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The Lie Symmetries of a Few Classes of Harmonic Functions

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THE LIE SYMMETRIES OF A FEW CLASSES OF HARMONIC FUNCTIONS

by

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ABSTRACT

THE LIE SYMMETRIES OF A FEW CLASSES
OF HARMONIC FUNCTIONS

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In a differential geometry setting, we can analyze the solutions to systems of differential equations in such a way as to allow us to derive entire classes of solutions from any given solution. This process involves calculating the Lie symmetries of a system of equations and looking at the resulting transformations. In this paper we will give a general background of the theory necessary to develop the ideas of working in the jet space of a given system of equations, applying this theory to harmonic functions in the complex plane. We will consider harmonic functions in general, harmonic functions with constant Jacobian, harmonic functions with fixed convexity and a few other subclasses of harmonic functions.

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1 Introduction

We want to analyze the Lie symmetries of several types of harmonic functions in the complex plane. Finding the symmetries will allow us to calculate new harmonic functions from old ones, essentially allowing us to "flow" from one function to another in such a way as to always force the interim functions to be harmonic.

In particular, we will consider the following classes: harmonic functions in general, harmonic functions with constant Jacobian, harmonic functions with fixed convexity and a few other subclasses of harmonic functions.

2 Harmonic Functions

2.1 Complex Numbers

We will use the following standard notations. We will denote the real numbers by $\mathbb{R}$ and the complex numbers by $\mathbb{C}$. When we write $z \in \mathbb{C}$, we will use the convention that $z = x + iy$ with $x, y \in \mathbb{R}$ and will define the real part of z to be x and the imaginary part of z to be y and use the notation $\mathrm{Re}(z) = x$ and $\mathrm{Im}(z) = y$.

Given a point $z \in \mathbb{C}$, we will make the identification of the complex plane with the Cartesian plane $\mathbb{R}^2$, graphing the point z in $\mathbb{R}^2$ by giving it the coordinates $(\mathrm{Re}(z), \mathrm{Im}(z)) \in \mathbb{R}^2$. Also, we define the conjugate of z, denoted $\overline{z}$, to be $\overline{z} = x - iy$ if $z = x + iy$. Two identities that will be very useful are $\mathrm{Re}(z) = x = \frac{(z+\overline{z})}{2}$ and $\mathrm{Im}(z) = y = \frac{(z-\overline{z})}{2}$. We will make constant use of these because they give us the ability to write the real and imaginary parts of z in terms of z and $\overline{z}$.

We will let $\mathbb{D}$ be the set of all complex numbers whose modulus is less than one, yielding $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, the open unit disk with $\partial\mathbb{D} = \{z \mid |z| = 1\}$. Also we will make the convention that a domain is an open connected subset of $\mathbb{C}$ and that $G \subset \mathbb{C}$ is a domain.

2.2 Harmonic and Analytic Functions

Let us now consider functions from G into $\mathbb{C}$. Let $f : G \to \mathbb{C}$ be defined by $\mathrm{Re}(f(z)) = u(z)$ and $\mathrm{Im}(f(z)) = v(z)$, where u and v are real valued functions taking values from G to $\mathbb{R}$, yielding $f(z) = u(z) + iv(z)$.

Definition 2.1. A real valued function $u : G \to \mathbb{R}$ is said to be harmonic if it has continuous second partial derivatives such that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Definition 2.2. A complex valued function $f = u + iv$ is said to be harmonic if u and v are real valued harmonic functions.

We will adopt the standard notation of $\frac{\partial f}{\partial x} = f_x$, $\frac{\partial f_x}{\partial y} = f_{xy}$, and so on, for the partial derivatives of the function f. Recall that Theorem 13.3 of [10] gives us that if f has continuous second derivatives then $f_{xy} = f_{yx}$. By the identities given in Section 2.1, we can write x and y in terms z and $\overline{z}$ allowing us to find f_z and $f_{\overline{z}}$ along with f_x and f_y. For example, if $f(z) = z + \overline{z}^2 = \left(x + x^2 - y^2\right) + i\,(y - 2xy)$ then $f_x = 1 + 2x - 2iy$, $f_y = -2y - 2ix$, $f_z = 1$ and $f_{\overline{z}} = 2\overline{z}$.

Example 2.3. Consider the real valued functions u, v_1 and v_2 of the complex variable $z = x + iy$. Let $u(z) = x^2 - y^2$, $v_1(z) = 2xy$ and $v_2(z) = -2xy$. We can see that

$$\begin{aligned} u_{xx} + u_{yy} &= 2 - 2 = 0 \\ v_{1xx} + v_{1yy} &= 0 + 0 = 0 \\ v_{2xx} + v_{2yy} &= 0 + 0 = 0 \end{aligned}$$

giving us that u, v_1 and v_2 are real valued harmonic functions. If we let $f_1(z) = u(z) + iv_1(z)$ and $f_2(z) = u(z) + iv_2(z)$ then simplifying, we get that $f_1(z) = x^2 - y^2 + i2xy = (x + iy)^2 = z^2$ and $f_2(z) = x^2 - y^2 - i2xy = (x - iy)^2 = \overline{z^2}$ and we have verified that f_1 and f_2 are complex valued harmonic functions.

We will not typically specify whether a function is real or complex valued harmonic because it will be clear from context. We will simply state that it is harmonic.

Definition 2.4. Two real valued harmonic functions u and v are said to be harmonic conjugates if they are harmonic and they satisfy the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Definition 2.5. If a function $f : G \to \mathbb{C}$ is continuously differentiable on G then we say f is analytic.

Theorem 2.6. *If $f = u + iv$ is analytic then u and v satisfy the Cauchy-Riemann equations.*

Fact 2.7. *Recall that if $f = u + iv$ is analytic, we can use the Cauchy-Riemann equations to get that*

$$\begin{aligned} f'(z) &= u_x(z) + iv_x(z) \\ &= u_x(z) - iu_y(z) \\ &= v_y(z) + iv_x(z) \\ &= v_y(z) - iu_y(z). \end{aligned}$$

Example 2.8. Consider f_1 and f_2 as defined in Example 2.3. From the Cauchy-Riemann equations we get that

$$\begin{aligned} u_x &= 2x &= v_{1y} \\ u_y &= -2y &= -v_{1x} \end{aligned}$$

and

$$\begin{aligned} u_x &= 2x &\neq v_{2y} \\ u_y &= -2y &\neq -v_{2x} \end{aligned}$$

and combining this with what we have from Example 2.3 we get that f_1 is harmonic and analytic but f_2 is strictly harmonic.

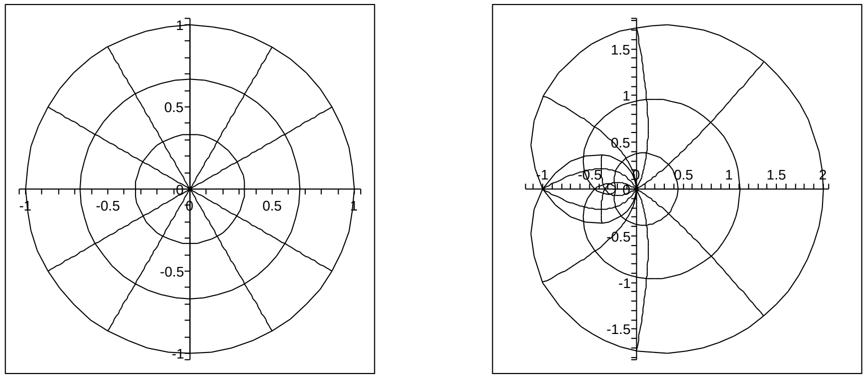
By Theorem 2.6, Definition 2.4 and Definition 2.5, we can see that any analytic function is harmonic. However, a harmonic function need not be analytic as shown in Example 2.8.

Often we will want to consider the image of $\mathbb{D}$ under a given function or map. To do this, we will look at the images of concentric circles and/or radial lines in the unit disk. Consider the examples given in Figure 1(a) and Figure 1(b).

Theorem 2.9. *Let u and v be real valued harmonic functions defined on G and suppose that u and v have continuous partial derivatives. Then $f : G \to \mathbb{C}$ given by $f = u + iv$ is analytic if and only if u and v are harmonic conjugates.*

Theorem 2.10. *If $u : G \to \mathbb{R}$ is a real valued harmonic function then u has a harmonic conjugate.*

Theorem 2.11. *If $f = u + iv$ is harmonic then there exist h and g analytic such that $f = h + \overline{g}$.*



(a) Image of concentric circles and radial lines in $\mathbb{D}$ under $f(z) = z$.

(b) Image of concentric circles and radial lines in $\mathbb{D}$ under $f(z) = z^2 - z$.

Figure 1: *Images of* $\mathbb{D}$ *under the maps* z *and* $z^2 - z$.

Proof. Since u and v are real valued harmonic functions then by Theorem 2.10 we have that u and v have harmonic conjugates; call them u_2 and v_2 respectively. Consider that

$$\begin{aligned} f &= u + iv \\ &= \frac{u + iu_2 + \overline{u + iu_2}}{2} + i\frac{v_2 + iv + \overline{v_2 + iv}}{2i} \\ &= \frac{u + iu_2 + v_2 + iv}{2} + \frac{\overline{u + iu_2 - (v_2 + iv)}}{2} \\ &= \frac{(u + v_2) + i(u_2 + v)}{2} + \frac{\overline{(u - v_2) + i(u_2 - v)}}{2} \\ &= h + \overline{g} \end{aligned}$$

if we let h and g be defined in the obvious way. Theorem 2.9 clearly shows that h and g are analytic. Therefore $f = h + \overline{g}$. □

Theorem 2.12. *If* h *and* g *are analytic then* $f = h + \overline{g}$ *is harmonic.*

Proof. Let $h = h^1 + ih^2$, $g = g^1 + ig^2$ and $f = h + \overline{g}$. Then $\text{Re}(f) = h^1 + g^1$ and $\text{Im}(f) = h^2 - g^2$. Since h and g are analytic then h^1, h^2, g^1 and g^2 are harmonic by Theorem 2.9 and Definition 2.4 which implies that $h^1 + g^1$ and $h^1 - g^2$ are harmonic. □

Example 2.13. Consider the function $f(z) = z + \overline{\frac{1}{3}z^3} = x + iy + \frac{1}{3}x^3 + ix^2y - xy^2 - \frac{1}{3}iy^3 = (x + \frac{1}{3}x^3 - xy^2) + i(y + x^2y - \frac{1}{3}y^3)$ recalling that $z = x + iy$. Let $u(z) = x + \frac{1}{3}x^3 - xy^2$ and let $v(z) = y + x^2y - \frac{1}{3}y^3$. This gives that $u_{xx} = 2x$, $u_{yy} = -2x$, $v_{xx} = 2y$ and $v_{yy} = -2y$ and it is easily verified that u and v are harmonic and therefore f is harmonic. Now if we let $h(z) = z = x + iy$ and $g(z) = \frac{1}{3}z^3 = \left(\frac{1}{3}x^3 - xy^2\right) + i\left(x^2y - \frac{1}{3}y^3\right)$, it is again easily verified that h and g are analytic therefore yielding that $f = h + \overline{g}$.

Theorem 2.14. *Suppose f is analytic in $\mathbb{D}$. Then f has a convergent power series expansion in $\mathbb{D}$ given by*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \quad \textit{where} \quad c_n = \frac{f^{(n)}(0)}{n!}.$$

Theorem 2.15. *If f is analytic in $\mathbb{D}$ then $f_{\overline{z}}(z) = 0$ for all $z \in \mathbb{D}$.*

Proof. Let f be analytic in $\mathbb{D}$. Then by Theorem 2.14 we have that $f(z) = \sum_{n=0}^{\infty} c_n z^n$ which implies that $f_{\overline{z}}(z) = 0$. □

Definition 2.16. A function G is said to be subordinate to a function F, written $G \prec F$, if

$$G(z) = F(\omega(z)), \;\; |z| < 1,$$

for some analytic function ω with $|\omega(z)| \leq |z|$ where F and G are analytic on $\mathbb{D}$.

Theorem 2.17. *Let $\mathbb{D}_r = \{z \mid |z| \leq r\}$. If $G \prec F$ then $G(\mathbb{D}_r) \subset F(\mathbb{D}_r)$ for all $r < 1$.*

For more details on subordinate functions see [8].

3 Schlicht Functions

In our analysis of the Lie symmetries of certain functional classes we will often restrict ourselves to the study of cases of schlicht functions. Schlicht functions are essentially normalized univalent complex valued harmonic functions. These classes have important applications in the field of geometric function theory, an area of complex analysis. For more information on univalent harmonic functions see [4] and [9].

Definition 3.1. A function $f : G \to \mathbb{C}$ is said to be one-to-one or univalent if $f(z_1) \neq f(z_2)$ for all $z_1, z_2 \in G$ with $z_1 \neq z_2$.

Definition 3.2. A harmonic function $f = h + \overline{g}$ is called sense-preserving at a point z_0 if $h'(z) \not\equiv 0$ and $\omega = \frac{g'}{h'}$ is analytic at z_0 and $|\omega(z_0)| < 1$. Similarly, f is sense-reversing at z_0 if $\overline{f} = g + \overline{h}$ is sense-preserving at z_0. A point z_0 is a singular point if f is neither sense-preserving nor sense-reversing at z_0.

Definition 3.3. Let $f(z)$ be defined on G. The Jacobian of f, denoted J_f, is the determinant of the matrix $\begin{bmatrix} \dfrac{\partial}{\partial x}\operatorname{Re}(f) & \dfrac{\partial}{\partial y}\operatorname{Re}(f) \\ \dfrac{\partial}{\partial x}\operatorname{Im}(f) & \dfrac{\partial}{\partial y}\operatorname{Im}(f) \end{bmatrix}$.

We say a function f is locally univalent on G if $J_f \neq 0$ on G. If $J_f > 0$ on G then f is locally univalent and sense-preserving and if $J_f < 0$ on G then f is locally univalent and sense-reversing.

