switch the role of  $H$  and  $G$  to get the correspondence  $K_j \rightarrow J_j, j \leq n$ . Similar to the above analysis, the structure constants for the  $B_i$ 's and the  $F_i$ 's differ by a factor of  $|N|$  and thus the structure constants for the  $B_i$ 's and  $D_i$ 's are the same as required.  $\square$

**Corollary 3.3.6.** *If  $H$  fuses to  $G$ , then there is a subgroup  $N$  containing  $H'$  such that  $H/N \cong G/G'$ .*

*Proof.* By Lemma 3.3.5, there is a normal subgroup  $N \subset H$  corresponding to  $G' \subset G$ . That same results shows that  $H/N$  fuses to  $G/G'$ . But the latter is an abelian group and by Corollary 3.3.1, we have  $H/N \cong G/G'$ . That  $N$  contains  $H'$  follows since  $H/N$  is abelian.  $\square$

**Lemma 3.3.7.** *Suppose that  $H$  fuses to  $G$  and let  $N$  be a normal subgroup of  $H$  such that  $\bar{N}$  is in the  $S$ -ring over  $H$ . Then there is a normal subgroup  $M$  of  $G$  such that  $H/N$  fuses to  $G/M$ .*

*Proof.* Let the conjugacy classes of  $G$  be  $C_1, \dots, C_t$  and the corresponding subsets in  $H$  be  $H_1, \dots, H_t$ . By hypothesis, we have  $\bar{N} = \sum_k \mu_k \bar{H}_k, \mu_k \in \{0, 1\}$ . Let  $M$  be the union of the  $C_k$ 's where  $\mu_k \neq 0$ . Then  $M$  is closed under multiplication and is normal as it is a union of conjugacy classes. By 3.3.5, we have that  $H/N$  fuses to  $G/M$ .  $\square$

# Chapter 4

## Groups which fuse from $D_{2n}$

In this chapter, we begin our investigation of groups  $G$  which fuse from a dihedral group.

### 4.1 The Dihedral Group

Recall that the dihedral group,  $D_{2n}$ , has the following presentation:

$$D_{2n} = \langle x, y : x^n = 1, y^2 = 1, y^{-1}xy = x^{-1} \rangle.$$

Here are some properties of this group that will prove useful:

The subgroup  $\langle x \rangle$  is cyclic of size  $n$  and index 2, and thus is normal in  $D_{2n}$ .

If  $n = 2m + 1$ , then the  $\frac{n+3}{2} = m + 2$  conjugacy classes of  $D_{2n}$  are:

$$C_i = \{x^i, x^{-i}\} (0 \leq i \leq m), C_{m+1} = \{x^j y | 1 \leq j \leq n\}.$$

Moreover,  $Z(D_{2n}) = \{1\}$ .

If  $n = 2m$ , the  $\frac{n}{2} + 3 = m + 3$  classes of  $D_{2n}$  are:

$$C_i = \{x^i, x^{-i}\} (0 \leq i \leq m), C_{m+1} = \{x^{2^j}y | 1 \leq j \leq m\}, C_{m+2} = \{x^{2^{j-1}}y | 1 \leq j \leq m\}.$$

In this case,  $Z(D_{2n}) = \langle x^{n/2} \rangle \cong \mathbb{Z}_2$ .

The question that we want to investigate is: which groups fuse from dihedral groups?

## 4.2 $D_{2n}$ fuses to $G$ where $n$ is odd

In this section, we investigate the case when  $n$  is odd and prove that  $G$  must be the dihedral group.

**Lemma 4.2.1.** *Let  $G$  be a finite group. Suppose  $G$  fuses from the dihedral group  $D_{2n}$ , where  $n$  is odd. Then  $G$  has a class  $C$  with  $n$  elements and all elements of  $C$  have order 2.*

*Proof.* Since  $D_{2n}$  has a class of size  $n$ ,  $G$  has a class  $C$  of size at least  $n$ . And since  $|C|$  divides  $|G|$  and  $C$  clearly doesn't contain the identity, we have that  $|C| = n$ . Let  $x \in C$ . Since  $G$  acts transitively on  $C$  by conjugation, we have

$$|G| = 2n = |C||Stab_G(x)| = n|Stab_G(x)|.$$

So the subgroup  $Stab_G(x)$  has size 2. But  $x$  and  $e$  are in  $Stab_G(x)$  and so we must have  $x^2 = e$ . Thus all elements of  $C$  have order 2.  $\square$

Next, we prove a result of Mann[Mann] which we will need later. If  $G$  is a finite group, we denote by  $k = k(G)$  the number of conjugacy classes of  $G$ . Let  $d_1, \dots, d_k$  be the degrees of the irreducible characters of  $G$ , and write  $T(G)$  for the sum of these degrees, that is,

$$T(G) = \sum_{i=1}^k d_i.$$

Then we also have

$$|G| = \sum_{i=1}^k d_i^2.$$

**Lemma 4.2.2.** *[Mann] For  $k$  and  $T(G)$  defined above, we have*

$$\sqrt{|G|} \leq T(G) \leq \sqrt{|G|k}.$$



*Proof.* The left-hand inequality is clear as

$$\sqrt{d_1^2 + \cdots + d_k^2} \leq \sqrt{(d_1 + \cdots + d_k)^2} \leq d_1 + \cdots + d_k.$$

Let  $d = (d_1, \dots, d_k)$  and  $v = (1, \dots, 1)$ . By the Cauchy Schwartz Inequality, we have:

$$|\langle d, v \rangle|^2 \leq \langle d, d \rangle \cdot \langle v, v \rangle,$$

which gives

$$T(G)^2 \leq |G|k,$$

and so

$$T(G) \leq \sqrt{|G|k},$$

as required. □

**Lemma 4.2.3.** [Mann] Let  $t(G)$  be the number of elements of  $G$  whose square is 1.

Then there exist numbers  $v_i = 1, 0, \text{ or } -1$  for  $i = 1, \dots, k$ , such that

$$t(G) = \sum_{i=1}^k v_i d_i.$$

*Proof.* See [Mann]. □

**Corollary 4.2.4.** [Mann]  $t(G) \leq T(G)$ .

*Proof.* From Lemma 4.2.3, we have

$$t(G) = \sum_{i=1}^k v_i d_i \leq \sum_{i=1}^k d_i = T(G),$$

as required. □

We apply these results to  $G$ . Since the class  $C$  of  $G$  has  $n$  involutions, we see that  $t(G) \geq n + 1$  and using Lemma 4.2.2, we have

$$n + 1 \leq t(G) \leq T(G) \leq \sqrt{2nk}.$$

This gives

$$(n + 1)^2 \leq 2nk,$$

and so

$$(n^2 + 2n + 1)/(2n) \leq k,$$

and finally,

$$[n/2 + 1 + 1/(2n)] \leq k. \tag{4.1}$$

By (4.1) and the fact that  $k$  is bounded above by the number of conjugacy classes of  $D_{2n}$  we have

$$n/2 + 1 + 1/(2n) \leq k \leq n/2 + 3/2.$$

As  $n$  is odd,  $k = n/2 + 3/2$ , which is the number of conjugacy classes for  $D_{2n}$ . Thus  $G$  has the same number of classes as  $D_{2n}$ . We have just proved:

**Proposition 4.2.5.** *Suppose  $G$  fuses from the dihedral group,  $D_{2n}$ , where  $n$  is odd. Then  $G$  has a class,  $C$ , of size  $n$  and every other conjugacy class of  $G$  has size 1 or 2. Further, the fusion of  $D_{2n}$  to  $G$  determines an isomorphism of the class algebras of  $D_{2n}$  and  $G$ .*

We will take advantage of the following three equivalent statements:

- 1)  $H$  fuses to  $G$  and  $G$  fuses to  $H$ .
- 2)  $Z(\mathbb{C}G) \cong Z(\mathbb{C}H)$ .
- 3)  $G$  and  $H$  have the same character table.

The equivalence of 1) and 2) is trivial. The equivalence of 2) and 3) can be found in [Feit, pg 42]. Now we prove Theorem 2.0.6 part i).

**Theorem 4.2.6.** *With the same hypotheses as Proposition 4.2.5,  $G$  is isomorphic to  $D_{2n}$ .*

Proof: By Proposition 4.2.5, we have  $Z(\mathbb{C}G) \cong Z(\mathbb{C}D_{2n})$ . Thus the character tables for the two groups are the same. By a result of [Cel], stated below,  $G$  is isomorphic to  $D_{2n}$ .

