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Kalman Consensus Strategies and Their Application to Cooperative Control

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Kalman Consensus Strategies and Their Application to Cooperative Control

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Abstract

In this paper, we propose discrete-time and continuous-time consensus update schemes motivated by the discrete-time and continuous-time Kalman filters. With certainty information encoded into each agent, the proposed consensus schemes explicitly account for relative confidence in the information that is communicated from each agent in the team. We show mild sufficient conditions under which consensus can be achieved using the proposed schemes in the presence of switching interaction topologies. The Kalman consensus scheme is shown to be input-to-state stable. We show how to exploit this fact in multi-agent cooperative control scenarios.

Index Terms

Information consensus, multi-agent systems, cooperative control, switched systems, Kalman filtering.

I. INTRODUCTION

During the last two decades there has been a dramatic paradigm shift in the way that computer systems are designed: moving from centralized mainframe computers to networks of less

capable, but much less expensive, personal computers. In much the same way, replacing large, expensive, monolithic vehicles with teams of networked vehicles, promises less expensive, more capable systems. In addition, there are applications where a team of vehicles can accomplish objectives that would be impossible for a single vehicle. For example, a formation of networked spacecraft could be used to synthesize a space-based interferometer with base-lines reaching tens to hundreds of kilometers [1], [2]. With teams of vehicles, much of the design complexity is shifted from mechanical hardware design to software that regulates the interaction of the team.

In recent years, there has been significant interest and research activity in the area of coordinated and cooperative control [3], [4], [5], [6], [7], [8], [9], [10]. Much of this work assumes the availability of global team knowledge, and/or the ability to plan group actions in a centralized manner.

Centralized coordination techniques are suitable if each member of the team has the ability to communicate to a centralized location or if the team is able to share information via a static fully connected network. On the other hand, real-world communication topologies are usually not fully connected. In many cases they depend on the relative position of the vehicles and on other environmental factors and are therefore dynamically changing in time. In addition, wireless communication channels are subject to multi-path, fading and drop-out. Therefore, cooperative control in the presence of real-world communication constraints, becomes a significant challenge.

In a recent article we argued that "shared information is a necessary condition for cooperation" [11]. Shared information may take the form of common objectives, common control algorithms, relative position information, or a world map. If this assertion is true, then information exchange becomes a central issue in cooperative control. In this article, we will refer to the information that is necessary for coordination as the *coordination information* or *coordination variable* [12]. In the presence of an unreliable, dynamically changing communication topology, it is not possible for all of the vehicles to have access to identical coordination information. Suppose that a particular cooperation strategy has been devised and shown to work if the team has global access to the coordination information. Cooperation will occur if each member on the team has access to the same information.

As an example, consider the meet-for-dinner problem introduced in [11]. In this problem, a group of friends decide to meet for dinner at a particular restaurant but fail to specify a precise time to meet. On the afternoon of the dinner appointment, each individual realizes that

they are uncertain about the time that the group will meet for dinner. A centralized solution to this problem is for the group to have a conference call, to poll each individual regarding their preferred time for dinner, and to average the answers to arrive at a time that the group will meet for dinner. However, this centralized solution requires that a conference line is available, and that the time of the conference call is known to the group. Since, whatever algorithm was used to convey the time of the conference call to the group, could also have been used to convey the time to meet for dinner, the central problem remains.

The information variable in this example is the time that the group will meet for dinner. The particular time is not what is important, but rather that each individual in the group has a consistent understanding of that information. A decentralized solution to the problem would be for each individual to call, one at a time, a subset of the group. Given his current estimate of the meeting time, the individual might update his estimate of the meeting time to be a weighted average of his current meeting time and that of the person with whom he is conversing. The question (which will be answered in this paper) is under what conditions this strategy will enable the entire team to converge to a consistent meeting time.

Therefore, if a centralized solution to a cooperation problem, with its associated coordination information, has been devised, then two additional questions must be addressed. First, what algorithms should be employed to ensure that the team is converging to a consistent view of the coordination information in the presence of an unreliable, dynamically changing communication topology? Second, if the action of the group is based on the (dynamically changing) coordination variable, will the cooperative control algorithm be robust with respect to the transient error in the coordination variable across the team?

Convergence to a consistent view of the coordination variable in the presence of an unreliable, dynamically changing communication topology is called the consensus problem. Consensus problems have recently been addressed in [13], [14], [15], [16], [17], [11], [18], [19], to name a few. In [14], sufficient conditions are given for consensus of the heading angles of a group of agents under undirected switching interaction topologies. In [15], average consensus problems are solved for a network of integrators using directed graphs. In [11] and [18], an algebraic graph approach is used to show necessary and/or sufficient conditions for consensus of information under time-invariant and switching interaction topologies respectively. In [16], a set-valued Lyapunov function approach is used to consider discrete-time consensus problems with unidirectional

time-dependent communication links.

Previous consensus seeking results reported in the literature do not explicitly account for agent confidence in their instantiation of the coordination variable. Most results assume that each individual in the group has identical confidence in their instantiation of the coordination variable. However, there are many cases where some individuals on the team will have access to better information than others. In cases like these, the consensus algorithm needs to be biased to favor agents with better information. For example, if a team of UAVs is tasked with tracking the location of a group of ground vehicles, the quality of information will be proportional to the relative sensing distance. UAVs that have recently flown close to a ground vehicle should be considered more reliable than those that are sensing from a greater distance, or whose information is old. As another example, in the meet-for-dinner problem described above, if one individual is considered more reliable than the others, his/her information should be weighted more heavily when making the team decision.

The primary contribution of this paper is to derive continuous-time and discrete-time consensus strategies, based on a Kalman-filter structure, that asymptotically achieves consensus in the presence of an unreliable, dynamically changing communication topology, giving proper weight to individuals with greater certainty in their coordination variable. In addition, we will show that the Kalman consensus scheme is input-to-state stable (ISS) where the input is the communication noise on each channel and the state is the consensus error between each pair of agents. The ISS property will be exploited to develop a distributed multi-vehicle cooperative control solution to the cooperative timing problem.

UAV cooperative timing problems have been investigated recently in the context of battlefield scenarios where the UAVs are required to converge on the boundary of a radar detection area to maximize the element of surprise [20], [12], [3], [21], [22]. Cooperative timing problems also arise in refueling scenarios, fire and hazardous material monitoring, moving area of regard problems, and continuous surveillance problems. In this paper we will investigate a simplified cooperative timing problem that must be accomplished in the presence of an unreliable, dynamically changing communication topology.

The paper is organized as follows. In Section II, we give an intuitive, non-rigorous derivation of the Kalman-like consensus strategies, and show their application to the meet-for-dinner problem. Section III contains the main technical results that shows that the proposed consensus strategies

are convergent under certain mild conditions. In Section IV we show that the Kalman consensus scheme is input-to-state stable. As a corollary, we show that most of the other consensus schemes proposed in the literature, are also ISS. In Section V the ISS property is exploited to develop a distributed solution to the cooperative timing problem. Finally, Section VI contains our conclusions.

II. KALMAN-FILTER APPROACH TO MULTI-AGENT CONSENSUS

The Kalman filter is used extensively to estimate a system's current state from imprecise measurement data [23], [24], [25]. It is well-known that the Kalman filter is an optimal estimator in the case of Gaussian statistics and that it is the best linear estimator in the case of other statistics [26]. Motivated by the Kalman filter scheme, we treat the final consensus value as the system state, which is unknown *a priori* but is the final equilibrium state that each agent in the group is expected to achieve. In the consensus problem, each agent has an estimate of the final consensus value. Communication from other agents regarding their estimate of the final consensus value will be regarded as measurement data. In this sense, each agent in the group performs its own estimate of the final consensus value based on the information available to it. Our goal is to guarantee that the information state of each agent achieves the final consensus value. In other words, the objective is to minimize the mean squared error between each agent's estimate of the coordination variable and the final consensus value. The error covariance matrix is interpreted as the confidence that each agent has in its current estimate of the coordination variable, where large covariance indicates low confidence, and small covariance indicates a high degree of confidence. In Section II-A we will derive the Kalman-consensus scheme for a continuous-time update scheme, and in Section II-B we will address the discrete-time case. Analytical properties of the algorithms will be derived in Section III.