If we let $f = h + \overline{g}$, where $h = h^1 + ih^2$ and $g = g^1 + ig^2$ are analytic, then f is harmonic and J_f is given by $J_f = |h'|^2 - |g'|^2$. To see this, use the Cauchy-Riemann equations, Fact 2.7 and Definition 3.3 to get the following:

$$
\begin{aligned}
J_f &= \begin{vmatrix} \dfrac{\partial}{\partial x}\operatorname{Re}(f) & \dfrac{\partial}{\partial y}\operatorname{Re}(f) \\ \dfrac{\partial}{\partial x}\operatorname{Im}(f) & \dfrac{\partial}{\partial y}\operatorname{Im}(f) \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial h^1}{\partial x} + \dfrac{\partial g^1}{\partial x} & \dfrac{\partial h^1}{\partial y} + \dfrac{\partial g^1}{\partial y} \\ \dfrac{\partial h^2}{\partial x} - \dfrac{\partial g^2}{\partial x} & \dfrac{\partial h^2}{\partial y} - \dfrac{\partial g^2}{\partial y} \end{vmatrix} \\
&= \begin{vmatrix} \dfrac{\partial h^1}{\partial x} + \dfrac{\partial g^1}{\partial x} & -\dfrac{\partial h^2}{\partial x} - \dfrac{\partial g^2}{\partial x} \\ \dfrac{\partial h^2}{\partial x} - \dfrac{\partial g^2}{\partial x} & \dfrac{\partial h^1}{\partial x} - \dfrac{\partial g^1}{\partial x} \end{vmatrix} \\
&= \left(\frac{\partial h^1}{\partial x} + \frac{\partial g^1}{\partial x}\right)\left(\frac{\partial h^1}{\partial x} - \frac{\partial g^1}{\partial x}\right) - \left(-\frac{\partial h^2}{\partial x} - \frac{\partial g^2}{\partial x}\right)\left(\frac{\partial h^2}{\partial x} - \frac{\partial g^2}{\partial x}\right) \\
&= \left(\left(\frac{\partial h^1}{\partial x}\right)^2 + \left(\frac{\partial h^2}{\partial x}\right)^2\right) - \left(\left(\frac{\partial g^1}{\partial x}\right)^2 + \left(\frac{\partial g^2}{\partial x}\right)^2\right) \\
&= |h'|^2 - |g'|^2.
\end{aligned}
$$

Theorem 3.4. *Let $f = h + \overline{g}$ be harmonic. Then f is locally univalent and sense-preserving on $\mathbb{D}$ if and only if $|g'(z)| < |h'(z)|$ for all $z \in \mathbb{D}$.*

Fact 3.5. *If f is sense-preserving and $h'(z_0) \neq 0$ then $J_f(z_0) > 0$ and if f is sense-reversing and $h'(z_0) \neq 0$ then $J_f(z_0) < 0$.*

Theorem 3.6 (Argument Principle for Harmonic Functions)**.** *Let f be harmonic in $\mathbb{D}$ such that f is continuous and non-zero on $\partial\mathbb{D}$. If f has no singular zeros in $\mathbb{D}$ then f is univalent on $\mathbb{D}$.*

Theorem 3.6 is proven in [7], it being a standard result from complex analysis.

Definition 3.7. The family of schlicht functions (that is normalized, analytic, univalent functions) is denoted by S and is defined by

$$S = \{f : \mathbb{D} \to \mathbb{C} \mid f \text{ is analytic and univalent with } f(0) = 0 \text{ and } f'(0) = 1\}.$$

If $f \in S$ then the power series expansion of $f(z)$ given by Theorem 2.14 has the general form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$

Example 3.8. Consider $f(z) = \dfrac{z}{1-z}$. We can see that $\dfrac{z_1}{1-z_1} = \dfrac{z_2}{1-z_2}$ implies that $z_1 - z_1z_2 = z_2 - z_1z_2 \;\Rightarrow\; z_1 = z_2$ which gives that f is univalent. Since f is analytic in $\mathbb{D}$, $f(0) = 0$ and $f'(z) = \dfrac{1}{(1-z)^2} \;\Rightarrow\; f'(0) = 1$ then $f \in S$. In this example, the image of $\mathbb{D}$ under f is the right half-plane with boundary at $x = -\frac{1}{2}$.

We can define a super class of S by relaxing the requirement of analyticity to complex valued harmonic. We will denote this class as S_H and define it by

$$S_H = \{f : \mathbb{D} \to \mathbb{C} \;\big|\; f \text{ is harmonic and univalent with } f(0) = 0 \text{ and } f_z(0) = 1\}.$$

Therefore, if $f \in S_H$ then $f = h + \overline{g}$ where h and g are analytic and we get that h and g have power series expansions of the form given by $h(z) = z + a_2z^2 + a_3z^3 + \ldots$ and $g(z) = b_1z + b_2z^2 + b_3z^3 + \ldots$ That is $f(z) = z + a_2z^2 + \ldots + b_1\overline{z} + b_2\overline{z}^2 + \ldots$

Now define S_H^O, a subclass of S_H by

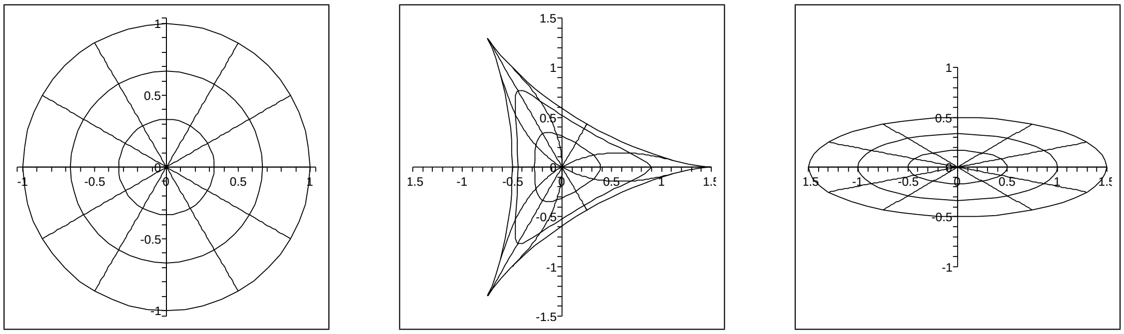
$$S_H^O = \{f \in S_H \;\big|\; f_{\overline{z}}(0) = 0\}.$$

For any $f \in S_H^O$ we get a power series expansion of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots + b_2\overline{z}^2 + b_3\overline{z}^3 + \ldots$

From these definitions, we can see that $S \subset S_H^O \subset S_H$. The following example will show us that the containments are proper.

Example 3.9. Let $f_1(z) = z$, $f_2(z) = z + \overline{\frac{1}{2}z^2}$ and $f_3(z) = z + \overline{\frac{1}{2}z}$. By the definitions of S, S_H^O and S_H and Theorem 3.6, we see that $f_1 \in S$, $f_2 \in S_H^O$ and $f_3 \in S_H$ and $f_2 \notin S$ and $f_3 \notin S_H^O$. Therefore $S \subsetneq S_H^O \subsetneq S_H$.

The graphs of the unit circle under each of f_1, f_2 and f_3 are given in Figure 2(a), Figure 2(b) and Figure 2(c), respectively.



(a) Image of concentric circles and radial lines in $\mathbb{D}$ under $f_1(z) = z$.

(b) Image of concentric circles and radial lines in $\mathbb{D}$ under $f_2(z) = z + \overline{\frac{1}{2}z^2}$.

(c) Image of concentric circles and radial lines in $\mathbb{D}$ under $f_3(z) = z + \overline{\frac{1}{2}z}$.

Figure 2: *Images of $\mathbb{D}$ under the maps f_1, f_2 and f_3.*

When we perform our analysis of the Lie symmetry groups, we will take into account several examples of schlicht functions and apply our findings to these classes of functions.

When considering functions in S_H, many times we will want to consider some geometric property of the image of $\mathbb{D}$ under the function. Here we define one of those properties.

Definition 3.10. A domain Ω is said to be convex in the direction of $e^{i\varphi}$ for a given $\varphi \in [0, \pi)$ if for every $a \in \mathbb{C}$ the set

$$\Omega \cap \{a + te^{i\varphi} \mid t \in \mathbb{R}\}$$

is either connected or empty. We say that Ω is convex in the direction of the real axis if every line parallel to the real axis has an empty or connected intersection with Ω.

We will call a function f convex in the direction of $e^{i\varphi}$, $f \in CD(\varphi)$ if $f(\mathbb{D})$ is convex in the direction of $e^{i\varphi}$. If a function f is convex in every direction then we will call it convex.

Definition 3.11. A domain Ω is said to be close-to-convex if the complement of Ω can be written as the union of non-crossing rays.

A function f will be said to be close-to-convex if $f(\mathbb{D})$ is close-to-convex. It is true that if f is convex in some direction, then it is close-to-convex. If f is convex then it is clearly convex in some direction and therefore close-to-convex.

Now consider the plot of concentric circles and radial lines for $f_2(z) = z + \overline{\frac{1}{2}z^2}$ as shown in Figure 2(b). Upon inspection, we can see that $f_2 \in CD(0)$ but $f_2 \notin CD\left(\frac{\pi}{2}\right)$ because $f(\mathbb{D}) \cap \{-0.6 + te^{i\frac{\pi}{2}} \mid t \in \mathbb{R}\}$ has two disjoint components. Each of Figure 2(a) and Figure 2(c) are convex in every direction giving us that $f_1(z) = z$ and $f_3(z) = z + \overline{\frac{1}{2}z}$ are convex in every direction as well.

In [4], J. Clunie and T. Sheil-Small state and prove Theorem 3.12, Theorem 3.13 and Theorem 3.14.

Theorem 3.12. *A harmonic function $f = h + \overline{g}$ which is locally univalent in $\mathbb{D}$ is a univalent mapping of $\mathbb{D}$ convex in the direction of the real axis if and only if $h - g$ is a univalent mapping of $\mathbb{D}$ convex in the direction of the real axis.*

Theorem 3.13. *Let $f = h + \overline{g}$ be locally univalent in $\mathbb{D}$ and suppose that for some ε with $|\varepsilon| \leq 1$ we have that $h + \varepsilon g$ is convex. Then f is univalent close-to-convex.*

Theorem 3.14. *A function $f = h + \overline{g}$ is in S_H and is convex if and only if the analytic functions*

$$h(z) - e^{2i\varphi}g(z) \quad \textit{with} \quad 0 \leq \varphi < \pi$$

are convex in the direction $e^{i\varphi}$ with f suitably normalized.

4 Fundamentals of Differential Geometry

In order to understand the process of finding the Lie symmetry groups of systems of differential equations, we need to understand some of the basics of differential geometry. This will allow us to interpret differential equations in a geometric setting. We begin with the basic definitions and a few examples to help familiarize ourselves with geometric structures. For a general text on differentiable manifolds, see [2].

4.1 Basic Differential Geometry

Definition 4.1. A topological manifold M of dimension n, or n-manifold, is a topological space with the following properties:

(i) M is Hausdorff,

(ii) M is locally Euclidean of dimension n, and

(iii) M has a countable basis of open sets.

Definition 4.2. If M is a topological n-manifold then a coordinate neighborhood or coordinate chart of M is a pair U, φ where U is an open set of M and φ is a homeomorphism of U to an open subset of $\mathbb{R}^n$. To $q \in U$ we assign the n coordinates $x^1(q), \ldots, x^n(q)$ of its image $\varphi(q)$ in $\mathbb{R}^n$ where each $x^i(q)$ is a real-valued function on U called the ith coordinate function.

When thinking about a topological n-manifold M, we will refer to the set $\varphi(U) \subset \mathbb{R}^n$ as the local coordinates of M allowing us to look at neighborhoods of M as if they were in $\mathbb{R}^n$ and not necessarily some abstract space.

Definition 4.3. A function $F : U \subset \mathbb{R}^n \to \mathbb{R}^m$ with U open, is a diffeomorphism if it is an infinitely differentiable homeomorphism and F^{-1} is also infinitely differentiable.

Definition 4.4. Let U, φ and V, ψ be coordinate neighborhoods of an n-manifold M. We say that U, φ and V, ψ are C^∞-compatible if $U \cap V \neq \emptyset$ implies that $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are diffeomorphisms of the open subsets $\varphi(U \cap V)$ and $\psi(U \cap V)$ of $\mathbb{R}^n$.

Definition 4.5. A differentiable or smooth structure on a topological manifold M is a family or coordinate atlas $\mathscr{U} = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that

(i) the U_α cover M,

(ii) for any α, β the neighborhoods U_α, φ_α and U_β, φ_β are C^∞-compatible,

(iii) any coordinate neighborhood V, ψ compatible with every $U_\alpha, \varphi_\alpha \in \mathscr{U}$ is itself in $\mathscr{U}$.

The set of all coordinate neighborhoods or charts is referred to as the coordinate atlas.

A differentiable or smooth manifold (or simply manifold) is a topological manifold together with a differentiable structure.

Theorem 4.6. *Let M be a Hausdorff space with a countable basis of open sets. If $\{V_\beta, \psi_\beta\}$ is a covering of M by C^∞-compatible coordinate neighborhoods, then there is a unique C^∞ structure on M containing these coordinate neighborhoods.*

A proof of Theorem 4.6 is given in Boothby ([2]) on page 54.

Definition 4.7. Let M and N be (smooth) manifolds of dimensions m and n, respectively. We say a function $F : M \to N$ is a smooth mapping or diffeomorphism if given any coordinate neighborhood U, φ of M and V, ψ of N, we have that $\psi \circ F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^m \to \mathbb{R}^n$ is a diffeomorphism as defined in Definition 4.3.

Definition 4.8. Let $F : M \to N$ be a smooth mapping from an m-dimensional manifold M to an n-dimensional manifold N. The rank of F at a point $x \in M$ is the rank of the $n \times m$ Jacobian matrix $[a_{ij}]$ where $a_{ij} = \dfrac{\partial F^i}{\partial x^j}$ at x, where $y = F(x)$ is expressed in any convenient local coordinates near x. The mapping F is of maximal rank on a subset $S \subset M$ if for each $x \in S$ the rank of F is as large as possible (i.e. the minimum of m and n).

Theorem 4.9. *Let $F : M \to N$ be of maximal rank at $x_0 \in M$ where the dimensions of M and N are m and n, respectively. Then there are local coordinates $x = (x^1, \dots, x^m)$ near x_0, and $y = (y^1, \dots, y^n)$ near $y_0 = F(x_0)$ such that in these coordinates F has the simple form*

$$y = (x^1, \dots, x^m, 0, \dots, 0), \quad \text{if } n > m,$$

or

$$y = (x^1, \dots, x^n), \quad \text{if } n \leq m.$$

For a proof of Theorem 4.9, see [2], pages 47 to 49.

Definition 4.10. Let M be a (smooth) manifold. A submanifold of M is a subset $N \subset M$, together with a smooth, one-to-one map $\varphi : \widetilde{N} \to N \subset M$ satisfying the maximal rank condition everywhere, where the parameter space $\widetilde{N}$ is some other manifold and N is the image of φ, that is $N = \varphi(\widetilde{N})$. In particular, the dimension of N is the same as that of $\widetilde{N}$ and does not exceed the dimension of M.

This map φ is called an immersion and defines a parametrization of the submanifold N. We may refer to N as an immersed submanifold.

Definition 4.11. A regular submanifold N of a manifold M is a submanifold parameterized by $\varphi : \widetilde{N} \to M$ with the property that for each $x \in N$ there exist arbitrarily small open neighborhoods U of x in M such that $\varphi^{-1}(U \cap N)$ is a connected open subset of $\widetilde{N}$.

Example 4.12. Let's consider an example of a submanifold of $\mathbb{R}^2$ that is not regular. Let φ be defined by $\varphi(t) = (\sin(2\arctan t), 2\sin(4\arctan t))$. Figure 3 shows $\varphi(\mathbb{R})$ and it can be verified that $\mathbb{R}, \varphi$ is a submanifold of $\mathbb{R}^2$. If we let B_r be an open ball of radius r centered at $(0,0)$ then for sufficiently small r we see that $\varphi^{-1}(B_r) = (-\infty, a) \cup (b, c) \cup (d, \infty)$ where $a < b < 0 < c < d$. This gives that $\varphi^{-1}(B_r)$ is disjoint for some r and therefore is not a regular submanifold of $\mathbb{R}^2$.