**Proposition 4.2.7.** *[Cel] If  $G$  is a finite group with the same character table as  $D_{2n}$  or  $Dic_{2n}$  then  $G$  is isomorphic to  $D_{2n}$  or  $Dic_{2n}$ . Moreover,  $D_{2n}$  and  $Dic_{2n}$  have the same character table if and only if  $n \equiv 0(4)$ .*

*Proof.* See [Cel]. □

### 4.3 $D_{2n}$ fuses to $G$ where $n$ is even

**Proposition 4.3.1.** *Suppose  $D_{2n}$  fuses to  $G$  where  $n$  is even. Then  $G$  does not have a class of size  $n$ .*

*Proof.* Suppose  $G$  does have a class  $C$  of size  $n$ . Let  $k$  be the number of classes of  $G$ . As in the case when  $n$  is odd, each element in  $C$  is an involution. By Lemma 4.2.2 and 4.2.4, we have

$$n + 1 \leq t(G) \leq T(G) \leq \sqrt{(2nk)}.$$

By the same argument, we conclude that

$$k \geq [n/2 + 1 + 1/(2n)].$$

Since  $n$  is even, we have

$$k \geq n/2 + 2. \tag{4.2}$$

If  $n$  is even, the center of  $D_{2n}$  has size 2 and thus by the fusion, the center of  $G$  must have size at most 2. Suppose first that  $Z(G)$  is trivial. Since  $G$  has a class of size  $n$ , it also has  $k - 2$  classes each with at least two elements so that

$$1 + 2(k - 2) \leq n.$$

Solving for  $k$  and using (4.2), we obtain

$$\frac{n}{2} + 2 \leq k \leq \frac{n}{2} + \frac{3}{2},$$

which is a contradiction. Thus  $Z(G)$  has order 2, that is,  $Z(G) = \{1, z\}$ ,  $z^2 = 1$ ,  $z \neq 1$ . But this implies that for any  $g \in C$ ,  $Stab_G(g)$  contains  $\{1, g, z, zg\}$ , which is also a

contradiction since this implies by the Orbit-Stabilizer Theorem that  $|G| = 4n$ .  $\square$

As a consequence of this proposition, we have Property 1.

We now prove Property 2.

**Lemma 4.3.2.** *If  $D_{2n}$  fuses to  $G$ ,  $n = 2m$  and  $Z(G) = \{1\}$ , then the normal subgroups  $D'_{2n}$  and  $G'$  correspond under the fusion. Thus,  $G/G'$  is isomorphic to  $\mathbb{Z}_2^2$ .*

*Proof.* The dihedral group  $D_{2n}$  has two classes  $C_1, C_2$  of involutions, each of size  $m$ . By Proposition 4.3.1, these two classes do not fuse. Thus  $G$  has two classes of size  $m$ . Let  $E_1, E_2$  be classes of  $G$  corresponding to  $C_1, C_2$  and  $h_i \in C_i, g_i \in E_i, i = 1, 2$ . In this case, we have  $\bar{C}_i = h_i \overline{D'_{2n}}, i = 1, 2$ , which implies  $\bar{C}_i^2 = m \overline{D'_{2n}}, i = 1, 2$ . Thus, because of the fusion, there is a normal subgroup  $N \subset G$  corresponding to  $D'_{2n}$  such that  $\bar{E}_i^2 = m\bar{N}, i = 1, 2$ . By Lemma 3.3.5,  $D_{2n}/D'_{2n}$  fuses to  $G/N$ . Since  $[D_{2n} : D'_{2n}] = 4$ , we have  $[G : N] = 4$ .

By Lemma 3.3.6, there is a subgroup  $M$  of  $D_{2n}$  containing  $D'_{2n}$  which corresponds to  $G'$  with  $D_{2n}/M \cong G/G'$ . We claim that  $M = D'_{2n}$  and  $N = G'$ . The quotient  $G/N$  has size 4. Thus  $G/N$  is abelian and therefore  $N$  contains  $G'$ . We then have the natural onto homomorphism  $G/G' \rightarrow G/N$ . And because  $D'_{2n} \subset M$ , we also have the natural homomorphism  $D_{2n}/D'_{2n} \rightarrow D_{2n}/M$ . Since these homomorphisms are onto, we must have  $|G/G'| \geq 4$  and  $|D_{2n}/M| \leq 4$ . We have

$$4 = |D_{2n}/D'_{2n}| \geq |D_{2n}/M| = |G/G'| \geq |G/N| = 4.$$

Thus,  $M = D'_{2n}$  and  $N = G'$ .  $\square$

**Lemma 4.3.3.** *Suppose  $n = 2m, m \geq 2$ . Then the dihedral group  $D_{2n} = \langle x, y | x^n, y^2, y^{-1}xy = x^{-1} \rangle$  has three subgroups of index 2. They are:*

$$M_1 = \langle x \rangle; \quad M_2 = \langle x^2, y \rangle; \quad M_3 = \langle x^2, xy \rangle.$$

Here,  $M_1$  is cyclic and  $M_2, M_3$  are dihedral groups of order  $n = 2m$ . In addition, for  $1 \leq i \neq j \leq 3$ ,  $M_i \cap M_j = \langle x^2 \rangle = D'_{2n}$ . If  $C_1, C_2$  are classes of  $D_{2n}$  of size  $m$ , we can reindex  $M_2, M_3$  so that  $C_1$  is a class in  $M_2$  and  $C_2$  is a class in  $M_3$ .

*Proof.* Clearly,  $M_i, i = 1, 2, 3$  has index 2. Conversely, any subgroup of index 2 is normal and contains the commutator subgroup,  $D'_{2n}$  with  $D_{2n}/D'_{2n} \cong \mathbb{Z}_2^2$ . Since  $\mathbb{Z}_2^2$  has three subgroups of index 2, by the Correspondence Isomorphism Theorem, there are three subgroups of  $D_{2n}$  of index 2 containing the commutator subgroup. The rest is obvious.  $\square$

**Lemma 4.3.4.** *Let  $N$  be a subgroup of  $G$  of index 2. Let  $x \notin N$  such that  $G = \langle N, x \rangle$ . Suppose  $g \in N$  and denote by  $C$  the class of  $g$  in  $N$  and  $C'$  the class of  $g$  in  $G$ . Then, either  $C' = C$  or  $C' = C \cup C^x$  with  $C \cap C^x = \emptyset$ . Thus either  $|C'| = |C|$  or  $|C'| = 2|C|$ .*

*Proof.* Since  $N$  has index 2 in  $G$ ,  $N$  is normal in  $G$ . Let  $\{1, a\}$  be the transversal for  $N$ . Thus,  $G = \langle N, a \rangle$ . For each  $h \in G \setminus N$ , we can write  $h = h_0a$ , with  $h_0 \in N$ . Then,  $C^h = C^{h_0a} = C^a$ . We now show that either  $C = C^a$  or  $C \cap C^a = \emptyset$ , or equivalently, if their intersection is nonempty, then  $C = C^a$ . Suppose  $u \in C \cap C^a$ . Then  $u \in C$  and  $u = w^a$ , with  $w \in C$ . We pick any  $v \in C$ . Since  $C$  contains both  $u, v$ , there is a  $y \in N$  such that  $v = u^y$ . We have,

$$v = u^y = (w^a)^y = w^{ay} = w^{yy^{-1}ay}.$$

But  $y^{-1}ay = za$  for some  $z \in N$ . Thus,

$$v = w^{y(za)} = (w^{yz})^a \in C^a.$$

Thus,  $C \subset C^a$  and  $C = C^a$  since they have the same size.  $\square$

**Theorem 4.3.5.** *Suppose  $D_{2n}$  fuses to  $G$ , with  $n = 2m$ , where  $m$  is odd. Then  $G \cong D_{2n}$ .*

*Proof.* By Lemma 4.3.2,  $G/G' \cong \mathbb{Z}_2^2$ . Now,  $\mathbb{Z}_2^2$  has three subgroups of index 2, all of which are normal. We denote their complete preimages in  $G$  by  $N_1, N_2, N_3$ . By Lemma 3.3.5, let  $M_1, M_2, M_3$  be the corresponding subgroups in  $D_{2n}$ . We can rename these groups to coincide with those in Lemma 4.3.3. Now consider the two dihedral subgroups of  $D_{2n}$ ,  $M_2$  and  $M_3$ . As  $m$  is odd,  $M_2, M_3$  each has a class of size  $m$ . Thus  $C_1$  and  $C_2$  are also classes in  $M_2$  and  $M_3$ , respectively. Under the fusion,  $N_2, N_3$  must each have a class of size at least  $m$ . Denote these classes by  $E_2, E_3$ . But since  $N_i$  has order  $2m$  and  $E_i \subset N_i$ , we have that  $|E_i| = m, i = 2, 3$ . It is then clear that  $E_2, E_3$  must correspond to  $C_1, C_2 \subset D_{2n}$ . Consider any element  $g_i \in E_i$ . By Lemma 4.3.4, the conjugacy class of  $g_i$  in  $N_i$  has size  $m$  or  $m/2$ . As  $m$  is odd, we see that  $E_i$  must also be a class in  $N_i, i = 2, 3$ . This implies that each element in  $E_i$  must have order 2. Hence, the  $E_i$ 's are classes of involutions.

Let  $E = G \setminus (N_2 \cup N_3)$ . Then  $E$  is a union of classes of  $G$ . Let  $E'$  be the corresponding union of classes of  $D_{2n}$ . Since  $E'$  is the complement of  $M_2 \cup M_3$  in  $D_{2n}$ , we have

$$E' = \{x, x^3, \dots, x^{2m-1}\}.$$

$E'$  has  $m$  elements and is its own inverse. Since  $x$  has order  $2m$  and  $m$  is odd,  $E'$  has an involution. Thus,  $E$  also has an involution. We then have  $t(G) \geq 2m + 2$ . As before, we have,

$$2m + 2 \leq t(G) \leq \sqrt{4mk},$$

where  $k$  is the number of classes of  $G$ . Thus,  $k \geq m + 2 + 1/m$ . But  $k$ , being an integer, implies  $k \geq m + 3$ . But for  $n = 2m$ ,  $D_{2n}$  has  $m + 3$  classes. Thus the centers of the two group algebras are isomorphic. By Proposition 4.2.7,  $G \cong D_{2n}$  or  $G \cong Dic_{2n}$ .