A. Continuous-time Consensus

The standard continuous-time Kalman filter is summarized in Table II-A [27]. The objective of this section is to show how the Kalman filter equations can be used to derive a decentralized information consensus scheme.

Let $\xi^* \in \mathbb{R}^m$ be the *a priori* unknown information state over which the team is to form consensus. In other words, each information state ξ_i will converge to the consensus value ξ^* as

---

System model and measurement model:

$$\dot{x} = Ax + Bu + Gw$$

$$z = Hx + v$$

$$x(0) \sim (\bar{x}_0, P_0), w \sim (0, Q), v \sim (0, R)$$

Assumptions:

$\{w(t)\}$ and $\{v(t)\}$ are white noise processes uncorrelated with $x(0)$

and with each other. $R > 0$.

Initialization:

$$P(0) = P_0, \hat{x}(0) = \bar{x}_0$$

Error covariance update:

$$\dot{P} = AP + PA^T + GQG^T - PH^T R^{-1} HP$$

Kalman gain:

$$K = PH^T R^{-1}$$

Estimate update:

$$\dot{\hat{x}} = A\hat{x} + Bu + K(z - H\hat{x})$$

TABLE I

CONTINUOUS-TIME KALMAN FILTER [27].

$t \to \infty$. Note that the consensus value will depend not only on interaction topologies but on the weighting factors in the update schemes. In this paper we will assume that the consensus state is a constant, which implies that the system dynamics are given by

$$\dot{\xi}^* = w,$$

where, with reference to Table II-A, $A = 0$, $B = 0$, $G = I_m$, and $E\{ww^T\} = Q$. In the following, we assume that $Q(t) > 0$ is uniformly lower and upper bounded.

Treat the ith information state ξ_i as the ith agent's estimate of ξ^* and suppose that the jth agent communicates ξ_j to the ith agent with transmission, or communication noise ν_{ij}. Also, let $g_{ij}(t)$ be a time-varying boolean variable that indicates the presence of an open communication channel from agent j to agent i at time t, i.e., $g_{ij}(t) = 1$ if information is communicated from j to i at time t and zero otherwise. Note that $g_{ii}(t) \overset{\triangle}{=} 1$. Using these definitions, it is clear that

the measurement model of the ith agent can be given by

$$
z_i = \begin{pmatrix} g_{i1}\left(\xi_1 + \nu_{i1}\right) \\ \vdots \\ g_{iN}\left(\xi_N + \nu_{iN}\right) \end{pmatrix}
$$

$$
= \begin{pmatrix} g_{i1}I \\ \vdots \\ g_{iN}I \end{pmatrix} \xi^* + \begin{pmatrix} g_{i1}\left(\xi_1 - \xi^* + \nu_{i1}\right) \\ \vdots \\ g_{iN}\left(\xi_N - \xi^* + \nu_{iN}\right) \end{pmatrix},
$$

where, with reference to Table II-A,

$$
H_i^T = \begin{pmatrix} g_{i1}I & \cdots & g_{iN}I \end{pmatrix}
$$

and

$$
v_i = \begin{pmatrix} g_{i1}\left(\xi_1 - \xi^* + \nu_{i1}\right) \\ \vdots \\ g_{iN}\left(\xi_N - \xi^* + \nu_{iN}\right). \end{pmatrix}
$$

If we define $P_i \triangleq E\{(\xi_i - \xi^*)(\xi_i - \xi^*)^T\}$ and assume that $E\{(\xi_i - \xi^*)(\xi_j - \xi^*)^T\} = 0$, where $i \neq j$, then

$$
R_i \triangleq E\{v_i v_i^T\}
$$

$$
= \begin{pmatrix} g_{i1}(P_1 + \Omega_{i1}) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & g_{iN}(P_N + \Omega_{iN}) \end{pmatrix},
$$

where $\Omega_{ij} \triangleq E\{\nu_{ij}\nu_{ij}^T\}$ is assumed to be upper bounded.

Therefore, the error covariance update in Table II-A becomes

$$
\dot{P}_i = -P_i H_i^T R_i^{-1} H_i P_i + Q
$$

$$
= -P_i \left[\sum_{j=1}^{N} g_{ij}(P_j + \Omega_{ij})^{-1}\right]P_i + Q.
$$

Similarly, the Kalman gain is given by

$$
K_i = P_i H_i^T R_i^{-1}
$$

$$
= \begin{pmatrix} g_{i1}P_i(P_1 + \Omega_{i1})^{-1} & \cdots & g_{in}P_i(P_N + \Omega_{iN})^{-1} \end{pmatrix},
$$

and the estimate update is given by

$$\dot{\xi}_i = K_i(z_i - H_i\xi_i)$$

$$= K_i\left[\begin{pmatrix} g_{i1}(\xi_1 + \nu_{i1}) \\ \vdots \\ g_{in}(\xi_N + \nu_{iN}) \end{pmatrix} - \begin{pmatrix} g_{i1}I_m \\ \vdots \\ g_{iN}I_m \end{pmatrix}\xi_i\right]$$

$$= \sum_{j=1}^{N} K_{ij}g_{ij}(\xi_j - \xi_i + \nu_{ij}),$$

where $K_i = [K_{i1}, K_{i2}, \cdots, K_{in}]$.

Summarizing, we have the following Kalman consensus scheme for the ith agent:

$$\dot{P}_i = -P_i\left[\sum_j g_{ij}(t)(P_j + \Omega_{ij})^{-1}\right]P_i + Q \tag{1}$$

$$K_{ij} = P_i(P_j + \Omega_{ij})^{-1} \tag{2}$$

$$\dot{\xi}_i = \sum_{j=1}^{n} g_{ij}(t)K_{ij}\left((\xi_j + \nu_{ij}) - \xi_i\right). \tag{3}$$

Note that Eq. (1) indicates that the certainty of information increases with communication but decreases with the size of the process noise. In addition, the rate of increase in certainty for the ith agent is inversely proportional to the certainty of the jth agent and the communication noise. Note also that the Kalman gain K_{ij} is reduced if either the communication noise is large, or if the certainty of the jth agent is small (hence P_j large). Note that Eq. (3) is similar to the continuous-time consensus schemes proposed in [14], [15], [11] except that the consensus gain K_{ij} is time-varying in (3), and the communication noise is explicitly included.

B. Discrete-time Consensus

The standard discrete-time Kalman filter is summarized in Table II-B [27]. Again assuming that ξ^* is constant we get

$$\xi^*[k+1] = \xi^*[k] + w[k],$$

where, with reference to Table II-A, $A[k] = I$, $B[k] = 0$, $G[k] = I_m$, and $E\{w[k]w[k]^T\} = Q[k]$.

Again letting $\nu_{ij}[k]$ represent the communication noise, the measurement model for the ith agent can be given by

System model and measurement model:

$$x[k+1] = A[k]x[k] + B[k]u[k] + G[k]w[k]$$

$$z[k] = H[k]x[k] + v[k]$$

$$x(0) \sim (\bar{x}_0, P_{x_0}), w \sim (0, Q[k]), v[k] \sim (0, R[k])$$

Assumptions:

$\{w[k]\}$ and $\{v[k]\}$ are white noise processes uncorrelated with x_0

and with each other. $R[k] > 0$.