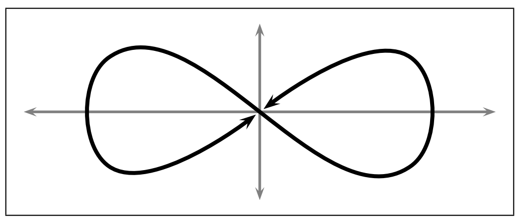


Figure 3: *Plot of* $\varphi(\mathbb{R})$

Theorem 4.13. *Let M be a smooth m-dimensional manifold, and $F : M \to \mathbb{R}^n$, $n \leq m$, be a smooth map. If F is of maximal rank on the subset $N = \{x \in M \mid F(x) = 0\}$, then N is a regular, $(m-n)$-dimensional submanifold of M.*

Example 4.14. The complex plane, $\mathbb{C}$, is a manifold of dimension 2 as shown in Figure 4. If we consider the open sets given by the standard Euclidean metric defined by the norm on $\mathbb{C}$, we can see that $\mathbb{C}$ is Hausdorff, locally Euclidean and has a countable basis therefore giving us a topological 2-manifold. Take $V = \mathbb{C}$ to be the open set and $\varphi : V \to \mathbb{R}^2$ given by $\varphi(x + iy) = (x, y)$ to be the homeomorphism from $\mathbb{C}$ to $\mathbb{R}^2$. By Theorem 4.6, we see that V, φ determines a unique differential structure on $\mathbb{C}$. It is clear that $\varphi \circ \varphi^{-1}(x, y) = (x, y)$ which is infinitely differentiable yielding a diffeomorphism from $\mathbb{R}^2$ to $\mathbb{R}^2$. Therefore, we can see that $\mathbb{C}$ is a manifold of dimension 2.

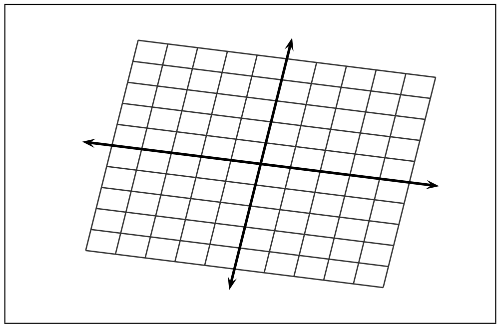


Figure 4: *Plot of* $\mathbb{C}$

Example 4.15. The unit sphere in $\mathbb{R}^3$ as shown in Figure 5 and defined by $S^2 = \{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$ is a manifold of dimension 2 with a possible set of coordinate charts given by $U = \{(x, y, z) \in S^2 \mid z \neq 1\}, \varphi$ and $V = \{(x, y, z) \in S^2 \mid z \neq -1\}, \psi$ where φ and ψ are the stereographic projections onto the plane $\{(x, y, 0) \mid (x, y) \in \mathbb{R}^2\} \subset \mathbb{R}^3$. We can see that φ is given by $\varphi(x, y, z) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$ and ψ is given by $\psi(x, y, z) = \left(\frac{x}{1+z}, \frac{y}{1+z}\right)$ where $(x, y, z) \in S^2$ and U and V are open in the Euclidean subspace topology of S^2 considered as a subspace of $\mathbb{R}^3$. Now choose any $(a, b) \in \mathbb{R}^2$ and consider the point $(x_1, y_1, z_1) = \left(\frac{2a}{a^2+b^2+1}, \frac{2b}{a^2+b^2+1}, \frac{a^2+b^2-1}{a^2+b^2+1}\right)$ in S^2. By the following calculation we can see that $(x_1, y_1, z_1) \in S^2$. Consider that

$$\begin{aligned} x_1^2 + y_1^2 + z_1^2 &= \frac{4a^2}{(a^2+b^2+1)^2} + \frac{4b^2}{(a^2+b^2+1)^2} + \frac{a^4+b^4+2a^2b^2-2a^2-2b^2+1}{(a^2+b^2+1)^2} \\ &= \frac{a^4+b^4+2a^2b^2+2a^2+2b^2+1}{(a^2+b^2+1)^2} \\ &= \frac{(a^2+b^2+1)^2}{(a^2+b^2+1)^2} = 1 \end{aligned}$$

Now consider that

$$\begin{aligned}
\varphi(x_1, y_1, z_1) &= \left(\frac{\frac{2a}{a^2+b^2+1}}{1 - \frac{a^2+b^2-1}{a^2+b^2+1}}, \frac{\frac{2b}{a^2+b^2+1}}{1 - \frac{a^2+b^2-1}{a^2+b^2+1}} \right) \\
&= \left(\frac{\frac{2a}{a^2+b^2+1}}{\frac{2}{a^2+b^2+1}}, \frac{\frac{2b}{a^2+b^2+1}}{\frac{2}{a^2+b^2+1}} \right) \\
&= (a, b)
\end{aligned}$$

This gives us that φ is surjective, that is, $\varphi(U) = \mathbb{R}^2$. A similar set of calculations give us that $\psi(V) = \mathbb{R}^2$. It is clear that $U \cup V = S^2$ and in order to finish verifying that S^2 is a manifold with the given coordinate atlas, we must verify that φ and ψ are compatible on $U \cap V$ meaning that $\varphi \circ \psi^{-1} : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}^2$ and $\psi \circ \varphi^{-1} : \mathbb{R}^2 \setminus (0,0) \to \mathbb{R}^2$ are infinitely differentiable and are therefore diffeomorphisms of $\mathbb{R}^2 \setminus (0,0)$ to $\mathbb{R}^2 \setminus (0,0)$. Consider that

$$\begin{aligned}
\varphi \circ \psi^{-1}(x, y) &= \varphi \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{1-x^2-y^2}{x^2+y^2+1} \right) \\
&= \left(\frac{\frac{2x}{x^2+y^2+1}}{1 - \frac{1-x^2-y^2}{x^2+y^2+1}}, \frac{\frac{2y}{x^2+y^2+1}}{1 - \frac{1-x^2-y^2}{x^2+y^2+1}} \right) \\
&= \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)
\end{aligned}$$

which we can see is infinitely differentiable on $\mathbb{R}^2 \setminus (0,0)$. Similarly we can show that

$$\psi \circ \varphi^{-1}(x, y) = \left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \right)$$

which yields that $\psi \circ \varphi^{-1}$ is infinitely differentiable on $\mathbb{R}^2 \setminus (0,0)$ as well. We have thus verified that φ and ψ are compatible on $U \cap V$ giving us that S^2 is a manifold with an atlas determined by the given atlas by Theorem 4.6.

Example 4.16. Let M and N be manifolds of dimensions m and n, respectively and let $\{U_\alpha, \varphi_\alpha\}$ and $\{V_\beta, \psi_\beta\}$ be smooth atlases for M and N, respectively. It is a

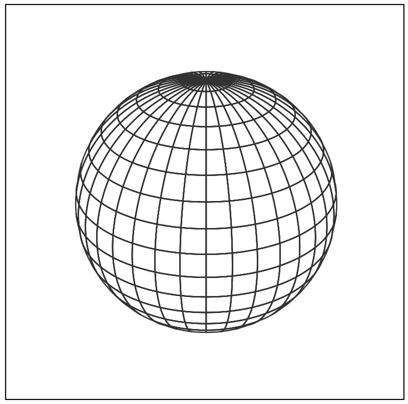


Figure 5: *Plot of* S^2

standard result from topology that the Cartesian product $M \times N$ is a topological manifold. Choose some $U, \varphi \in \{U_\alpha, \varphi_\alpha\}$ and $V, \psi \in \{V_\beta, \psi_\beta\}$. Define W, θ on $M \times N$ by $W = U \times V$ and $\theta : W \to \mathbb{R}^{m+n}$ by $\theta(p, q) = \varphi(p) \times \psi(q)$ for $(p, q) \in W$ where $\varphi(p) \times \psi(q) = \big(\varphi^1(p), \ldots, \varphi^m(p); \psi^1(q), \ldots, \psi^n(q)\big)$. Choose $U_1, \varphi_1; U_2, \varphi_2 \in \{U_\alpha, \varphi_\alpha\}$ and $V_1, \psi_1; V_2, \psi_2 \in \{V_\beta, \psi_\beta\}$ such that $U_1 \cap U_2 \neq \emptyset$ and $V_1 \cap V_2 \neq \emptyset$ let $W_1 = U_1 \times V_1$ and $W_2 = U_2 \times V_2$ and note that $W_1 \cap W_2 \neq \emptyset$. Let θ_1 and θ_2 be the corresponding local coordinate maps from W_1 and W_2 to $\mathbb{R}^{m+n}$ respectively defined by $\theta_1(p, q) = \varphi_1(p) \times \psi_1(q)$ and $\theta_2(p, q) = \varphi_2(p) \times \psi_2(q)$. Consider the map

$$\theta_1 \circ \theta_2^{-1} : \theta_2(W_1 \cap W_2) \subset \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}.$$

Choose some $(p, q) \in \theta_2(W_1 \cap W_2)$ and consider that

$$\begin{aligned} \theta_1 \circ \theta_2^{-1}(p, q) &= (\varphi_1 \times \psi_1) \circ (\varphi_2 \times \psi_2)^{-1}(p, q) \\ &= (\varphi_1 \times \psi_1) \circ (\varphi_2^{-1}(p) \times \psi_2^{-1}(q)) \\ &= (\varphi_1 \circ \varphi_2^{-1}(p)) \times (\psi_1 \circ \psi_2^{-1}(q)) \end{aligned}$$

which is infinitely differentiable since φ_1 is smoothly compatible with φ_2, and similarly ψ_1 with ψ_2, thus giving us that $\theta_1 \circ \theta_2^{-1}$ is a diffeomorphism. A similar calculation will show that $\theta_2 \circ \theta_1^{-1}$ is a diffeomorphism as well. Together, these give us that W_1, θ_1 and W_2, θ_2 are C^∞-compatible. Therefore, $M \times N$ is a manifold with coordinates atlas as defined above.

4.2 Vector Fields on Manifolds

When we have an abstract manifold M of dimension m, we may not have an ambient space that M is living in. We want to develop the ideas that allow us to talk about paths and tangent vectors on M without requiring that M be strictly embedded in $\mathbb{R}^n$.

Suppose Γ is a smooth curve on a manifold M of dimension m, parameterized by $\phi : I \to M$ where I is a subinterval of $\mathbb{R}$. In local coordinates $x = (x^1, \ldots, x^m)$, Γ is given by m smooth functions $\phi(\varepsilon) = (\phi^1(\varepsilon), \ldots, \phi^m(\varepsilon))$ of the real variable ε.

Definition 4.17. With the above considerations, at each point $x = \phi(\varepsilon)$ of Γ the curve has a tangent vector, namely the derivative $\dot{\phi}(\varepsilon) = \dfrac{d\phi}{d\varepsilon} = (\dot{\phi}^1(\varepsilon), \ldots, \dot{\phi}^m(\varepsilon))$ which has the notation of

$$\mathbf{v}|_x = \dot{\phi}(\varepsilon) = (\dot{\phi}^1(\varepsilon), \ldots, \dot{\phi}^m(\varepsilon)) = \dot{\phi}^1(\varepsilon)\frac{\partial}{\partial x^1} + \dot{\phi}^2(\varepsilon)\frac{\partial}{\partial x^2} + \ldots + \dot{\phi}^m(\varepsilon)\frac{\partial}{\partial x^m}.$$

This notation may seem strange but will prove itself extremely useful when we introduce Lie derivatives. This notation will also allow us to distinguish between the tangent vectors and the local coordinates of the manifold. Simply think of the differential operators as basis vectors, that is, when we're considering $\mathbb{R}^2$ as a manifold, any tangent vector can be written as a linear combination of the basis vectors $\left\{\dfrac{\partial}{\partial x}, \dfrac{\partial}{\partial y}\right\}$ just as any element in $\mathbb{R}^2$ can be written as a linear combination of the basis elements $\{(1,0),(0,1)\}$.

Example 4.18. Consider the helix as a 1-dimensional submanifold of $\mathbb{R}^3$. We can describe the helix by $\phi(\varepsilon) = (\cos\varepsilon, \sin\varepsilon, \varepsilon)$ in $\mathbb{R}^3$ with coordinates (x, y, z). The tangent vector $\dot{\phi}(\varepsilon)$ is given by

$$\begin{aligned}\dot{\phi}(\varepsilon) &= \frac{d}{d\varepsilon}\left(\phi^1(\varepsilon)\right)\frac{\partial}{\partial x} + \frac{d}{d\varepsilon}\left(\phi^2(\varepsilon)\right)\frac{\partial}{\partial y} + \frac{d}{d\varepsilon}\left(\phi^3(\varepsilon)\right)\frac{\partial}{\partial z} \\ &= -\sin\varepsilon\frac{\partial}{\partial x} + \cos\varepsilon\frac{\partial}{\partial y} + \frac{\partial}{\partial z} \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\end{aligned}$$

at the point $(x, y, z) = \phi(\varepsilon) = (\cos\varepsilon, \sin\varepsilon, \varepsilon)$.

Definition 4.19. The collection of all tangent vectors to all possible curves passing through a given point $x \in M$ is called the tangent space to M at x and is denoted by $TM|_x$.

If M is an m-dimensional manifold then $TM|_x$ is an m-dimensional vector space over $\mathbb{R}$ with $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^m}\right\}$ providing a basis for $TM|_x$ in the given local coordinates.

Definition 4.20. A vector field $\mathbf{v}$ on an m-dimensional manifold M assigns a tangent vector $\mathbf{v}|_x \in TM|_x$ to each point $x \in M$, with $\mathbf{v}|_x$ varying smoothly from point to point.

In local coordinates $(x^1, \ldots, x^m)$, a vector field has the form

$$\mathbf{v}|_x = \xi^1(x)\frac{\partial}{\partial x^1} + \xi^2(x)\frac{\partial}{\partial x^2} + \ldots + \xi^m(x)\frac{\partial}{\partial x^m},$$

where each $\xi^i(x)$ is a smooth function of x. There is a subtlety here that we will ignore and that is namely that each $\frac{\partial}{\partial x^i}$ in the above expression is an element of $TM|_x$ and therefore changes for each $x \in M$ but this distinction will be clear from context.

Definition 4.21. An integral curve of a vector field $\mathbf{v}$ is a smooth parameterized curve $x = \phi(\varepsilon)$ whose tangent vector at any point is equal to the value of $\mathbf{v}$ at that point. This means that $\dot{\phi}(\varepsilon) = \mathbf{v}|_{\phi(\varepsilon)}$ for all ε.

If we consider an integral curve $\phi(\varepsilon) = (\phi^1(\varepsilon), \ldots, \phi^m(\varepsilon))$ in local coordinates, we get that $\phi(\varepsilon)$ must be a solution to the autonomous system of ordinary differential equations given by

$$\frac{dx^i}{d\varepsilon} = \xi^i(x), \quad i = 1, \ldots, m,$$

where $\xi^i(x)$ are the coefficients of $\mathbf{v}$ at x. Since $\xi^i(x)$ are smooth functions then the standard existence and uniqueness theorems apply with initial condition given by $\phi(0) = x_0$. This implies that there is a unique maximal integral curve passing through any given point $x_0 \in M$, where maximal means that no other integral curve through x_0 properly contains this unique curve.

Definition 4.22. If $\mathbf{v}$ is a vector field on an m-dimensional manifold M, we denote the parameterized maximal integral curve passing through $x \in M$ by $\Psi(\varepsilon, x)$ and call Ψ the flow generated by $\mathbf{v}$.

From Definition 4.22 we see that for each $x \in M$ and interval I_x containing 0 with $\varepsilon \in I_x$, $\Psi(\varepsilon, x)$ will be a point on the integral curve passing through $x \in M$. The flow of a vector field has the basic properties that

$$\Psi(\delta, \Psi(\varepsilon, x)) = \Psi(\delta + \varepsilon, x), \quad x \in M$$

for all $\delta, \varepsilon \in \mathbb{R}$ such that both sides of the above equation are defined,

$$\Psi(0, x) = x,$$

and

$$\frac{d}{d\varepsilon}\Psi(\varepsilon, x) = \mathbf{v}|_{\Psi(\varepsilon,x)}$$

for all ε where defined.