Now we prove that  $G$  can't be isomorphic to the dicyclic group. Since  $m$  is odd, by Proposition 4.2.7, the character tables of the dihedral and dicyclic groups are different and hence  $Z(\mathbb{C}D_{2n}) \not\cong Z(\mathbb{C}Dic_{2n})$ . Thus,  $G \cong D_{2n}$ .  $\square$

We prove Property 3.

**Proposition 4.3.6.** *Suppose that  $D_{2n}$  fuses to  $G$ , with  $n = 2m$ ,  $m$  is even and  $Z(G) = \{1\}$ . Then  $G$  has a unique class  $C$  of odd size. Moreover,  $C$  is contained in  $G'$  and is a class of involutions.*

*Proof.* Since  $n$  is even,  $D_{2n}$  has nontrivial center of size 2, say  $Z(D_{2n}) = \{1, z\}$ . Let  $C_1, C_2$  be the two classes of size  $m$ . By assumption, we have  $Z(G) = \{1\}$ , so that the class  $\{z\}$  of  $D_{2n}$  must fuse with some of the other classes of  $D_{2n}$ . By order consideration, since classes have orders dividing the group, it can't fuse with  $C_1$  or  $C_2$ . Thus the class  $\{z\}$  must fuse with a nonzero number of classes of  $D_{2n}$  all of which have size 2. The other fusions will consist of classes of order 2 and thus have even size. Moreover,  $C$  is contained in the subgroup,  $N_1$ , of  $G$  corresponding to the full rotational subgroup of  $D_{2n}$ . But since  $\{1\}$  and  $C$  are the only classes of odd size contained in  $N_1$ , we must have that  $C \subset G'$ . Since  $|C|$  is the only nontrivial class of odd size, we have that  $C^{-1} = C$  and thus there must be at least one element in  $C$  which is its own inverse, i.e an involution. Since all elements in  $C$  have the same order,  $C$  is a class of involutions.  $\square$

Now we prove Properties 4-7.

**Proposition 4.3.7.** *Suppose that  $D_{2n}$  fuses to  $G$ , where  $n$  is even. Then the character degrees of  $G$  are*

$$1, 1, 1, 1, d_5, \dots, d_k,$$



where  $d_5, \dots, d_k$  are even. Also,  $r$ , the number of non-linear characters of  $D_{2n}$  which fuse to give a character of  $G$ , is a square.

*Proof.* Since  $G/G'$  has size 4,  $G$  has 4 linear characters. By the Magic Rectangle Condition, the non-linear characters of  $G$  are formed by fusing  $r$  of the degree 2 characters of  $D_{2n}$ . By using the first column, i.e. the column of the identity class, the degree of any non-linear character of  $G$  is

$$\frac{\sum_{m=1}^{s_i} d_{im} \chi_{im}(C_{jk})}{\sum_{m=1}^{s_i} d_{im}^2} \cdot \sqrt{\sum_{m=1}^{s_i} d_{im}^2} = \frac{\sum_{i=1}^r 2(2)}{\sum_{i=1}^r 2^2} \cdot \sqrt{\sum_{i=1}^r 2^2} = 2\sqrt{r},$$

which is even as required.  $\square$

**Lemma 4.3.8.** *Suppose that  $D_{4m}$  fuses to  $G$  where  $Z(G) = \{1\}$ . Then  $G'$  is not abelian.*

*Proof.* Suppose that  $G'$  is abelian. Let  $x \in G'$ . Then  $G' \subset C_G(x)$ . Since  $G/G' \cong \mathbb{Z}_2^2$ , the conjugacy class containing  $x$  must have 2 or 4 elements. But this is a contradiction as we have proven that there is a class of odd size in  $G'$ .  $\square$

**Lemma 4.3.9.** *Suppose that  $D_{4m}$  fuses to  $G$  where  $Z(G) = \{1\}$ . Then  $f = \min\{\chi(1) \mid \chi \in \text{Irr}(G), \chi(1) > 1\} = 2$ .*

*Proof.* By Proposition 4.3.7,  $f$  is even. Suppose  $f \geq 4$ . Then  $4 = |G/G'| \leq f$  and thus by [Isaacs(5.14)(b); pg 75], we have  $G'$  is abelian, which is a contradiction to Lemma 4.3.8.  $\square$

**Lemma 4.3.10.** *If  $D_{2n}$  fuses to  $G$ , then each character of  $G$  is real.*

*Proof.* Since each element of  $D_{2n}$  is conjugate to its inverse, the entries in the character table for  $D_{2n}$  are real. But the character table of  $G$  can be obtained from the character

table of  $D_{2^n}$  by fusing using the Magic Rectangle Condition. Thus each character of  $G$  is real.  $\square$

**Proposition 4.3.11.** *Suppose that  $D_{4m}$  fuses to  $G$ . Then every  $g \in G$  is conjugate to its inverse.*

*Proof.* Since each character of  $G$  is real by Lemma 4.3.10, each element of  $G$  is conjugate to its inverse.  $\square$

We prove Properties 8-9.

**Proposition 4.3.12.** *Suppose that  $D_{4m}$  fuses to  $G$ . Then  $G$  is 2-nilpotent and solvable.*

*Proof.* Since all the non-linear characters of  $G$  are even, then by [Huppert, page 313],  $G$  is 2-nilpotent, that is, the Sylow 2-group of  $G$  has a normal complement. Let  $N$  be the normal subgroup of odd order and index a power of 2. By Feit-Thompson,  $N$  is solvable. And since  $G/N$  is a 2-group, it is solvable. Thus  $G$  is solvable.  $\square$

We call  $G$  a  $p$ -group of maximal class if  $G$  has order  $p^n$  and nilpotent class  $n - 1$ . We state two theorems from Finite Groups by Huppert[Hup]. First, a *quasidihedral group* is the group with the following presentation,

$$\langle r, s \mid r^{2^{n-1}} = s^2 = 1, srs = r^{2^{n-2}-1} \rangle.$$

**Theorem 4.3.13.** *If  $G = 2^n$  and of maximal class, then  $G$  is dihedral, dicyclic or quasidihedral.*

**Theorem 4.3.14.** *Let  $G$  be a  $p$ -group of order  $p^n$ ,  $n \geq 3$ .  $G$  is a  $p$ -group of maximal class iff there exists an element in  $G$  whose conjugacy class size is  $p^{n-2}$ .*

**Lemma 4.3.15.** *If  $G$  possesses an element  $x$  with  $|C_G(x)| = 4$ , then the Sylow 2-group,  $P$ , of  $G$  is dihedral, dicyclic or quasidihedral. In particular,  $|P/P'| = 4$  and  $P$  has a cyclic group of order  $|P|/2$ .*

*Proof.* Let  $P$  be a Sylow 2-group of  $G$  of order  $2^n$ . Since  $|C_G(x)| = 4$ ,  $C_G(x)$  is a 2-group. Without loss of generality, we may assume  $C_G(x) \subset P$ . By definition,  $C_P(x) = P \cap C_G(x)$ , thus  $|C_P(x)| = 4$ . The class of  $x$  in  $P$  then has size  $2^{n-2}$  and thus by Theorem 4.3.14,  $P$  is a 2-group of maximal class and  $P$  is a dihedral, dicyclic or quasidihedral by Theorem 4.3.13. The rest follows.  $\square$

Now, we prove Properties 10-13.

**Corollary 4.3.16.** *Suppose that  $D_{4m}$  fuses to  $G$ , where  $m$  is even and  $Z(G) = \{1\}$ . The Sylow 2-group of  $G$  is dihedral, dicyclic or quasidihedral.*

*Proof.* Consider  $C_1$ , the class of size  $m$  in  $G$ . Pick any  $x \in C_1$ . Since  $|C_G(x)| = 4$ , the result follows from Lemma 4.3.15.  $\square$

**Definition 4.3.17.** *Suppose  $G$  is a finite group and  $M, N$  are proper normal subgroups with  $N \leq M$ . We say that  $(G, M, N)$  is a Camina triple if the conjugacy classes in  $G \setminus M$  are union of cosets of  $N$ . If  $M = N$ ,  $(G, N)$  is a Camina pair.*

**Proposition 4.3.18.** *Suppose  $H$  fuses to  $G$ . Suppose that  $(H, M, N)$  is a Camina triple and suppose that there are normal subgroups  $M', N'$  of  $G$  which correspond to  $M$  and  $N$ , respectively. Then  $(G, M', N')$  is a Camina triple.*

*Proof.* Consider a conjugacy class  $C' \subset G \setminus M'$ . We prove that  $C'$  is a union of cosets of  $N'$ . Let  $C'$  corresponds to  $C \subset H \setminus M$  under the fusion. Notice that  $C$  is a union of cosets of  $N$ . By Lemma 3.3.5,  $H/N$  fuses to  $G/N'$ . By the proof of the same lemma, this fusion of quotient groups is determined by the fusion of  $H$  to  $G$  and since  $C$  is a union of cosets of  $N$ ,  $C'$  must be also.  $\square$