Initialization:

$$P[0] = P_{x_0}, \hat{x}_0 = \bar{x}_0$$

Time update: (effect of system dynamics)

error covariance: $P[k+1]^- = A[k]P[k]A[k]^T + G[k]Q[k]G[k]^T$

estimte: $\hat{x}[k+1]^- = A[k]\hat{x}[k] + B[k]u[k]$

Measurement update: (effect of measurement $z[k]$)

error covariance: $P[k+1] = [(P[k+1]^-)^{-1} + H[k+1]^T R[k+1]^{-1} H[k+1]]^{-1}$

estimate: $\hat{x}[k+1] = \hat{x}[k+1]^- + P[k+1]H[k+1]^T R[k+1]^{-1}(z[k+1] - H[k+1]\hat{x}[k+1]^-)$

TABLE II

DISCRETE-TIME KALMAN FILTER [27].

$$z_i[k] = \begin{pmatrix} g_{i1}[k]\left(\xi_1[k] + \nu_{i1}[k]\right) \\ \vdots \\ g_{iN}[k]\left(\xi_N[k] + \nu_{iN}[k]\right) \end{pmatrix}$$

$$= \begin{pmatrix} g_{i1}[k]I \\ \vdots \\ g_{iN}[k]I \end{pmatrix} \xi^*[k] + \begin{pmatrix} g_{i1}[k]\left(\xi_1[k] - \xi^*[k] + \nu_{i1}[k]\right) \\ \vdots \\ g_{iN}[k]\left(\xi_N[k] - \xi^*[k] + \nu_{iN}[k]\right) \end{pmatrix},$$

where, with reference to Table II-A,

$$H_i^T[k] = \begin{pmatrix} g_{i1}[k]I & \dots & g_{iN}[k]I \end{pmatrix}$$

and

$$v_i = \begin{pmatrix} g_{i1}[k]\left(\xi_1[k] - \xi^*[k] + \nu_{i1}[k]\right) \\ \vdots \\ g_{iN}[k]\left(\xi_N[k] - \xi^*[k] + \nu_{iN}[k]\right) \end{pmatrix}.$$

If we define $P_i[k] \triangleq E\{(\xi_i[k] - \xi^*[k])(\xi_i[k] - \xi^*[k])^T\}$ and assume that $E\{(\xi_i[k] - \xi^*[k])(\xi_j[k] - \xi^*[k])^T\} = 0$, where $i \neq j$, then

$$R_i[k] \triangleq E\{v_i[k]v_i[k]^T\}$$

$$= \begin{pmatrix} g_{i1}[k](P_1[k] + \Omega_{i1}[k]) & \ldots & & 0 \\ & \vdots & \ddots & & \vdots \\ & 0 & & \ldots & g_{iN}[k](P_N[k] + \Omega_{iN}[k]) \end{pmatrix},$$

where $\Omega_{ij}[k] \triangleq E\{\nu_{ij}[k]\nu_{ij}[k]^T\}$.

Therefore, the time update in Table II-B becomes

$$P_i^-[k+1] = P_i[k] + Q[k]$$

$$\xi_i^-[k+1] = \xi_i[k].$$

The measurement update is given by

$$P_i[k+1] = \left[(P_i^-[k+1])^{-1} + H_i^T[k+1]R_i^{-1}[k][k+1]H_i[k+1]\right]^{-1}$$

$$= [(P_i[k] + Q[k])^{-1} + \sum_{j=1}^{n} g_{ij}[k](P_j[k] + \Omega_{ij}[k])^{-1}]^{-1},$$

$$\xi_i[k+1] = \xi^-[k+1] + P_i[k+1]H_i^T[k+1]R_i^{-1}[k+1]\left(z_i[k+1] - H_i[k+1]\xi^-[k+1]\right)$$

$$= \xi_i[k] + P_i[k+1]\left(\sum_{j=1}^{n} \left[g_{ij}[k](P_j[k] + \Omega_{ij})^{-1}[k]\left((\xi_j[k] + \nu_{ij}[k+1]) - \xi_i[k]\right)\right]\right).$$

Summarizing, we have the following discrete-time Kalman consensus scheme for the ith agent:

$$P_i[k+1] = [(P_i[k] + Q[k])^{-1} + \sum_{j=1}^{n} g_{ij}[k](P_j[k] + \Omega_{ij}[k])^{-1}]^{-1}, \tag{4}$$

$$\xi_i[k+1] = \xi_i[k] + P_i[k+1]\left(\sum_{j=1}^{n} \left[g_{ij}[k](P_j[k] + \Omega_{ij})^{-1}[k]\left(\xi_j[k] + \nu_{ij}[k+1]\right) - \xi_i[k]\right)\right]. \tag{5}$$

C. Meet for Dinner Example

To illustrate, consider the meet-for-dinner problem discussed in the introduction. Suppose that there are $N = 10$ agents who communicate with exactly one other individual, chosen randomly from the group, for a random length of time. After the communication has expired, the process
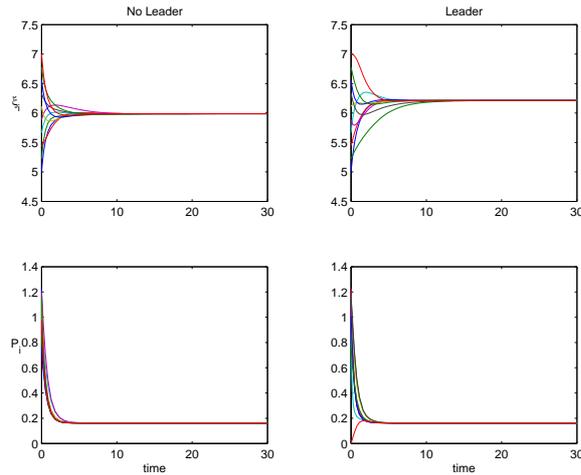
Fig. 1.   Continuous-time meet-for-dinner simulation. The subplot in the upper left shows the evolution of the coordination variable assuming that all agents begin with equally confident covariance. The subplot in the lower left shows the associated covariance. The subplots on the right show identical data where the agent with initial time $\xi_i = 7$ has an initial covariance of $P_i = 0.001$.

is repeated. Figure 1 shows the state and variance plots under the continuous Kalman consensus scheme (1)–(3) where the initial state is uniformly assigned. The subplots on the left show the arrival times and variance when the initial variances are uniformly assigned. The subplots on the right show the arrival times and variances when the variance of the agent with initial arrival time $\xi_i = 7$ is given an initial variance of $P_i = 0.001$, which is significantly lower than the other agents. Note that in this case, the final consensus value is influenced to a greater degree by this agent. Figure 2 shows similar plots using the discrete Kalman consensus scheme (4)–(5). Both the continuous-time and discrete-time simulations use the values $\Omega_{ij} = 0.1$, $Q = 0.1$.

III. CONVERGENCE RESULTS

The objective of this section is to state some technical properties of the algorithms given in Eqs. (1)–(3) and Eqs. (4)–(5). For notational simplicity, we will focus on the case where each information state ξ^* is a scalar. The vector case reduces to the scalar case if P_{i0} is a diagonal matrix. The general case where P_{i0} is non-diagonal is currently a topic of research. In Section III-A, we will introduce some notation and results from graph theory and non-negative matrices that will be used in the convergence arguments. In Section III-B we analyze the continuous-time case and in Section III-C we analyze the discrete-time case.
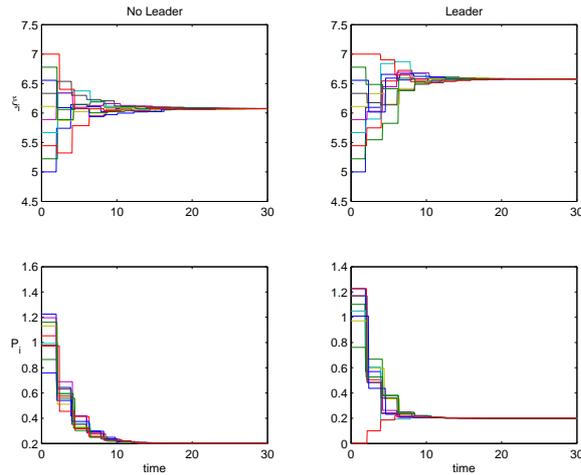
Fig. 2.   Discrete-time meet-for-dinner simulation. The subplot in the upper left shows the evolution of the coordination variable assuming that all agents begin with equally confident covariance. The subplot in the lower left shows the associated covariance. The subplots on the right show identical data where the agent with initial time $\xi_i = 7$ has an initial covariance of $P_i = 0.001$.