5 Lie Symmetry Groups

The vast majority of the information in this section was taken from Applications of Lie Groups to Differential Equations by Peter J. Olver [12]. In this book he gives the groundwork which will be employed in this paper for our analysis of the Lie symmetries of some classes of harmonic functions. Another resource for general information about Lie symmetry groups is [14].

5.1 Abstract Groups

Definition 5.1. A group is a set G together with a group operation $\star$ such that for any two elements $g, h \in G$, the product $g \star h$ is again an element of G. The group operation is required to satisfy the following axioms:

(i) Associativity. If $g, h, k \in G$ then $g \star (h \star k) = (g \star h) \star k$.

(ii) Identity Element. There is a unique element $e \in G$ called the identity element, such that $e \star g = g \star e = g$ for all $g \in G$.

(iii) Inverses. For each $g \in G$, there exists an inverse in G, denoted g^{-1} such that $g \star g^{-1} = g^{-1} \star g = e$.

Often we denote the product of $g, h \in G$ by gh, $g \cdot h$ or $g + h$, depending on the group.

Definition 5.2. A group is said to be abelian if the group operation is commutative, meaning that $g \cdot h = h \cdot g$ for all $g, h \in G$.

5.2 Lie Symmetry Groups

Definition 5.3. An r-parameter Lie group is a group G with operation $\cdot$ which also carries the structure of an r-dimensional (smooth) manifold in such a way that both the group operation

$$m : G \times G \to G, \;\; m(g, h) = g \cdot h, \;\; g, h \in G$$

and the inversion

$$i : G \to G, \;\; i(g) = g^{-1}, \;\; g \in G$$

are smooth (infinitely differentiable) maps between manifolds.

Definition 5.4. Let G and H be groups with operations $*$ and $\star$, respectively. A function $\varphi : G \to H$ is said to be a group homomorphism if $\varphi(g_1 * g_2) = \varphi(g_1) \star \varphi(g_2)$ for all $g_1, g_2 \in G$. If φ is injective and surjective, then φ is called an isomorphism and we say that G and H are isomorphic if there exists an isomorphism between them. If G and H are isomorphic, we will notate this as $G \cong H$. For two Lie groups to be isomorphic, the isomorphism must be a smooth map between them.

Example 5.5. Consider the manifold $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x > 0\}$ and consider it as a group under multiplication. The multiplication map $m : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $m(x, y) = x \cdot y$ and the inversion map $i : \mathbb{R}^+ \to \mathbb{R}^+$ is defined by $i(x) = 1/x$, both maps are seen to be diffeomorphisms, the first one being a diffeomorphism from $\mathbb{R}^+ \times \mathbb{R}^+$ (a 2-manifold by Example 4.16) onto $\mathbb{R}^+$ and the second from $\mathbb{R}^+$ onto $\mathbb{R}^+$. Therefore, by definition $\mathbb{R}^+$ is a Lie Group. With the homomorphism $\varphi(t) = \ln t$ we can show that $\mathbb{R}^+$ is isomorphic to $\mathbb{R}$ as a group under $+$. Clearly $\mathbb{R}$ is a Lie group under $+$ and we can see that given $s, t \in \mathbb{R}^+$ we have that

$$\varphi(s \cdot t) = \ln(s \cdot t) = \ln s + \ln t = \varphi(s) + \varphi(t)$$

showing that φ is a homomorphism. It is easily verified that φ is surjective and injective and that it is a smooth map yielding that $\mathbb{R}^+$ and $\mathbb{R}$ are isomorphic as Lie groups.

Example 5.6. Consider the manifold $SO(2)$, the set of all 2×2 real orthogonal matrices of determinant 1. $SO(2)$ is given by the set

$$SO(2) = \left\{ \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \middle| \, \theta \in \mathbb{R} \right\}$$

under the operation of matrix multiplication. So given

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \in SO(2)$$

we have that

$$\begin{aligned} &\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \cdot \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \\ &\qquad = \begin{bmatrix} \cos\theta\cos\phi - \sin\theta\sin\phi & -\cos\theta\sin\phi - \sin\theta\cos\phi \\ \sin\theta\cos\phi + \cos\theta\sin\phi & \cos\theta\cos\phi - \sin\theta\sin\phi \end{bmatrix} \\ &\qquad = \begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} \end{aligned}$$

With this multiplication in mind, we see that there is a natural isomorphism from the groups $\{SO(2), \cdot\}$ onto $\{\mathbb{R}_{2\pi}, +\}$ where $\mathbb{R}_{2\pi} = \{\theta \pmod{2\pi} \mid \theta \in \mathbb{R}\}$. The group $\mathbb{R}_{2\pi}$ is simply $\mathbb{R}$ under addition with the modular arithmetic defined in such a way as to make $2\pi k = 0$ for all $k \in \mathbb{Z}$, for example, $\pi + 3\pi = 0$ in $\mathbb{R}_{2\pi}$. The group isomorphism is given by

$$\varphi\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right) = \theta.$$

Given $x = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}, y = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \in SO(2)$, from the previous calculation we can see that

$$\varphi(x \cdot y) = \varphi\left(\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix}\right) = \theta + \phi = \varphi(x) + \varphi(y)$$

showing that φ is indeed a homomorphism of these two groups. It can be verified that φ is well-defined, surjective and injective and is a smooth map between $SO(2)$ and $\mathbb{R}_{2\pi}$ being careful to consider equivalence classes as the elements of $\mathbb{R}_{2\pi}$ and not the individual elements of $\mathbb{R}$. Therefore φ is an isomorphism of Lie groups.

Fact 5.7. *There are only two connected 1-parameter Lie groups, namely* $\{SO(2), \cdot\} \cong \{\mathbb{R}_{2\pi}, +\}$ *and* $\{\mathbb{R}, +\} \cong \{\mathbb{R}^+, \cdot\}$.

We will make extensive use of the Lie group $\{\mathbb{R}, +\}$. When we consider Lie symmetry groups of certain differential equations, this is the Lie group we will use.

As with any group, there is a natural definition of a Lie subgroup.

Definition 5.8. A Lie subgroup H of a Lie group G is given by a submanifold $\varphi : \widetilde{H} \to G$ where $\widetilde{H}$ itself is a Lie group, $H = \varphi(\widetilde{H})$ is the image of φ and φ is called a Lie group homomorphism.

Theorem 5.9. *Suppose G is a Lie group. If H is a closed subgroup of G, then H is a regular submanifold of G and hence a Lie group in its own right. Conversely, any regular Lie subgroup of G is a closed subgroup.*

Many times the Lie group that we are considering may not globally define a group structure but in some neighborhood of the identity element we have a group structure. We call this a local Lie group and give the following definition.

Definition 5.10. An r-parameter local Lie group consists of connected open subsets $V_0 \subset V \subset \mathbb{R}^r$ containing the origin 0, and smooth maps

$$m : V \times V \to \mathbb{R}^r,$$

defining the group operation, and

$$i : V_0 \to V$$

defining the group inversion, with the following properties.

(i) Associativity. If $x, y, z \in V$ and also $m(x, y)$ and $m(y, z)$ are in V, then

$$m(x, m(y, z)) = m(m(x, y), z).$$

(ii) Identity Element. For all $x \in V$, $m(0, x) = x = m(x, 0)$.

(iii) Inverses. For each $x \in V_0$, $m(x, i(x)) = 0 = m(i(x), x)$.

We will continue to denote a local group operation by xy, $x \cdot y$ or $x + y$ for $m(x, y)$ and x^{-1} or $-x$ for $i(x)$.

Note that our definition of a local Lie group is only defined on open neighborhoods of the origin in $\mathbb{R}^r$ for an r-parameter local Lie group. For the abstract manifold which is a global Lie group, there is a natural way of forming a local Lie group if the space is not already $\mathbb{R}^r$. That is, take the coordinate chart of the identity and this will give a local Lie group that is a subset of $\mathbb{R}^r$ in local coordinates. It is a fact that every local Lie group arises from this construction meaning that every local Lie group is locally isomorphic to a neighborhood of the identity of some global Lie group.

Theorem 5.11. *Let $V_0 \subset V \subset \mathbb{R}^r$ be a local Lie group with multiplication $m(x, y)$ and inversion $i(x)$. Then there exists a global Lie group G and a coordinate chart $\chi : U^* \to V^*$, where U^* contains the identity element, such that $V^* \subset V_0$, $\chi(e) = 0$, and*

$$\chi(g \cdot h) = m(\chi(g), \chi(h))$$

whenever $g, h \in U^*$ and

$$\chi(g^{-1}) = i(\chi(g))$$

whenever $g \in U^*$. Also, there is a unique connected, simply-connected Lie group G^* having the above properties. If G is any other such Lie group, there exists a covering map $\pi : G^* \to G$ which is a local Lie group isomorphism.

Example 5.12. Let $V = \{x \in \mathbb{R} \mid |x| < 1\}$ with group multiplication defined by

$$m(x, y) = \frac{2xy - x - y}{xy - 1}, \quad x, y \in V$$

and inversion defined by

$$i(x) = \frac{x}{2x - 1}$$

which is only defined for $x \in V_0 = \{x \mid |x| < \frac{1}{2}\}$. Associativity and the identity element can be easily verified. Since $i(x)$ only holds in V_0 then this is a local Lie group and not a global one.

Example 5.13. By Theorem 5.11, we get that the local Lie group given in Example 5.12 must have come from a global Lie group. Since there is only one connected, simply-connected one-parameter Lie group, $\mathbb{R}$, then the local Lie group in Example 5.12 must coincide with some coordinate chart of 0 of $\mathbb{R}$. If we let $\chi : U^* \to V^* \subset \mathbb{R}$ with

$$\chi(t) = \frac{t}{t - 1}, \quad t \in U^* = \{t < 1\},$$

then we can see that

$$\chi(t+s) = m(\chi(t), \chi(s)) = \frac{2\chi(t)\chi(s) - \chi(t) - \chi(s)}{\chi(t)\chi(s) - 1}$$

and

$$\chi(-t) = i(\chi(t)) = \frac{\chi(t)}{2\chi(t) - 1}$$

where defined. Therefore χ is the coordinate chart of $\mathbb{R}$ that satisfies Theorem 5.11 and we can see that the above local Lie group did in fact arise from a global Lie group.

Theorem 5.14. *Let G be a connected Lie group and $U \subset G$ a neighborhood of the identity. Let $U^k = \{g_1 \cdot g_2 \cdot \ldots \cdot g_k \mid g_i \in U\}$. So U^k is the set of all possible products of k elements of U. Then*

$$G = \bigcup_{k=1}^{\infty} U^k.$$

Proof. We will show that $\bigcup_{k=1}^{\infty} U^k$ is both open and closed in G and since G is connected then this will imply that $\bigcup_{k=1}^{\infty} U^k = G$.

Choose $V^* \subset G$ open such that the identity element e of G is in V^*. Let $V = V^* \cap i(V^*)$. Note that $e \in V^* \Rightarrow e = e^{-1} \in i(V^*) \Rightarrow e \in V$ and therefore $V \neq \emptyset$. Since V^* is open and the inversion map is a diffeomorphism then $i(V^*)$ is open implying that V is open, being the intersection of two open sets. Note that $V \subset V^*$.

Now choose some $g \in V$. This implies that $g \in i(V^*)$ which then implies that $g^{-1} \in V^*$. Now since $g \in V^* \Rightarrow i(g) = g^{-1} \in i(V^*)$ and therefore $g^{-1} \in V^* \cap i(V^*) = V$. So we have shown that V is open containing e, nonempty and that every element in V has its inverse in V as well.

Let $V_g = \{g \cdot v \mid v \in V\}$. We can clearly see that $g \in V_g$ since $g = g \cdot e \in V_g$. Choose some $b \in V_g$ such that b is a boundary point of V_g. Since G is Hausdorff and b is a boundary point of V_g then we may form a sequence $\{b_n\} \to b$ such that $b \neq b_n$ and $b_n \notin V_g$ of all n. Since $b \in V_g$, there exists a $v \in V$ such that $b = g \cdot v \;\Rightarrow\; g^{-1} \cdot b = v$. By the continuity of the group operation we have that $\{g^{-1} \cdot b_n\} \to g^{-1} \cdot b = v \in V$ and since V is open, there exists some i such that $g^{-1} \cdot b_i \in V$. Thus $b_i \in V_g$ which contradicts our supposition. Therefore V_g does not contain any of its boundary points which implies that V_g is open. We have shown that if V^* is any open set containing $e \in G$ then V_g is also open in G.

It can be easily verified that $U^k = \{g_1 \cdot g_2 \cdot \ldots \cdot g_k \mid g_i \in U\} = \bigcup_{g_i \in U} g_i U^{k-1}$. Since $U^1 = U$ and U is open then a simple inductive argument gives that U^k is open for any k. Therefore $\bigcup_{k=1}^{\infty} U^k$ is open in G since each U^k is.

Now let U be an open set containing e and let $\mathscr{U} = \bigcup_{k=1}^{\infty} U^k$ and choose some element in $p \in \overline{\mathscr{U}}$. From what has been shown previously we see that since U is open and contains e then U_p is an open set containing p and since $p \in \overline{\mathscr{U}}$ then $U_p \cap \mathscr{U} \neq \emptyset$ which implies that there is a $p_0 \in U_p \cap \mathscr{U}$. Since $p_0 \in \mathscr{U}$ then $p_0 = g_1 \cdot g_2 \cdot \ldots \cdot g_n \in U^n$ for some n and $g_i \in U$ for all $i = 1 \ldots n$. Since $g_1 \cdot g_2 \cdot \ldots \cdot g_n \in U_p$ then $g_1 \cdot g_2 \cdot \ldots \cdot g_n = p \cdot u$ for some $u \in U$ which implies that $p^{-1} \cdot g_1 \cdot g_2 \cdot \ldots \cdot g_n = u \in U \;\Rightarrow\; p = g_1 \cdot g_2 \cdot \ldots \cdot g_n \cdot (p^{-1} \cdot g_1 \cdot g_2 \cdot \ldots \cdot g_n)^{-1} \in U^{n+1} \subset \mathscr{U}$ and therefore $p \in \mathscr{U}$ which implies that $\mathscr{U} = \overline{\mathscr{U}}$. Therefore we have shown that $\mathscr{U} = \bigcup_{k=1}^{\infty} U^k$ is closed. Now since $\bigcup_{k=1}^{\infty} U^k$ is both open and closed in the connected set G then $G = \bigcup_{k=1}^{\infty} U^k$. □

Theorem 5.14 gives us that any open set U of a Lie group G containing the identity element can generate a global Lie group because any element in G is simply a finite product of elements in U. This in turn allows us, in conjunction with Theorem 5.11, to generate a global Lie group from a local one simply by taking all possible words from some open set containing the identity element.

When working with Lie groups, local or global, in general we are not mainly concerned with the group in and of itself as a self-contained entity but more importantly on how the Lie group acts on another manifold. This is the main idea of the infinitesimal analysis of a Lie group. We need a few definitions to consider these examples. Ultimately it is these group actions that will allow us to analyze the symmetries of sets of differential equations.