**Corollary 4.3.19.** *Suppose that  $D_{4m}$  fuses to  $G$ , where  $m$  is even and  $Z(G) = \{1\}$ . Let  $M$  be the normal subgroup of  $G$  corresponding to  $\langle x \rangle \leq D_{4m}$  under the fusion. Then  $(G, M, G')$  is a Camina triple.*

*Proof.* Since  $(D_{4m}, \langle x \rangle, D'_{4m})$  is a Camina triple, by Proposition 4.3.18,  $(G, M, G')$  is a Camina triple as required.  $\square$

**Proposition 4.3.20.** *Suppose that  $D_{4m}$  fuses to  $G$ , where  $m$  is even.  $G$  is not a direct product.*

*Proof.* Suppose that  $G$  is a direct product of nontrivial subgroups,  $H$  and  $K$ . Then  $D_{2n}$  is also a direct product of nontrivial subgroups. But this is only true if  $m$  is odd, contradicting our hypothesis.  $\square$

**Proposition 4.3.21.** *Suppose that  $D_{2n}$  fuses to  $G$  where  $n = 2m$ . Then  $G$  has at most  $d(n) + 3$  normal subgroups, where  $d(n)$  is the number of divisors of  $n$ . Furthermore, for each divisor  $d < n$  of  $n$ ,  $G$  has unique normal subgroup of order  $d$ .*

*Proof.* By Lemma 3.3.5, each normal subgroup  $N$  of  $G$  corresponds to a normal subgroup in  $D_{2n}$ . Thus the number of normal subgroups of  $G$  is no more than the number of normal subgroups of  $D_{2n}$ , which is  $d(n) + 3$ . The rest follows since with the exception of  $M_2$  and  $M_3$  of Lemma 4.3.3, the only other normal subgroups of  $D_{2n}$  are the cyclic subgroups.  $\square$

## 4.4 $D_{2n}$ fuses to $G$ where $|G| = 8p$

Now we prove our main theorem part iii).

**Theorem 4.4.1.** *Suppose that  $D_{2n}$  fuses to  $G$  where  $|G| = 8p$ . Then  $G \cong D_{2n}$  or  $G \cong Dic_{2n}$ .*

*Proof.* We use the classification of groups of order  $8p$  by [Western]. There are 15 isomorphism classes. We can eliminate types 1)-3) since they are abelian. By Proposition 4.3.20, we can also eliminate types 4), 5), 7)-9) and 14). The commutator subgroup of types 6), 13) and 15) has index at least 8 and hence can also be eliminated by Property 2, Proposition 4.3.2. Type 11) violates Properties 6 and 7 and hence can also be eliminated. We have eliminated all but types 10) and 12) which are the dihedral and dicyclic groups. □

# Chapter 5

## Schur Rings over Cyclic Groups

### 5.1 Classification of S-rings over Cyclic Groups

In this section, we introduce three types of S-ring and note a result of [LMan] which says that an S-ring over  $\mathbb{Z}_n$  is one of these three types.

Suppose  $S$  is an S-ring over a cyclic group  $\mathbb{Z}_m$  for some  $m$ . Then  $S$  is one of the following three types:

**Type 1:** A given *S-ring*  $S$  over  $G$  is of this type if we can find  $H \leq \text{Aut}(\mathbb{Z}_m)$  such that principal subsets of  $S$  are orbits of  $H$ .

**Type 2:** Suppose a cyclic group  $\mathbb{Z}_m$  is a direct product of two subgroups,  $H$  and  $K$ . Let  $H'$  and  $K'$  be S-ring over  $H$  and  $K$ , respectively. Let  $\{D_1, \dots, D_t\}$  and  $\{E_1, \dots, E_s\}$  be the principal subsets of these S-ring. Then the S-ring of this type is generated by the following principal subsets:  $\{\overline{D_i \times E_j} : 1 \leq i \leq t, 1 \leq j \leq s\}$ . We call  $S$  the *dot product* of  $H'$  and  $K'$ .

**Lemma 5.1.1.** [LMan] Let  $H, K$  be subgroups of  $G$  with  $H$  a normal subgroup and  $H \subset K$ . Let  $S$  be an S-ring over  $K$ . Suppose  $H$  is an S-subgroup and  $\rho : G \rightarrow G/H$  is

the natural mapping. Then  $D(\rho^*(S)) = \{\rho(D) : D \in D(S)\}$  generates an S-ring over  $K/H$ .

*Proof.* This is a direct consequence of parts i) and ii) of Lemma 3.3.3.  $\square$

Reversing this process, suppose  $S_{G/H}$  is a S-ring over  $G/H$  with  $D(S_{G/H}) = \{E_1, \dots, E_r\}$  where  $E_1 = \{H\}$ . We define  $D_0 = \{e\}$ ,  $D_1 = H \setminus \{e\}$  and  $D_i = \rho^{-1}(E_i)$  for  $i = 2, \dots, r$ . Then  $\{D_0, D_1, \dots, D_r\}$  generates an S-ring over  $G$ . Moreover,  $\rho^*(\rho^{-1}(S_{G/H})) = S_{G/H}$ .

**Type 3:** To describe our next construction, we'll restate Lemma 3.3.3.

**Lemma 5.1.2.** [LMa] Suppose there is a normal subgroup  $H$  of  $G$  such that  $\overline{H} \in S$ . Let  $\rho : G \rightarrow G/H$  be the natural epimorphism.

i) If  $D = g_1A_1 \cup \dots \cup g_kA_k$  is an S-principal subset of  $G$  where the  $A_i$ 's are nonempty subsets of  $H$  and  $g_1H, \dots, g_kH$  are distinct cosets of  $H$ , then  $|A_1| = \dots = |A_k|$ .

ii) If  $D_1, D_2$  are S-principal subsets of  $G$ , then either  $\rho D_1 \cap \rho D_2 = \emptyset$  or  $\rho D_1 = \rho D_2$ .

**Proposition 5.1.3.** [LMan] Using the notation defined above, suppose  $K \supset H$  is a subgroup in  $G$  with  $K/H$  being an  $S_{G/H}$ -subgroup. For any S-ring  $S_K$  of  $K$  with  $\overline{H} \in S_K$  and  $\rho^*S_K = (\mathbb{Z}[K/H]) \cap S_{G/H}$ , there is an S-ring  $S$  over  $G$  such that

$$D(S) = D(S_K) \cup \{\rho^{-1}(E) : E \in D(S_{G/H}) \text{ with } E \not\subseteq K/H\}.$$

Furthermore,  $S \cap \mathbb{Z}[K] = S_K$  and  $\rho^*S = S_{G/H}$ .

*Proof.* See [LMan, Proposition 1.4].  $\square$

The S-ring  $S$  constructed above is the *wedge product* of  $S_K$  and  $S_{G/H}$  and will be denoted as  $S_K \wedge S_{G/H}$ .

Thus for any given S-ring  $S$  over  $G = \mathbb{Z}_m$  of this type, there exist cyclic subgroups  $H$  and  $K$  satisfying the above proposition with  $S = S_K \wedge S_{G/H}$ .

**Example 5.1.4.**

Suppose  $G = A_4$ . Let

$$H = K = A'_4 = \{1, (12)(34), (13)(24), (14)(23)\}$$

be subgroups of  $G$ . Let

$$D(S_{G/H}) = \{\{A'_4\}, \{(123)A'_4\}, \{(132)A'_4\}\}.$$

Trivially,  $K/H$  is an  $S_{G/H}$ -subgroup. Let

$$D(S_K) = \{\{1\}, \{(12)(34), (13)(24), (14)(23)\}\}$$

be the principal subsets which generate the  $S$ -ring over  $K$ . Clearly, we have that  $\overline{H} \in S_K$  and  $\rho^* S_K = (\mathbb{Z}[K/H]) \cap S_{G/H}$ . Then the principal subsets which generate the wedge product over  $G$  are

$$\{\{1\}, \{(12)(34), (13)(24), (14)(23)\}, \{(123)A'_4\}, \{(132)A'_4\}\}.$$

This is the class algebra of  $A_4$  which we have shown to be a wedge product. We now state some important results from [LMan] about S-ring over cyclic groups.

Throughout this section, we assume  $G$  is cyclic and  $|G|$  is not prime. Let  $S$  be the S-ring over  $G$ . Assume  $S$  is nontrivial and imprimitive. Since  $S$  is nontrivial, we can consider all the proper  $S$ -subgroups of  $G$ , partially order them and pick  $H$  the minimal, nontrivial subgroup. Then  $S \cap \mathbb{Z}[H]$  is a primitive  $S$ -ring over  $H$  by choice of  $H$ . Suppose further that  $|H|$  is not prime. Then  $S \cap \mathbb{Z}[H]$  must be trivial by Corollary 5.2.4. Thus  $H \setminus \{e\}$  is a principal subset. Let  $p$  be a divisor of  $|H|$  and the unique subgroup of  $H$  of order  $p$  be denoted  $P_1$ .



With this setup, we have the following consequences.