A. Preliminaries

Let $\mathcal{A} = \{A_i | i \in \mathcal{I}\}$, where $\mathcal{I} = \{1, 2, \cdots, n\}$, be a set of n agents among whom consensus is desired. A directed graph $\mathcal{G}$ will be used to model the interaction topology among these agents. In $\mathcal{G}$, the ith vertex represents the ith agent A_i and a directed edge from A_i to A_j denoted as (A_i, A_j) represents a unidirectional information exchange from A_i to A_j, that is, agent j receives information from agent i, $(i, j) \in \mathcal{I}$. If the information flows from agent i to agent j, agent i is called the parent of j, and agent j is called the child of i. A directed path in graph $\mathcal{G}$ is a sequence of edges $(A_{i_1}, A_{i_2}), (A_{i_2}, A_{i_3}), (A_{i_3}, A_{i_4}), \cdots$ in that graph. Graph $\mathcal{G}$ is called strongly connected if there is a directed path from A_i to A_j and A_j to A_i between any pair of distinct vertices A_i and A_j, $\forall (i, j) \in \mathcal{I}$. A directed tree is a directed graph, where every node, except the root, has exactly one parent. A spanning tree of a directed graph is a directed tree formed by graph edges that connect all the vertices of the graph [28]. We say that a directed graph has a spanning tree if there exists a spanning tree that is a subset of the directed graph. Fig. 3 shows a directed graph with more than one possible spanning trees. The double arrows denote one possible spanning tree with A_5 as the parent. Spanning trees with A_1 and A_4 as the parent, are also possible. As a comparison, Figs. 4 shows two cases where the graph does not have a spanning tree.
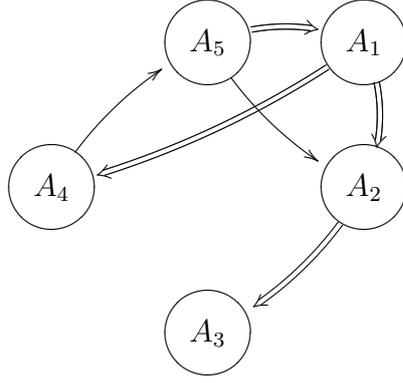
Fig. 3. A directed graph that has more than one possible spanning trees, but is not strongly connected. One possible spanning tree is denoted with double arrows.

Fig. 4. (a) A directed graph that has two leaders, and hence does not contain a spanning tree. (b) A directed graph that has two isolated groups, and hence does not contain a spanning tree.

The interaction topology may change dynamically. Let $\bar{\mathcal{G}} = \{\mathcal{G}_1, \mathcal{G}_2, \cdots, \mathcal{G}_M\}$ denote the set of all possible directed interaction graphs defined for $\mathcal{A}$. It is obvious that $\bar{\mathcal{G}}$ has a finite number of elements and that $\mathcal{G}(t) \in \bar{\mathcal{G}}$. The union of a set of directed graphs $\{\mathcal{G}_{i_1}, \mathcal{G}_{i_2}, \cdots, \mathcal{G}_{i_m}\} \subset \bar{\mathcal{G}}$ is a directed graph with vertices given by A_i, $i \in \mathcal{I}$ and edge set given by the union of the edge sets of $\mathcal{G}_{i_j}$, $j = 1, \cdots, m$. We will assume throughout the paper that the interaction topology does not switch infinitely fast.

Let $M_n(\mathbb{R})$ represent the set of all $n \times n$ real matrices. Given a matrix $A = [a_{ij}] \in M_n(\mathbb{R})$, the directed graph of A, denoted by $\Gamma(A)$, is the directed graph on n vertices V_i, $i \in \mathcal{I}$, such that there is a directed edge in $\Gamma(A)$ from V_j to V_i if and only if $a_{ij} \neq 0$ [29]. For example, the

directed graph of the matrix

$$
A = \begin{bmatrix}
0 & 0 & 0 & 0 & a_{15} \\
a_{21} & 0 & 0 & 0 & a_{25} \\
0 & a_{32} & 0 & 0 & 0 \\
a_{41} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & a_{54} & 0
\end{bmatrix},
$$

where $a_{ij} \neq 0$, corresponds to the graph in Fig. 3.

A matrix $A = [a_{ij}] \in M_n(\mathbb{R})$ is nonnegative, denoted as $A \geq 0$, if all its entries are nonnegative. Furthermore, if all its row sums are $+1$, A is said to be a (row) stochastic matrix [29]. A stochastic matrix P is called indecomposable and aperiodic (SIA) if $\lim_{n \to \infty} P^n = \mathbf{1}y^T$, where y is a column vector, and $\mathbf{1}$ denotes an $n \times 1$ column vector with all the entries equal to 1 [30]. For nonnegative matrices, $A \geq B$ implies that $A - B$ is a nonnegative matrix. It is easy to verify that if $A \geq \rho B$, for some $\rho > 0$, then the directed graph of B is a subset of the directed graph of A.

Two $n \times n$ nonnegative matrices are said to be of the same type if their zero elements are in the same locations [30]. We will use the notation $P \sim Q$ to denote that P and Q are of the same type.

Lemma 3.1: Given $n \times n$ nonnegative matrices P, Q, R, and S, if $P \sim R$ and $Q \sim S$, then $(P + Q) \sim (R + S)$ and $PQ \sim RS$. Moreover, if a time-varying nonnegative matrix $M(t)$ with continuous entries is of a fixed type for $t \in [t_1, t_2]$, where $t_1 < t_2$, then $M(t) \sim \int_{t_1}^{t_2} M(t)dt$.

Proof: Trivial.   ■

Let $\xi_i \in \mathbb{R}$, $i \in \mathcal{I}$, represent the ith information state associated with the ith agent. The set of agents $\mathcal{A}$ is said to achieve consensus asymptotically if for any $\xi_i(0)$, $i \in \mathcal{I}$, $\|\xi_i(t) - \xi_j(t)\| \to 0$ as $t \to \infty$ for each $(i, j) \in \mathcal{I}$.

B. Continuous-time Consensus

The following theorem is our main technical result.

Theorem 3.2: Given switching interaction topologies and zero transmission or communication noise, the Kalman consensus scheme given in Eqs. (1)–(3) achieves asymptotic consensus if there exist infinitely many consecutive uniformly bounded time intervals such that the union of the interaction graph across each interval has a spanning tree.

The proof of this theorem depends upon the following five lemmas.

Lemma 3.3: Let $C(t) = [c_{ij}(t)] \in M_n(\mathbb{R})$ be piecewise continuous, where $c_{ij} \geq 0$, $i \neq j$, and $\sum_j c_{ij} = 0$. Let $\Phi_C(t, t_0)$ be the corresponding transition matrix. Then $\Phi_C(t, t_0)$ is a stochastic matrix with positive diagonal entries for any $t \geq t_0$.

Proof: From [31], we know that

$$\Phi_C(t, t_0)$$
$$= I + \int_{t_0}^t C(\sigma_1) \, d\sigma_1 + \int_{t_0}^t C(\sigma_1) \int_{t_0}^{\sigma_1} C(\sigma_2) \, d\sigma_2 \, d\sigma_1 + \cdots. \tag{6}$$

Noting that $C(t)\mathbf{1} = 0$, where $\mathbf{1}$ is a column vector of ones, we can verify that $\Phi_C(t, t_0)\mathbf{1} = \mathbf{1}$.