Definition 5.15. Let M be a smooth manifold. A local group of transformations acting on M is given by a (local) Lie group G with identity e, an open set $\mathscr{U}$, with

$$\{e\} \times M \subset \mathscr{U} \subset G \times M$$

which is the domain of definition of the group action, and a smooth map $\Psi : \mathscr{U} \to M$ with the following properties:

(i) If $(h, x) \in \mathscr{U}$ and $(g, \Psi(h, x)) \in \mathscr{U}$ and $(g \cdot h, x) \in \mathscr{U}$, then

$$\Psi(g, \Psi(h, x)) = \Psi(g \cdot h, x).$$

(ii) For all $x \in M$,

$$\Psi(e, x) = x.$$

(iii) If $(g, x) \in \mathscr{U}$ then $(g^{-1}, \Psi(g, x)) \in \mathscr{U}$ and

$$\Psi(g^{-1}, \Psi(g, x)) = x.$$

Generally we will notate $\Psi(g, x)$ as $g \cdot x$. It will be clear from context that we are considering a group action and not the group operation of the Lie group. From Definition 5.15, it can be shown that each individual group transformation is a diffeomorphism, where defined, from the M component of $\mathscr{U}$ into M. That is to say that for any fixed $g \in G$, $\psi_g(x) = \Psi(g, x)$ is a diffeomorphism of $M^* \subset M$ into M where $M^* = \{x \in M \mid (g, x) \in \mathscr{U}\}$. Also, if $\mathscr{U} = G \times M$ then we call G a global group of transformations.

We can form local Lie groups for each $x \in M$ by looking at the local Lie group

$$G_x \equiv \{g \in G \mid (g, x) \in \mathscr{U}\}.$$

It can be verified that G_x is in fact a local Lie group.

Definition 5.16. We say a local group of transformations G acting on M is called connected if:

(i) G is a connected Lie group and M is a connected manifold,

(ii) the domain of definition $\mathscr{U} \subset G \times M$ is a connected open set, and

(iii) for each $x \in M$, the local group G_x is connected.

From this point on, we will only consider connected groups of transformations.

Definition 5.17. A set $\mathscr{O} \subset M$ is an orbit of a local transformation group G provided that

(i) if $x \in \mathscr{O}$, $g \in G$ and $g \cdot x$ is defined, then $g \cdot x \in \mathscr{O}$.

(ii) if $\widetilde{\mathscr{O}} \subset \mathscr{O}$ and $\widetilde{\mathscr{O}}$ satisfies part (i) then either $\widetilde{\mathscr{O}} = \mathscr{O}$ or $\widetilde{\mathscr{O}} = \emptyset$.

It is true that the orbits of a Lie group of transformations are submanifolds of M, but they may have differing dimensions and may not necessarily be regular. Also, if there is only one orbit then the group action is said to be transitive and that single orbit would then be M, meaning that we could flow from any one element on M to any other element.

Definition 5.18. Let G be a local Lie group of transformations acting on a manifold M and let $S \subset M$. The set S is said to be G-invariant if given any $g \in G$ and $x \in S$ such that $g \cdot x$ is defined then $g \cdot x \in S$.

It is clear that the orbit of any element $x \in M$ under the action of a group G is a G-invariant set and any union of orbits will also be a G-invariant set. It can also be verified that any G-invariant set is the union of a collection of orbits.

From the definition of a flow, Definition 4.22, we can see that the flow, $\Psi(\varepsilon, x)$, generated by a vector field, $\mathbf{v}$, on M is the same as a local group action of the Lie group $\mathbb{R}$ on the manifold M. The vector field $\mathbf{v}$ is called the infinitesimal generator of the action. The orbits of the group action are the maximal integral curves of $\mathbf{v}$.

If $\Psi(\varepsilon, x)$ is any one-parameter Lie group of transformations acting on an m-dimensional manifold M, then the infinitesimal generator of the action is obtained by

$$\mathbf{v}|_x = \sum_{i=1}^{m} \left(\frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} \Psi^i(\varepsilon, x) \right) \frac{\partial}{\partial x^i}, \tag{1}$$

where $\Psi(\varepsilon, x) = (\Psi^1(\varepsilon, x), \ldots, \Psi^m(\varepsilon, x))$ is the image of $x \in M$ under the action of $\varepsilon \in G$. Since it will be clear that we are referring to the infinitesimal generator at a point $x \in M$, we will usually denote the infinitesimal generator by $\mathbf{v}$ as opposed to $\mathbf{v}|_x$.

The process of finding the one-parameter group of transformations from the infinitesimal generator $\mathbf{v}$ is called exponentiating. If we make the definition that

$$\exp(\varepsilon \mathbf{v})x \equiv \Psi(\varepsilon, x)$$

for $\varepsilon \in G$ then $\exp(\varepsilon \mathbf{v})x$ is the flow of x under the Lie symmetry group. The exponentiation is well defined since $\Psi_{\mathbf{v}}(\varepsilon, x) = \Psi_{t\mathbf{v}}\left(\frac{\varepsilon}{t}, x\right)$ for any $t \neq 0$. Using this notation we have the properties that

$$\exp[(\delta+\varepsilon)\mathbf{v}]x = \exp(\delta\mathbf{v})\exp(\varepsilon\mathbf{v})x$$

whenever defined,

$$\exp(0\mathbf{v})x = x, \tag{2}$$

and

$$\frac{d}{d\varepsilon}[\exp(\varepsilon\mathbf{v})x] = \mathbf{v}|_{\exp(\varepsilon\mathbf{v})x} \tag{3}$$

for all $x \in M$. We can see the parallels between the exponentiation of the infinitesimal generator and the properties of the exponential function, hence the name. One common notation we will adopt for the flow of a particular element x will be $\widetilde{x}$, that is, $\Psi(\varepsilon, x) = \exp(\varepsilon\mathbf{v})x = \widetilde{x}$.

Equations (2) and (3) lead us to the following technique for exponentiating an infinitesimal generator $\mathbf{v}$. Let M be an m-dimensional manifold with a one-parameter Lie group G acting on M. Let the infinitesimal generator of $\varepsilon \in G$ be $\mathbf{v} = \sum_{i=1}^{m} \xi^i(x^1, \ldots, x^m)\frac{\partial}{\partial x^i}$. In order to exponentiate $\mathbf{v}$, we must solve the system of differential equations

$$\begin{aligned}
\frac{d\widetilde{x}^1}{d\varepsilon} &= \xi^1(\widetilde{x}^1, \ldots, \widetilde{x}^m), \\
\frac{d\widetilde{x}^2}{d\varepsilon} &= \xi^2(\widetilde{x}^1, \ldots, \widetilde{x}^m), \\
&\vdots \\
\frac{d\widetilde{x}^m}{d\varepsilon} &= \xi^m(\widetilde{x}^1, \ldots, \widetilde{x}^m)
\end{aligned}$$

subject to the initial conditions $\widetilde{x}^1|_{\varepsilon=0} = x^1, \ldots, \widetilde{x}^m|_{\varepsilon=0} = x^m$ which we will often notate simply as $\widetilde{x}^i|_0 = x^i$. The solution for $\widetilde{x}^i$ will be the flow from x^i under the group action and we will call this the exponentiation of x.

Example 5.19. Let's consider a few examples of vector fields and associated flows found through exponentiation.

(a) Let $M = \mathbb{R}$ with coordinate given by x. Let $\mathbf{v} = \dfrac{\partial}{\partial x}$. In order to exponentiate this we must satisfy the differential equation

$$\frac{d}{d\varepsilon}(\widetilde{x}) = 1$$

subject to the initial condition $\widetilde{x}|_0 = x$. This has solution $\widetilde{x} = x + \varepsilon$ giving us the exponentiation of

$$\widetilde{x} = \exp\left(\varepsilon \frac{\partial}{\partial x}\right) x = x + \varepsilon$$

which is globally defined. If we were to define $\mathbf{v}$ as $\mathbf{v} = x\dfrac{\partial}{\partial x}$ then we get that $\widetilde{x}$ must satisfy $\dfrac{d}{d\varepsilon}(\widetilde{x}) = \widetilde{x}$ subject to $\widetilde{x}|_0 = x$ which yields

$$\widetilde{x} = \exp\left(\varepsilon x \frac{\partial}{\partial x}\right) x = e^{\varepsilon} x.$$

(b) Let $M = \mathbb{R}^2$ with coordinates given by (x, y). Let $\mathbf{v} = -y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$. If we let $(\widetilde{x}, \widetilde{y}) = \exp\left[\varepsilon\left(-y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}\right)\right](x, y)$ then in order to exponentiate $\mathbf{v}$ we must solve the system of differential equations

$$\frac{d}{d\varepsilon}(\widetilde{x}) = -\widetilde{y}$$
$$\frac{d}{d\varepsilon}(\widetilde{y}) = \widetilde{x}$$

subject to the initial conditions $\widetilde{x}|_0 = x$ and $\widetilde{y}|_0 = y$. The solution to this system is

$$\widetilde{x} = x\cos\varepsilon - y\sin\varepsilon$$

$$\widetilde{y} = y\cos\varepsilon + x\sin\varepsilon$$

which gives that $\Psi(\varepsilon, (x, y)) = (x\cos\varepsilon - y\sin\varepsilon, y\cos\varepsilon + x\sin\varepsilon)$.

(c) Let $M = \mathbb{R}^2$ with coordinates given by (x, y). Let $\mathbf{v} = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}$. Again let $\widetilde{x}$ and $\widetilde{y}$ be the transformed x and y coordinates respectively. Now we must satisfy the system

$$\frac{d}{d\varepsilon}(\widetilde{x}) = \widetilde{x}^2$$
$$\frac{d}{d\varepsilon}(\widetilde{y}) = \widetilde{x} \cdot \widetilde{y}$$

subject to the initial conditions $\widetilde{x}|_0 = x$ and $\widetilde{x}|_0 = y$. This yields a solution of

$$\widetilde{x} = \frac{x}{1 - \varepsilon x}$$
$$\widetilde{y} = \frac{y}{1 - \varepsilon x}$$

which gives that $\Psi(\varepsilon, (x, y)) = \left(\frac{x}{1 - \varepsilon x}, \frac{y}{1 - \varepsilon x} \right)$.

In trying to solve the systems of differential equations generated by exponentiating a given infinitesimal generator, often we will employ a computer mathematics software package such as Maple. In the above examples, they generated simple enough systems of differential equations to be solved by hand but in the calculation of many Lie symmetries we will have systems of more than fifty or so equations necessitating the usage of a computer generated solution.

Example 5.20. Here are a few examples of group transformations with their associated infinitesimal generators.

(a) Let $G = SO(2)$ and $M = S^2$. As shown in Example 4.15, we see that M is a manifold where $(x, y, z) \in M$ if $x^2 + y^2 + z^2 = 1$. Choose some

$$\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \in SO(2).$$

Define the group action by

$$\Psi(\theta,(x,y,z)) = (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta, z).$$

Notate the action of an element $\theta \in G$ on an element $x \in M$ by $(\widetilde{x},\widetilde{y},\widetilde{z})$ or by $\theta \cdot (x,y,z)$. Consider that

$$\begin{aligned}
&(x\cos\theta - y\sin\theta)^2 + (y\cos\theta + x\sin\theta)^2 + z^2\\
&= x^2\cos^2\theta - 2xy\cos\theta\sin\theta + y^2\sin^2\theta\\
&\quad + y^2\cos^2\theta + 2xy\cos\theta\sin\theta + x^2\sin^2\theta\\
&\quad + z^2\\
&= x^2(\cos^2\theta + \sin^2\theta) + y^2(\cos^2\theta + \sin^2\theta) + z^2\\
&= x^2 + y^2 + z^2\\
&= 1
\end{aligned}$$

which therefore shows that $\theta \cdot (x,y,z) \in SO(2)$. We can verify that the other axioms of Definition 5.15 hold for this group action definition as well. In this case, $SO(2)$ is a global group action on M. If we fix some height $\alpha \in [-1,1]$ and form the set $S_\alpha = \{(x,y,\alpha) \mid x^2+y^2+\alpha^2 = 1\} \in M$, we can check that S_α is an orbit of the group action and in fact the collection of sets $\mathscr{S} = \{S_\beta \mid \beta \in [-1,1]\}$ forms the collection of all orbits of this group action. Since S_α is an orbit, it is simply the flow of an element $(x,y,\alpha) \in M$ under the group G or in other words, S_α is an integral curve of the group action. Since $\mathscr{S}$ has more than one element then we can see that the group action is not transitive. Six of the orbits are shown in Figure 6. We can see that each orbit S_α is a G-invariant set and any union of S_α's also forms a G-invariant set. For example, the union of the six orbits shown in Figure 6 forms a G-invariant set. We can calculate the infinitesimal generator by taking the derivatives of each of the coordinate functions as given by (1) on page 32 yielding

$$\begin{aligned}\mathbf{v} &= \left(\frac{d}{d\theta}\bigg|_{\theta=0} (x\cos\theta - y\sin\theta) \right) \frac{\partial}{\partial x} + \left(\frac{d}{d\theta}\bigg|_{\theta=0} (y\cos\theta + x\sin\theta) \right) \frac{\partial}{\partial y} \\ &\quad + \left(\frac{d}{d\theta}\bigg|_{\theta=0} (z) \right) \frac{\partial}{\partial z} \\ &= (-x\sin 0 - y\cos 0)\frac{\partial}{\partial x} + (-y\sin 0 + x\cos 0)\frac{\partial}{\partial y} \\ &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}.\end{aligned}$$

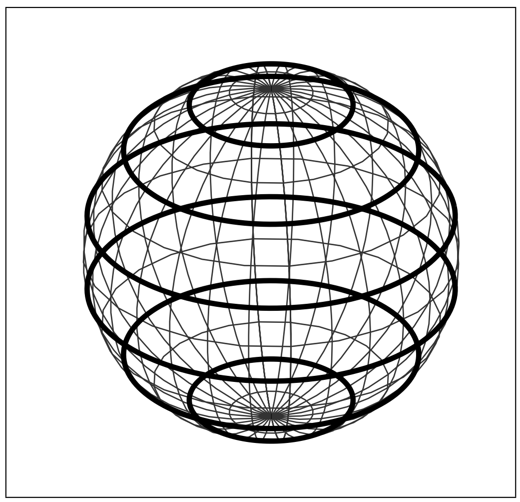


Figure 6: *Plot of S^2 with the orbits of the group action*

As shown in Example 5.19(b), if we were to exponentiate $\mathbf{v} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, we get that $\widetilde{x} = x\cos\theta - y\sin\theta$ and $\widetilde{y} = y\cos\theta + x\sin\theta$ and $\widetilde{z} = z$ arising from $\frac{d\widetilde{z}}{d\theta} = 0, \widetilde{z}|_0 = z$. This gives us the group action that we started with $\theta \cdot (x, y, z) = (x\cos\theta - y\sin\theta, y\cos\theta + x\sin\theta, z)$.