**Theorem 5.1.5.** *[LMan, Thm 3.4] Suppose  $H \setminus \{e\}$  is a principal subset of  $S$  and the order of  $H$  is not prime. For any principal subset  $D$  which is not a union of  $H$ -cosets,  $D = E$  or  $D = E \cdot (H \setminus \{e\})$ , where  $E$  is a principal subset such that  $\langle E \rangle \cap H = \{e\}$ .*

*Proof.* See [LMan, Theorem 3.4]. □

We now prove a theorem which describes  $S$  using smaller subgroups. Consider

$$\{K \mid K \text{ is an } S\text{-subgroup with } K \cap H = \{e\}\}.$$

This set is nonempty as  $\{e\}$  is in it. Pick  $K$  with the property that  $|K|$  has the largest order.

**Theorem 5.1.6.** *[LMan, Thm 3.5] Suppose  $|H|$  is not a prime and  $H \setminus \{e\}$  is a principal subset. Let  $K$  be as above. Then  $S(KH) = S(K) \cdot S(H)$ . In particular, if  $G = KH$ , then  $S = S(K) \cdot S(H)$ . And if  $HK \neq G$ , and  $\rho : G \rightarrow G/H$  is the natural homomorphism, then  $S = S(HK) \wedge \rho^*S$ .*

*Proof.* Clearly,  $|H|, |K|$  are relatively prime. If  $D \subset K$ , we're done since  $D = D \cdot \{e\}$ . If  $D$  is any principal subset in  $KH \setminus K$ , then  $D$  can't be a union of  $H$ -cosets since  $KH$  is precisely the union of  $kH$ -cosets. By Theorem 5.1.5,  $D = E \cdot (H \setminus \{e\})$  or  $E$ , where  $E$  is a principal subset whose generated group intersects  $H$  trivially. Since  $E \subset HK$ ,  $\langle E \rangle \subset K$ . Thus  $S(KH) = S(K) \cdot S(H)$ .

To show the last part of the theorem, we need only show that  $D$  is a union of  $H$ -cosets if  $D$  is not in  $HK$ . Suppose not, then by 5.1.5,  $D = E \cdot (H \setminus \{e\})$  or  $E$ . Since  $D \not\subset HK$ , we have that  $E \not\subset K$ . But this implies  $\langle E \rangle K$  is a  $S$ -subgroup properly containing  $K$ , which also intersects  $H$  trivially, contradicting our choice of  $K$ . Thus  $D$  must be a union of  $H$ -cosets. □

**Example 5.1.7.**

Suppose  $G = \mathbb{Z}_{12} = \langle x \rangle$ . Suppose the S-ring  $S$  over  $G$  is generated by the following principal subsets:

$$\{\{e\}, \{x^4, x^8\}, \{x^3, x^6, x^9\}, \{x, x^2, x^5, x^7, x^{10}, x^{11}\}\}.$$

Let  $H = \langle x^3 \rangle$ . Clearly,  $H$  satisfies the conditions of the theorem. In this case,  $K = \langle x^4 \rangle$ . We then see that  $S = S(K) \cdot S(H)$ , where  $S(K) = \{\{e\}, \{x^4, x^8\}\}$  and  $S(H) = \{\{e\}, \{x^3, x^6, x^9\}\}$ .

**Example 5.1.8.**

Let  $G$  be as above. Let  $S$  be the S-ring generated by

$$\{\{e\}, \{x^3, x^6, x^9\}, \{x, x^4, x^7, x^{10}\}, \{x^2, x^5, x^8, x^{11}\}\}.$$

Let  $H = \langle x^3 \rangle$  so that  $H$  satisfies the conditions of the theorem. In this case,  $K$  is the trivial subgroup. Clearly,  $HK \neq G$ . Let  $\rho : G \rightarrow G/H$  be the natural homomorphism. We see then that  $S = S(HK) \wedge \rho^* S$ , where  $S(HK) = S(H) = \{\{e\}, \{x^3, x^6, x^9\}\}$ .

We now state the two main results of [LMan]. As before,  $G$  is cyclic and  $S$  is the nontrivial S-ring over  $G$ . The first result [LMan, Cor. 4.3 and Thm 4.5] states that if  $G$  has at least one Sylow subgroup which is not an  $S$ -subgroup then  $S$  is a dot or wedge product of smaller subgroups.

The second and main result of [LMan] is that if there is a subgroup  $H$  which is not an  $S$ -subgroup, then  $S$  is a dot product or a wedge product of smaller subgroups. It is the second result which we wish to be precise.

Suppose  $S$  is a nontrivial S-ring over a cyclic group  $G$ . Suppose there exists a subgroup  $H$  which is not an  $S$ -subgroup. By Corollary 4.3 and Theorem 4.5 of [LMan], we assume  $H$  is not a  $p$ -Sylow subgroup and that every  $p$ -Sylow subgroup is an  $S$ -subgroup. Suppose  $|H| = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ , where each  $p_i$  is prime and  $r_i \geq 1$ . Let  $H_i$

denote the unique subgroup of order  $p_i^{r_i}$ . Since  $H$  is not an  $S$ -subgroup, not all  $H_i$  are  $S$ -subgroups. Without loss of generality, assume  $H_1$  is not an  $S$ -subgroup. To simplify our notation, we write  $p$  for  $p_1$ ,  $r$  for  $r_1$  and  $P_r$  for the  $p$ -Sylow subgroup of  $G$ . Suppose  $\{e\} \leq P_{a_1} \leq \cdots \leq P_{a_k} = P_r$  are all  $S$ -subgroups in  $P_r$ , where  $0 = a_0 < a_1 < a_2 < \cdots < a_k = r$ . Since  $H_1$  is a  $p$ -group and  $H_1$  is not an  $S$ -subgroup,  $a_{i+1} - a_i \neq 1$  for some  $i$ . Let  $i$  be the smallest integer such that  $a_{i+1} - a_i \neq 1$ . Set  $a_i = s$  and  $a_{i+1} = t$ . We then have the following lemma.

**Lemma 5.1.9.** *[LMan, Lemma 5.1] Suppose all Sylow-subgroups of  $G$  are  $S$ -subgroups and there exists a nontrivial subgroup  $H$  which is not an  $S$ -subgroup. Then there exists a prime divisor of  $G$ , integers  $s, t$  with  $s + 2 \leq t$  such that  $P_t \setminus P_s$  is a principal subset.*

We assume the conditions of the previous lemma. Let  $|G| = p^r m$ , where  $p$  doesn't divide  $m$ . Let  $M$  be the subgroup of order  $m$ . Since every Sylow subgroup of  $G$  is an  $S$ -subgroup and  $M$  is a product of Sylow subgroups,  $M$  is also an  $S$ -subgroup.

**Theorem 5.1.10.** *[LMan, Thm 5.2] With the same assumptions and notation of the previous lemma, we have,*

*i) If  $s = 0, t = r$ , then  $S = S(P_r) \cdot S(M)$ .*

*ii) If  $r > t, s = 0$ , then  $S = S(P_t M) \wedge \rho^* S$ , where  $\rho : G \rightarrow G/P_t$  is the natural homomorphism.*

*iii) If  $s \geq 1$ , then  $S = S(P_s M) \wedge \rho^* S$ , where  $\rho : G \rightarrow G/P_s$  is the natural homomorphism.*

*Proof.* Part i) and ii) follows directly from 5.1.6. □

## 5.2 Application to Fusion

Now we apply the results from the previous section to our problem.

**Proposition 5.2.1.** *Suppose that  $D_{4m}, m = 2k$  fuses to  $G$  where  $Z(G) = \{1\}$ . Let  $S$  denote the  $S$ -ring on  $\mathbb{Z}_{2m} = \langle x \rangle \leq D_{4m}$  which fuses to  $F_G(M_1)$  and  $T$  the  $S$ -ring on  $\mathbb{Z}_m = \langle x^2 \rangle = D'_{4m}$  that fuses to  $F_G(G')$ . Then  $\mathbb{Z}_m$  has a nontrivial subgroup  $H$  such that  $\bar{H} \notin S$  and  $\bar{H} \notin T$ .*

*Proof.* Since  $m$  is even, there is an element in  $\mathbb{Z}_m$  of order 2. Let  $H$  be the subgroup generated by this element. Suppose  $\bar{H} \in T$ , then  $\bar{H} - 1$  is an element of  $T$ , implying that the  $S$ -ring on  $D_{4m}$  has a non-identity class with one element. As  $Z(G)$  is trivial, this is a contradiction.  $\square$

**Proposition 5.2.2.** *Suppose that  $D_{4m}, m = 2k$  fuses to  $G$  where  $Z(G) = \{1\}$ . Let  $S$  denote the  $S$ -ring on  $\mathbb{Z}_{2m} = \langle x \rangle \leq D_{4m}$  which fuses to  $F_G(M_1)$  and  $T$  the  $S$ -ring on  $\mathbb{Z}_m = \langle x^2 \rangle = D'_{4m}$  that fuses to  $F_G(G')$ . Then  $S$  and  $T$  are either dot products or wedge products of strictly smaller rings.*

*Proof.* By Theorem 5.2.4, since  $2m$  and  $m$  are not primes and  $2m - 1$  and  $m - 1$  do not divide  $4m$ ,  $S$  and  $T$  must be imprimitive. By Proposition 5.2.1, Lemma 5.1.9, and Theorem 5.1.10, the result follows.  $\square$

An  $S$ -ring  $S$  on  $G$  is *primitive* if for every  $S$ -principal set  $D$ ,  $\langle D \rangle = G$  or  $\langle D \rangle = \{1\}$ . If  $S$  is not primitive,  $S$  is *imprimitive*. The following lemma will lead us to a fundamental result of Wielandt.

