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Determining High School Geometry Students' Geometric Understanding Using van Hiele Levels: Is There a Difference Between Standards-based Curriculum Students and NonStandards-based Curriculum Students?

Rebekah Loraine Genz
Brigham Young University - Provo

DETERMINING HIGH SCHOOL GEOMETRY STUDENTS' GEOMETRIC UNDERSTANDING USING VAN HIELE LEVELS: IS THERE A DIFFERENCE BETWEEN STANDARDS-BASED CURRICULUM STUDENTS AND NONSTANDARDS-BASED CURRICULUM STUDENTS?

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Rebekah Genz

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of a thesis submitted by

Rebekah Genz

This thesis has been read by each member of the following graduate committee and by majority vote has been found to be satisfactory.

Date	Blake E. Peterson, Chair
Date	Keith R. Leatham
Date	Gary R. Lawlor

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______________________ ______________________________
Date

Blake E. Peterson
Chair, Graduate Committee

Accepted for the Department

____
Charles N. Walter
Graduate Coordinator

Accepted for the College

____
Thomas W. Sederberg
Associate Dean
College of Physical and Mathematical Science

ABSTRACT

DETERMINING HIGH SCHOOL GEOMETRY STUDENTS'

GEOMETRIC UNDERSTANDING USING VAN HIELE

LEVELS: IS THERE A DIFFERENCE BETWEEN

STANDARDS-BASED CURRICULUM STUD-

ENTS AND NONSTANDARDS-BASED

CURRICULUM STUDENTS?

Rebekah Genz

Department of Mathematics Education

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Research has found that students are not adequately prepared to understand the concepts of geometry, as they are presented in a high school geometry course (e.g. Burger and Shaughnessy (1986), Usiskin (1982), van Hiele (1986)). Curricula based on the National Council of Teachers of Mathematics (NCTM) *Standards* (1989, 2000) have been developed and introduced into the middle grades to improve learning and concept development in mathematics. Research done by Rey, Reys, Lappan and Holliday (2003) showed that Standards-based curricula improve students' mathematical understanding and performance on standardized math exams.

Using van Hiele levels, this study examines 20 ninth-grade students' levels of geometric understanding at the beginning of their high school geometry course. Ten of the students had been taught mathematics using a Standards-based curriculum, the Connected Mathematics Project (CMP), during grades 6, 7, and 8, and the remaining 10 students had been taught from a traditional curriculum in grades 6, 7, and 8. Students with a Connected Mathematics project background tended to show higher levels of geometric understanding than the students with a more traditional curriculum (NONcmp) background.

Three distinctions of students' geometric understanding were identified among students within a given van Hiele level, one of which was the students' use of language. The use of precise versus imprecise language in students' explanations and reasoning is a major distinguishing factor between different levels of geometric understanding among the students in this study.

Another distinction among students' geometric understanding is the ability to clearly verbalize an infinite variety of shapes versus not being able to verbalize an infinite variety of shapes.

The third distinction identified among students' geometric understanding is that of understanding the necessary properties of specific shapes versus understanding only a couple of necessary properties for specific shapes.

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Chapter 1-- Introduction

Those who have taught high school geometry are familiar with statements such as, "A rectangle is a stretched out square," or with students claiming that a square is not a rectangle "because it has four equal sides." These are examples of the types of understanding with which students generally enter high school geometry.

A mathematics teacher from the Netherlands, Pierre M. van Hiele, along with his wife Dina M. van Hiele, developed a learning theory for geometry. The van Hiele theory (1986) sets forth a learning model in which students pass through five different levels of thinking as they develop from a holistic understanding of geometric figures to an understanding of formal deductive geometric proof. The van Hiele theory provides a structure for understanding the how students develop an understanding of geometric concepts through appropriate learning experiences.

According Hoffer (1981), high school geometry course curricula are presented at a higher level than most students are capable of upon entering high school geometry (p. 14). Other researchers (e.g., Usiskin (1982), Geddes et al. (1988), and Burger and Shaughnessy (1986)) confirm these assertions. Burger and Shaughnessy (1985) explain that high school geometry as it is taught in most high schools is taught at a deductive level, but most students are only capable of reasoning informally about geometric concepts upon entrance into geometry (p. 427).

The van Hiele theory (1986) asserts that students at a lower level of thinking cannot be expected to understand instruction presented at a higher level of thinking: "This is the most important cause of bad results in the education of mathematics" (van Hiele, 1986, p.66). According to Teppo (1991), the van Hiele theory emphasizes that "a systematically developed field of knowledge must be gained in all aspects of geometry before a student is capable of reaching the theoretical level" (p. 213). Burger and Shaughnessy (1986) found no high school students reasoning at a level of deduction in their research.

Burger and Shaughnessy (1986) used student interviews to study students' van Hiele levels. They explain, "The results of the interviews and the van Hiele theory have implications for the way geometry is taught in the school and for the way students learn geometric concepts" (p. 420). Activities that encourage development through the van Hiele levels need to be incorporated into geometry curriculum are recommended by Burger and Shaughnessy. Burger and Shaughnessy believe that geometry course material does not course material that promotes the development from one level to the next (p. 426). Geometry concepts need to be introduced in mathematics classes in the elementary and middle grades as well. "Many students have only had brief encounters with geometric concepts during their elementary school years…we must allow students to explore geometric concepts and shapes informally for many years prior to a high school course in geometry" (Burger & Shaughnessy, 1985, p. 426).

Teppo (1991) explains that systematic geometry instruction in the middle grades is necessary to prevent students from entering high school at low levels of geometric concept development (p. 217). Systematic geometry instruction would engage students in sequential learning activities during the middle grades that would help students enter high school geometry at a level at which they can comprehend the material, and be prepared to learn deductive geometric proof. Usiskin (1982) believes that systematic geometry instruction before high school is necessary to promote students' success in a geometry course.

The suggestions given by Burger and Shaughnessy (1986), Usiskin (1982), and Teppo were given over 20 years ago, and since that time many moves to improve mathematics curricula have been made. The National Council of Teachers of Mathematics (NCTM) has published several documents with recommendations and guidelines that provide a framework and set of goals for mathematics curriculum improvement. Among these publications are the *Curriculum and Evaluation Standards* (1989) and the *Principles and Standards for School Mathematics* (2000). Standards-based curricula have been developed to try to meet the recommendations made by mathematics education researchers and the needs of mathematics students. The intent of the *Standards* documents (1989, 2000) and Standards-based curricula is that students will explore, conjecture, to reason logically and use several different methods to solve problems. However, does a Standards-based curricula assist students in understanding geometry concepts, and in overcoming the issues with the van Hiele levels previously discussed?

This study explored the relationship between students' geometric understanding, using the van Hiele theory (1986), and the curricula the students used. The level of understanding students acquired from a Standards-based curriculum background will be compared to the level of understanding students acquired with a nonStandards-based curriculum background.

Chapter 2-- Theoretical Framework and Literature Review

This study is framed by components of the van Hiele theory of levels of understanding in geometry (1986). Parallels between the van Hiele theory and the NCTM *Standards* (1989, 2000) are drawn and incorporated into the framework to determine the effects of Standards-based curricula.

The van Hiele Theory

The van Hiele theory (1986) is a learning model that describes the geometric thinking students go through as they move from a holistic perception of geometric shapes to a refined understanding of geometric proof.

Pierre M. van Hiele, and his wide Dina M. van Hiele, developed this theory out of the frustrations both they and their students experienced with the teaching and learning of geometry. van Hiele (1986) explains that when teaching his geometry students, "it always seemed as though I were speaking a different language" (p. 39). van Hiele wanted to know why students experienced difficulty in learning geometry and how he could remedy those difficulties. The solution van Hiele found for his students' frustrations was the theory of different levels of thinking. The following are the van Hiele levels that are used and referred to in this study:

<u>Level 0</u> (Visualization): The student reasons about basic geometric concepts such as simple shapes, primarily by means of visual considerations of the concept as a whole without explicit regard to properties of its components.

<u>Level 1</u> (Analysis): The student reasons about geometric concepts by means of informal analysis of component parts and attributes. Necessary properties of the concept are established.

Level 2 (Abstraction): The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.

Level 3 (Deduction): The student reasons formally within the context of a mathematical system, complete with undefined theorems, axioms, an underlying logical system, definition, and theorems.

Level 4 (Rigor): The student can compare systems based on different axioms and can study various geometries in the absences of concrete models.

(Burger & Shaunghnessy, 1986, p. 31)

A study carried out by Burger and Shaughnessy used the van Hiele levels to interpret interviews conducted with 14 students ranging from kindergarten to college age. The interviews consisted of eight open-ended tasks dealing with geometric shapes in the following categories: drawing, identifying and defining, sorting, and logical reasoning. Using the task-based interviews, Burger and Shaughnessy identified what students are capable of at the first four van Hiele levels. They developed a list of specific characteristics of students' thinking exhibited at each of level. Burger and Shaughnessy call this list the Level Indictors. DeVilliers (2003) summarized these Level Indicators as follows:

> At level 0 students will use irrelevant properties to identify, compare, classify, and describe geometric figures; they will refer to visual prototypes of figures; they become confused with orientation of geometric figures; they do not consistently classify figures; they use irrelevant properties to sort figures; they

cannot conceive of the notion of an infinite number of a particular geometric figure; they define geometric figures using visual attributes of a figure only.

At level 1 students do not make class inclusions between different classes of figure; they make explicit comparisons of figures using their underlying properties; they sort geometric figures in terms of one property; they will use too many properties to define a geometric figure; they will not use definitions from the text or the teacher, they prefer their own definition; they try to prove the truth of a statement using empirical methods, sketches for example.

At level 2 students can formulate correct definitions according to sufficient conditions; they can use definitions other than their own and accept different equivalent definitions; geometric figures can be classified hierarchically; they can use logical if…then statements to formulate conjectures; they are still unsure about axioms, definitions and proof.

At level 3 students can understand the role of axioms, definitions and proof. They are able to make conjectures and prove them.

(DeVilliers, 2003, p. 12)

Learning Phases. Progress through the van Hiele levels occurs by way of instructional *learning phases* within the van Hiele theory (1986). Teppo (1991) explained that, “students progress from one level to the next as the result of purposeful instruction organized into five phases of sequenced activities that emphasize exploration, discussion, and integration” (p. 212). Each instructional learning stage builds upon and adds to the thinking of the previous level. The instruction at each learning phase fully and clearly defines that which was implied at the previous phase.

The learning phases mapped out by van Hiele (1986) are as follows:

1. In the first stage, that of *information*, pupils get acquainted with the working domain.

2. In the second stage, that of *guided orientation*, they are guided by tasks (given by the teacher, or made by themselves) with different relations of the network that has to be formed.

3. In the third stage, that of *explication*, they become conscious of the relations, they try to express them in words, they learn the technical language accompanying the subject matter.

4. In the fourth stage, that of *free orientation*, they learn by general tasks to find their own way in the network of relations.

5. In the fifth stage, that of *integration*, they build an overview of all they have learned of the subject, of the newly formed network of relations now at their disposal.

   (van Hiele, 1986, p. 53)

Table 1 shows the learning phases students must progress through to acquire the next level.

Table 1 van Hiele's Model of Instruction

Abstraction (Level 2)	Student understands properties of geometric shapes, forms definitions, and understands necessary and sufficient properties.	
↑ **Learning Period 2**	<u>Phases of Learning</u> Integration Free Orientation Explication Guided Orientation Information	↑

Analysis (Level 1)	Student reasons about geometric concepts by an informal analysis of component parts.
Learning Period 1	<u>Phases of Learning</u> Integration Free Orientation Explication Guided Orientation Information
Visual (Level 0)	Student reasons about basic geometric shapes by visual considerations.

Note. Adapted from "Van Hiele Levels of Geometric Thought Revisited," by A. Teppo, 1991, Mathematics Teacher, 84, p. 210. Each level of understanding is separated by a learning period in which instruction is divided into five phases of learning, which allows students to develop to the next level of understanding.

According to the van Hiele theory (1986) knowledge is strengthened and added to within the learning phases between each level. Learning should build upon and add to the previous knowledge learned. This type of learning and development as prescribed in the learning phases is evident in the NCTM *Standards,* "In a coherent curriculum, mathematical ideas are linked to and build on one another so that students' understandings, and knowledge deepens and their ability to apply mathematics expands" (NCTM, 2000, p. 14-15).

Standards-based Curricula

Standards-based curricula refer to teaching materials that implement recommendations put forth by the NCTM *Standards* documents (1989, 2000) regarding mathematics curricula.

Several Standards-based curriculum programs have been developed for the middle grades and funded by the National Science Foundation (NSF). Each of the Standards-based curriculum programs funded by NSF is a program based on the principle that the mathematics content and teaching methods of the middle grades should identify and explore mathematical concepts that will prepare students to continue to study mathematics in high school. One such project is the Connected Mathematics Project (CMP), which is designed for the 6^{th}-, 7^{th}-, and 8^{th}-grades. “Research results consistently show that CMP students outperform other students on tests of problem-solving ability, conceptual understanding, and proportional reasoning” (Lappan et al., 2002, Key features section, para. 6).