(b) Let $G = \mathbb{R}$ and $M = \mathbb{R}^2$. Given $g \in G$, define the action of g on an element (x, y) by $g \cdot (x, y) = (x + g, y - g)$. We can verify that this defines a group action of G on M. The orbit of any element $(x_0, y_0) \in \mathbb{R}^2$ is the line with slope

-1 passing through the point (x_0, y_0) where five orbits are shown in Figure 7. Also each of these orbits is an integral curve or flow of the group action on the element $(x_0, y_0) \in M$. Each of these orbits forms a G-invariant set and so does any union thereof. For example, the union of the five orbits shown in Figure 7 forms a G-invariant set. Again, we can calculate the infinitesimal generator of this group action.

$$\begin{aligned}\mathbf{v} &= \left(\frac{d}{dg}\bigg|_{g=0} (x+g) \right) \frac{\partial}{\partial x} + \left(\frac{d}{dg}\bigg|_{g=0} (y-g) \right) \frac{\partial}{\partial y} \\ &= \frac{\partial}{\partial x} - \frac{\partial}{\partial y}\end{aligned}$$

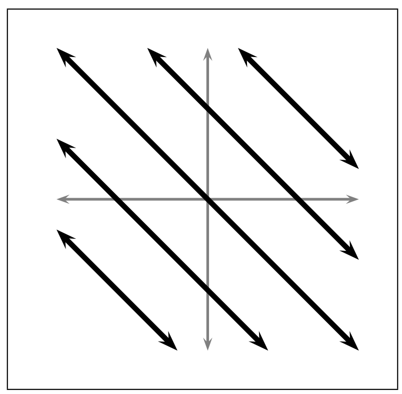


Figure 7: *Plot of* $\mathbb{R}^2$ *with the orbits of the group action*

5.3 Lie Algebras

The concept of G-invariance is an essential idea for finding the symmetries of differential equations. Once we are able to consider the set of all solutions to a system of differential equations as a manifold in a particular Euclidean space, we will calculate the group action that leaves this manifold invariant and thus form a solution set from any given solution. Before beginning our infinitesimal analysis, let's establish a few definitions necessary to form a vector space of infinitesimal generators for a particular group, being the Lie algebra relative to the Lie group being studied.

Let $F : M \to N$ be a smooth map between manifolds. The map F induces a map from TM to TN, the tangent bundles of M and N, respectively. This map is called the differential of F, denoted dF. We can think of dF as a function $dF : TM|_x \to TN|_{F(x)}$ given by

$$dF(\mathbf{v}|_x) = \sum_{j=1}^{n} \mathbf{v}(F^j(x)) \frac{\partial}{\partial y^j}$$

where $F(x) = (y^1, \ldots, y^n)$ in local coordinates. We can also define dF by

$$dF(\mathbf{v}|_x)f(y) = \mathbf{v}(f \circ F)(x), \quad y = F(x)$$

for all $\mathbf{v}|_x \in TM|_x$ and all smooth $f : N \to \mathbb{R}$.

Definition 5.21. Let $\mathbf{v}$ and $\mathbf{w}$ be vector fields on a manifold M. Then their Lie bracket $[\mathbf{v}, \mathbf{w}]$ is the unique vector field satisfying

$$[\mathbf{v}, \mathbf{w}](f) = \mathbf{v}(\mathbf{w}(f)) - \mathbf{w}(\mathbf{v}(f))$$

for all smooth $f : M \to \mathbb{R}$. If

$$\mathbf{v} = \sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i}, \quad \mathbf{w} = \sum_{i=1}^{m} \eta^i(x) \frac{\partial}{\partial x^i},$$

then

$$[\mathbf{v}, \mathbf{w}] = \sum_{i=1}^{m} \{\mathbf{v}(\eta^i) - \mathbf{w}(\xi^i)\} \frac{\partial}{\partial x^i}.$$

Definition 5.22. Let G be a Lie group. For any group element $g \in G$, the right multiplication map

$$R_g : G \to G$$

defined by

$$R_g(h) = h \cdot g$$

is a diffeomorphism, with inverse given by

$$R_{g^{-1}} = (R_g)^{-1}.$$

A vector field $\mathbf{v}$ on G is called right-invariant if

$$dR_g(\mathbf{v}|_h) = \mathbf{v}|_{R_g(h)} = \mathbf{v}|_{hg}$$

for all $g, h \in G$. The set of all right-invariant vector fields forms a vector space over $\mathbb{R}$.

Definition 5.23. A Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear operation

$$[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

called the Lie bracket for $\mathfrak{g}$, satisfying the axioms,

(i) Bilinearity

$$[c\mathbf{v} + c'\mathbf{v}', \mathbf{w}] = c[\mathbf{v}, \mathbf{w}] + c'[\mathbf{v}', \mathbf{w}]$$

and

$$[\mathbf{v}, c\mathbf{w} + c'\mathbf{w}] = c[\mathbf{v}, \mathbf{w}] + c'[\mathbf{v}, \mathbf{w}']$$

(ii) Skew-Symmetry

$$[\mathbf{v}, \mathbf{w}] = -[\mathbf{w}, \mathbf{v}],$$

(iii) Jacobi Identity

$$[\mathbf{u}, [\mathbf{v}, \mathbf{w}]] + [\mathbf{w}, [\mathbf{u}, \mathbf{v}]] + [\mathbf{v}, [\mathbf{w}, \mathbf{u}]] = 0,$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{v}', \mathbf{w}, \mathbf{w}' \in \mathfrak{g}$ and $c, c' \in \mathbb{R}$.

With this vocabulary, when we calculate the infinitesimal generators of the Lie symmetries of a system of differential equations, we can look at the Lie algebra spanned by those generators. Any vector field in the span will also be the infinitesimal generator of a Lie symmetry.

5.4 Infinitesimal Analysis

We want to consider the effect a flow $\exp(\varepsilon \mathbf{v})$ has on a function defined on the manifold M of the Lie group action. In a broad sense, we want to know how a function will change under a flow, that is, what is the function of the flow of a point $x \in M$ instead of simply what is the function of $x \in M$. In order to do this, we need to define the Lie derivative of a function defined on M.

Let $f : M \to \mathbb{R}$ and let $\mathbf{v}$ have coordinates functions given by $\mathbf{v} = \displaystyle\sum_{i=1}^{m} \xi^i(x) \frac{\partial}{\partial x^i}$. We define the Lie derivative as follows.

Definition 5.24. The Lie derivative of a function f under an infinitesimal generator $\mathbf{v}$, denoted $\mathbf{v}(f)$, is a function from M into $\mathbb{R}$ given by

$$\mathbf{v}(f)(x) = \sum_{i=1}^{m} \xi^i(x) \frac{\partial f}{\partial x^i}(x),$$

where the $\xi^i(x)$ are the local coordinate functions of $\mathbf{v}$ and $\dfrac{\partial f}{\partial x^i}$ is the standard partial derivative of f with respect to x^i.

From the chain rule, we can see that

$$\frac{d}{d\varepsilon} f(\exp(\varepsilon \mathbf{v})x) = \sum_{i=1}^{m} \xi^i(\exp(\varepsilon \mathbf{v})x) \frac{\partial f}{\partial x^i}(\exp(\varepsilon \mathbf{v})x) = \mathbf{v}(f)[\exp(\varepsilon \mathbf{v})x]$$

which gives us that

$$\left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} f(\exp(\varepsilon \mathbf{v})x) = \mathbf{v}(f)(x).$$

This justifies the notation for the basis vectors of the tangent space. In essence, we can think of the Lie derivative as simply a directional derivative. To illustrate this, let's consider an integral curve Γ relative to some infinitesimal generator $\mathbf{v}$. If we apply a function $f : M \to \mathbb{R}$ to every point $x \in \Gamma$ and ask how f changes as we flow along Γ infinitesimally, the answer would be the derivative of f in the direction of Γ. This directional derivative is the Lie derivative.

In order to calculate the symmetry group of a system of differential equations, we need to interpret the system geometrically as a manifold embedded canonically in a nice Euclidean space. To do this, begin with a system $\mathscr{S}$ of differential equations involving p independent variables $x = (x^1, \ldots, x^p)$ and q dependent variables $u(x) = (u^1(x), \ldots, u^q(x)) = (u^1(x^1, \ldots, x^p), \ldots, u^q(x^1, \ldots, x^p))$. A solution to $\mathscr{S}$ will have the form $u = f(x)$. Now let $X = \mathbb{R}^p$ with coordinates $(x^1, \ldots, x^p)$, represent the space of independent variables and let $U = \mathbb{R}^q$ with coordinates $u = (u^1, \ldots, u^q)$, represent the space of dependent variables. Let $M \subset X \times U$ be some open subset of the cartesian product of X and U.

In order to understand how we transform functions under a local Lie group of transformations, we must understand how a group transforms the graph of a function. Let g be an element of a Lie group G of transformations of the space $X \times U$ and let $u = f(x)$. We will identify the function $u = f(x)$ with its graph

$$\Gamma_f = \{(x, f(x)) \bigm| x \in \Omega\} \subset X \times U,$$

where $\Omega \subset X$ is the domain of definition of f. Note that Γ_f is a p-dimensional submanifold of $X \times U$. If Γ_f is a subset of the domain of definition of the group transformation g, then g will transform $f(x)$ by transforming the graph Γ_f. This gives us that

$$g \cdot \Gamma_f = \{(\widetilde{x}, \widetilde{u}) = g \cdot (x, u) \bigm| (x, u) \in \Gamma_f\}.$$

Note that the set $g \cdot \Gamma_f$ is not necessarily the graph of another function $\widetilde{u} = \widetilde{f}(\widetilde{x})$ for all g. Since G acts smoothly on Γ_f and the identity fixes Γ_f we may shrink the domain of definition of the group transformation in such a way as to ensure that $g \cdot \Gamma_f$ is the graph of a function. That is $g \cdot \Gamma_f = \Gamma_{\widetilde{f}}$ and we call the function $\widetilde{f}$ the transform of f by g.

In order to explicitly find $\widetilde{f}$, we will need to eliminate the variable x in the expression of $\widetilde{f}$ and solve in terms of $\widetilde{x}$. We will deal strictly with projectable transformations, meaning that the transformation can be given by

$$g \cdot (x, u) = (\Xi_g(x), \Phi_g(x, u)),$$

where Ξ_g and Φ_g are both smooth functions. Here we get that $\widetilde{x} = \Xi_g(x)$ implying that $x = \Xi_g^{-1}(\widetilde{x}) = \Xi_{g^{-1}}(\widetilde{x})$. We get the inverse transformation simply from the inverse element g^{-1} of g because of the properties given in Definition 5.22. This allows us to solve $\widetilde{f}$ as a function of x. We can see that $\widetilde{f}(\widetilde{x}) = \Phi_g(x, u) = \Phi_g(\Xi_{g^{-1}}(\widetilde{x}), u)$. Often we will adopt the notation of $\widetilde{f}(x)$ instead of the technically correct $\widetilde{f}(\widetilde{x})$. This is done for simplicity in notation and it will be clear from context what is meant.

Example 5.25. Consider the one-parameter group of transformations where

$$g_\varepsilon : (x, t, u) \mapsto (x + 2\varepsilon t, t, e^{-\varepsilon x - \varepsilon^2 t} u), \quad \varepsilon \in \mathbb{R}.$$

This example arises as a symmetry of the heat equation. If $u = f(x, t)$ is a solution then its transform by ε is given by

$$\widetilde{u} = e^{-\varepsilon x - \varepsilon^2 t} u = e^{-\varepsilon x - \varepsilon^2 t} f(x, t)$$

which must be solved in terms of $\widetilde{x}$ and $\widetilde{t}$. Solving for $\widetilde{x}$ and $\widetilde{t}$ simply involves taking $-\varepsilon$ as the variable of the transform. This yields the transformed function

$$\widetilde{u} = e^{-\varepsilon(\widetilde{x} - 2\varepsilon\widetilde{t}) - \varepsilon^2 \widetilde{t}} f\left(\widetilde{x} - 2\varepsilon\widetilde{t}, \widetilde{t}\right) = e^{-\varepsilon\widetilde{x} + \varepsilon^2 \widetilde{t}} f\left(\widetilde{x} - 2\varepsilon\widetilde{t}, \widetilde{t}\right)$$

which can be simply written as $e^{-\varepsilon x + \varepsilon^2 t} f(x - 2\varepsilon t, t)$.

This leads us to the most important definition of this paper.

Definition 5.26. Let $\mathscr{S}$ be a system of differential equations. A symmetry group of the system $\mathscr{S}$ is a local group of transformations G acting on an open subset $M \subset X \times U$ of the space of independent and dependent variables for the system with the property that whenever $u = f(x)$ is a solution of $\mathscr{S}$, and whenever $g \cdot f$ is defined for $g \in G$, then $u = g \cdot f(x)$ is also a solution of the system.

In order to actually calculate the symmetries for a system of differential equations, we must extend or prolong the space of independent and dependent variables to a space called the jet-space, consisting of the independent variables, the dependent variables and all possible partial derivatives of each dependent variable with respect to each independent variable up to the highest order given in the system $\mathscr{S}$.

Let X and U be the Euclidean spaces of p independent and q dependent variables respectively as defined above recalling that $X \cong \mathbb{R}^p$ and $U \cong \mathbb{R}^q$. We need a way of representing all possible partial derivatives of u up to a certain order, say k. Make the following definitions by considering their coordinates, where $u^i_{x_j}$ is the standard partial derivative of the coordinate function u^i with respect to x_j.

$$\begin{aligned}
U_1 &= \{u^{(1)} = (u^1_{x_1}, \dots, u^q_{x_1}; u^1_{x_2}, \dots, u^q_{x_2}; \dots; u^1_{x_p}, \dots, u^q_{x_p})\} \\
U_2 &= \{u^{(2)} = (u^1_{x_1x_1}, \dots, u^q_{x_1x_1}; u^1_{x_1x_2}, \dots, u^q_{x_1x_2}; \dots; u^1_{x_px_p}, \dots, u^q_{x_px_p})\} \\
&\vdots \\
U_k &= \{u^{(k)} = (u^1_{x_1\dots x_1}, \dots, u^q_{x_1\dots x_1}; \dots; u^1_{x_p\dots x_p}, \dots, u^q_{x_p\dots x_p})\}
\end{aligned} \tag{4}$$

Essentially the coordinates of U_i are all possible i-th order partial derivatives of u. With these in consideration we define $U^{(k)}$ to be

$$U^{(k)} = U \times U_1 \times U_2 \times \dots \times U_k,$$

the Cartesian product of the U_i as defined by (4). Note that $U^{(k)}$ is a Euclidean space of dimension

$$q \begin{pmatrix} p+n \\ n \end{pmatrix} \equiv qp^{(n)}.$$

For example, if the independent variable were (x, y) and the dependent variable were (u, v) then we would have:

$$u^{(1)} = (u_x, u_y; v_x, v_y)$$

$$u^{(2)} = (u_{xx}, u_{xy}, u_{yy}; v_{xx}, v_{xy}, v_{yy})$$

$$u^{(3)} = (u_{xxx}, u_{xxy}, u_{xyy}, u_{yyy}; v_{xxx}, v_{xxy}, v_{xyy}, v_{yyy})$$

Since each of the partial derivatives is continuous then we have that $u_{xy} = u_{yx}$, $v_{xy} = v_{yx}, \ldots$ and we need only consider one of these possibilities in the prolongation.

With the sets U_i as defined by (4), we can make the following definition.

Definition 5.27. The n-th order jet space of the underlying space $X \times U$ is the space $X \times U^{(n)}$ whose coordinates represent the space of independent variables, dependent variables and all possible partial derivatives of the dependent variables of orders up to n.

If $M \subset X \times U$ then we can define the n-jet space of M by

$$M^{(n)} \equiv M \times U_1 \times U_2 \times \ldots \times U_n.$$

Now that we've defined the prolongation of a particular space into its corresponding n-th order jet space, we need to define the prolongation of a function $u = f(x)$ where $f : X \to U$. The function f will induce a function, $u^{(n)} = \mathsf{pr}^{(n)} f(x)$, called the n-th prolongation of f, from X to the space $U^{(n)}$ defined by all the partial derivatives of f at the point x.