**Lemma 5.2.3.** *Let  $H$  be an abelian group and let  $p$  be a prime dividing  $|H|$ . Then  $G = \langle x^p | x \in H \rangle$  is a proper subgroup of  $H$ .*

*Proof.* Define  $\rho : H \rightarrow H$  by  $\rho(x) = x^p$ . Since  $H$  is abelian,  $\rho$  is a homomorphism and  $G = \text{Im } \rho$ . By Cauchy's theorem, there is at least one element of order  $p$  and thus  $\text{Ker } \rho \neq \{e\}$ . Then  $|G| = |\text{Im } \rho| < |H|$ .  $\square$

**Theorem 5.2.4.** [Scott] *If  $H$  is a finite Abelian group not of prime order, and  $P \neq E$  is a cyclic Sylow  $p$ -subgroup, then there is no nontrivial, primitive  $S$ -ring  $R$  over  $H$ .*

*Proof.* By Cauchy's theorem, there exists a subgroup  $K$  of  $H$  of order  $p$ . By Sylow's Theorem,  $K$  lies in some conjugate of  $P$  and hence in  $P$  as  $P$  is the unique Sylow  $p$ -subgroup. Since  $P$  is cyclic,  $K$  is unique. Suppose  $R$  is a nontrivial, primitive  $S$ -ring over  $H$ , and let  $\bar{t} = \sum x_i \neq 1, x_i \in H$ , be a basis element of  $R$ . By the Binomial Theorem,  $\bar{t}^p \equiv \sum x_i^p \pmod{p}$ . Suppose  $\bar{t}^p \not\equiv je \pmod{p}$ , for some integer  $j$  and  $e$  the identity of  $H$ . But this implies  $Q = \{x_i^p\}$  generates a nontrivial subgroup of  $H$ , a contradiction since  $\bar{Q} \in R$ .

Thus,  $\bar{t}^p \equiv \sum x_i^p \equiv je \pmod{p}$ . For all  $h \in H$ , either  $h$  has no  $p$ th roots in  $H$  or an entire  $xK$  coset of them. To see this, suppose  $h = x^p$  some  $x \in H$ . Then  $(xk)^p = x^p k^p = h$ , for any  $k \in K$ . Conversely, if  $x^p = y^p = h$ , then  $(xy^{-1})^p = 1$  and  $xy^{-1} \in K$ . Thus,  $xK = yK$ .

For all  $i$ , either  $x_i^p = e$  or  $x_i^p = g$  for some  $g \neq e$ . In the first case,  $x_i \in K$ , else  $K$  is not the unique subgroup of order  $p$ . In the latter case, there must be  $p - 1$  other  $x_i$ 's with  $x_i^p = g$  else  $\sum x_i^p \not\equiv je \pmod{p}$ . Thus we can divide up the elements of  $t$  into two groups: elements in  $K$  and  $xK$  cosets of  $p$ th roots. Hence,

$$\bar{t} = a + b\bar{K}, a \in \mathbb{Z}K, b \in \mathbb{Z}H.$$

And since  $\langle t \rangle = H$  and  $H \neq K$ , we see that  $b \neq 0$ .

Claim: There is a basis element of the form  $\bar{t} = b\bar{K}$ . Suppose not, pick a basis element  $\bar{t}_1 = a_1 + b_1\bar{K}$  such that  $L(b_1)/L(a_1)$  is a maximum and another basis element

$\bar{t}_2 = a_2 + b_2\bar{K} \neq t_1^*$ , possibly  $t_2 = t_1$ . We then have

$$\bar{t}_1\bar{t}_2 = a_1a_2 + (a_1b_2 + b_1a_2 + b_1b_2|K|)\bar{K},$$

where the identity is not in the right term (since  $t_2 \neq t_1^*$ ) and thus

$$\frac{L(a_1b_2 + b_1a_2 + b_1b_2|K|)}{L(a_1a_2)} > \frac{L(b_1a_2)}{L(a_1a_2)} = \frac{L(b_1)}{L(a_1)}.$$

But this is a contradiction since  $\bar{t}_1\bar{t}_2$ , being a linear combination of  $t_j \neq e$ , means

$$\begin{aligned} \bar{t}_1\bar{t}_2 &= \lambda_3\bar{t}_3 + \cdots + \lambda_k\bar{t}_k \\ &= \lambda_3(a_3 + b_3\bar{K}) + \cdots + \lambda_k(a_k + b_k\bar{K}) \\ &= (\lambda_3a_3 + \cdots + \lambda_ka_k) + (\lambda_3b_3 + \cdots + \lambda_kb_k)\bar{K} \end{aligned}$$

And,

$$\frac{L(\lambda_3b_3 + \cdots + \lambda_kb_k)}{L(\lambda_3a_3 + \cdots + \lambda_ka_k)} \leq \frac{\frac{L(\lambda_3b_1a_3)}{L(a_1)} + \cdots + \frac{L(\lambda_ka_kb_1)}{L(a_1)}}{\lambda_3L(a_3) + \cdots + \lambda_kL(a_k)} = \frac{L(b_1)}{L(a_1)},$$

since  $\frac{L(b_n)}{L(a_n)} \leq \frac{L(b_1)}{L(a_1)}$  for all  $n$ . Thus,  $\bar{t} = b\bar{K}$  is a basis element.

Let

$$M = \{h|h \in H, th = t\}$$

and

$$N = \{h \in H|h \text{ occurs with coefficient } L(t) \text{ in } t^*t\}.$$

Then, as proven in Lemma 5.2.5,  $M = N$ . Clearly,  $K \subset M$ , and  $M$  is a proper subgroup of  $H$ , else  $H = K$ . But  $\bar{M} \in R$ , contradicting the assumption that  $R$  is primitive.  $\square$

**Lemma 5.2.5.** *Suppose*

$$M = \{h|h \in H, th = t\}$$

and

$$N = \{h \in H | h \text{ occurs with coefficient } L(t) \text{ in } t^*t\}$$

as above. Then  $M = N$ .

*Proof.* Suppose  $t = \sum_{i=1}^r x_i$ , so that  $L(t) = r$ . Let  $n \in N$ . Then,

$$t^*t = L(t)n + \dots = rn + \dots$$

So for all  $x_i^{-1} \in t^*$ , there exists a unique  $x_{j_i} \in t$  such that  $x_i^{-1}x_{j_i} = n$ . Hence,

$$\begin{aligned} tn &= (x_1 + \dots + x_r)n \\ &= x_1n + \dots + x_rn \\ &= x_1x_1^{-1}x_{j_1} + \dots + x_rx_r^{-1}x_{j_r} \\ &= x_{j_1} + \dots + x_{j_r} \\ &= t. \end{aligned}$$

Thus  $n \in M$  and  $N \subset M$ . Conversely, let  $m \in M$ . Then

$$\begin{aligned} mt &= m(x_1 + \dots + x_r) \\ &= (x_1 + \dots + x_r). \end{aligned}$$

Thus, for all  $j$ , there exists a unique  $i_j$ , such that  $mx_{i_j} = x_j$ , that is,  $m = x_jx_{i_j}^{-1}$ . And

so,

$$\begin{aligned}
t^*t &= (x_1^{-1} + \cdots + x_r^{-1})(x_1 + \cdots + x_r) \\
&= x_1x_{j_1}^{-1} + \cdots + x_rx_{j_r}^{-1} + \cdots \\
&= m + \cdots + m + \cdots \\
&= rm + \cdots .
\end{aligned}$$

Hence,  $M \subset N$ . □

The following corollary is thus trivial.

**Corollary 5.2.6.** *Any  $S$ -ring  $S$  on a cyclic group satisfies one of:*

1.  $\dim(S) = 1, 2$ ;
2.  $S$  is imprimitive;
3.  $|G|$  is prime.

**Proposition 5.2.7.** *Suppose that  $D_{4m}$  fuses to  $G$ . Then there is a non-trivial normal subgroup  $H$  of  $G$  which corresponds to a subgroup  $N \cong \mathbb{Z}_h$  of  $\mathbb{Z}_n \subset D_{2n}$  such that  $D_{2n}/N \cong D_{2n/h}$  fuses to  $G/H$ .*

*Proof.* By Lemma 4.3.2,  $D'_{2n} \cong \mathbb{Z}_m$  fuses to  $F_G(G')$ . Let  $S$  be the  $S$ -ring on  $D'_{2n}$  determined by this fusion. We claim  $S$  is imprimitive. Suppose not, then by Corollary 5.2.6, our only case is (1). Clearly,  $\dim(S) \neq 1$ . If  $\dim(S) = 2$ , we have two classes of sizes 1 and  $m - 1$ . This means  $G$  has a class of size  $m - 1$ . But  $(m - 1) \mid |G|$  implies  $m - 1 \leq 4$ , contrary to hypothesis. Thus  $S$  is imprimitive as claimed.