A study conducted by Reys, Reys, Lapan and Holliday (2003) assessed the impact of Standards-based curricula on students in the middle grades. Their study looked at two different standards-based curricula programs funded by the NSF, MATH Thematics (Billstein et al., 1999) and the Connected Mathematics Project (Lappan et al., 2002). The study examined the achievement of eighth graders because they were the students who would be taking the Missouri Assessment Program mathematics exam (MAP), and because eighth graders had studied mathematics using the standards-based curricula materials for at least two years (in grades 6 and 7). Students who had studied mathematics using a Standards-based curriculum, compared to students who did not study under a Standards-based curriculum showed improvements in standardized tests scores. According to this study, “Significant differences on the MAP were identified between students using the Standards-based curriculum materials and students from comparison districts using other curriculum materials. All significant differences reflected higher performance for students using NSF Standards-based materials" (p.87). Therefore, research has shown that Standards-based curriculum materials do improve students' mathematical understanding.

Instructional Aspects of the van Hiele Theory and its Relationship with Standards-based Curricula

NCTM *Standards* documents (1989, 2000) address issues regarding the teaching and learning of geometry. The methodology of the van Hiele theory (1986) is clearly evident in the *Standards* (1989, 2000). The *Standards* (1989) state:

> Evidence suggests that the development of geometric ideas progresses through a hierarchy of levels. Students first learn to recognize whole shapes and then analyze the relevant properties of shape. Later they can see relationships between shapes and make simple deductions. Curriculum development and instruction must consider this hierarchy. (p. 48)

The *Standards* (1989, 2000) reiterate the van Hiele theory in with regard to how geometry can be effectively taught. The *Standards* stress the importance of sequential learning as expressed by van Hiele's theory, "A school mathematics curriculum should provide a road map that helps teachers guide students to increasing levels of sophistication and depths of knowledge" (NCTM, 2000, p. 16).

Students develop greater understanding through the types of tasks and learning experiences they engage in. According to the *Standards* (2000), "Students learn mathematics through the experiences that teachers provide. Students' understanding of mathematics, their ability to use it to solve problems, their confidence, and their disposition toward mathematics are all shaped by the teaching they encounter in school"

(p. 16). The van Hiele theory (1986) emphasizes the same relationship between teaching and learning; it emphasizes the use of appropriate instructional experiences: "The transition from one level to the following is not a natural process; it takes place under the influence of a *teaching-learning program*" (van Hiele, 1986, p. 50, italics added). A teaching-learning program is a curriculum in which teachers use appropriate instructional experiences to engage the students in an active, conceptually rich approach to learning. van Hiele (1986) believed that students need to be actively engaged in "a suitable choice of exercises" (p. 39). The use of appropriate tasks and experiences is emphasized in the *Standards* (2000) as well, "The kinds of experiences teachers provide clearly play a major role in determining the extent and quality of students' learning. Students' understanding of mathematical ideas can be built throughout their school years if they actively engage in *tasks* and *experiences* designed to deepen and connect their knowledge" (p. 20, italics added). Appropriate tasks are problematic to the students. Students become confident in choosing solutions paths and trying new ideas, and students ultimately persevere in solving new types of problems (NCTM, 2000, p. 21). Within this teaching-learning program, students develop from one level to the next, and students' knowledge and understanding of geometric concepts builds and develops in the hierarchical fashion put forth in the *Standards* documents (1989, 2000) and by van Hiele (1986).

A vital feature of the van Hiele theory (1986) is that students at a lower level of thinking cannot understand information/instruction presented to them at a higher level, and this, according to van Hiele, is the reason students struggle so much in mathematics and particularly in geometry: "The ways of thinking of the base level, the second level, and the third level have a hierarchic arrangement. Thinking at the second level is not possible without that of the base level; thinking at the third level is not possible without thinking at the second level." (van Hiele, 1986, p. 51). Students who develop through the instructional learning stages that lead to each level in the correct sequence develop mathematical understanding of concepts at each level more thoroughly. By way of this process students come to have a full understanding of geometric concepts. A mathematics curriculum should be organized in such a way that mathematical ideas are presented and integrated so that students understand how the ideas and concepts build upon and connect to each other:

> In planning individual lessons, teachers should strive to organize the mathematics so that fundamental ideas form an integrated whole. Big ideas encountered in a variety of contexts should be established carefully, with important elements such as terminology, definitions, notation, concepts, and skills emerging in the process. (NCTM, 2000, p. 15)

The arrangement of the geometry standards within the *Standards* document (1989, 2000) coincides with the hierarchical arrangement of the van Hiele (1986) levels. Each geometry standard builds upon the information presented in the previous standard. Each grade band within the *Standards* (2000) document has the same basic structure; each has the same set of "instructional programs," or goals to be accomplished within the specific grade band. Of course, from one grade band to the next the instructional programs develop in sophistication. The following are the four instructional programs outlined for each grade band for geometry.

1. Analyze characteristics and properties of two-and three- dimensional geometric shapes and develop mathematical arguments about geometric relationships
2. Specify locations and describe spatial relationships using coordinate geometry and other representational systems
3. Apply transformations and use symmetry to analyze mathematical situations
4. Use visualization, spatial reasoning, and geometric modeling to solve problems (NCTM, 2000, p. 96)

Within these instructional programs are specific expectations for students to achieve within each grade band. Table 2 outlines a subset of the expectations under one instructional program for each grade band as it is outlined in the 2000 NCTM geometry standards, and shows how the van Hiele levels are developed from one grade band to the next.

Table 2 Example of the Correlation Between the Geometry Standards and the van Hiele Levels

<u>Instructional Program:</u> Analyze characteristics and properties of two- and three-dimensional geometric shapes and develop mathematical arguments about geometric relationships.	
Expectations for Grade Bands	**van Hiele Level**
Grades Pre-K- 2: Students should be able to "recognize, name, build, draw, compare, and sort two- and three- dimensional shapes; describe attributes, and parts of two- and three-dimsional shapes" (p. 96).	**Level 0:** Students will use irrelevant properties to identify, compare, classify, and describe geometric figures; they will refer to visual prototypes of figures; they become confused with orientation of geometric figures; they do not consistently classify figures; they use irrelevant properties to sort figures; they cannot conceive of the notion of an infinite number of a particular geometric figure, and they also define geometric figures using visual attributes of a figure only. (DeVilliers, 2003, p. 12)
Grades 3-5: Students should be able to "identify, compare, and analyze attributes, of two- and three- dimensional shapes and develop vocabulary to describe these	**Level 1:** Students do not make class inclusions between different classes of figure; they make explicit comparisons of figures using their underlying properties;

attributes; classify two- and three- dimensional shapes according to their properties and develop definitions of classes of shapes such as triangles and pyramids" (p. 164).	they will sort geometric figures in terms of one property; they will use too many properties to define a geometric figure; they will not use definitions from the text or the teacher, they prefer their own definition; they will try to prove the truth of a statement using empirical methods, sketches for example. (DeVilliers, 2003, p. 12)
Grades 6-8: Students should be able to "precisely describe, classify, and understand relationships among types of two- and three- dimensional objects using their defining properties; create and critique inductive and deductive arguments concerning geometric ideas and relationship, such as congruence, similarity, and the Pythagorean relationship" (p. 232).	**Level 2:** Students can formulate correct definition according to sufficient conditions; they can use definitions other than their own and accept different equivalent definitions; geometric figures can be classified hierarchically; they can use logical if…then statements to formulate conjectures, but they are still unsure about axioms, definitions and proof. (DeVilliers, 2003, p. 12)
Grades 9-12: Students should be able to " analyze properties and determine attributes of two- and three- dimensional objects; establish the validity of geometric conjectures using deduction, prove theorems, and critique arguments made by others" (p. 308).	**Level 3:** Students can understand the role of axioms, definitions and proof. They are able to make conjectures and prove. (DeVilliers, 2003, p. 12)

The expectations, outlined for one instructional program of the geometry standards, build upon each other from one grade band to the next. The expectations are organized so that the fundamental ideas will form an integrated whole at the end of each grade band and thus at the end of high school. The important elements such as terminology, definitions, notation, concepts, and skills emerge across the grade bands and develop in sophistication as students develop through grades. The *Standards* (1989, 2000) provide a roadmap of sequential learning that allows students to acquire higher levels of knowledge. Therefore, Standards-based curricula should prepare students for their high school geometry course by sequentially leading them through the van Heile (1986) levels.

The Acquisition of a Language in Developing Geometric Understanding

van Hiele (1986) believed that language is a crucial part of the learning process as students progress through the levels of thinking:

> The science to be studied is defined by the context in which the language symbols will have to be developed. The teacher must try to help the child with the development of those language symbols and he must do this just in the context belonging to the science he wants to introduce. (van Hiele 1986, p. 98)

A language whose context lies in a specific level likewise characterizes each van Hiele level. van Hiele believed that each level is associated with its own language.

Within the learning phases between van Hiele (1986) levels and within the geometry standards, new vocabulary and mathematical symbols are introduced to clearly define and discuss new topics/objects of study. For example, language will be introduced at level 1 to define a geometric figure. This language is not appropriate to use at level 0 because students are not yet capable of understanding it, and thus will not understand an explanation or task using that language. As students progress between van Hiele levels and the geometry standards, their language and use of it will develop as they create and use their definitions and explanations.

Van Hiele (1986) stressed the importance of the introduction, use and acquisition of language at each level of thinking. Language is of particular importance within the learning phases. Regarding the first learning phase, *information*, van Hiele says, "The teacher holds a conversation with the pupils, in well-known language symbols, in which the context he wants to use becomes clear" (p. 97). Teachers need to introduce and use the appropriate words and symbols when introducing a new concept. Within the second learning phase, *directed orientation*, van Hiele explains that students need to use the new language they have been introduced to, although it may not be completely understood, using the language or symbols appropriately within carefully chosen tasks, the student will begin to understand the language and symbols related to the concept being learned. Within the third learning phase, *explication*, van Hiele explains that it is important to make explicit the concepts students are involved in learning, which is developed class and group discussions. Within the conversations teachers engage their students in, "the teacher takes care that technical language is developed" (p. 97). Finally, within the fourth learning phase*, free orientation*, students now understand and make connections among the relationships they see and have worked with on tasks, and the students "now know the relevant language symbols" (p. 97). Students are comfortable speaking of, and using language and language symbols appropriately for the geometric concept they have been studying. Students clarify and reorganize their thoughts and understanding of geometric concepts through talking about them, and using the language specifically related to these concepts.

The *Standards* documents (1989, 2000) also purport the importance of language in teaching conceptual understanding. Selecting tasks that allow mathematical communication to occur includes not only developing precise language but also, "guiding classroom discussion on the basis of what is learned" (NCTM, 2000, p. 270).

There are consequences of the association of a language with each van Hiele (1986) level. Teachers often give students tasks or present material to students using the language of the third level, deduction. Unfortunately, students at levels 0-1 are not yet able to understand the material teachers are giving them because they have yet to acquire the language of the second level. Fuys and Geddes (1984) explained best the belief that van Hiele asserts about the acquisition of appropriate language through the development of the levels:

> Language structure is a critical factor in the movement through the van Hiele level –from global (concrete) structures (level 0) to visual geometric structures (level 1-2) to abstract structures (level 3-4). In stressing the importance of language, van Hiele notes that many failures in teaching geometry result from a language barrier- the teacher using the language of a higher level than is understood by the student. (p. 3)

Burger and Shaughnessy (1986) have found this consequence to be true within their own research when studying the discussion between teachers and their students. Thus, students must acquire the language of the level the learning activities are presented at before they can even comprehend the discussion or instruction the teacher is engaging them in; only in this way can students be conversant about the material and concepts at that level. For example, a student at level 2, abstraction, may regard a rhombus as a special parallelogram, but students at lower van Hiele levels cannot understand this concept. "The types of communication in which students can engage are constrained by their current mathematical understandings" (Sfard, 2003, p. 237).

A study conducted by Fuys and Geddes (1984) reveals the importance of language in the acquisition of geometric understanding. Sixth and ninth graders were given open-ended geometry tasks with subsequent geometric instruction in a clinical interview setting. The students' geometric understanding was assessed using van Hiele levels (1986). The study found that the type of language, precise or imprecise, differed among students assessed at different van Hiele levels. Level 0 students had difficulty explaining their thoughts using precise geometric terms. The sixth graders assessed at van Hiele level 0 had a "lack of familiarity with basic geometric concepts and terminology, and poor language (vocabulary and grammar) both generally and in mathematics, especially expressive language" (Fuys & Geddes, 1984, p. 7). Even the ninth graders assessed at van Hiele level 0 demonstrated an inability to use language appropriately: "Particularly noticeable was their poor language, i.e. their inability to express an idea clearly in a complete sentence" (p. 8).