Note that $C(t)$ can be written as $B(t) - \mu I_n$, where $B(t)$ is a nonnegative matrix and μ is a constant greater than $\max_{\tau \in [t_0, t]} \max_{i \in \mathcal{I}} |c_{ii}(\tau)|$. It is straightforward to see that

$$\frac{d}{dt}\Phi_C(t, t_0) = C(t)\Phi_C(t, t_0)$$

and

$$\frac{d}{dt}[\Phi_B(t, t_0)e^{-\mu(t-t_0)}]$$
$$= B(t)\Phi_B(t, t_0)e^{-\mu(t-t_0)} - \mu\Phi_B(t, t_0)e^{-\mu(t-t_0)}$$
$$= (B(t) - \mu I_n)\Phi_B(t, t_0)e^{-\mu(t-t_0)}$$
$$= C(t)\Phi_B(t, t_0)e^{-\mu(t-t_0)},$$

and that $\Phi_C(t_0, t_0) = \Phi_B(t_0, t_0)e^{-\mu(t_0-t_0)} = I$. Therefore, we obtain $\Phi_C(t, t_0) = \Phi_B(t, t_0)e^{-\mu(t-t_0)}$. From Eq. (6), it is straightforward to see that $\Phi_B(t, t_0)$ is nonnegative and has positive diagonal entries. Therefore, it follows that $\Phi_C(t, t_0)$ is nonnegative and has positive diagonal entries. Combining these arguments implies that the transition matrix $\Phi_C(t, t_0)$ is a stochastic matrix with positive diagonal entries. ■

Lemma 3.4: Let $C(t) = [c_{ij}(t)] \in M_n(\mathbb{R})$ and $\tilde{C} = [\tilde{c}_{ij}(t)] \in M_n(\mathbb{R})$ be continuous on $t \in [\tau, s]$, where $s > \tau$ such that $c_{ij}(t) \geq 0$ and $\tilde{c}_{ij}(t) \geq 0$, $\forall i \neq j$, and $\sum_{j=1}^n c_{ij}(t) = \sum_{j=1}^n \tilde{c}_{ij}(t) = 0$. Let $\Phi_C(s, \tau)$ and $\Phi_{\tilde{C}}(s, \tau)$ be the corresponding transition matrices. Also let the graph associated with $C(t)$ be fixed for $t \in [\tau, s]$ and suppose that $\tilde{C}(t)$ corresponds to the same fixed graph as $C(t)$. Then the graph of $C(t)$ is a subset of the graph of $\Phi_C(s, \tau)$ and $\Phi_C(s, \tau) \sim \Phi_{\tilde{C}}(s, \tau)$.

Proof: Let $C(t) = B(t) - \mu I_n$, where $B(t)$ is a nonnegative matrix and μ is a constant greater than $\max_{t \in [\tau,s]} \max_{i \in \mathcal{I}} |c_{ii}(t)|$. Following Lemma 3.3, we know that $\Phi_C(s, \tau) = \Phi_B(s, \tau) e^{-\mu(s-\tau)}$. Note that the graphs associated with $C(t)$ and $B(t)$ are the same, so are the graphs associated with $\Phi_C(s, \tau)$ and $\Phi_B(s, \tau)$. Therefore from Eq. (6), we can see that $\Phi_B(s, \tau) \geq \int_\tau^s B(\sigma_1) d\sigma_1$, where $\int_\tau^s B(\sigma_1) d\sigma_1 \sim B(t)$ for $t \in [\tau, s]$, or in other words, the graph associated with $B(t)$ for $t \in [\tau, s]$ is a subset of the graph associated with $\Phi_B(s, \tau)$. Therefore, the graph associated with $C(t)$ for $t \in [\tau, s]$ is a subset of the graph associated with $\Phi_C(s, \tau)$.

Note that $\Phi_{\tilde{C}}(s, \tau) = \Phi_{\tilde{B}}(s, \tau) e^{-\tilde{\mu}(s-\tau)}$, where $\tilde{C} = \tilde{B} - \tilde{\mu} I_n$. In order to show that Φ_C is of the same type as $\Phi_{\tilde{C}}$, we need to show that Φ_B is of the same type as $\Phi_{\tilde{B}}$. Note that B and $\tilde{B}$ are of the same type since they correspond to the same graph. By writing Φ_B and $\Phi_{\tilde{B}}$ as in Eq. (6) and comparing each term, Lemma 3.1 implies that each corresponding term is of the same type, which in turn implies that $\Phi_B(s, \tau)$ and $\Phi_{\tilde{B}}(\tilde{s}, \tilde{\tau})$ are of the same type. ∎

Lemma 3.5: Let $S_A = \{A_1, A_2, \cdots, A_\ell\}$ be a set of stochastic matrices with positive diagonal entries. If the graph associated with A_i has a spanning tree, then A_i is SIA. If the union of the graphs of matrices A_i, $i = 1, \cdots, \ell$, has a spanning tree, then the matrix product $\Pi_{i=1}^\ell A_i$ is SIA.

Proof: The first statement is shown in Corollary 3.5 and Lemma 3.7 in [18]. For the second statement, note that the product of stochastic matrices is still a stochastic matrix. Also note that $\Pi_{i=1}^\ell A_i \geq \gamma \sum_{i=1}^\ell A_i$ for some $\gamma > 0$ according to Lemma 2 in [14]. Since the union of the graphs of matrices in S_A has a spanning tree, it is obvious that the graph associate with $\sum_{i=1}^\ell A_i$ has a spanning tree. Therefore, it can be seen that the graph associated with the matrix product has a spanning tree, which in turn implies, from the first statement of the Lemma, that the matrix product is SIA. ∎

Lemma 3.6: Let $C(t) = [c_{ij}(t)] \in M_n(\mathbb{R})$ be piecewise continuous for $t \in [\tau, s]$, where $s > \tau$ is bounded, $c_{ij} \geq 0$, $i \neq j$, and $\sum_j c_{ij} = 0$. If the union of the directed graphs of matrix $C(t)$ for $t \in [\tau, s]$ has a spanning tree, then the transition matrix $\Phi_C(s, \tau)$ is SIA.

Proof: Note that $\Phi_C(s, \tau) = \Phi_C(s, t_\ell)\Phi_C(t_\ell, t_{\ell-1}) \cdots \Phi_C(t_1, \tau)$, where t_j, $j = 1, \cdots, \ell$, denotes the times when $C(t)$ is discontinuous. From Lemma 3.4, we know that the graph associated with $C(t)$ for each $t \in [t_{i-1}, t_i]$ is a subset of the graph associated with $\Phi_C(t_i, t_{i-1})$, $i = 1, \cdots, \ell+1$. In other words, if the union of the directed graphs of matrix $C(t)$ has a spanning tree, so does the union of the directed graphs of the corresponding transition matrices. Also note from Lemma 3.4 that each $\Phi_C(t_i, t_{i-1})$ is a stochastic matrix with positive diagonal entries. The proof then follows

from Lemma 3.5. ∎

Before moving on, we need the following definition from [30]. Given a stochastic matrix $S = [s_{ij}] \in M_n(\mathbb{R})$, define

$$\lambda(S) = 1 - \min_{i_1, i_2} \sum_j \min(s_{i_1 j}, s_{i_2 j}).$$

Note that $\lambda(S) \le 1$ for any stochastic matrix S. If $\lambda(S) < 1$, S is called a scrambling matrix. $\lambda(S) = 0$ if and only if the rows of S are identical. The introduction of λ will be useful for the proof of Theorem 3.2.

Lemma 3.7: (See [30].) Let $\mathcal{S} = \{S_1, S_2, \cdots, S_k\}$ be a finite set of SIA matrices with the property that for each sequence $S_{i_1}, S_{i_2}, \cdots, S_{i_j}$ of positive length, the matrix product $S_{i_j} S_{i_{j-1}} \cdots S_{i_1}$ is SIA. Then for each infinite sequence $S_{i_1}, S_{i_2}, \cdots$ there exists a column vector ν such that

$$\lim_{j \to \infty} S_{i_j} S_{i_{j-1}} \cdots S_{i_1} = \mathbf{1} \nu^T. \tag{7}$$

In addition, in the case that $\mathcal{S}$ is an infinite set, $\lambda(W) < 1$, where $W = S_{k_1} S_{k_2} \cdots S_{k_{N_t+1}}$ and N_t is defined as the number of different types of all $n \times n$ SIA matrices. Furthermore, if there exists a constant $0 \le d < 1$ satisfying $\lambda(W) \le d$, then Eq. (7) also holds.

Proof: See Lemma 4 and the concluding remarks in [30]. ∎

Proof of Theorem 3.2:

From Eq. (1), we see that $P_i > 0$ is uniformly lower bounded since Q_i is uniformly lower bounded. Also noting that $-P_i[\sum_j g_{ij}(t)(P_j + \Omega_{ij})^{-1}]P_i \le -P_i^2/(P_i + \Omega_{ii})$, we know that P_i is uniformly upper bounded. From Eq. (2), we can see that $K_{ij}(t) > 0$, $\forall i \ne j$, is uniformly lower and upper bounded.