For example, if we have two independent variables (x, y) and two dependent variables (u, v) as defined in the previous example, then $\mathsf{pr}^{(2)} f(x, y)$ would be defined by

$$(u, v; u_x, u_y, v_x, v_y; u_{xx}, u_{xy}, u_{yy}, v_{xx}, v_{xy}, v_{yy}) =$$
$$\left(f^1, f^2; \frac{\partial f^1}{\partial x}, \frac{\partial f^1}{\partial y}, \frac{\partial f^2}{\partial x}, \frac{\partial f^2}{\partial y}; \frac{\partial^2 f^1}{\partial x^2}, \frac{\partial^2 f^1}{\partial x \partial y}, \frac{\partial^2 f^1}{\partial y^2}, \frac{\partial^2 f^2}{\partial x^2}, \frac{\partial^2 f^2}{\partial x \partial y}, \frac{\partial^2 f^2}{\partial y^2}\right).$$

We can let a system $\mathscr{S}$ of l n-th order differential equations in p independent and q dependent variables be given as a system of equations

$$\Delta_v(x, u^{(n)}) = 0, \quad v = 1, \ldots, l.$$

The function $\Delta(x, u^{(n)}) = (\Delta_1(x, u^{(n)}), \ldots, \Delta_l(x, u^{(n)}))$ will be assumed to be smooth in its arguments which will allow us to consider Δ as a smooth map from the jet space $X \times U^{(n)}$ to some l-dimensional Euclidean space. That is,

$$\Delta : X \times U^{(n)} \to \mathbb{R}^l.$$

The differential equations tell where Δ vanishes on $X \times U^{(n)}$ and therefore determine a subvariety

$$\mathscr{S}_\Delta = \{(x, u^{(n)}) \mid \Delta(x, u^{(n)}) = 0\} \subset X \times U^{(n)} \tag{5}$$

of the total jet space.

The main point of prolongation is simply this: if we make the identification of the system of differential equations with its corresponding subvariety then this will allow us to interpret the differential equations as a manifold, giving us the geometric interpretation necessary to find the symmetries of the system. Prolongation gives us the necessary space where we can calculate the Lie symmetries of the subvariety thus giving us the Lie symmetries of a system of differential equations.

A solution $u = f(x)$ to the system $\Delta = 0$ is one such that

$$\Delta_v(x, \mathsf{pr}^{(n)} f(x)) = 0, \quad v = 1, \ldots, l$$

whenever x lies in the domain of f. This means that the graph $\Gamma_f^{(n)}$ of the prolongation $\mathsf{pr}^{(n)} f(x)$ must lie entirely within the subvariety $\mathscr{S}_\Delta$ determined by the system where

$$\Gamma_f^{(n)} \equiv \{(x, \mathsf{pr}^{(n)} f(x))\} \subset \mathscr{S}_\Delta = \{\Delta(x, u^{(n)}) = 0\}.$$

Now that we are able to prolong functions, we must define an equivalent way of prolonging group actions so that the prolongation of a group action will act on the prolongation of a function so as to agree with any lower dimensional prolongation. Choose some point $(x_0, u_0^{(n)}) \in M^{(n)}$ and choose any smooth function $u = f(x)$ defined in some neighborhood of x_0 whose graph lies in M and satisfies $u_0^{(n)} = \mathsf{pr}^{(n)} f(x_0)$. Let g be an element of a local group G acting on M near the identity. This implies that $g \cdot f$ is defined near (x_0, u_0) with $(\widetilde{x_0}, \widetilde{u_0}) = g \cdot (x_0, u_0)$ and $u_0 = f(x_0)$.

Definition 5.28. Define the n-th prolongation of the action of g on f by

$$\mathsf{pr}^{(n)} g \cdot (x_0, u_0^{(n)}) = (\widetilde{x_0}, \widetilde{u_0}^{(n)})$$

where

$$\widetilde{u_0}^{(n)} \equiv \mathsf{pr}^{(n)} (g \cdot f)(\widetilde{x_0}).$$

The following theorem will establish the connection between the group action on the subvariety and the symmetries of the corresponding system of differential equations.

Theorem 5.29. *Let M be an open subset of $X \times U$ and suppose that $\Delta(x, u^{(n)}) = 0$ is an n-th order system of differential equations defined over M, with corresponding subvariety $\mathscr{S}_\Delta \subset M^{(n)}$. Suppose G is a local group of transformations acting on M whose prolongation leaves $\mathscr{S}_\Delta$ invariant, meaning that whenever $(x, u^{(n)}) \in \mathscr{S}_\Delta$, we* *have that* $\mathsf{pr}^{(n)} g \cdot (x, u^{(n)}) \in \mathscr{S}_\Delta$ *for all* $g \in G$ *such that this is defined. Then* G *is a symmetry group of the system of differential equations as defined in Definition 5.26.*

Proof. Suppose $u = f(x)$ is a local solution to $\Delta(x, u^{(n)}) = 0$. Therefore

$$\Gamma_f^{(n)} = \{(x, \mathsf{pr}^{(n)} f(x))\}$$

of the prolongation $\mathsf{pr}^{(n)} f$ lies entirely within $\mathscr{S}_\Delta$. If $g \in G$ is such that $g \cdot f$ is well defined then the graph of its prolongation is simply the transform of the graph of $\mathsf{pr}^{(n)} f$ because

$$\Gamma_{g \cdot f}^{(n)} = \mathsf{pr}^{(n)} g(\Gamma_f^{(n)}).$$

Since $\mathscr{S}_\Delta$ is invariant under $\mathsf{pr}^{(n)} g$ then the graph of $\mathsf{pr}^{(n)}(g \cdot f)$ is contained in $\mathscr{S}_\Delta$ which implies that $g \cdot f$ is a solution to the system $\Delta = 0$. □

Now we want to define a way of prolonging the infinitesimal generator of a group action so that the prolongation of the group action agrees with that of the associated infinitesimal generator.

Definition 5.30. Let $M \subset X \times U$ be open and suppose $\mathbf{v}$ is a vector field on M, with corresponding one-parameter group action $\exp(\varepsilon \mathbf{v})$. The n-th prolongation of $\mathbf{v}$, denoted $\mathsf{pr}^{(n)} \mathbf{v}$, will be a vector field on the n-jet space $M^{(n)}$ and is defined to be the infinitesimal generator of the corresponding prolonged one-parameter group $\mathsf{pr}^{(n)}[\exp(\varepsilon \mathbf{v})]$. The formula for $\mathsf{pr}^{(n)} \mathbf{v}$ is given by

$$\mathsf{pr}^{(n)} \mathbf{v}|_{(x, u^{(n)})} = \left.\frac{d}{d\varepsilon}\right|_{\varepsilon=0} \mathsf{pr}^{(n)}[\exp(\varepsilon \mathbf{v})](x, u^{(n)})$$

for any $(x, u^{(n)}) \in M^{(n)}$.

In order to simplify our notation of the prolongation of a function we will define a total derivative.

Definition 5.31. Let $P(x, u^{(n)})$ be a smooth function defined on an open subset $M^{(n)} \subset X \times U^{(n)}$. The total derivative of P with respect to x^i is the unique smooth function $D_i P(x, u^{(n+1)})$ defined on $M^{(n+1)}$ and depending on derivatives of u up to order $n+1$, with the property that if $u = f(x)$ is any smooth function then

$$D_i P(x, \mathsf{pr}^{(n+1)} f(x)) = \frac{\partial}{\partial x^i} \left(P \left(x, \mathsf{pr}^{(n)} f(x) \right) \right).$$

This means that $D_i P$ is obtained from P by differentiating P with respect to x^i while treating all the u^α and their derivatives as functions of x.

We can see that for the total derivative of ϕ we get that

$$\phi^j(x, u^{(1)}) = D_j \phi(x, u) = \frac{\partial \phi}{\partial x^j} + u_j \frac{\partial \phi}{\partial u}.$$

This definition of total derivative extends naturally to functions depending on the variables $x = (x^1, \ldots, x^p)$, $u = (u^1, \ldots, u^q)$ and the derivatives u^α_J of u as defined below.

Theorem 5.32. *Given $P(x, u^{(n)})$, the i-th total derivative of P has the general form*

$$D_i P = \frac{\partial P}{\partial x^i} + \sum_{\alpha=1}^{q} \sum_J u^\alpha_{J,i} \frac{\partial P}{\partial u^\alpha_J},$$

where, for $J = (j_1, \ldots, j_k)$,

$$u^\alpha_{J,i} = \frac{\partial u^\alpha_J}{\partial x^i} = \frac{\partial^{k+1} u^\alpha}{\partial x^i \partial x^{j_1} \partial x^{j_2} \ldots \partial x^{j_k}}.$$

The sum is over all J's of order $\#J$, where $0 \le \#J \le n$, and n is the highest order derivative appearing in P.

Essentially a total derivative arises as a direct application of the chain rule. Let's consider the following example of a total derivative. Let P be a function of (x, y) and $(u, v; u_x, u_y, v_x, v_y)$. Then we have the following total derivatives:

$$D_x P = \frac{\partial P}{\partial x} + \frac{\partial P}{\partial u} \cdot u_x + \frac{\partial P}{\partial v} \cdot v_x + \frac{\partial P}{\partial u_x} \cdot u_{xx} + \frac{\partial P}{\partial u_y} \cdot u_{yx} + \frac{\partial P}{\partial v_x} \cdot v_{xx} + \frac{\partial P}{\partial v_y} \cdot v_{yx}$$

and

$$D_yP = \frac{\partial P}{\partial y} + \frac{\partial P}{\partial u}\cdot u_y + \frac{\partial P}{\partial v}\cdot v_y + \frac{\partial P}{\partial u_x}\cdot u_{xy} + \frac{\partial P}{\partial u_y}\cdot u_{yy} + \frac{\partial P}{\partial v_x}\cdot v_{xy} + \frac{\partial P}{\partial v_y}\cdot v_{yy}$$

So if we had a function $P = x^2 + y^2u_xu_y + uv$ then we would have the following total derivatives:

$$\begin{aligned} D_xP &= 2x + v\cdot u_x + u\cdot v_x + y^2u_y\cdot u_{xx} + y^2u_x\cdot u_{yx} + 0\cdot v_{xx} + 0\cdot v_{yx} \\ &= 2x + vu_x + uv_x + y^2u_yu_{xx} + y^2u_xu_{yx} \end{aligned}$$

and

$$\begin{aligned} D_yP &= 2yu_xu_y + v\cdot u_y + u\cdot v_y + y^2u_y\cdot u_{xy} + y^2u_x\cdot u_{yy} + 0\cdot v_{xy} + 0\cdot v_{yy} \\ &= 2yu_xu_y + vu_y + uv_y + y^2u_yu_{xy} + y^2u_xu_{yy}. \end{aligned}$$

Higher order total derivatives can be taken and are simply defined recursively using Definition 5.31 meaning that $D_{xy}P = D_x(D_yP)$ and because of the smoothness of the functions we have that $D_{xy}P = D_{yx}P$.

The following theorem will give us the general prolongation formula, a formula for prolonging general vector fields. A complete proof can be found in [12].

Theorem 5.33. *Let*

$$\mathbf{v} = \sum_{i=1}^{p}\xi^i(x,u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q}\phi_\alpha(x,u)\frac{\partial}{\partial u^\alpha}$$

be a vector field defined on an open subset $M \subset X \times U$. The n-th prolongation of $\mathbf{v}$ *is the vector field*

$$\mathsf{pr}^{(n)}\mathbf{v} = \mathbf{v} + \sum_{\alpha=1}^{q}\sum_{J}\phi_\alpha^J(x,u^{(n)})\frac{\partial}{\partial u_J^\alpha}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$, the second summation being over all (unordered) multi-indices $J = (j_1,\ldots,j_k)$, with $1 \le j_k \le p$, $1 \le k \le n$

where p is the order of $x = (x^1, \ldots, x^p)$ and q is the order of $u = (u^1, \ldots, u^q)$. The coefficient functions ϕ_α^J of $\mathsf{pr}^{(n)}\mathbf{v}$ are given by the formula:

$$\phi_\alpha^J(x, u^{(n)}) = D_J\left(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha\right) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha,$$

where $u_i^\alpha = \dfrac{\partial u^\alpha}{\partial x^i}$ and $u_{J,i}^\alpha = \dfrac{\partial u_J^\alpha}{\partial x^i}$.

This prolongation formula will allow us to symbolically represent the n-th prolongation of a general infinitesimal generator.

Now we need a way of discussing the rank of a system of differential equations.

Definition 5.34. Let

$$\Delta_v(x, u^{(n)}) = 0, \quad v = 1, \ldots, l,$$

be a system of differential equations. The system is said to be of maximal rank if the $l \times (p + qp^{(n)})$ Jacobian matrix defined by

$$\mathsf{J}_\Delta(x, u^{(n)}) = \left(\frac{\partial \Delta_v}{\partial x^i}, \frac{\partial \Delta_v}{\partial u_J^\alpha}\right)$$

of Δ with respect to all the variables $(x, u^{(n)})$ is of rank l whenever $\Delta(x, u^{(n)}) = 0$, where p are q are the numbers of independent and dependent variables, respectively, and $p^{(n)} = \begin{pmatrix} p+n \\ n \end{pmatrix}$.

The next theorem will give us the infinitesimal criterion for an infinitesimal generator, allowing us to calculate the Lie symmetries of a system of differential equations.

Theorem 5.35. *Suppose*

$$\Delta_v(x, u^{(n)}) = 0, \quad v = 1, \ldots, l,$$

is a system of differential equations of maximal rank defined over $M \subset X \times U$. If G is a local group of transformations acting on M, and

$$\mathsf{pr}^{(n)}\mathbf{v}[\Delta_v(x, u^{(n)})] = 0, \quad v = 1, \ldots, l \quad \textit{whenever} \quad \Delta(x, u^{(n)}) = 0,$$

for every infinitesimal generator $\mathbf{v}$ *of G, then G is a symmetry group of the system of differential equations.*

Theorem 5.36. *Let $\Delta = 0$ be a system of differential equations of maximal rank defined over $M \subset X \times U$. The set of all infinitesimal symmetries of this system forms a Lie algebra of vector fields on M as defined in Section 5.3. Moreover, if this Lie algebra is finite-dimensional, the (connected component of the) symmetry group of the system is a local Lie group of transformations acting on M.*

Now we have all the tools necessary to calculate the Lie symmetries of the harmonic functions of interest to us. Also notice that if we can satisfy Theorem 5.35 by the set of infinitesimal generators that span the associated Lie algebra given by Theorem 5.36, then any infinitesimal generator will hold in the infinitesimal criterion because it is simply some combination (scalar multiplication, addition and Lie bracket) of the spanning set of infinitesimal generators, which each individually hold in the criterion.

6 Overview of Main Results

The Lie symmetries for functions satisfying Laplace's equation have been known for some time. We are interested in a subfamily of this. In particular, we will consider the symmetries of planar harmonic functions expressed in the form $f = h + \overline{g}$ and some of their subclasses. We often make the convention that if $f = f^1 + if^2$ is analytic then $f^1(z) = f^1(x, y)$ and $f^2(z) = f^2(x, y)$ are harmonic conjugates. Since f^1 and f^2 are real valued harmonic, we will consider f^1 and f^2 as maps from $\mathbb{R}^2$ to $\mathbb{R}$ or as maps from $\mathbb{C}$ to $\mathbb{R}$ interchangeably without explanation when the substitution is made.