Hence there is some principal set  $E \subset \mathbb{Z}_m$ , such that  $N = \langle E \rangle \neq \{1\}, \mathbb{Z}_m$ . By [Wielandt, Proposition 23.6],  $\overline{N} \in S$ . Let  $H \leq G'$  be the corresponding subgroup



promised by Lemma 3.3.7. Thus we have  $N = \mathbb{Z}_h$  cyclic and

$$D_{4m}/N = D_{4m}/\mathbb{Z}_h \cong D_{4(m/h)}$$

is the dihedral group which fuses to  $G/H$  by Lemma 3.3.5. □

**Proposition 5.2.8.** *Suppose that  $D_{4m}$  fuses to  $G$  with  $m$  even and  $Z(G) = \{1\}$ . Let  $S = F_G(D_1)$  be the  $S$ -ring for the dihedral group  $D_1 \cong D_{2m}$ . Then  $S$  is not a direct product of  $S$ -rings.*

*Proof.* If  $S$  is a dot product of  $S$ -rings,  $S = S_H S_K$ , then  $D_{2m} = HK$ , for nontrivial subgroups  $H, K$ , where  $m = 2k$ , with  $k$  odd. In fact,  $H$  and  $K$  must be  $\mathbb{Z}_2$  and  $D_{2k}$ , respectively. But this implies  $e \neq x \in H$  must be a principal subset, contradicting the fact that  $Z(G)$  is trivial. □

### 5.3 $D_{2n}$ fuses to $G$ where $|G| = 8p^2$

**Proposition 5.3.1.** *Suppose that  $D_{4m}$ ,  $m$  even, fuses to  $G$  and let  $C'$  be a class in  $G$  with corresponding subset  $C \subset \mathbb{Z}_m \subset D_{4m}$ . Suppose that  $|C|$  is odd and that  $C$  is invariant under the action of  $\text{Aut}(\mathbb{Z}_m)$  so that  $C$  is a union  $C = O_1 \cup \dots \cup O_t$  of  $\text{Aut}(\mathbb{Z}_m)$  orbits. Since  $C$  is odd, we may assume that  $O_1 = \{x^m\}$ . Let  $\zeta$  be a primitive  $2m$ th root of unity and let  $\chi^{(h)}$  be the character of  $D_{4m}$  such that,*

$$\chi^{(h)}(x^u) = \zeta^{hu} + \zeta^{-hu}, \chi^{(h)}(yx^u) = 0.$$

Let  $n_v$  be the order of an element in  $O_v$  and let  $\mu$  denote the Mobius function,  $\phi$  the Euler function. Let the index  $i$  correspond to the partition of the characters containing  $\chi^{(h)}$  and the index  $j$  to  $C$ . Then for  $\tau_{ij}$  defined in the Magic Rectangle Section, we have,

$$\tau_{ij} = \frac{1}{|C|} \left( (-1)^h + \sum_{v=2}^t \frac{\phi(n_v)}{\phi(\frac{n_v}{\gcd(h, n_v)})} \mu\left(\frac{n_v}{\gcd(h, n_v)}\right) \right). \quad (5.1)$$

The character value of  $G$  is  $\tau_{ij}(2\sqrt{w})$  where  $w$  is the number of characters in that part of the partition which contains  $\chi^{(h)}$ , in this case  $w$  is a square and is an integer.

*Proof.* If  $Y = Y_n$  is the set of all primitive  $n$ th roots of unity, then  $\bar{Y} = \mu(n)$ . [NivenZuckerman, page 197] If  $m \in \mathbb{N}$ , we have,

$$\bar{Y}^{(m)} = \frac{\phi(n)}{\phi(\frac{n}{\gcd(n, m)})} \bar{Y}_{n/\gcd(n, m)}.$$

Since  $C \subset \mathbb{Z}_m$ , each conjugacy class not equal to  $O_1$  has 2 elements. From the Magic

Rectangle Condition, Equation 3.1,

$$\tau_{ij} = \frac{\sum_{m=t_1}^{t_{rj}} |C_m| X_{w_k}(C_m)}{[\sum_{m=t_1}^{t_{rj}} |C_m|] d_i},$$

we see that the denominator is  $2|C|$ . From the character table of the dihedral group, we have  $\chi^{(h)}(x^m) = 2(-1)^h$ . Each orbit  $O_v$ ,  $v > 1$ , is a union of elements of order  $n_v$ . It is also a union of classes of  $D_{4m}$ . The character value for each class  $\{x^u, x^{-u}\}$  is  $\zeta^{hu} + \zeta^{-hu}$ ,  $1 \leq h \leq m-1$ . Since  $O_v$  is a union of elements of order  $n_v$ , the value of Equation 3.1 on  $O_v$  is,

$$2 \sum (\zeta^{hu} + \zeta^{-uh}) = 2\bar{Y}_{n_v}^{(h)}.$$

Thus the numerator of  $\tau_{ij}$  is,

$$\begin{aligned} 2(-1)^h + 2 \sum_{v=2}^t \bar{Y}_{n_v}^{(h)} &= 2(-1)^h + 2 \sum_{v=2}^t \frac{\phi(n_v)}{\phi(\frac{n_v}{\gcd(n_v, h)})} \bar{Y}_{\frac{n_v}{\gcd(n_v, h)}} \\ &= 2(-1)^h + 2 \sum_{v=2}^t \frac{\phi(n_v)}{\phi(\frac{n_v}{\gcd(n_v, h)})} \mu(\frac{n_v}{\gcd(n_v, h)}). \end{aligned}$$

Finally, since the column of the character table of  $D_{4m}$  corresponding to the non-trivial class of size one is  $(1, 1, 1, 1, \pm 2, \dots, \pm 2)$ , the value of  $\tau_{ij}$  corresponding to that class is a sum of 2's and  $-2$ 's. Thus the corresponding entry in the character table of  $G$  from the Magic Rectangle Condition is,

$$\begin{aligned} \eta_i \tau_{ij} &= \tau_{ij} \sqrt{\sum_{m=1}^{s_i} d_{i_m}^2} \\ &= \tau_{ij} \sqrt{4w} = 2\sqrt{w} \tau_{ij}, \end{aligned}$$

as required. □

**Theorem 5.3.2.** *Suppose that  $G$  fuses from a dihedral group and that  $|G| = 8p^2$  where  $p$  is an odd prime. Then we have either, i)  $G$  is dihedral or dicyclic; or ii)  $p=3$  and  $G$  is the group of order 72.*

*Proof.* Let  $C \subset \mathbb{Z}_{2p^2}$  corresponds to the class  $C'$  of  $G$  which has odd size given by Property 3, Proposition 4.3.6. Then  $|C|$  is  $p$  or  $p^2$  as these are the only odd divisors of  $|G|$ . We first show that  $C$  must be a union of automorphism classes of  $\mathbb{Z}_{2p^2}$ , that is, we show that if  $C$  contains an element of a certain order, it must contain all of the elements of that order in  $\langle x^2 \rangle = D'_{8p^2} \cong \mathbb{Z}_{2p^2}$ .

For every principal subset  $T$ , we define  $T^{(n)} = \{g^n | g \in T\}$ . By [Wielandt, Theorem 23.9], a subset  $T \subset D'_{8p^2} \cong \mathbb{Z}_{2p^2}$  is a principal subset if and only if  $T^{(n)}$  is a principal subset for  $n$  prime to  $2p^2$ . Suppose  $g \in C$  and  $h \in \mathbb{Z}_{2p^2}$  with  $|g| = |h| = r$ . We show  $h \in C$ . Since  $|g| = |h| = r$ , there exists a  $t$  with  $\gcd(t, 2p^2) = 1$  such that  $g^t = h$ . By [Wielandt, Thm 23.9],  $C^{(t)}$  is also a principal subset. Since  $t$  is odd and  $a = x^{2p^2} \in C$ ,  $a^t = x^{2p^2} \in C^{(t)}$ . But  $C^{(t)}$  is a principal subset implies  $C^{(t)} = C$ . Thus  $g^t = h \in C$ .

We first claim that  $C = C_2 \cup C_{2p}$ . There are two cases to consider:  $|C| = p$  or  $|C| = p^2$ .

If  $|C| = p$ , then  $C$  is either  $C_2 \cup C_p$  or  $C_2 \cup C_{2p}$  where  $C_i$  denotes the set of elements in  $D'_{2n}$  of order  $i$ . We show that the first case yields a contradiction.

Clearly,  $C$  is invariant under the action of  $Aut(\mathbb{Z}_m)$  and  $|C|$  is odd. We can thus apply Proposition 5.3.1. The character value of  $G$  for the column corresponding to  $C$  and the character corresponding to the fusion of characters containing  $\chi^h$  is

$$\frac{1}{p} \left( (-1)^h + \frac{\phi(p)}{\phi(p/\gcd(h, p))} \mu(p/\gcd(h, p)) \right) 2\sqrt{w}.$$

If we consider the partition containing  $\chi^h$ , where  $h$  is odd and  $\gcd(h, p^2) = p$ , in the

character table for  $D_{2n}$ , then in the fusion the character table value for  $G$  for the rows of the partition of characters which contain  $\chi^h$  and the column of  $C'$  is  $2(p-2)\sqrt{w}/p$ . By Proposition 5.3.1, this is an integer, hence  $w \geq p^2$ . But  $1 \leq h < 2p^2$ , thus  $h$  can take on exactly  $p$  values. Then  $w \leq p$ , which is a contradiction.

If  $|C| = p^2$ , then we have four cases: i)  $C = C_2 \cup C_p \cup C_{p^2}$ , ii)  $C = C_2 \cup C_p \cup C_{2p^2}$ , iii)  $C = C_2 \cup C_{2p} \cup C_{p^2}$ , iv)  $C = C_2 \cup C_{2p} \cup C_{2p^2}$ .