Let $t_0, t_1, \cdots$ be an infinite time sequence corresponding to the times at which graph $\mathcal{G}(t)$ switches topology. Since the interaction topology cannot switch infinitely fast, we assume that $t_i - t_{i-1} \ge t_L$, $\forall i = 1, 2, \cdots$. Note that each interval $[t_{i-1}, t_i)$ can be divided into finite or infinite number of subintervals such that the length of each subinterval is greater than or equal to t_L but less than or equal to $t_M = 2t_L$ and the graph on each subinterval is fixed. Relabel these subintervals as $s_0, s_1, \cdots$.

Without transmission or communication noise, Eq. (3) can be rewritten in matrix form as $\dot{\xi} = \Lambda(t)\xi$, where $\xi = [\xi_1, \cdots, \xi_n]^T$ and $\Lambda(t) = [\lambda_{ij}(t)]$. As mentioned above, the solution

can be denoted as $\xi(t) = \Phi(t, s_j)\Phi(s_j, s_{j-1})\cdots\Phi(s_1, s_0)\xi(s_0)$, where Φ is the transition matrix. Noting that $\lambda_{ij}(t) = g_{ij}(t)K_{ij}(t)$, $\forall j \neq i$, and $\sum_j \lambda_{ij}(t) = 0$, we know that $\Lambda(t)$ is continuous and satisfies the hypothesis of Lemma 3.4 for $t \in [s_{j-1}, s_j]$. Noting that $K_{ij}(t)$ is uniformly lower and upper bounded, we know that each nonzero, that is, positive, entry λ_{ij}, where $i \neq j$, satisfies the property that $\lambda_{ij} \in [\lambda_L, \lambda_M]$, which is a compact set. In addition, $\lambda_{ii} = -\sum_{j \neq i} \lambda_{ij}$, which is also in a compact set. In the case that the interaction topology is switching with time, there are a finite number of possible interaction topologies. For each possible interaction topology, note that matrix $\Lambda(t)$ has the same structure in the sense that positive, zero, and negative entries are in the same places for $t \in [s_{j-1}, s_j]$. From Lemma 3.4, each transition matrix $\Phi(s_j, s_{j-1})$ is a stochastic matrix, where $t_L \leq s_j - s_{j-1} \leq t_M$, and $\Phi(s_j, s_{j-1})$ is of constant type over this interval, for each possible interaction topology. Combining the above arguments with the fact that $\Phi(s_j, s_{j-1})$ is a continuous function of $\lambda_{ij}(t)$ for $t \in [s_{j-1}, s_j]$, we see that each nonzero entry of $\Phi(s_j, s_{j-1})$ is lower bounded for each possible interaction topology. It is straightforward to see that there are only finitely many types for $\Phi(s_j, s_{j-1})$. We know that there exists a sequence of unions of the directed interaction graphs across some time intervals and each union is uniformly bounded and has a spanning tree. Thus the transition matrix $\Phi^{(k)}$ for each union is a product of finitely many matrices $\Phi(s_{k_i}, s_{k_{i-1}})$. From Lemma 3.1, the type of $\Phi^{(k)}$ is uniquely decided by the order and type of each element in its product. Also, from Lemma 3.6, we know that each $\Phi^{(k)}$ is SIA. In addition, noting that the graph associated with each $\Phi^{(k)}$ has a spanning tree, we see that any number of products of $\Phi^{(k)}$ is also SIA according to the second part of Lemma 3.5. Noting that $\Phi^{(k)}$ can only have finitely many types, we see that for each type of $\Phi^{(k)}$ its nonzero entries are lower bounded. Let $W = \Phi^{(j_1)}\Phi^{(j_2)}\cdots\Phi^{(j_{N_t+1})}$. From the second part of Lemma 3.7, we know that $\lambda(W) < 1$. Note that W can only have finite many types, denoted as W_t. In order to show that $\lambda(W) \leq d < 1$, it is sufficient to show that for each type, there exists a $0 \leq d_i < 1$ such that $\lambda(W) \leq d_i$. This can be verified by noting that the nonzero entries of W are lower bounded for each type. Let $d = \max\{d_1, d_2, \cdots, d_{W_t}\}$. It is obvious that $\lambda(W) \leq d$. From Lemma 3.7, we can show that $\xi(t) \to \mathbf{1}\nu^T\xi(0)$, where ν is a nonnegative column vector. ∎

In previous results on consensus [14], [18], the coefficient matrix $C(t)$ was assumed to be piecewise constant with finite dwell time, and elements drawn from a finite set. The following corollary of Theorem 3.2 shows that these conditions can be relaxed.

Corollary 3.8: Let $\dot{\xi} = C(t)\xi$, where $C(t) = [c_{ij}(t)] \in M_n(\mathbb{R})$ is piecewise continuous,

$c_{ij} \geq 0$, $i \neq j$, $\sum_j c_{ij} = 0$, and each nonzero entry c_{ij}, $i \neq j$, is both uniformly lower and upper bounded. Under switching interaction topologies, ξ_i achieves consensus if there exist infinite many consecutive uniformly bounded time intervals such that the union of the interaction graph across each such interval has a spanning tree.

C. Discrete-time Consensus

Theorem 3.9: Given switching interaction topologies and zero transmission or communication noise, the discrete-time Kalman consensus scheme listed in Eq. (4)–(5) achieves asymptotic consensus if there exist infinitely many consecutive uniformly bounded time intervals such that the union of the interaction graph across each interval has a spanning tree.

Proof: Without transmission or communication noise, Eq. (5) can be written as

$$
\xi_i[k+1] = \left[1 - P_i[k+1] \sum_{j \neq i} g_{ij}[k](P_j[k] + \Omega_{ij})^{-1} \right] \xi_i[k]
$$
$$
+ P_i[k+1] \sum_{j \neq i} \left[g_{ij}[k](P_j[k] + \Omega_{ij})^{-1} \xi_j[k] \right]. \tag{8}
$$

Note that each weighting factor of ξ_ℓ is less than or equal to 1 and the sum of the weighting factors of ξ_ℓ is equal to 1, where $\ell \in \mathcal{I}$. Letting $\xi = [\xi_1, \cdots, \xi_n]^T$, we can rewrite Eq. (8) as $\xi[k+1] = D[k]\xi[k]$, where it can be verified that $D[k]$ is a stochastic matrix with positive diagonal entries. In addition, for each possible interaction topology, $D[k]$ is of the same type and its nonzero entries are lower bounded.

We know that there exists a sequence of unions of the directed interaction graphs across some time intervals and each union is uniformly bounded and has a spanning tree. Let $D^{(i)}$ be the product of matrices $D[k]$ over the ith union. Note that each $D^{(i)}$ is SIA from Lemma 3.5. As a result, the proof follows the same reasoning as the proof of Theorem 3.2 with $D^{(i)}$ playing the role of $\Phi^{(k)}$. ∎

IV. CONSENSUS SCHEMES ARE INPUT-TO-STATE STABLE

We are primarily interested in the application of consensus algorithms to cooperative control problems. In this section we will explore a control architecture where a consensus algorithm is in cascade with a coordination algorithm, as shown in Figure 5. Our purpose in this section
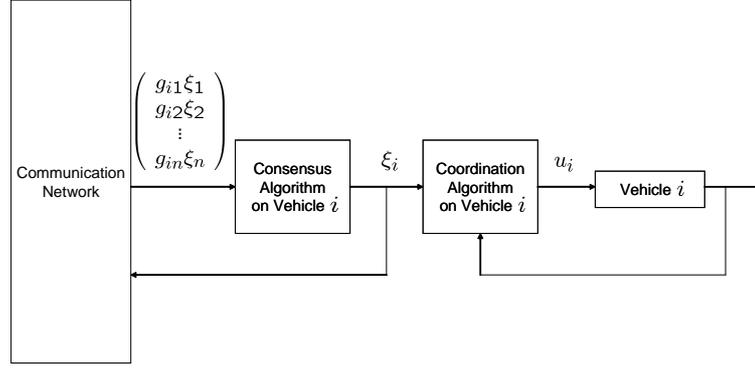
Fig. 5. The control architecture consists of a consensus algorithm in cascade with a coordination algorithm. The consensus algorithm receives information from the communication network to produce a value of the coordination variable ξ_i. The coordination algorithm uses the coordination variable ξ_i to produce a command to the vehicle u_i. We assume that identical consensus and coordination algorithms are implemented on each vehicle.

is to derive conditions on the consensus and coordination algorithms that guarantee that the cooperation objective is achieved. Toward that end, rewrite Eq. (3) as

$$\dot{\xi}_i = \sum_{j=1}^{n} g_{ij}(t) K_{ij} \left(\xi_j - \xi_i \right) + \sum_{j=1}^{n} g_{ij}(t) K_{ij} \nu_{ij}. \tag{9}$$

Letting $x_{ij} = \xi_i - \xi_j$ and $x = (x_{11}, x_{12}, \ldots, x_{1n}, x_{21}, \ldots, x_{nn})^T$, we get the state-space model

$$\dot{x} = A(t)x + B(t)\nu \tag{10}$$

where ν is a column vector created by stacking the communication noise terms ν_{ij}, and the elements of $A(t)$ and $B(t)$ are linear combinations of $g_{ij}K_{ij}(t)$ and can be easily constructed from Eq. (9). The vector x represents the total consensus error.