Fuys and Geddes (1984) incorporated an instructional period into their study and evaluated how students' geometric understanding developed with instruction. Regarding the improvement of language skills Fuys and Geddes explain, "It was only after some instruction that students began to express themselves more precisely in terms of properties of shapes" (p. 9). This reiterates that not only is carefully determined instruction important for developing higher levels of understanding, but it improves students' language skills in their abilities to explain their reasoning.

van Hiele (1986) not only found language and language symbols of importance for geometry, but for any type of mathematics or science to be studied. van Hiele explained "The teacher must try to help the child with the development of those language symbols and he must do this just in the context belonging to the science he wants to introduce" (p. 96). Other research has found that the learning of a new language and language symbols is imperative to attain a full understanding of a particular mathematical concept.

A study conducted by Sfard (2000) purports the importance of the acquisition of language and language symbols in the developing a full understanding of mathematical concepts. Sfard's research lies in the context of algebraic functions. She found that students were beginning their study of algebraic functions by studying the graph of a function; this is what Sfard calls the *object* level. Sfard came to realize that students really had no understanding of where the graph, or object, came from, that ordered pairs of a function were points on the graph, and that the ordered pairs were determined from the function expression. Knowledge of ordered pairs of a function is at the level of understanding Sfard calls *process*. Sfard believed that if the of teaching of functions began at the process level, students' development of object knowledge would follow more naturally, but she learned that one cannot teach students at the process level without teaching at the objects level simultaneously and vice versa. In fact, Sfard found that process knowledge and object knowledge build and develop together. This is where language begins to play a role. A teacher may introduce the graph of a function and discuss it with her students before they have any understanding of what the graph is, but with discussion, or social interactions, and use of the language and language symbols of the graph, students will begin to develop meaning of the graph as a language symbol and how it relates to the function expression. Thus, process and object knowledge develop together, through the social interactions with other students, with the teacher, with the use of the textbook, to create a full-fledged understanding of the mathematical concept. Language or language symbols must be introduced before students have an understanding of what that expression, or symbol means. But as Sfard explains, the process of introducing new language to students and students' acquisition of the language with understanding is "like two legs that make moving forward possible due to the fact that they are never in exactly the same place, and at any given time one of them is ahead of the other" (p. 56). Sfard believes the introduction and use of language and language symbols when teaching for understanding of algebraic mathematical concepts is imperative for students' conceptual development. The introduction and use of language is similarly important to the teaching for understanding of geometric concepts.

Sfard (2000) explained, "Today's student is usually thrown straight into a predetermined mathematical conversation, governed by a set of ready-made rules" (p. 55). Burger and Shaughnessy (1986), Usiskin (1982), and Geddes et al. (1988) all found that in most high school geometry courses, the teaching of geometric concepts begins at van Hiele (1986) level 3, deduction; because few, if any, high school students are at level 3, geometry begins in a language foreign to the students. The van Hiele theory explains that when presenting geometric concepts or tasks to students teachers "use the language of the third level and the pupils are not even able to use the language of the second level" (van Hiele, 1986, p. 90).

The *Standards* (1989, 2000) also support the importance of the acquisition of a language within the learning process contained between the van Hiele levels.

> The Curriculum Standards present a dynamic view of the classroom environment. They demand a context in which students are actively engaged in developing mathematical knowledge by exploring, discussing, describing, and demonstrating.

> Integral to this social process is *communication*. Ideas are discussed through talking, writing, speaking, listening, and reading.
>
> (NCTM, 1989, p. 214, italics added)

The *Standards* document (2000) contains a communication standard in which a learning goal for both the 6-8th grade band and 9-12th grade band is “Use the language of mathematics to express mathematical ideas precisely” (p. 268, 348). Of course as with the progression of the van Hiele levels, language develops and progresses through the grade bands. The language acquired and used by students in high school will be more sophisticated than that of used by students in the middle grades. According to the Standards (2000), “when students in grades 6-8 explain their thinking, they can be held to standards that are more stringent than would likely be applied to younger students, though not as demanding as might be applied in high school” (p. 268).

Summary, Uniqueness and Significance of This Study

Evidence from a variety of sources makes it clear that students are not learning geometry concepts appropriately in order to prepare them for success in their high school geometry course. Researchers have made for the improvement of geometry curricula prior to high school geometry in the hope that students will be more adequately prepared to understand the concepts taught in high school geometry. The suggestions include developing activities that will help students develop through the van Hiele (1986) levels, meaning that more geometry should be introduced to students in the elementary and middle grades, and that systematic geometry instruction should be incorporated into curricula to help students develop to higher levels of understanding.

Research conducted by Burger and Shaughenssy (1986) and van Hiele (1986) are made reference to in the 2000 NCTM Standards document. Their recommendations and research have impacted the development of Standards-based curricula. Research has shown that Standards-based curricula programs have made improvements in students' mathematical understanding.

This study investigated whether Standards-based curricula better prepare students for high school geometry.

This study was designed to answer the following research questions:

1. At what van Hiele level are high school students entering geometry?
2. Is there a difference in van Hiele levels of students, at the beginning of their high school geometry course, who have participated in standards-based curriculum and students who have not participated in Standards-based curriculum prior to high school?
3. What differences, if any, can be identified among students within a given van Hiele level?

Chapter 3—Research Methods

Participants

The participants in this study were twenty 9th-grade geometry students; ten students were chosen from Sunnyside Junior High School in Union Town, Utah, and ten students were chosen from Lincoln Junior High School in Colton, Utah. Both schools are in the Whitman school district. IN this school district, 6th-grade is taught the elementary schools and 7th- and 8th- grade are taught in the junior high schools. Students from these two junior high schools were chosen because Sunnyside Junior High School, which has 7th- and 8th-grade, uses the Standards-based curriculum, The Connected Mathematics Project (CMP). The students chosen from Sunnyside Junior High School also came from elementary schools that used CMP in the 6th-grade. Lincoln Junior High School uses a traditional curriculum, textbooks that are not affiliated with the NSF in the 7th- and 8th-grade. Similarly, students chosen from Lincoln Junior High School also came from elementary schools that used textbooks that are not affiliated with NSF. The twenty students were randomly selected for participation based on their responses to the Participant Questionnaire included in Appendix A. The participant questionnaire allowed only those students who had had either three years of the Standards-based curriculum in the 6th -, 7th - and 8th -grade, or three years of the non Standards-based curriculum in the 6th -, 7th -, and 8th -grade into the pool of students from which the participants were randomly chosen.

The schools were chosen based on information acquired from the Whitman school district; according to the school district the two participating schools are extremely similar according to the socioeconomic status of the students and other comparable traits of the schools, such as the student population size and the elementary schools that the students funneled into the schools from. Sunnyside Junior High School had fully implemented CMP and Lincoln Junior High School did not use any type of reform mathematics curriculum (NONcmp). Thus, the main difference accounted for between these schools is the mathematics curriculum used in the grades preceding geometry.

Students were referred to in the study and the analysis of the data by the pseudonym assigned to them and by the type of curriculum background they have (i.e. CMP or NONcmp).

The Interview Process

An assistant researcher, who had been trained to do the interviews appropriately, and I interviewed twenty students, individually in one-on-one interviews. The assistant researcher and I each interviewed students at both schools, alternating between schools. Thus, each of us interviewed ten students. The interviews took place in the students' geometry teachers' classroom immediately after school under the supervision of the students' geometry teachers. Each interview took about 50 minutes to complete. The interviews took place September 6th-21st, 2005, which was two weeks after the first day of school for the fall term.

The interview consisted of giving the students eight open-ended tasks, developed by Burger and Shaughnessy (1986), which they answered to the best of their ability. The interview followed a script, written by Burger and Shaughnessy, designed to prevent any influence of the interviewer from skewing the results of the interview, and to give the interviewer control over the line of questioning. Each interviewer followed the script as closely as possible. Following the script prevented any major discrepancies between each of the interviews, and thus, the interviews are as similar as possible among the twenty students.

Tasks

Burger and Shaughnessy (1986) developed the tasks that were used in this study to assess students' geometric understanding at a specific van Hiele level. According to Burger and Shaughnessy, these tasks were developed to evaluate students' basic geometric skills. The tasks are open-ended and were designed to provide interpretation at several different van Hiele (1986) levels since students are at varying levels of geometric understanding according to the van Hiele theory. There are three triangle tasks and five quadrilateral tasks with the following content: drawing shapes, identifying and defining shapes, sorting shapes, and logical reasoning about geometric shapes. The tasks and script for each task are provided in Appendix B.

Data Analysis

The data in this study consisted of the students' written work from the tasks, the interviewer's field notes, and the videotaped interviews. The students' work and discussion/explanation for each task were analyzed using the same process developed by Burger and Shaughnessy (1986) in conjunction with their Level Indicators (Appendix C).

Burger and Shaughnessy (1986) developed "analysis of interview forms" (p. 37) for each task. The analysis of interview forms include the following: specific observations will be made and certain questions answered about the student's responses to each task; student responses will be tabulated; an overall summary made about the student's performance on the task; any confounding factors about the interview was cited; an overall van Hiele (1986) level was assigned for the task based on the behaviors which were observed according to the Level Indicators. Each task was analyzed extensively and a form for each task in the interview was filled out. Samples of some of the Analysis of Interview Forms are in Appendix D.

The assistant researcher and I analyzed each interview separately then compared our responses to the interviews for consistency, and agreed on a final level assignment for each student.

Once each task was evaluated for each student, the analysis forms were compiled into a single summary for each student. The final summary assigned a predominant van Hiele (1986) level.

After the students had been assigned to specific van Hiele (1986) levels, the students within a given van Hiele level were more carefully examined. This time a grounded theory approach was taken. According to Strauss and Corbin (1998) "Grounded theories, because they are drawn from data, are likely to offer insight, enhance understanding, and provide a meaningful guide to action" (p. 12).

I viewed the interviews of the students, within a given van Hiele (1986) level, a second time. With this second viewing I was specifically looking for variations of students' reasoning/understanding within the given van Hiele level. Episodes containing dialogue specific to the central features of the given van Hiele level were noted by recording time it took place within the interview. A brief description of the situation in which the dialogue took place and the nature of the student dialogue were recorded. For example, for students at level 1, episodes containing dialogue about "[reasoning] about geometric concepts by means of informal analysis of component parts and attributes" (Burger and Shaughnessy, p. 31, 1986) were noted.

Once the central ideas in which students varied within the given van Hiele (1986) level were identified, another analysis of the data was conducted. This third analysis was done by primarily studying the specific episodes within the videotaped interviews of student dialogue identified in the previous analysis. The purpose of this analysis was to confirm the codings of the previous analysis or to search for alternate, more appropriate codings. The objects identified in the analysis were then organized into categories reflecting the rationale for each as reflected in the episodes of the student interviews. The identified categories were then scrutinized by reviewing the student interviews again to determine if the categories could be refined or if other episodes within the interviews would fit within the categories. This was done, of course, to answer the question, "What differences, if any, can be identified among students within a given van Hiele level?"

During the analysis, the video data of the student interviews and the student work from the interviews were reviewed often to find relevant dialogue and examples that reflected the findings and to check the accuracy of the findings.

Statistical Analysis

A t-test was performed on the data in this study. A t-test was used to determine if there was a statistical significance between the mean van Hiele level (1986) of the CMP students and the mean van Hiele level of the NONcmp students. A t-test was performed on the data because t-tests are recommended for determining statistical significance when comparing two populations with small samples sizes. This type of statistical test was also chosen because it allowed us to compare the results of two population samples with different treatments, or mathematics curricula.

Chapter4-- Results

Burger and Shaughnessy's (1986) Level Indicators and Analysis of Interview Forms were used to analyze the 20 student interviews in the first analysis of the data; two students were assessed to be at van Hiele level 0 and eighteen students were assessed at van Hiele (1986) level 1 (see table 3). The results answer the first research question: At what van Hiele levels are ninth grade students entering geometry? The initial results of the student interviews showed that the majority of students, at the beginning of ninth grade geometry, are at a van Hiele level 1.

Table 3
van Hiele Levels of Students

Van Hiele Level 0	Van Hiele Level 1
Kelly, NONcmp	Rachel, NONcmp
Abbey, NONcmp	Becky, NONcmp
	Jack, NONcmp
	Minny, NONcmp
	Trent, NONcmp
	Trevor, NONcmp
	Patty, NONcmp
	Evan, NONcmp
	Alice, CMP
	Susan, CMP
	Jeremy, CMP
	Katie, CMP
	Mia, CMP
	Joe, CMP
	Adrian, CMP
	Kara, CMP
	Steve, CMP
	Trish, CMP

Note. Students were assessed at a specific van Hiele level according to the criteria, developed by Burger and Shaughnessy (1986), in the Level Indicators.