Case (i):  $C = C_2 \cup C_p \cup C_{p^2}$ ; as before the character value is

$$\frac{1}{p^2} \left( (-1)^h + \frac{\phi(p)}{\phi(p/\gcd(h,p))} \mu(p/\gcd(h,p)) + \frac{\phi(p^2)}{\phi(p^2/\gcd(h,p^2))} \mu(p^2/\gcd(h,p^2)) \right) 2\sqrt{w}.$$

If  $h$  is odd and  $\gcd(p^2, h) = 1$ , this equates to  $-4\sqrt{w}/p^2$ . As this value must be an integer, we have  $w \geq p^4$  but there can only be at most  $2p^2 - 1$  such values of  $h$ , which again is a contradiction.

Case ii):  $C = C_2 \cup C_p \cup C_{2p^2}$ ; the character value for this case is,

$$\frac{1}{p^2} \left( (-1)^h + \frac{\phi(p)}{\phi(p/\gcd(h,p))} \mu(p/\gcd(h,p)) + \frac{\phi(2p^2)}{\phi(2p^2/\gcd(h,2p^2))} \mu(2p^2/\gcd(h,2p^2)) \right) 2\sqrt{w}.$$

If  $h$  is odd and  $\gcd(p^2, h) = 1$ , this equals  $-4\sqrt{w}/p^2$ . Since this value is an integer,  $w \geq p^4$  but only  $p$  such values of  $h$  satisfy this condition, which is a contradiction.

Case iii):  $C = C_2 \cup C_{2p} \cup C_{p^2}$ ; the character value is

$$\frac{1}{p^2} \left( (-1)^h + \frac{\phi(2p)}{\phi(2p/\gcd(h,2p))} \mu(2p/\gcd(h,2p)) + \frac{\phi(p^2)}{\phi(p^2/\gcd(h,p^2))} \mu(p^2/\gcd(h,p^2)) \right) 2\sqrt{w}.$$

If  $h$  is odd and  $\gcd(p^2, h) = p$ , the above value is  $-4\sqrt{w}/p^2$ . But this gives  $w \geq p^4$ , a contradiction.

Case iv):  $C = C_2 \cup C_{2p} \cup C_{2p^2}$ ; the character value is

$$\frac{1}{p^2} \left( (-1)^h + \frac{\phi(2p)}{\phi(2p/\gcd(h, 2p))} \mu(2p/\gcd(h, 2p)) + \frac{\phi(2p^2)}{\phi(2p^2/\gcd(h, 2p^2))} \mu(2p^2/\gcd(h, 2p^2)) \right) 2\sqrt{w}.$$

If  $\gcd(h, p^2)$  is not  $p^2$ , our value is 0 and if  $\gcd(h, p^2) = p^2$ , that is, when  $h = p^2$ , our value is  $-2$ . Thus we see that case iv) is the only possible case. And in this case, the common value of  $\chi^h$  for the columns of the classes making up  $C$  by the Magic Rectangle Condition is 0 for all  $h$  except when  $h = p^2$  in which case we get  $-2$ . Thus the character  $\chi^h$  of  $D_{8p^2}$  does not fuse with any other character and is a character of degree 2. Notice we can also check by using the Magic Rectangle Equation 3.2 directly to get  $-2$ . Moreover, for any  $h \neq p^2$ , since  $\chi^h$  gives the value 0,  $\chi^h$  must fuse with at least one other degree 2 character. Thus  $G$  only has one irreducible character of degree 2.

By Property 8, Proposition 4.3.12, the Sylow 2-subgroup of  $G$  has a normal complement, that is,  $G = NS$ . Consider the  $S$ -ring  $F_G(N)$  on  $N$  determined by the fusion. If this  $S$ -ring is imprimitive, we can find a nontrivial subgroup  $J$  of  $N$  with  $\bar{J} \in F_G(N)$ . Then  $|J| = p$  and by Lemma 3.3.5,  $D_{8p}$  fuses to  $G/J$ . Thus  $G/J$  is dihedral or dicyclic by Theorem 4.4.1. But, assuming  $p > 2$ ,  $G/J$  has more than one irreducible representation of degree 2. By lifting these irreducible characters to those of  $G$ , we have that  $G$  also has more than one irreducible character of degree 2 which is a contradiction.

Thus the  $S$ -ring on  $N$  is primitive and fuses from a cyclic group of order  $p^2$ . And by [Wielandt Theorem 25.4], since  $p^2$  isn't prime,  $F_G(N)$  is trivial. Thus  $G$  must have a class of size  $p^2 - 1$ . Then we must have that  $(p^2 - 1) | 8$  and the only prime which satisfies this divisibility is 3. Thus  $G$  is the group of order 72 in this case.

We can now assume that  $C = C_2 \cup C_{2p}$ . In [Yuanda, pg. 77-93], Yuanda classified groups of order  $8p^2$  using 6 lemmas. By Property 10, the Sylow-2 group of  $G$  must be

dihedral, dicyclic or abelian of type [4,2]. We can thus eliminate all the groups from Lemmas 2 and 4. We can also eliminate groups from Lemma 3 since the index of the commutator subgroup of these groups is at least 8. Groups from Lemma 1 can also be eliminated by Property 10 and the requirement that  $G$  has trivial center. From the remaining lemmas, there are only 2 groups with trivial center and  $|G/G'| = 4$ . These two groups can be eliminated since they each have a class of size  $p^2$  contradicting Property 3 in which we proved that  $G$  has a unique nontrivial class of odd size.

Now, suppose  $G$  has nontrivial center. Then  $Z(G) \cong \mathbb{Z}_2$ . As before, Property 10 will help us eliminate all the groups from Lemmas 2 and 4. There are three groups from Lemma 1 with  $Z(G) \cong \mathbb{Z}_2$ . Two of these groups are dihedral and dicyclic. The last one has presentation:

$$G = \langle a, x, y | a^{p^2} = x^4 = y^2 = 1, x^y = x^{-1}, a^x = a^{-1} = a^y \rangle.$$

But this group violates Property 6 and 7. The element  $axy$ , for example, is not conjugate to its inverse. Finally, there are 5 groups from Lemmas 5 and 6 with  $Z(G) \cong \mathbb{Z}_2$ . But all five violates Property 13 since each has at least 2 distinct normal subgroups of size  $p$ .

We have eliminated all groups in the classification except for the two groups listed on page 79 of [Yuanda], which are the dihedral and dicyclic groups. □

# Bibliography

- [Cel] Ozdem Celik. *On the Character Tables of Dihedral Groups and Dicyclic Groups*. Mitt. Math. Sem Giessen 98 (1973), 1-3.
- [Feit] Feit, W., *Characters of Finite Groups*. New York, 1967.
- [Gorenstein] Gorenstein, Daniel. *Finite Groups*. Chelsea Publishing Company, New York, N.Y.
- [Hup] Huppert, B. *Endliche Gruppen I* I., Die Grundlehren der Mathematischen Wissenschaften, vol 134, Springer-Verlag, Berlin-New York, 1967.
- [Huppert] Huppert, B. *Character Theory of Finite Groups*. New York : de Gruyter, 1998. page 313.
- [HumphriesJohnson] Stephen Humphries and Kenneth Johnson. *Fusions of Character Tables and Schur Rings of Abelian Groups*, preprint. Communications in Algebra(2006).
- [Smith] Smith, Jonathan D.H. *An Introduction to Quasigroups and Their Representations*. Taylor and Francis Group. Boca Raton, Florida 2007.
- [LMa] Ka Hin Leung and Siu Lun MA. *The Structure of Schur Rings Over Cyclic Groups*. Journal of Pure and Applied Algebra 66(1990) 287-302.



- [LMan] Ka Hin Leung and Shing Hing Man. *On Schur Rings Over Cyclic Groups*. Israel Journal of Mathematics 106(1998), 251-267. —. *On Schur Rings over Cyclic Groups, II*. Journal of Algebra 183, 273-285(1996).
- [Mann] Mann, Avinoam. *Finite Groups Containing Many Involutions*. Proceedings of the American Mathematical Society (1994) Volume 122, Number 2.
- [Muzychuk] Muzychuk, Mikhail. *On the Structure of Basic sets of Schur Rings over Cyclic Groups*. Journal of Algebra 169, 655-678(1994).
- [NivenZuckerman] Ivan Niven, Herbert Zuckerman, Hugh Montgomery. *An Introduction to the Theory of Numbers*. John Wiley and Sons, Inc. New York, New York 1960.
- [Qian] Guohua Qian, Wujie Shi, Xingzhong You. *Conjugacy Class Outside a Normal Subgroup*. Communications in Algebra Vol 32, No. 12, pp. 4809-4820, 2004.
- [Scott] Scott, William Raymond. *Group Theory*. Englewood Cliffs, New Jersey. Prentice Hall, 1964.
- [Western] Western, A. E. *Groups of Order  $p^3q$* . Proc. London Math. Society., 30(1898), pg. 209-263.
- [Wielandt] Wielandt, Helmut *Finite Permutation Groups*. Academic Press, Inc. New York, New York, 1964.
- [Yuanda] Yuanda, Zhang. *The Structures of Groups of Order  $8p^2$* . Chinese Annals of Mathematics Vol 4 Series B.