Theorem 4.1: Under the hypothesis of Theorem 3.2, the Kalman consensus scheme given by Eqs. (1), (2), and (10) is input-to-state stable.

The proof of this theorem requires the following two lemmas.

Lemma 4.2: Under the hypothesis of Theorem 3.2, if the communication error ν is zero, then the consensus error x is uniformly stable.

Proof: Note that $\xi(t) = \Phi(t, t_0)\xi(t_0)$, where $\Phi(t, t_0)$ is a stochastic matrix according to Lemma 3.3. As a result, we see that the ith coordination variable $\xi_i(t)$ is equal to a weighted average of all agents' initial coordination variables communicating with agent i. Since a weighted

average can never be greater (or smaller) than any one of the components in the average, we know that $\xi_i(t) \in [\min_j \xi_j(t_0), \max_j \xi_j(t_0)]$ for all t and i. Then it is straightforward to see that

$$\|x(t)\|_\infty \leq \|x(t_0)\|_\infty, \quad \text{for} \quad t \geq t_0.$$

$\blacksquare$

Lemma 4.3: The norm of $B(t)$ in Eq. (10) is bounded.

Proof: Since $B(t)$ is composed of linear combinations of $K_{ij}(t)$, if $\|K_{ij}(t)\|$ is bounded for each (i, j), then $\|B(t)\|$ will also be bounded. $\|K_{ij}\|$ was shown to be bounded in the proof of Theorem 3.2. $\blacksquare$

Proof of Theorem 4.1: By Lemma 4.2, the Kalman consensus error is uniformly stable. By Theorem 3.2, $\|\xi_i - \xi_j\| \to 0$ as $t \to \infty$ for all (i, j). Since each element of $x \to 0$, then $\|x\| \to 0$ as $t \to \infty$ and we conclude uniform asymptotic stability. Any linear system that is uniformly asymptotically stable is also uniformly exponentially stable [31]. Additionally, linear uniformly exponentially stable systems with $\|B(t)\| < \beta$ for finite β are bounded-input bounded-output stable [31]. Since the Kalman consensus error governed by Eq. (10) is a linear uniformly asymptotically stable system with $\|B(t)\|$ bounded, it is ISS. $\blacksquare$

Corollary 4.4: If the continuous-time consensus schemes presented in [11], [18], [14], and [15] are augmented with communication noise, then the representation of these schemes that is equivalent to Eq. (10) is ISS.

Proof: The difference between each of these schemes and Eq. 3 is that the consensus gain $K_{ij}(t)$ is time invariant. Therefore, from the proof of Theorem 4.1 it is clear that they are ISS. $\blacksquare$

Referring to Figure 5 we see that the combination of the communication network and the consensus scheme is an ISS system. It is well known that the cascade combination of two ISS systems is also ISS. Therefore if the feedback loop containing the coordination algorithm and the ith vehicle is ISS from the consensus error to the cooperation objective, then the total system will be ISS from the communication noise to the cooperation objection. This concept is shown schematically in Figure 6 and can be summarized by the following Theorem.

Theorem 4.5: If a consensus scheme is ISS from the communication noise to the consensus error and a coordination scheme is ISS from the consensus error to the cooperation objective, then the cascade interconnection of the two (see Fig. 6) is ISS from the communication noise
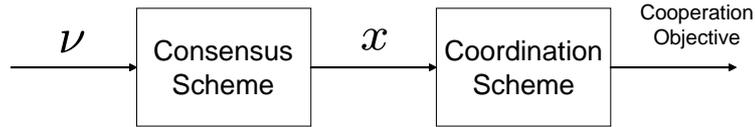
Fig. 6.   The distributed cooperative control problem can be thought of as a cascade connection between the consensus algorithm and the coordination algorithm. If both are ISS, then the cascade system will be ISS

to the cooperation objective.

V. ILLUSTRATIVE EXAMPLE - DISTRIBUTED COOPERATIVE TIMING FOR A TEAM OF UAVs

Suppose that a team of UAVs, flying at distinct altitudes, is tasked to simultaneously visit a pre-specified location. For simplicity, also assume that paths with appropriate velocities have been precomputed for each UAV as shown in Figure 7. Algorithms that achieve this functionality are described in [21].
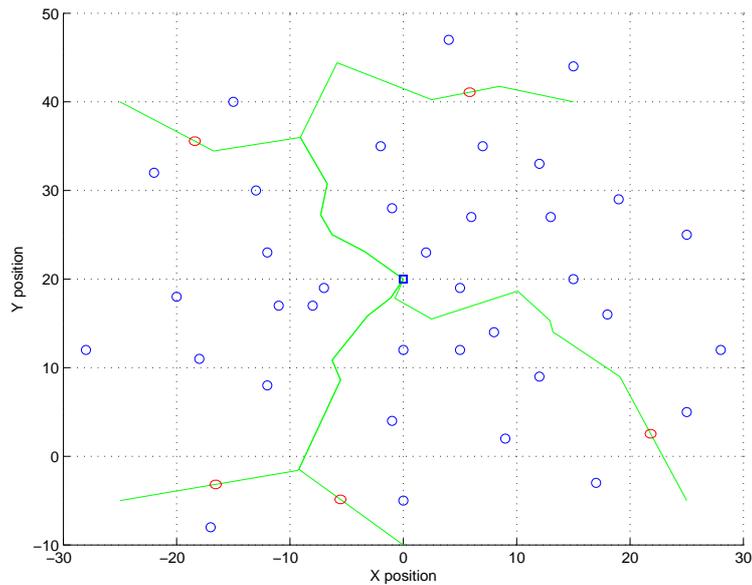

Fig. 7.   Cooperative timing scenario with five UAVs.

We will also assume that each UAV has autopilot functionality that maintains the UAV on its pre-defined path, but that the velocity along the path can be adjusted to meet the simultaneous arrival objective [32], [33]. We will assume that the velocity hold autopilot has been designed

such that

$$\dot{v}_i = \alpha_i \left(v_i^c - v_i \right) \tag{11}$$

where $\alpha_i > 0$, v_i is the velocity, and v_i^c is the commanded velocity for the ith UAV. Let L_i denote the length of the path remaining to the target, then

$$\dot{L}_i = -v_i.$$

Given L_i and v_i, the ith UAV can estimate its expected time-of-arrival as

$$\tau_i = \frac{L_i}{v_i}.$$

Therefore

$$\dot{\tau}_i = \frac{v_i \dot{L}_i - L_i \dot{v}_i}{v_i^2}$$

$$= -1 - \alpha_i \tau_i \left(\frac{v_i^c - v_i}{v_i} \right).$$

The cooperation objective for this problem is that each UAV arrives at its destination simultaneously, i.e. $\tau_i - \tau_j = 0$ for each (i, j). The coordination variable for this problem is chosen as the arrival time. Therefore ξ_i represents the ith UAVs understanding of the team arrival time. Letting

$$v_i^c = v_i + \frac{v_i}{\alpha_i \tau_i} \left(\gamma \tau_i - \gamma \xi_i - 1 \right) \tag{12}$$

we get that

$$\dot{\tau}_i = -\gamma \tau_i + \gamma \xi_i.$$

Note that

$$(\dot{\tau}_i - \dot{\tau}_j) = -\gamma \tau_i + \gamma \xi_i + \gamma \tau_j - \gamma \xi_j$$

$$= -\gamma \left(\tau_i - \tau_j \right) + \gamma \left(\xi_i - \xi_j \right),$$

and that the system $\dot{\phi} = -\gamma \phi + \gamma u$ is input-to-state stable. In fact we have that

$$|\phi(t)| \leq e^{-\gamma(t-t_0)} \phi(t_0) + \sup_{t_0 \leq \sigma \leq t} |u(\sigma)|.$$

Therefore, from Theorem 4.5, the combination of the consensus strategy given by Eqs. (1)–(3) and the velocity controller given by Eq. (12) is input-to-state stable with the input being

communication noise and the state consisting of both the consensus discrepancy $\xi_i - \xi_j$ and the UAV arrival discrepancy $\tau_i - \tau_j$.