These results also answered the second research question for this study: Is there a difference in van Hiele levels of students at the beginning of their geometry course, who have participated in Standards-based curriculum versus students who have participated in a nonStandards-based curriculum? Having used Burger and Shaughnessy's (1986) criteria (Level Indicators) to assess student geometric understanding, there is not a difference between the CMP students and the NONcmp students.

From the first analysis of the student interviews, it was clear that all students within van Hiele (1986) level 1 "[reason] about geometric concepts by means of informal analysis of component parts and attributes," (p. 31) but through the initial analysis of the student interviews it became apparent that students do this at varying levels of sophistication. Some students have tendencies to analyze geometric shapes and their component parts, "[using] imprecise properties (qualities) to compare drawings and to identify, characterize and sort shapes" (Burger and Shaughnessy, 1986, p. 31) while other students are able to analyze the attributes of the components of shapes using precise language to describe and discuss the components of the shapes. But what was clear among the students is that they were all analyzing properties and components of geometric shapes, which, according to the van Hiele levels described earlier, is the distinguishing characteristic of van Hiele level 1 reasoning.

Students at van Hiele level 1 also understand the necessary properties for shapes. The students assessed at level 1 demonstrated a spectrum of understanding of the necessary properties of specific geometric shapes. Some students within level 1 understood that there are specific, necessary properties that determine a certain geometric shape. Other students understood that there are some specific properties for certain shapes, but they did not consider, or yet know, enough of these properties when determining a specific shape; they may only consider one property when characterizing a specific shape or sorting shapes.

Therefore, it was very apparent that students were at differing levels of understanding of geometric concepts within van Hiele level 1. A second analysis of the student interviews within level 1 was performed. From this second analysis of the student interviews, three major distinctions among the students within level 1 were identified:

1. <u>Infinite Variety:</u> Students have a clear understanding of an infinite variety of shapes versus students who cannot verbalize an infinite variety of shapes, but their understanding of an infinite variety of shapes is near.
2. <u>Necessary Properties:</u> Necessary properties of shapes are clearly understood versus necessary properties of specific shapes are still being formulated.
3. <u>Precise Language:</u> Students use precise language to discuss/describe the components and properties of shapes versus students use imprecise, sometimes ambiguous, visual descriptions to discuss components of shapes.

The three distinctions described above divide the students initially assessed at van Hiele level 1 into three subsets within level 1: Level 1A, Level 1B, and Level 1C. Students were divided into these subsets according to their understanding of these three distinctions. For example, students within *level 1C* understand an infinite variety of geometric shapes, understand necessary properties of geometric shapes, and use precise language to discuss geometric concepts. One the other hand, students within *level 1A* cannot verbalize an infinite variety of geometric shapes, necessary properties of geometric shapes are still being formulated within the student's understanding, and they tend to use descriptive language rather than precise language to discuss geometric concepts. Table 4 shows the distribution of students among these three subsets of van Hiele level 1, including the two level 0 students.

Table 4 Students Within Each Subset of van Hiele Level 1

Level 0	Level 1A	Level 1B	Level 1C
Kelly, NONcmp	Rachel, NONcmp	Trent, NONcmp	Patty, NONcmp
Abbey, NONcmp	Becky, NONcmp	Trevor, NONcmp	Evan, NONcmp
	Jack, NONcmp	Jeremy, CMP	Adrian, CMP
	Minny, NONcmp	Katie, CMP	Kara, CMP
	Alice, CMP	Mia, CMP	Steve, CMP
	Susan, CMP	Joe, CMP	Trish, CMP

Level 1A

Students within this subset were initially evaluated at van Hiele level 1 because the majority of their reasoning and responses demonstrated on the geometry tasks reflected level 1 reasoning according to the criteria laid out in the Level Indicators (Burger and Shaughnessy, 1986, p. 34). But upon closer examination, in a subsequent analysis of the interviews, it became clear that these students differed in their sophistication of reasoning within level 1 compared to other students within level 1.

Infinite Variety. Students cannot verbalize an infinite variety of shapes, but an understanding of an infinite variety of shapes seems to be near. Students have difficulty knowing how to discuss how many triangles/quadrilaterals they can draw. Students tend to say “ a lot,” but they are not comfortable giving a specific number. For example, Jack explains how many triangles he can draw: “a lot…depends on how many angles I could do.” Another student, Becky, explains how many triangles she could draw, “As many as there are degrees because each degree makes it a different triangle, I don’t know, 179?” Another student, Alice, explains how many different triangles she can draw, “tons…different angles and different sides…small and big.”

Similarly with quadrilaterals, when asked how many quadrilaterals she could draw Rachel asks, “Do I have to say a number?…a lot…maybe different sizes of sides or different types of shapes.” When asked how many quadrilaterals John could draw he explains, “a lot…more than five…perimeter would be different.”

Thus, students within this subset are not able to verbalize an infinite variety of triangles and quadrilaterals, but they understand that there is more than a small variety of shapes, and that geometric shapes differ from each other according to their component parts.

Necessary Properties. Students are beginning to understand necessary properties of shapes. These students tend to list 1-2 properties for triangles and specific quadrilaterals. For example, Minny explains the properties she sees necessary for a shape to be a square: “four equal sides that don’t have a hole in them.”[1] Alice lists the properties she sees necessary for a triangle: “Three sides, three lines and they like all connect.” Because Minny and Alice do not realize or consider other necessary properties of squares and triangles they tend to label shapes inappropriately. For example, Minny’s characterization of square allows her to consider rhombi as squares. Alice’s characterization of triangle allows her to label three-sided shapes with curved sides as triangles. Thus, necessary properties are not quite in the grasp of these students’ understanding, but they are beginning to understand that there are specific necessary properties to guarantee certain shapes.

Rachel explains the properties necessary for a rectangle: “two short sides that are equal to each other and two long sides that are equal to each other that are opposites.” Becky explains the properties necessary for a square: “a square has four sides and each

[1] For Minny, “that don’t have a hole in them” means that the shape is closed, all sides are connected.

sides is the same length as all the others." Neither of these students verbalizes that rectangles must have four equal angles, or four right angles. Yet, both of these students generally labeled rectangles appropriately and did not include other shapes as rectangles. This leads one to understand that students within this subset are still developing the necessary properties of shapes and the verbalization of these properties; these students are very good at identifying shapes correctly while also considering some of the properties of the shapes.

It should be noted that students within this subset could generally only discuss properties of triangles, squares and rectangles. Parallelograms and rhombi are not familiar at all, even though students may say that they have heard the word "parallelogram" or "rhombi" before. For example, Becky described a parallelogram as "almost like a triangle except shorter, it's cut off." Clearly, she does not know what a parallelogram is, at least by name, and she ends up describing a trapezoid.
For Jack, a rhombus has, "four sides and it's not really any other shape."

Precise Language. Students tend to use descriptive, sometimes, imprecise language to describe/discuss specific shapes and the attributes of their components. These students are clearly analyzing the components of shapes, but they do not use precise language to describe specific attributes of shapes. Words such as acute, obtuse, parallel, congruent, side lengths, etc. are not used. The exception to this is that these students are comfortable discussing "right angles," and "right triangles."

When explaining why three triangles are similar Minny says, "all have one much longer side; they're not just three of the same side." Alice explains why several triangles are similar, "All kind of have this longer side…angles aren't equal on all sides." Rachel explains how two quadrilaterals are similar, “they have two little sides that are equal to each other and two longer sides that are equal to each other, like a rectangle.”

Some students tend to use ambiguous, visual descriptions to explain what they have identified among shapes. Alice describes a rectangle as having “two sides that are longer that connect to two shorter sides.” Alice’s description is ambiguous because it is unclear what “longer” or “shorter” means. Susan explains how three triangles are similar by discussing the sides of each triangle:

> These two [sides] are the same length as these two [sides], but on this [triangle]…to me this one is like stretched out and so like this one could be like the same as this one if it wasn’t stretched out. So it could be like the other ones, it’s just stretched out.

Susan’s description is a mixture of analyzing the attributes of the sides of triangles, but she also used visually imprecise language such as “stretched out” to describe similarities among triangles.

Level 1C

Students within this subset were initially evaluated at van Hiele level 1 because their reasoning demonstrated on the geometry tasks reflected level 1 reasoning according to the criteria laid out in the Level Indicators (Burger and Shaughnessy, 1985, p. 34). These students did not demonstrate any level 0 characteristics. Thus, their level of reasoning was evaluated at level 1. Once again, upon closer examination of all the interviews of the students within level 1, the students within this subset of level 1 exhibited a difference in their reasoning compared to the other students within level 1.

The students within this subset of level 1 demonstrate clear understanding of the three distinctions described above.

Infinite Variety. Students understand that there is an infinite variety of shapes and they can clearly verbalize this understanding. For example, Evan explains how many triangles he could draw, “unlimited, I could keep going forever.” Evan further explains how all the triangles would differ from each other, “different sizes, different angles, and different side lengths.” Adrian explains his understanding of an infinite variety of triangles by saying, “there’s right triangles, obtuse, acute, and then there’s the isosceles and equilateral and scalene. Just talking about the different angles…that’s infinity so it’s not that hard.” Patty explains how many triangles she can draw and how they would differ from each other, “You can draw different sizes of them, and that infinitely…you can also draw different angles, that’s another way.”

Similarly with quadrilaterals, Trish explains how many quadrilaterals she can draw, “infinite…you can have different angles, different like squares rectangles, rhombuses and um, different sides or different lengths.” Kara explains the number of possible quadrilaterals, “there’s like no end…different angles or different sizes of lines or something.” Adrian also explains an infinite variety of quadrilaterals and their differences, “Just about as many as you want…size, shape, or angles.”

Necessary Properties. Students within this subset tend to list 2-3 properties of shapes when characterizing a certain shape. These students understand that there are specific necessary properties that define certain geometric shapes; they look for these particular properties when identifying shapes.

For example, Steve lists the properties he sees as necessary for triangles, “three straight sides, three angles, lines are all connected.” Trish lists several properties for a triangle, “Three angles, has to be a closed figure…there has to be three lines, there can only be one right angle…it won’t be closed if there’s two…all angles have to equal 180.” Patty explains, “A triangle has three sides, three corners, sides are straight and they all connect.”

Similarly for quadrilaterals, Kara lists the properties necessary for a rectangle, “the opposite of each line has to be the same length as the line...and then there has to be right angles.” Adrian explains the properties necessary for a parallelogram, “ Acute and obtuse angles, and the opposite side is parallel and the same.” Evan explains the properties he sees as necessary for squares, “ the sides were all equal and it had four right angles.”

It should be noted that students within this subset are able to discuss squares, rectangles and parallelograms. Rhombi are generally not familiar to these students. Some students ended up, perhaps by process of elimination, describing a trapezoid as a rhombus. For example, Patty demonstrates how students tend to revert back to a lower level of reasoning when shapes or concepts are unfamiliar. She explains the properties for a rhombus, “two sides are exactly the same, only the other two sides, one is bigger and one is smaller.” Trish explains the properties of a rhombus as “it’s got four sides, two of them are parallel to each other, but two are not and it’s got two angles that are equal.” Clearly, theses students are confused and unfamiliar with rhombi, even unfamiliar with the way rhombi “look” since they don’t describe even the basic shape of rhombi in imprecise, descriptive language.

Precise Language. When students discuss the attributes of the components of specific shapes, they tend to use precise language such as "obtuse," or "acute" to describe the angles or they refer to the type of shape they are discussing such as "isosceles," "rectangle," or "parallelogram."

For example, when sorting triangles into similar groups, the use of precise language is particularly apparent. Kara identifies the common characteristic of groups of shapes using precise language such as "right triangles," "both isosceles," "scalene." Similarly, Evan sorts quadrilaterals into similar groups by identifying the common characteristics using precise language such as "they are parallelograms," "all have obtuse angles," "both have obtuse angles, both parallelograms, both have acute angles." Steve identifies common characteristics among triangles as, "all obtuse," "acute," "sides aren't all equal."

These students use precise language consistently throughout the interview. Not only in discussing how shapes are similar or different from each other, but when asked about other types of geometric concepts such as class inclusions. For example, Kara explains, using precise language, why she allows the class inclusion Squares $\subseteq$ Rectangles, "The opposite line for each line is the same length and there's right angles." Similarly, Trish explains why she allows the class inclusion Rectangles $\subseteq$ Parallelograms, "Opposite angles are equal, opposite sides are parallel, it's got four angles and four sides." Adrian uses precise language to describe the differences among triangles he drew. He describes triangles as being "equilateral," "obtuse," or "isosceles." When describing the properties of a parallelogram, Trish says, "Opposite side are parallel to each other, they won't intersect at any point, it's got opposite angles equal to each other, it's got four sides and four angles."

Level 1B

Students within this subset were initially evaluated at van Hiele level 1 because their reasoning demonstrated on the geometry tasks reflected level 1 reasoning according to the criteria laid out in the Level Indicators (Burger and Shaughnessy, 1985, p. 34). On a few of the tasks these students worked through, they were considered level 0 according to the criteria for level 0 in the Level Indicators, but the majority of their reasoning within the tasks were evaluated at level 1. Thus, the dominant level of reasoning of each of these students was evaluated at level 1. But upon closer examination, in a subsequent analysis of the interviews, it became clear that these students differed in their reasoning abilities compared to the other students evaluated at level 1. Students within this subset are a mixture of the three distinctions described above.

Below are specific examples of students who demonstrate a mixture of the three distinguishing characteristics identified.

Trent. Students within this subset may be able to verbalize an infinite variety of shapes, but tend to use imprecise, ambiguous language to describe/discuss the attributes they see among shapes. For example, Trent clearly understands an infinite variety of shapes. He explains how many triangles he can draw, "infinitely many because of the different angle combinations and the size."

Trent also seems to be developing an understanding of the necessary properties of specific shapes. Properties listed for a square, Trent says, "four sides that are the same and four angles that are the same." When describing the properties for a parallelogram, Trent explains, “two angles measures that are the same, the other two are the same but different than the other two, and it’s like a squashed rectangles or square.” This shows how Trent is developing the concepts of necessary properties of shapes, but he still reverts back to visual considerations and imprecise descriptions.

Trent often reverts back to describing how shapes look to him using ambiguous, visual language. For example, Trent describes a rectangle as being “a square that’s been stretched out.” Yet when identifying similarities among specific quadrilaterals, Trent refers to the quadrilaterals by their type name consistently, “trapezoids,” “parallelograms,” “rectangles.” Trent clearly demonstrates a mixture of precise and imprecise language use.

Trevor. Students may understand an infinite variety of shapes, but tend to use descriptive, imprecise language to discuss shapes, and the necessary properties of shapes may still be under formulation within their understanding. For example, Trevor understands an infinite variety of shapes. He explains that there is an “infinite” variety of four-sided figures he could draw and they would all differ by “size, shape, angle.”

The necessary properties are still being formulated within Trevor’s understanding. When asked to list properties for a rectangle, Trevor lists, “four corners, it’s basically just a stretched square.” When asked to list properties for a triangle, Trevor explains, “It has to be a shape with three corners.” Trevor’s characterizations clearly show that the necessary properties of specific shapes are not yet clear in his understanding; he considers an inadequate list of properties to characterize/determine specific shapes, and he stills wants to incorporate visual considerations of the shape.

Trevor's descriptions of geometric shapes are clearly based on an analysis of component parts of shapes, but his descriptions are ambiguous and sometimes visually based. When describing similarities among triangles Trevor explains, "the bottom is one length but the two sides appear to be the same length up towards the top." Yet, Trevor does make clear, precise observations about some triangles, "at least two of the sides are the same length." Trevor describes a parallelogram, "it's a square that's sort of been tilted and then stretched; these two sides are parallel, then these two sides are parallel." These are clear examples of how Trevor is still developing in his use of precise language. He has the ability to use clear, precise language, but he often uses imprecise language in his discussions.

Joe. Students may use precise language to analyze the components of shapes, but they may not be able to verbalize an infinite variety of shapes, and the necessary properties may are still be in formulation within their understanding. For example, Joe's ability to verbalize an infinite variety of shapes is near, but not quite fully understood. He explains that he could draw "three" different triangles, "equilateral, acute and obtuse." Joe is clearly reasoning about how many different *types* of triangles that are possible, and yet when asked again how many triangles he could draw, he only says, "a lot."

The necessary properties for specific geometric shapes are still being formulated within Joe's understanding. When asked to list properties for a square he lists, "all the sides are equal." Properties listed for rectangles are "two of the sides that are opposite are equal, and the opposite sides from each other are equal." Properties listed for triangles, "three sides, they're straight." Joe seems to understand the concept of triangles and the necessary properties for them somewhat better than the properties for quadrilaterals. The inadequate lists of properties for squares and rectangles allow Joe to consider rhombi as squares and parallelograms as rectangles.

Although Joe is still formulating his understanding of several geometric concepts, he uses clear, precise language in the majority of his discussions. Joe describes triangles as having "right angles," as being "acute triangles," or as being "obtuse." For quadrilaterals, Joe uses clear, precise language to describe similarities, "all their sides are equal," "squares," "the two sides that are facing each other are the same length on both sides."

Katie. Katie cannot verbalize an infinite variety of shapes, but her understanding is developing towards an understanding of this concept. She explains how many triangles she can draw:

> A lot of each different kind of triangle but there's only three different kind like right triangles, isosceles, and acute…they have different numbers of angles, all the angles wouldn't be the same, but would look alike but wouldn't be the same.

Similarly with quadrilaterals, Katie explains how many quadrilaterals she can draw, "probably more than triangles. Four-sides figures are easier than triangles…all different sides lengths different positions on the paper, like rotated differently."

The necessary properties of specific shapes seem to be established in Katie's understanding. Katie describes the properties necessary for a triangle, "probably it has three straight sides and three angles and all the sides have to be connected together." The properties she lists for rectangles are, "all the angles are the same, and two sides are parallel and the other two sides are parallel." The properties Katie describes for a parallelogram demonstrate how she is still developing from a visual level 0 for some concepts; she explains that parallelograms "look kind of like a rectangle, but two of the angles are the same and two of the angles are different, and two sides are parallel and the other two sides are parallel."

The language Katie uses throughout the interview is a mixture of precise and descriptive. As evidenced in the dialogue above, Katie used precise language to explain properties of geometric shapes. But Katie has a tendency to use descriptive language. For example, when describing similarities among triangles, she explains, "they look similar, like they have the longest side on the bottom then they have the two shorter sides up on the top." Yet, Katie will describe similarities among other shapes more precisely, "all parallelograms…all have two larger than 90-degree angles and two smaller than 90-degree angles."

Students within this subset are all capable of discussing squares and rectangles and occasionally students are able to discuss parallelograms, but rhombi are not familiar enough to discuss. For example, Jeremy explains that a rhombus is "a diamond-shaped figure that's not parallel." Jeremy clearly has an idea of what a rhombus looks like, but he does not understand the properties of a rhombus. Similarly, Mia explains that a rhombus "looks like a kite."

Statistical Results

A t-test was performed to compare the average van Hiele level of the CMP students to the average van Hiele level of the NONcmp students. To determine the average van Hiele levels of each group of students, numerical values were assigned to each level. Level 0 was assigned the value 0, level 1A was assigned the value 0.5, level 1B was assigned the value 1, and level 1C was assigned the value 1.5. By considering the number of CMP students within each of these levels, the average level for the CMP students was determined to be 1.1. Similarly for the NONcmp students, the average level was determined to be 0.7.

The t-test performed determined whether the difference between the mean van Hiele (1986) levels of the two groups of students, 1.1 - 0.7 = 0.4, is statistically significant or not. The value of the t-statistic determined from this data is .3923, with 9 degrees of freedom. The results of the t-test showed that this difference is not statistically significant; the P-value for this data is greater than 0.25. Thus, there is not enough convincing evidence to show statistically that the Standards-based curriculum, CMP, is more effective in preparing students for high school geometry than the nonStandards-based curriculum is.

Chapter 5—Conclusions

Overview and Conclusions

This research was conducted to determine if a Standards-based curriculum better prepares students for high school geometry compared to a nonStandards-base curriculum. This study used the van Hiele (1986) theory of levels of thinking to answer this question. The instruments used to assess the van Hiele level of twenty geometry students were developed by two researchers, Burger and Shaughnessy (1986).

This study had three research questions, which are:

1. At what van Hiele level are high school students entering geometry?
2. Is there a difference in van Hiele levels of students, at the beginning of their high school geometry course, who have participated in standards-based curriculum and students who have not participated in Standards-based curriculum prior to high school?
3. What differences, if any, among students within a given van Hiele level can be identified?

Using the criteria determined by Burger and Shaugnessy's (1986) study, the Level Indicators, the twenty students were assessed at specific van Hiele levels. Two students were found to be at van Hiele (1986) level 0, and the remaining 18 students were at van Hiele level 1. The answer to the second research question was clearly *no* with these initial results; there was no difference between the Standards-based curriculum students and the nonStandards-based curriculum students.

With a subsequent analysis of the student interviews, specifically looking at the students within level 1, more conclusions were made and seen regarding any differences between the Standards-based curriculum students and the nonStandards-based curriculum students. From the subsequent analysis of the student interviews, three distinct differences are seen among the level 1 students. The three distinctions identified are summarized as follows: Students' understanding of and ability to explain an *infinite variety* of shapes; students' understanding of and ability to explain the *necessary properties* of specific shapes; and students' use of *language* in their explanations, either precise or imprecise.

These three distinctions divide students into three subsets of level 1: Level 1A, Level 1B, and Level 1C. The three distinctions show us that there is a spectrum of understanding within level 1 students. Having divided the students into four levels of understanding, including the level 0 students, it was reasonable to again ask the question of whether there is a difference between the levels of understanding of CMP students and NONcmp students.

A t-test was conducted on the data to determine if there was a statistical significance between the mean van Hiele levels of the CMP students and the NONcmp students. The average van Hiele level for the CMP students is 1.1; the average van Hiele level for the NONcmp students is 0.7. There is not enough numerical evidence to show that the difference between these means, 0.4 is statistically significant using the t-test described earlier.

Although statistical significance cannot be shown, there other significant conclusions that can be made about the results of this study. The three distinctions of level 1 abilities, and thus the three subsets of van Hiele (1986) level 1, identified in this study has significance in van Hiele level research. The three subsets of level 1 demonstrate that students are at varying levels of understanding within a certain van Hiele level. The three distinctions identified allow us to be more specific about the abilities students have within level 1. Burger and Shaughnessy (1986) identified several specific abilities students have at each van Hiele level within their Level Indicators, but this research has shown that the abilities identified by Burger and Shaaghnessy can be further refined. Furthermore, these findings would make it reasonable to conclude that there are subsets of understanding within all the van Hiele levels.

The subsequent analysis of the level 1 students reveals something more about the differences between students within the different subsets of level 1. The third distinction identified is precise language: Students use precise language to discuss/describe the components and properties of shapes versus students use imprecise, sometimes ambiguous, visual descriptions to discuss components of shapes. Within this study, it was found that students within level 1C used precise language ("isosceles," "parallelogram," "opposite angles," etc.) in their explanations. Students within level 1B used precise language within the majority of their explanations, but they also used descriptive, imprecise language ("stretched out," "this angle is bigger," etc.). Level 1A students used precise language almost equally as much as using descriptive, imprecise language.

The use of language was a large distinguishing factor between level 1 students, and even between the two different levels, level 0 and level 1. The type of language, precise or imprecise, that students used has not been a focus of past research studies involving van Hiele (1986) levels. For example, Burger and Shauhgnessy (1986) did not make language the focus of their Level Indicators; the Level Indicators list several different capabilities and understandings students have within specific van Hiele levels.

Furthermore, the type of language used by students at specific levels was only *one* of several characteristics listed within the Level Indicators.

Fuys and Geddes (1984) found that language plays a role in the learning of geometry concepts with sixth and ninth graders. According to Fuys and Geddes, progress within and between levels was influenced by "instruction and ability, in particular, language ability" (p. 10). The study conducted by Fuys and Geddes also found that it was only after specific instruction on particular concepts that students began to be able to express their thoughts "precisely in terms of properties of shapes" (Fuys & Geddes, 1984, p. 9).

Research, within other areas of mathematics such as the study of algebraic functions, has argued the importance of the acquisition and use of language within the learning process. Sfard (2000) believes that the introduction and use of precise language and language symbols within the learning process of algebraic functions is imperative for the student to acquire a full understanding of the concept of function. Sfard believes that precise language and language symbols must be introduced when the concept is initially introduced to the students, and only through practice with discourse using the language will the student develop a full understanding of function. The conclusions Sfard has come to are similar to the findings and claims made by van Hiele (1986) in his belief about the acquisition of language in developing geometric understanding.

van Hiele (1986) believed that the introduction and acquisition of a new language is imperative for developing understanding within geometry. The learning phases within the van Hiele theory clearly show that language is an important part of developing from one level to the next. van Hiele believed that students need to be introduced to the precise language, they need to use it in the tasks they work on, and they need to use it in conversations with each other. Only through this process can students learn the language of the subject, geometry, and gain a full understanding of geometric concepts. The belief of the importance of language is evident in other sources of mathematics research and teaching guidelines, specifically the NCTM *Standards* documents (1989, 2000).

The *Standards* documents (1989, 2000) express the importance of students' acquisition of a new language when developing a full understanding of mathematical concepts. According to the Standards students should be able to "use the language of mathematics to express mathematical ideas *precisely*" (NCTM, 2000, p. 348, italics added).