The cooperative timing scenario was simulated with an unreliable switching communication topology. The team is connected in the graph shown in Fig. 8 where each link is only available 70 percent of the time. When an agent receives communication it updates its estimate of ξ using
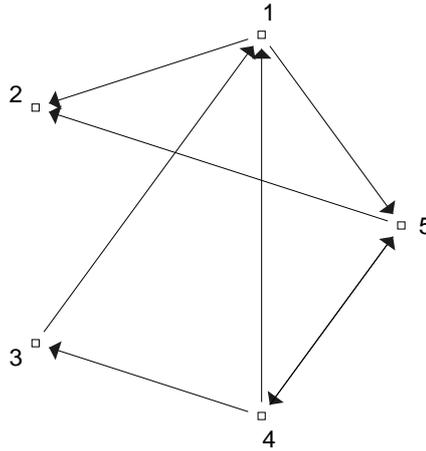

Fig. 8.   Union of possible communication topologies.

the Kalman consensus scheme of Section II-A. In between consensus updates, agents control their velocity using Equation (12). Five agents were given a single target at which to arrive simultaneously. Fig. 7 shows the application scenario, where each red circle represents an agent, the blue circles represent threats, the blue square represents the target, and the green lines are the waypoint paths.

In the first case, communication noise was set to zero and each agent started with approximately the same confidence in its estimate of ξ. A plot of ξ for each vehicle is shown in Fig. 9(a) and Fig. 9(b) shows τ for each vehicle. As can be seen, each agent in the team achieves agreement using consensus, adjusts its velocity to match ξ_i, and arrives at the target in approximately 20 seconds.

In the second case, significant communication noise is added. ξ is shown for each vehicle in Fig. 10(a) and τ for each vehicle is shown in Fig. 10(b). As can be seen, each agent in the team achieves approximate agreement using consensus where the error in agreement is due to the communication noise.
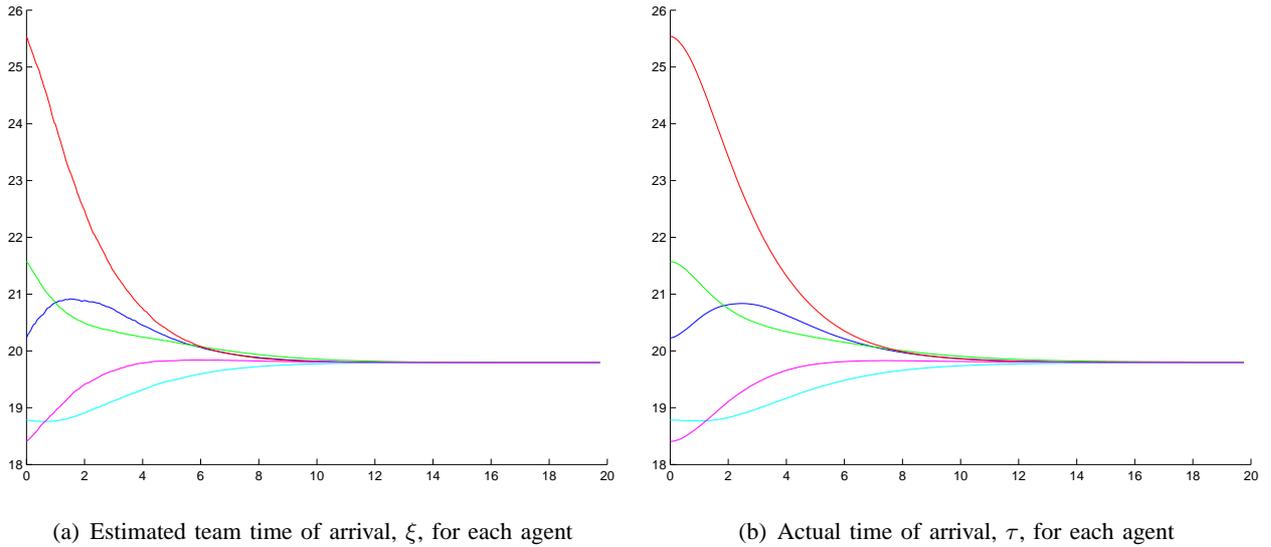
(a) Estimated team time of arrival, ξ, for each agent   (b) Actual time of arrival, τ, for each agent

Fig. 9. Cooperative timing with no communication noise.

(a) Estimated team time of arrival, ξ, for each agent   (b) Actual time of arrival, τ, for each agent
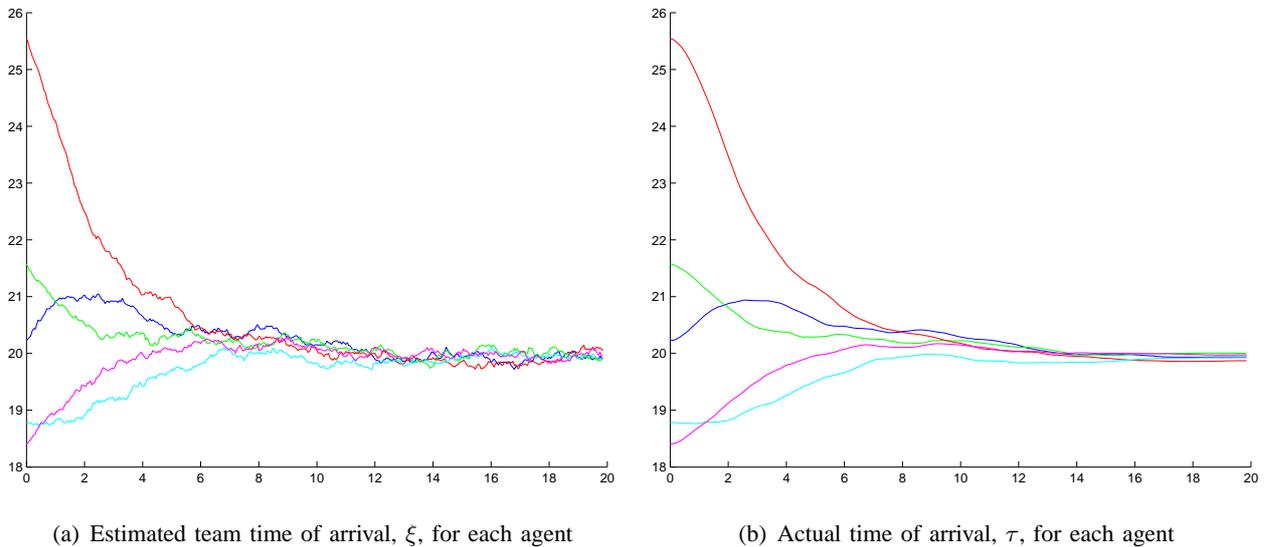
Fig. 10. Cooperative timing with significant communication noise.

VI. CONCLUSION

This paper has considered the problem of consensus seeking with relative uncertainty in distributed multi-agent systems. We have proposed discrete-time and continuous-time Kalman filter-like consensus schemes that are appropriate when different agents in the group may have different confidences about their information state. Sufficient conditions have been shown for consensus seeking using the proposed consensus schemes under switching interaction topologies.

Consensus schemes were shown to be input-to-state stable from the communication noise to the consensus error. This fact was exploited in an application to a UAV distributed cooperative timing scenario.

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