Future Research

There are some implications of this study and suggestions for future research that warrant mentioning. One of the first issues to address is the lack of statistical significance of the mean van Hiele (1986) levels between the two groups of students. First, the way in which the Connected Mathematics Project curriculum was implemented in the schools may be a reason for finding no statistical significance. The CMP students chosen in this study may have had varying experiences with the Connected Mathematics curriculum. According to the school district the schools the CMP students came through were all fully implemented in the Connected Mathematics curriculum, meaning that the schools were using the curriculum as it is outlined the curriculum materials. Of course, fully implementing the curriculum means different things to different teachers. It is difficult to judge whether all CMP students had an equal experience with the Connected Mathematics curriculum, and whether they had the type of experience with the curriculum which the authors of the CMP curriculum would consider as fully implemented according to their recommendations.

The sample of students interviewed in this study is quite small, only twenty students. If at all possible, it would be better to interview a larger sample of students. This would make it possible to conduct statistical tests of significance on the data with better chances of finding some sort of statistical significance.

It should be noted that in this study the timing of the interviews did not make a difference in students' assignment of van Hiele levels. Specifically, students in level 1C were not all interviewed at the end of the interview period, and likewise, students assigned level 0 were not the first students to be interviewed. Thus, the timing of the interviews, whether at the beginning or the end, did not skew the results of this study. However, a consideration for future research should be the time frame in which the student interviews are conducted. The student interviews in this study began exactly two weeks into the school year, and the interviews took 10 days to accomplish. Quite a bit of teaching occurred within the first two weeks of class, and continued as we interviewed students during those 10 days. Future research could be improved by interviewing students at the end of 8^{th}-grade where no students have any more experience with specific geometric concepts than others.

A final thought for future research; the trend in the data found in this study is that CMP students tend to be in the middle to upper range of level 1, and the NONcmp students are in the lower to middle range of level 1. Even a slightly larger sample size could reflect a more even spread of both populations of students across level 0 and level 1. Perhaps one reason for this may be that the Connected Mathematics Project (CMP) does not address geometry in the 6^{th}-, 7^{th}-, and 8^{th}-grades as in depth as anticipated. Past research has shown great improvements in the standardized state math exams, as is reflected in the research conducted by Reys, Reys, Lapan, and Holliday (2003), but these exams are not as focused on geometry concepts as they are on algebra concepts.

Of Course, the issue of whether CMP addresses geometry concepts as in depth as anticipated may not be that the curriculum has not made provisions for teaching geometry; after all, the authors of the CMP curriculum have clearly outlined specific geometry goals that students should have achieved by the end of each of 6^{th}, 7^{th}, and 8^{th} grade. A reason for the lack of geometry in CMP may be the implementation of the area of geometry. Some teachers may focus less on geometry concepts than on algebra concepts. When visiting with some of the schools in the Whitman school district to determine which school to conduct this study in, teachers explained that there simply was not enough time in the school year to get to all the content outlined in the CMP curriculum. Most often, the majority of geometry concepts and activities outlined in the curriculum were saved until the last few weeks of class.

Summary

This study set out to show that the van Hiele theory (1986) and research done in the area of geometric understanding using the van Hiele theory has implications for the way students should be taught geometric concepts prior to a geometry course in high school. Research shows that students are inadequately prepared to understand the concepts presented in a high school geometry course, which is presented at a level of deduction, of proof. From the time that a great deal of van Hiele related research had been conducted, in the 1980's, Standards-based curriculum programs have been developed and implemented into the middle grades. This study has shown that there is a significant relationship between the van Hiele theory, the implications for teaching geometric concepts that research has left us with, and the NCTM Standards documents (1989, 2000). Research has shown that students who study under the Standards-based curriculum programs have higher standardized test scores, and thus, in general better conceptual understanding. But the question still remained; do Standards-based curricula improve students' conceptual understanding in geometry?

Through one-on-one interviews with 9^{th}-grade geometry students this study has determined that most students enter geometry at a van Hiele level 1, and further more, there are specific levels of understanding within van Hiele level 1. van Hiele level 1 can be divided into three subsets; level 1-A, level 1-B, and level 1-C. Students were divided into the subsets of level 1 according to their understanding of an infinite variety of shapes, their understanding of the necessary properties of shapes, and the type of language students use, precise or imprecise.

Of course, this study was designed to determine if there was a difference in levels of understanding between Standards-based curriculum students and nonStandards-based curriculum students. This study has concluded that there is no significant difference between the two samples of students.

Clearly, there are more questions to ask and considerations to make regarding the teaching and learning of geometry concepts. The fact, that research conducted long ago showed, still remains: students need to be better prepared for high school geometry so that they may be successful in the course.

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Appendix A

Participant Questionnaire

Participant Questionnaire

If you are willing to be considered to be a participant in this study, please fill this questionnaire out and return it with the consent form.

Name: ____________________________

What school did you attend in 6th grade? _______________________________________

Who was your math teacher in 6th grade? ______________________________________

What school did you attend in 7th grade? _______________________________________

Who was your math teacher in 7th grade? ______________________________________

What school did you attend in 8th grade? _______________________________________

Who was your math teacher in 8th grade? ______________________________________

If you are chosen to participate, a sign up sheet will be sent around in class for you to sign up for a time to be interviewed after school.

Thank you!

Appendix B

Burger & Shaughnessy Tasks/Scripts

TRIANGLE ACTIVITIES

Activity 1: Drawing Triangles

Purpose: To discover what attributes (shape, size, proportion, orientation, etc.) the student attends to when drawing distinct triangles (student-generated triangles).

Script:

1. *Draw a triangle. Let's label that #1.*
2. *Draw another triangle that is different in some way from #1. Let's call it #2.*
3. *Draw another triangle that is different from #1 and #2 (call it number 3).*
4. *Can you draw another triangle different from #1, #2, and #3? (If so, draw it and call it #4.)*
5. *How many different triangles could you draw?*
6. *How is #2 different from #1?*
7. *How is #3 different from #1 and #2?*
8. *How is #4 different from #1, #2, and #3?*
9. *How many different triangles do you think you could draw? How would they all be different from each other?*

If the student focuses on only 1 attribute, say orientation, ask, *"Can you find some other way to make a different triangle, other than just turning it (or making it larger/smaller, etc.)."*

Note: Have students put their names on all pages they use during the interview, including sketches.

Activity 2: Identifying and Defining Triangles

Part A.

Purpose: To determine whether the student can identify certain triangles.

Script:	*Put a T on each triangle on this sheet.*

Part B.

Purpose: To determine the properties that the student focuses on when identifying triangles.

Script:

1. *Why did you put a T on* ______________ *?* (Pick out at least 3/4 of those marked.) Be sure to include all "unusual" responses.
2. *Are there any triangles in #12 ?* If so, *"how many do you see ?"*
3. *Are there any triangles in #10 ?* If so, *"how many do you see ?"*
4. Pick out at least 4 (if possible) not marked as triangles. Ask, *Why did you not put a T on* ______________ *?* (for each one)

Part C.

Purpose: To elicit properties the student perceives as <u>necessary</u> for a figure to be a triangle.

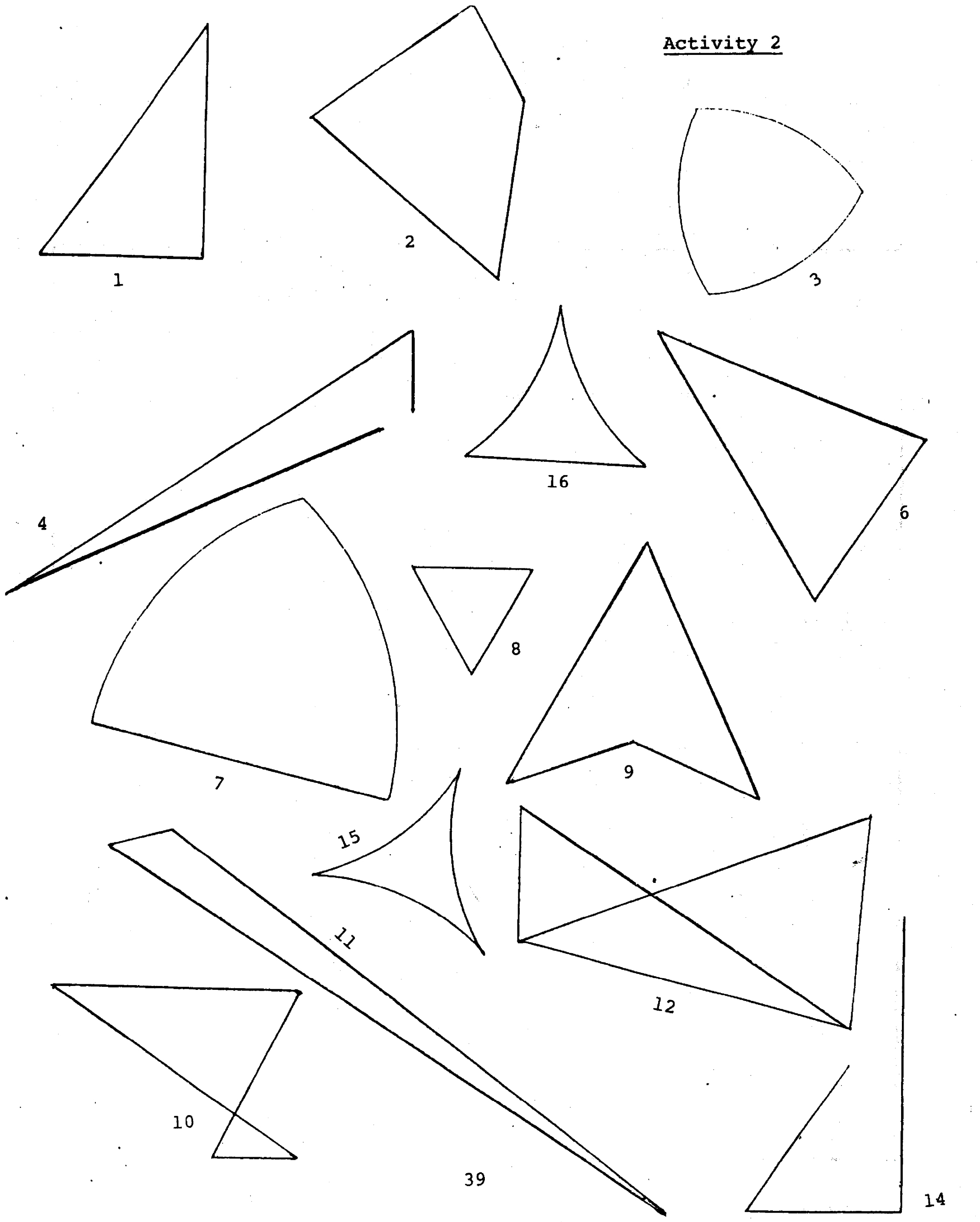
Script:	*What would you tell someone to look for to pick out all the triangles on a sheet of figures?*

Part D.

Purpose: To elicit properties the student perceives as <u>necessary</u> and <u>sufficient</u> for a figure to be a triangle.

Script:	*What is the shortest list of things you could tell someone to look for to pick out all the triangles on a sheet of figures?*

Activity 2



Activity 3: Sorting Triangles

Part A.

Purpose: To determine what properties the student focuses on when comparing triangles.

Script: (Place cutouts on the table.)

1. *Put some of these together that are alike in some way.* (Record the grouping.)

   How are they alike?

2. (Put the cutouts back together.)

   Can you put some together that are alike in another way? (Record the grouping.)

   How are they alike?

3. (Repeat as long as sortings appear useful. Remind students, if necessary, that they can reuse figures.)

Part B.

Purpose: To determine the student's ability to distinguish common properties of preselected triangles.

Script:

1. (Interviewer selects a set of triangles that have some common property: all isosceles, all right triangles, all obtuse, etc.).

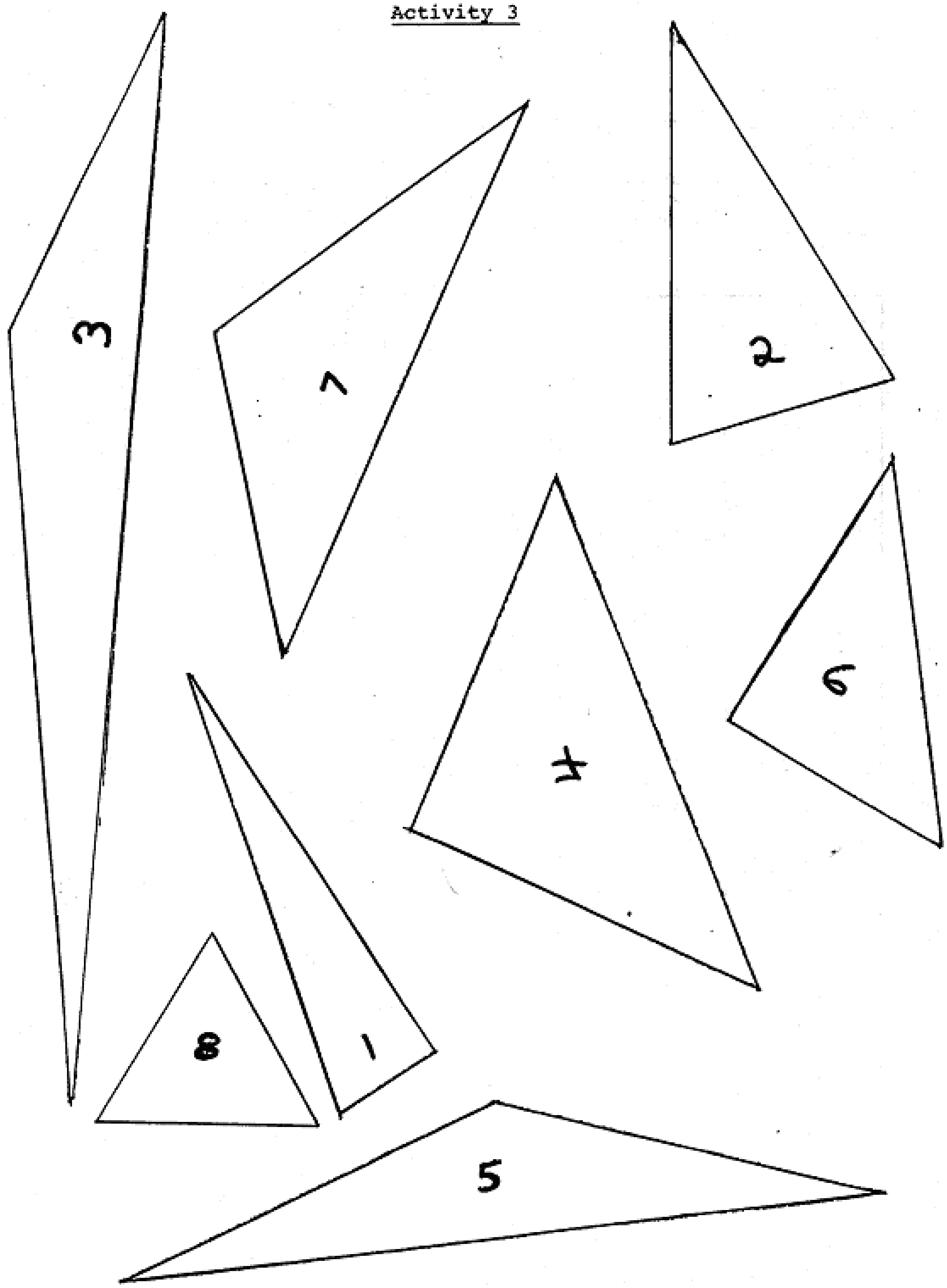
   All of these shapes are alike in some way.
   How are they alike?

   (The student may find a property that the shapes share, but which does not distinguish them from the others. If this happens, praise can be given, and the student can be told, *"There is another way -- can you find it?"*)

2. Repeat part 1 with a different sorting rule. (Make sure at least one of the sortings contains more than two shapes.)

3. Include at least one sorting using a group that the student had formed in part A.

Activity 3



QUADRILATERAL ACTIVITIES

Activity 1: Drawing Quadrilaterals

Purpose: To discover what attributes (shape, size, proportion, orientation, etc.) the student attends to when drawing distinct quadrilaterals (student-generated quadrilaterals).

Script:

1. *Draw a 4 sided figure. Let's label that #1.*
2. *Draw another 4 sided figure that is different in some way from the first one (call it #2).*
3. *Draw another 4 sided figure that is different in some way from #1 and #2.*
4. *Draw another 4 sided figure that is different in some way from the others.*
5. *Draw another 4 sided figure that is different in some way from the others.*
6. *How many different four sided figures could you draw?*
7. *How is #2 different from #1 ?*
8. *How is #3 different from the first two?*
9. *How is #4 different from the first three?*
10. *How is #5 different from the first four?*
11. *How many different 4 sided figures could you draw? How would they all be different from each other?*

    If the student focuses only on the attribute of orientation or the attribute of size, ask: *"Can you find some other way to make them different other than just turning them (or just making them bigger, smaller)?"*

Note: Have students write their names on <u>all</u> pages they use during the interview, including sketches they make.

Activity 2: Identifying and Defining Quadrilaterals

Part A.

Purpose: To discover whether the student can identify certain 4-sided shapes.

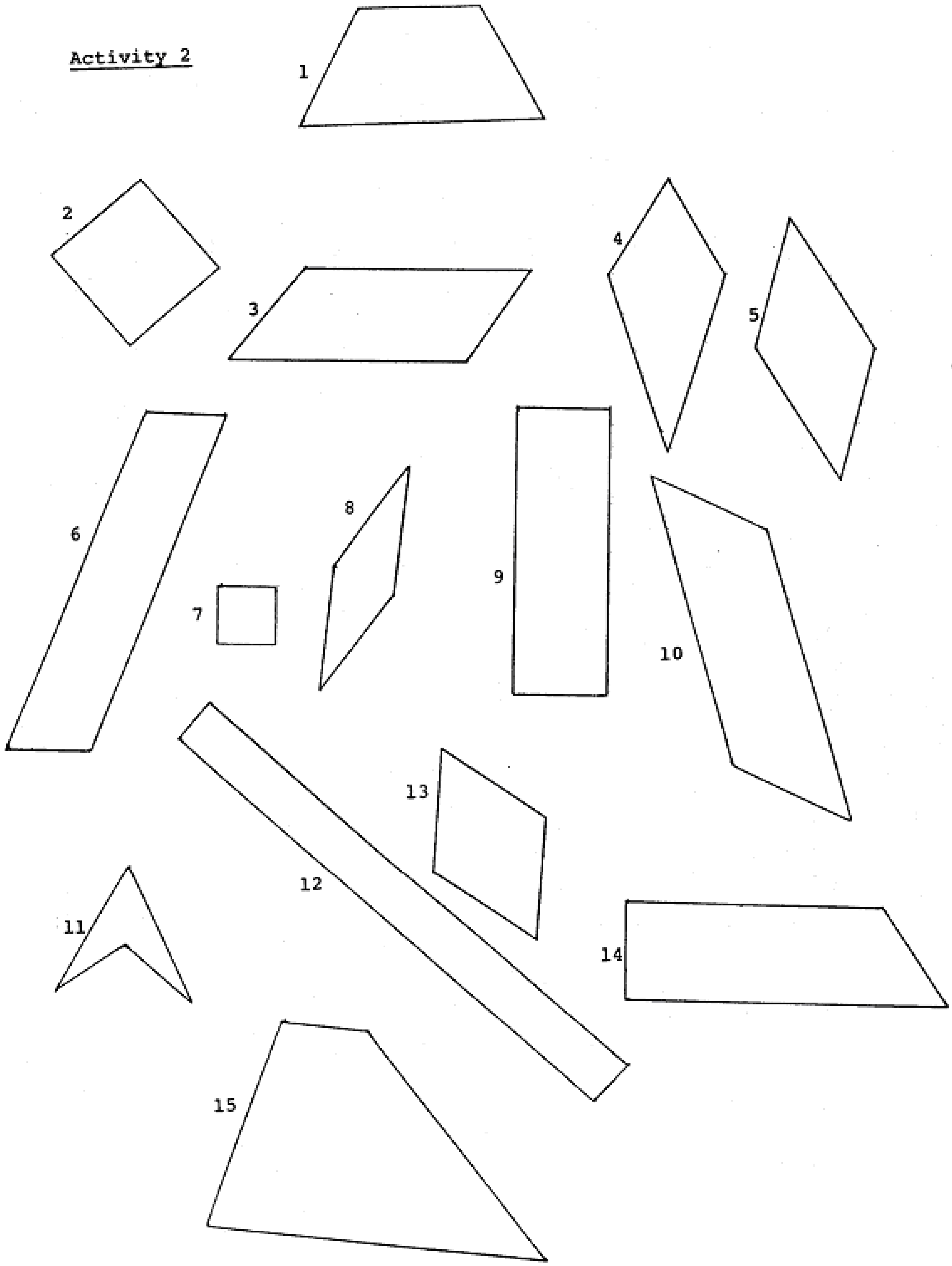
Script:
Have you ever heard the word "square"? (If so:) *Put an S on each square.* (Stop and let the student proceed.) *Have you ever heard the word "rectangle"?* (If so:) *Put an R on each rectangle.* (Stop again.) *Have you ever heard the word "parallelogram"?* (If so:) *Put a P on each parallelogram.* (Stop again.) *Have you ever heard the word "rhombus"?* (If so:) *Put a B on each rhombus.* (Stop again.)

Part B.

Purpose: To determine the properties of figures which the student focuses on when identifying quadrilaterals.

Script:
Why did you put an S on ____________ *?* (Pick out two or so.) Repeat the same question for each shape in Part A familiar to the student. Ask about any "unusual" responses that may have occurred in the lettering.

Activity 2



Activity 2 (continued)

Part C.

Purpose: To elicit properties that the student perceives as necessary for certain figures.

Script:

What would you tell someone to look for to pick out all the squares on a sheet of figures?

(Repeat for rectangles, if familiar.)

(Repeat for parallelograms, if familiar.)

(Repeat for rhombs, if familiar.)

Part D.

Purchase: To elicit properties that the student perceives as necessary and sufficient for determining certain quadrilaterals.

Script:

(If "square" if familiar:)

What is the shortest list of things you could tell someone to look for to pick out all the squares on a sheet of paper?

(Repeat for rectangles, if familiar.)

(Repeat for parallelograms, if familiar.)

(Repeat for rhombs, if familiar.)

Activity 2 (continued)

Part E.

Purpose: To examine whether the student applies his/her own properties of quadrilaterals consistently.

Script:

(If "rectangle" is familiar:)
Is #2 a rectangle? Why?

(If "parallelogram" is familiar:)
Is #9 a parallelogram? Why?

(If "rhombus" is familiar:)
Is #7 a rhombus? Why?

If the student changes his/her mind and decides that squares are rectangles, or rectangles are parallelograms, etc., then ask: *"If you went back and marked these shapes all over again, would you do it the same way?"* (If so, have the student reletter underneath the figures on the same sheet.)

(If there are inconsistencies in the relettering -- i.e., R & P under #9 but only R on #12 -- ask why he/she did not put the second/third letter on that figure.)

Activity 3: Sorting Quadrilaterals

Part A.

Purpose: To determine what properties the student focuses on when comparing quadrilaterals.

Script: (Place cutouts on table.)

1. *Put some of these together that are alike in some way.* (Record the grouping.)

2. *How are they alike?*

   (Put them all together again.)

3. *Can you put some together so that they are alike in another way?*

   (Repeat as long as sortings appear useful. Remind students, if necessary, that they can reuse figures.)

Part B.

Purpose: To determine the student's ability to distinguish properties of preselected quadrilaterals.

Script:

1. (Interviewer selects a set of quadrilaterals that have some common property: number of parallel sides, exactly 2 pairs of equal sides, a right angle, etc.)

   All of these shpaes are alike in some way. How are they alike?

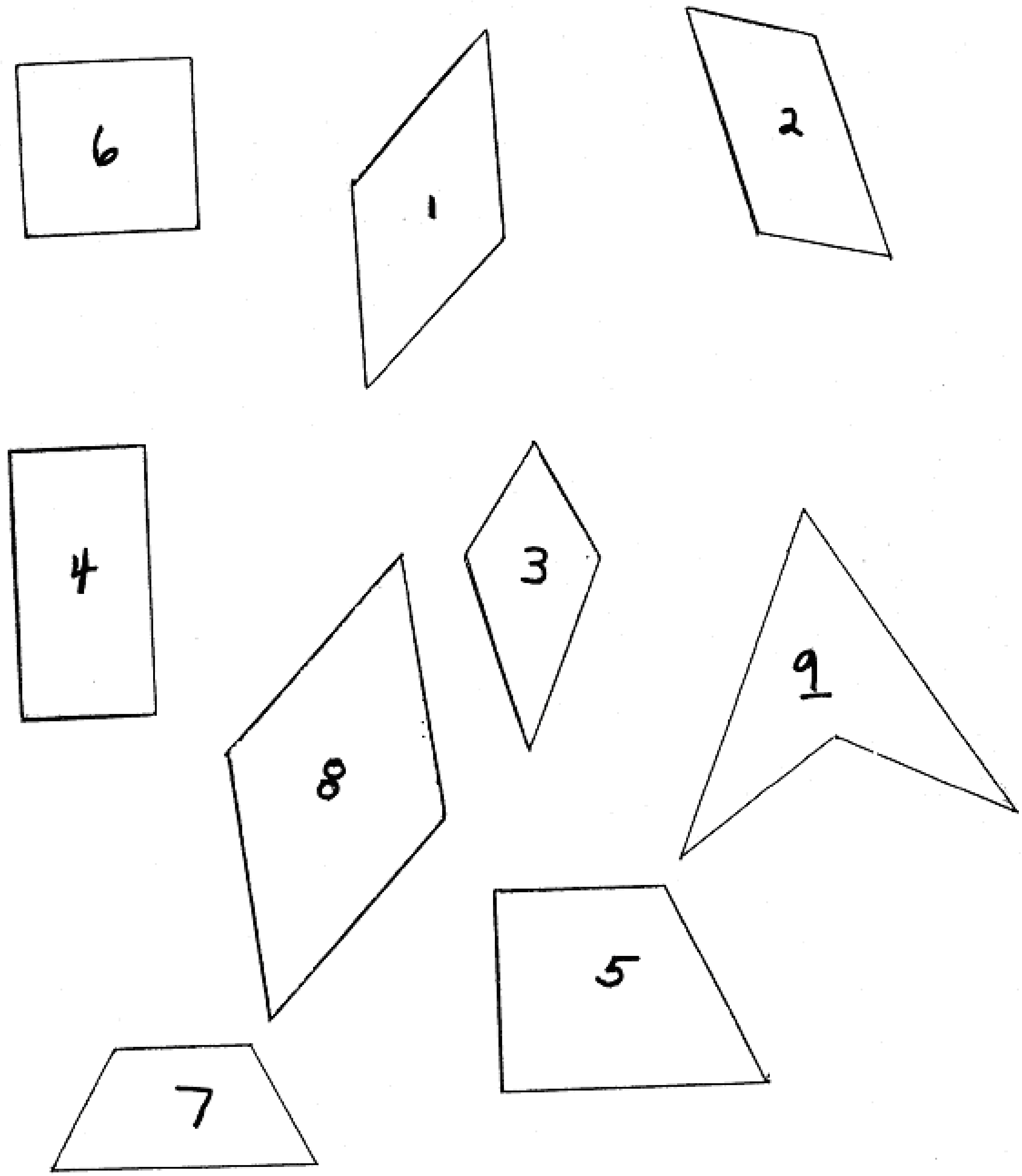
   (The student may find a property that the shapes share, but which does not distinguish them from the others. If this happens, praise can be given and the student can be told, "*There is another way -- can you find it?*" Continue as long as the search seems fruitful.)

2. Repeat part 1 with a different sorting rule.

   (Make sure at least one of the sortings contains more than two shapes.)

3. Include at least one sorting using a group that the student had formed in part A.

Activity 3



Activity 4: **What's My Shape?**

Purpose: To determine the properties and their interrelationships that the student perceives as sufficient to determine a shape among types, i.e., rectangle, trapezoid, etc.

Script: (Carefully give the directions in Part A.)

Part A.

1. *I'm going to show you a sheet of paper with some clues about a certain shape. I will uncover the clues one at a time.*

2. *Stop me when you have just enough clues to know for sure what type of shape it is. Ask for another clue if you want one.*

3. *Make a drawing of the shape if you want to. Think out loud if you want to, and tell me what you are thinking about.*

Part B.

Begin the "game" by uncovering clue #1 for shape A. Continue uncovering clues as the student requests. Reassure the student, if necessary, that the clues are consistent.

If the student makes drawings, have him/her go from top to bottom of a sheet of paper and label the drawings, shape A, B, or C.

4. If after one clue he/she says "quadrilateral", say, *"Good, let us look at another clue or so, and see what kind."*

5. Once he/she has decided for sure, ask *"Why?"* Then ask *"Is there any other shape you could draw that fits these clues?"*

   If the student says no, ask *"If I show you another clue, could it change your mind?"*

6. Uncover another clue and ask, *"Does this one change your mind?"*

   If the student decides he/she made an incorrect decision, replay the game until you are through and repeat all the questions in Part B.

SHAPE A.

1. It is a closed figure with 4 straight sides.

2. All the sides are the same length.

3. One of the angles is 60°.

4. One of the angles is 120°.

5. Another angle is 60°.

6. Another angle is 120°.

7. Two sides are parallel.

8. Two other sides parallel.

9. The diagonals are perpendicular.

10. The diagonals bisect each other.

SHAPE B.

1. It is a closed figure with 4 straight sides.
2. It has 2 long sides and 2 short sides.
3. It has a right angle.
4. The 2 long sides are parallel.
5. It has 2 right angles.
6. The 2 long sides are not the same length.
7. The 2 short sides are not the same length.
8. The 2 short sides are not parallel.
9. The 2 long sides make right angles with one of the short sides.
10. It has only 2 right angles.

SHAPE C.

1. It is a closed figure with 4 straight sides.

2. It has 2 long sides and 2 short sides.

3. The 2 long sides are the same length.

4. The 2 short sides are the same length.

5. One of the angles is larger than one of the other angles.

6. Two of the angles are the same size.

7. The other 2 angles are the same size.

8. The 2 long sides are parallel.

9. The 2 short sides are parallel.

Activity 5. Equivalent Definitions of "Parallelogram"

Purpose: To determine if the student can establish the logical equivalence of two definitions of parallelogram.

Script:

A. *Suppose we have a quadrilateral in which the opposite sides are parallel. Can we be sure that the opposite sides are congruent?*
(If not, *draw one.* If so: *How can we show this?*)

B. *Suppose we have a quadrilateral with opposite sides congruent. Do they have to be parallel?*
(If not, *show an example.* If so: *How can we be sure?*)

C. 1. *Have you ever heard the word "theorem"?*

Have you ever heard the word "axiom"?

Have you ever heard the word "postulate"?

2. *Can you give an example for each of these terms that you have heard of?*

3. *What is the difference between a theorem and a postulate?* (or *an axiom and a theorem* if familiar, etc. Contrast all pairs of familiar terms.)

Appendix C

Level Indicators

Level Indicators

Level 0

1. Use of imprecise properties (qualities) to compare drawings and to identify, characterize, and sort shapes.
2. References to visual prototypes to characterize shapes.
3. Inclusion of irrelevant attributes when identifying and describing shapes, such as orientation of the figure on the page.
4. Inability to conceive of an infinite variety of shapes.
5. Inconsistent sortings; that is, sortings by properties not shared by the sorted shapes.
6. Inability to use properties as necessary conditions to determine a shape; for example, guessing the shape in the mystery shape task after far too few clues, as if the clues triggered a visual image.

Level 1

1. Comparing shapes explicitly by means of properties of their components.
2. Prohibiting class inclusions among several general types of shapes, such as quadrilaterals.
3. Sorting by single attributes, such as properties of sides, while neglecting angles, symmetry and so forth.
4. Application of a litany of necessary properties instead of determining sufficient properties when identifying shapes, explaining identifications, and deciding on a mystery shape.
5. Descriptions of types of shapes by explicit use of their properties, rather than by type names, even if known.
6. Explicit rejection of textbook definitions of shapes in favor of personal characterizations.
7. Treating geometry as physics when testing the validity of a proposition; for example relying on a variety of drawings and making observations about them.
8. Explicit lack of understanding of mathematical proof.

Level 2

1. Formation of complete definitions of types of shapes.
2. Ability to modify definitions and immediately accept and use definitions of new concepts.
3. Explicit references to definitions.
4. Ability to accept equivalent forms of definitions.
5. Acceptance of logical partial ordering among types of shapes, including attributes.
6. Ability to sort shapes according to a variety of mathematically precise attributes.
7. Explicit use of “if, then” statements.
8. Ability to form correct informal deductive arguments, implicitly using such logical forms as the chain rule (if p implies q and q implies r, then p implies r) and the law of detachment (modus ponens).

9. Confusion between the roles of axiom and theorem.

<u>Level 3</u>

1. Clarifications of ambiguous questions and rephrasing of problem tasks into precise language.
2. Frequent conjecturing and attempts to verify conjectures deductively.
3. Reliance on proof as the final authority in deciding the truth of a mathematical proposition.
4. Understanding the roles of the components in a mathematical discourse, such as axioms, definitions, theorems, and proof.
5. Implicit acceptance of the postulates of Euclidean geometry.

(Burger & Shaughnessy, 1986, p. 43)

Appendix D

Analysis of Interview Forms

1. Triangle Task 1
2. Quadrilateral Task 2
3. Overall Level Summary

TRIANGLE ACTIVITIES

Δ1

1. Drawing Triangles

1.	Attributes S Varied		Table 1
Attribute	Varied?	Verbalized?	Mathematically Precise?
Side lengths			
Angle measures			
Orientation			
Similarity			
Standard classes			
Equilateral			
Right			
Isosceles			
Acute			
Scalene			
Obtuse			
# Triangles first time	# Triangles second time		
Probe necessary to stimulate S to vary attributes?			
If so, what additional attributes were varied?			

1. Drawing Triangles (continued)

1. continued	Table 1 (continued)
Comments	

Level of reasoning predominant ______________________

Supporting data:

Confounding data:

3

2. Identifying and Defining Quadrilaterals

2A.	Part A Identifications		Table 9
SHAPE	FAMILIAR?	CORRECTLY IDENTIFIED? (circled)	EXTRA INCLUSIONS
Square		2,7	
Rectangle		2,7,9,12	
Parallelogram		2,3,5,6,7,8,9,10,12,13	
Rhombus		2,7,8,13	
(Other)			
Trapezoid		1,14,15	
Kite		2,4,7,8,11,13	

Did S. change any identifications?

If so, describe changes by figure number:

When were changes made?

Who initiated discussion?

4

2. Identifying and Defining Quadrilaterals (continued)

2B.	S's Part B reasons for marking shapes	Table 10
SQUARE:		
RECTANGLE:		
PARALLELOGRAM:		
RHOMBUS:		

2C.	Part C Characterizations of Shapes	Table 11
SQUARE		
Additional properties needed for definition:		
Did S list only necessary properties?	Did S list sufficient properties?	
Comments:		

2C.	Part C Characterizations of Shapes continued	Table 12
RECTANGLE		
Additional properties needed for definition:		
Did S list only necessary properties?	Did S list sufficient properties?	
Comments :		

2. Identifying and Defining Quadrilaterals (continued) 6

2C.	Part C Characterizations of Shapes continued	Table 13
PARALLELOGRAM		
Additional properties needed for definition:		
Did S list only necessary properties?	Did S list sufficient properties?	
Comments:		

2C.	Part C Characterizations of Shapes continued	Table 14
RHOMBUS		
Additional properties needed for definition:		
Did S list only necessary properties?	Did S list sufficient properties?	
Comments		

2. Identifying and Defining Quadrilaterals (continued) 7

2D.	Part D Minimal Characterizations of Shapes	Table 15
SQUARE:		
Same as Part C?	Alteration from Part C:	
Additional properties needed for definition:		
Redundant properties:		
Were S's properties necessary?	Were S's properties sufficient?	Were S's properties minimal?
Comments:		

2. Identifying and Defining Quadrilaterals (continued) ▱8

2D.	Part D Minimal Characterizations of Shapes	Table 16

RECTANGLE:

Same as Part C?	Alteration from Part C:

Additional properties needed for definition:

Redundant properties:

Were S's properties necessary?	Were S's properties sufficient?	Were S's properties minimal?

Comments:

2. Identifying and Defining Quadrilaterals (continued) 9

2D.	Part D Minimal Characterizations of Shapes	Table 17

PARALLELOGRAM:

Same as Part C?	Alteration from Part C:

Additional properties needed for definition:

Redundant properties:

Were S's properties necessary?	Were S's properties sufficient?	Were S's properties minimal?

Comments:

2. Identifying and Defining Quadrilaterals (continued) 10

2D.	Part D Minimal Characterizations of Shapes	Table 18

RHOMBUS:

Same as Part C?	Alteration from Part C:

Additional properties needed for definition:

Redundant properties:

Were S's properties necessary?	Were S's properties sufficient?	Were S's properties minimal?

Comments:

/_7 11

2. Identifying and Defining Quadrilaterals (continued)

2E.	Part E Class Inclusions	Table 19

Inclusion	Did S permit the inclusion?
Squares $\subseteq$ Rectangles	
Rectangles $\subseteq$ Parallelograms	
Squares $\subseteq$ Rhombi	

Describe inconsistencies and omissions in Part E. (How did S explain them?)

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2. Identifying and Defining Quadrilaterals (continued)

Level of reasoning predominant ________________________________

Supporting data:

Confounding data:

SUMMARY

Record the predominant levels for the Triangle Activities as an ordered triple, and for Quadrilateral Activities as an ordered quintuple.

Triangle Activities (, ,)   Quadrilateral Activities (, , , ,)

Describe the overall level of reasoning used by S throughout the interview, if one seems evident. Include excerpts from the interview to support your choice.

Did any particular misconceptions appear about the shapes and their properties? If so, describe them.

What statements or ideas expressed by S seemed especially confounding in view of your selection of overall level? (If there were none, say so.)