The results in the following sections are the application of all the background information given up to this point. We have applied these techniques to systems of differential equations that represent certain properties such as harmonic, harmonic area-preserving or harmonic and convex in some direction. All calculations are given in the appendices. The broad picture now is that we want to analyze the transformed function $\widetilde{f}$ which arises as the transform of f under the Lie symmetries. Our transformed function $\widetilde{f}$ is guaranteed to have the properties described by the system of differential equations $\Delta = 0$ because it is in the flow of one function that satisfies $\Delta = 0$. The broad picture becomes more and more clear as we look at the results.

In the following sections, we consider the symmetries of harmonic functions, area-preserving harmonic functions and harmonic functions with certain convexity. Along with the general symmetries we consider a few important subcases which have applications to some previously published results.

7 Harmonic Univalent Functions

Let $f = h + \overline{g}$ where h and g are analytic, notating $\mathrm{Re}(h) = h^1$, $\mathrm{Im}(h) = h^2$, $\mathrm{Re}(g) = g^1$ and $\mathrm{Im}(g) = g^2$ with $z = x + iy$. Let Δ be defined as

$$\Delta = \begin{pmatrix} h^1_x - h^2_y \\ h^1_y + h^2_x \\ g^1_x - g^2_y \\ g^1_y + g^2_y \end{pmatrix}.$$

We see that $\Delta = 0$ represents the Cauchy-Riemann equations for h and g giving that h and g are analytic. Now let $\mathbf{v}$ be the infinitesimal generator of the Lie symmetries of Δ defined on the space $X \times U$ where X has coordinates $\{(x, y)\}$ and U has coordinates $\{(h^1, h^2, g^1, g^2)\}$; that is $X \times U$ is the 0-th order jet space, the space of independent and dependent variables. Let $a^1, a^2, \ldots, a^6$ be general functions of x, y, h^1, h^2, g^1 and g^2. In generic form, $\mathbf{v}$ is given by

$$\mathbf{v} = a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y} + a^3 \frac{\partial}{\partial h^1} + a^4 \frac{\partial}{\partial h^2} + a^5 \frac{\partial}{\partial g^1} + a^6 \frac{\partial}{\partial g^2}.$$

From the calculations of $\mathbf{v}$ as given explicitly in Appendix A, we can see that $a^1 + ia^2$, $a^3 + ia^4$ and $a^5 + ia^6$ are each analytic in z, h and g where we define analytic in h to mean that $a^i_{h^1} = a^{i+1}_{h^2}$ and $a^i_{h^2} = -a^{i+1}_{h^1}$ for $i = 1, 3, 5$ and analytic in g meaning the same only in terms of $g = g^1 + ig^2$. Therefore we are defining analytic in z, h and g to mean that a^i and a^{i+1} for $i = 1, 3, 5$ satisfy the Cauchy-Riemann equations in the complex variables $x + iy$,$h^1 + ih^2$ and $g^1 + ig^2$. A simple example of this is the function

$$(a^1 + ia^2)(z, h) = z^2 + h^2 = x^2 - y^2 + (h^1)^2 - (h^2)^2 + i\left(2ixy + 2ih^1h^2\right)$$

yielding

$$a^1(z, h) = x^2 - y^2 + (h^1)^2 - (h^2)^2$$

and

$$a^2(z,h) = 2xy + 2h^1h^2.$$

It is easily verified that

$$\begin{aligned} a^1_x &= a^2_y \\ a^1_y &= -a^2_x \\ a^1_{h^1} &= a^2_{h^2} \\ a^1_{h^2} &= -a^2_{h^2} \end{aligned}$$

thus showing that $a^1 + ia^2$ is analytic in z and h. In this context, we are thinking of h and g as though they are independent of z making the above informal definition of "analytic in" appropriate.

Since a^1, a^2 and a^3, a^4 and a^5, a^6 are each as a pair independent of the other pairs then we can represent a spanning set of generators for the Lie algebra of the Lie symmetries by the generators

$$\begin{aligned} \mathbf{v}_1 &= a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y} \\ \mathbf{v}_2 &= a^3\frac{\partial}{\partial h^1} + a^4\frac{\partial}{\partial h^2} \\ \mathbf{v}_3 &= a^5\frac{\partial}{\partial g^1} + a^6\frac{\partial}{\partial g^2} \end{aligned}$$

where $a^1 + ia^2, a^3 + ia^4$ and $a^5 + ia^6$ have the properties as described above.

7.1 Reparametrization of the Domain

Let $a^1 + ia^2$ be an arbitrary analytic function of only $z = x + iy$. Consider

$$\mathbf{v}_1 = a^1(x,y)\frac{\partial}{\partial x} + a^2(x,y)\frac{\partial}{\partial y}$$

as an infinitesimal generator of a subalgebra of the Lie algebra of the associated symmetries. This is not the most general case calculated by the infinitesimal criterion but it is the first case that we want to consider. If we let $\widetilde{x}$ and $\widetilde{y}$ be the transformed independent variables under the action of $\varepsilon \in \mathbb{R}$ then to solve for $\widetilde{x}$ and $\widetilde{y}$ we need to solve the ordinary system of differential equations

$$\frac{d\widetilde{x}}{d\varepsilon} = a^1(\widetilde{x}, \widetilde{y})$$
$$\frac{d\widetilde{y}}{d\varepsilon} = a^2(\widetilde{x}, \widetilde{y})$$

subject to the initial conditions $\widetilde{x}|_{\varepsilon=0} = x$ and $\widetilde{y}|_{\varepsilon=0} = y$. This may or may not be easily solved. Whatever the solution may be, we can see that the transform $z \mapsto \widetilde{z}$ where $\widetilde{z} = \widetilde{x} + i\widetilde{y}$ is simply a reparametrization of the domain of the transformed function. So if $f(z) = h(z) + \overline{g(z)}$ is a harmonic function then under the Lie symmetries we get that $(f \circ \widetilde{z})(z)$ also satisfies the system $\Delta = 0$, meaning that $f \circ \widetilde{z}$ also is a harmonic function of z. Let's consider a few examples showing that the transformed function is harmonic. We denote the transformed functions by $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$.

Example 7.1. For each of the following examples, we will calculate the transformed function $\widetilde{f}$ under the infinitesimal generator $\mathbf{v}_1$.

(a) Let $a^1(z) = x$ and $a^2(z) = y$ implying that $(a^1 + ia^2)(z) = x + iy = z$ which is analytic. So we are considering the infinitesimal generator $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and in order to exponentiate this we must solve

$$\frac{d\widetilde{x}}{d\varepsilon} = \widetilde{x}$$
$$\frac{d\widetilde{y}}{d\varepsilon} = \widetilde{y}$$

with $\widetilde{x} = x$ and $\widetilde{y} = y$ when $\varepsilon = 0$. The above system has solution $\widetilde{x} = xe^{\varepsilon}$ and $\widetilde{y} = ye^{\varepsilon}$ and hence $\widetilde{z} = e^{\varepsilon}(x + iy) = e^{\varepsilon}z$ which is a scaling of the domain.

Therefore if $f(z)$ is a harmonic function then $\widetilde{f}(z) = f(e^{\varepsilon} z)$ is also harmonic. When $\varepsilon = 0$ we can see that $\widetilde{f} = f$ which is the initial element in the flow of the group action on the function f. As ε flows in some interval around $0 \in \mathbb{R}$, the group action guarantees that $\widetilde{f}$ is harmonic.

(b) Let's consider another similar example. Let $a^1(z) = -y$ and $a^2(z) = x$, implying that $(a^1 + ia^2)(z) = (-y + ix) = i(x + iy) = iz$, which is analytic. The corresponding infinitesimal generator is $-y\dfrac{\partial}{\partial x} + x\dfrac{\partial}{\partial y}$. Exponentiating this infinitesimal generator gives us that $\widetilde{x} = x\cos\varepsilon - y\sin\varepsilon$ and $\widetilde{y} = y\cos\varepsilon + x\sin\varepsilon$. We can easily see that $\widetilde{x} = x$ and $\widetilde{y} = y$ when $\varepsilon = 0$ and that appropriately differentiating we get the infinitesimal generator given above. Therefore the transform of z is

$$\begin{aligned}
\widetilde{z} &= \widetilde{x} + i\widetilde{y} \\
&= x\cos\varepsilon - y\sin\varepsilon + i(y\cos\varepsilon + x\sin\varepsilon) \\
&= x(\cos\varepsilon + i\sin\varepsilon) + iy(\cos\varepsilon + i\sin\varepsilon) \\
&= e^{i\varepsilon}(x + iy) \\
&= e^{i\varepsilon} z,
\end{aligned}$$

which gives a rotation of the domain. This gives us that if $f(z)$ is a harmonic function then $\widetilde{f}(z) = f(e^{i\varepsilon} z)$ is also a harmonic function.

(c) Since the set of infinitesimal generators forms a Lie algebra over $\mathbb{R}$ we can combine these generators through addition, scalar multiplication and the Lie bracket. This allows us to compose these two transformations. In the first example we get that $z \mapsto e^{\varepsilon_1} z$ and in the second $z \mapsto e^{i\varepsilon_2} z$. Since these flows are independent of each other then the epsilons do not relate and we will notate this by assigning subscripts. Therefore, through composing these we get that $z \mapsto e^{i\varepsilon_2}z \mapsto e^{\varepsilon_1}\left(e^{i\varepsilon_2}z\right) = e^{\varepsilon_1 + i\varepsilon_2}z$ or simply $z \mapsto re^{i\varepsilon}z$ where $r > 0$ and r and ε are independent of each other. Therefore if $f(z)$ is a harmonic function then $\widetilde{f}(z) = f\left(re^{i\varepsilon}z\right)$ is also harmonic.

Of course in Example 7.1 we have only considered very basic functions $a^1 + ia^2$, but regardless of what analytic function we let this be, it will simply be a reparametrization of the domain of the function that we want to transform in the "direction" of $a^1 + ia^2$ in the jet-space.

7.2 Flow in h and g

Let's consider $\mathbf{v}_2 = a^3\frac{\partial}{\partial h^1} + a^4\frac{\partial}{\partial h^2}$ and $\mathbf{v}_3 = a^5\frac{\partial}{\partial g^1} + a^6\frac{\partial}{\partial g^2}$, two infinitesimal generators of the Lie algebra of associated symmetries. Since a^3 and a^4 are independent of a^5 and a^6 we can consider $\mathbf{v}_2$ and $\mathbf{v}_3$ completely independent of each other. Let's first consider the exponentiation of $\mathbf{v}_2$. Let $a^3 + ia^4$ be an arbitrary analytic function $b(z)$. The exponentiation of $\mathbf{v}_2$ would yield that $\widetilde{h}^1 = \varepsilon a^1 + h^1$ and $\widetilde{h}^2 = \varepsilon a^2 + h^2$ which would imply that

$$\begin{aligned}\widetilde{h} &= \widetilde{h}^1 + i\widetilde{h}^2 \\ &= \varepsilon a^1 + h^1 + i(\varepsilon a^2 + h^2) \\ &= \varepsilon b + h.\end{aligned}$$

Similarly if we let $a^5 + ia^6$ be an arbitrary analytic function $c(z)$ then exponentiating $\mathbf{v}_3$ gives us that $\widetilde{g} = \varepsilon c + g$. We may compose these just as we did in Example 7.1 (c). In order to simplify the composition, we need not consider an ε_1 and ε_2 but just simply ε in the composition even though they are independent of each other. Therefore if $f = h + \overline{g}$ is a harmonic function then

$$\widetilde{f}(z) = \varepsilon b(z) + h(z) + \overline{\varepsilon c(z) + g(z)} = h(z) + \overline{g(z)} + \varepsilon d(z) = f(z) + \varepsilon d(z)$$

where $d(z)$ is any arbitrary harmonic function. This shows us that the group action is transitive. Given any two harmonic functions $f_1(z) = h_1(z) + \overline{g_1(z)}$ and $f_2(z) = h_2(z) + \overline{g_2(z)}$, we can flow from one to the other. Consider, for example, $d = f_2 - f_1$ which is clearly harmonic. We can flow from f_1 to f_2 by letting ε flow from 0 to 1. With $\widetilde{f_1}(z) = f_1(z) + \varepsilon\,(f_2(z) - f_1(z))$, we can see that if $\varepsilon = 0$ then $\widetilde{f_1}(z) = f_1(z)$ as it should be and if $\varepsilon = 1$ we get that $\widetilde{f_1}(z) = f_2(z)$ thus giving us the desired flow. This result is trivial since the above flow can easily be proven to be harmonic for any ε independent of the fact that it is a flow in a Lie symmetry which guarantees its harmonicity but it is meant to be a simple result of looking at the Lie symmetries of harmonic functions.

7.3 Flow with a Normalization

Let's consider the vector field $\mathbf{v} = g^1 \frac{\partial}{\partial h^1} + g^2 \frac{\partial}{\partial h^2} + h^1 \frac{\partial}{\partial g^1} + h^2 \frac{\partial}{\partial g^2}$. This arises from the general infinitesimal generator by letting $a^1 + ia^2 \equiv 0$, $(a^3 + ia^4)(g) = g^1 + ig^2 = g$ and $(a^5 + ia^6)(h) = h^1 + ih^2 = h$ each of which are clearly analytic in g and h and therefore satisfy the requirements of the coordinate functions as given in Section 7. Therefore the exponentiation will yield a harmonic function. The exponentiation of $\mathbf{v}$ yields

$$\widetilde{h}(z) = h(z)\cosh\varepsilon + g(z)\sinh\varepsilon$$

$$\widetilde{g}(z) = g(z)\cosh\varepsilon + h(z)\sinh\varepsilon$$

giving us that

$$\begin{aligned}
\widetilde{f}(z) &= \widetilde{h}(z) + \overline{\widetilde{g}(z)} \\
&= h(z)\cosh\varepsilon + g(z)\sinh\varepsilon + \overline{g(z)\cosh\varepsilon + h(z)\sinh\varepsilon} \\
&= \cosh\varepsilon\left(h(z) + g(z)\tanh\varepsilon + \overline{g(z) + h(z)\tanh\varepsilon}\right)
\end{aligned}$$

is harmonic. Since the scalar multiple of any harmonic function is harmonic then $\widehat{f}(z) = h(z) + g(z)\tanh\varepsilon + \overline{g(z) + h(z)\tanh\varepsilon}$ is harmonic. Let's define $\widehat{h}(z) = h(z) + g(z)\tanh\varepsilon$ and $\widehat{g}(z) = g(z) + h(z)\tanh\varepsilon$ thus giving us that $\widehat{f} = \widehat{h} + \overline{\widehat{g}}$ is harmonic. With this definition, let's examine what properties $\widehat{f}$ has given certain properties of f.

Suppose that $f = h + \overline{g} \in S_H^O$ then we have that $f(0) = 0, f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$. We claim that $\widehat{f} \in S_H$. First, let's show that $\widehat{f}$ is univalent if f is univalent. Suppose that there exist two distinct points $z_1, z_2 \in \mathbb{D}$ such that $\widehat{f}(z_1) = \widehat{f}(z_2)$. If we equate real and imaginary parts we get that

$$h^1(z_1)(1+\tanh\varepsilon) + g^1(z_1)(1+\tanh\varepsilon) = h^1(z_2)(1+\tanh\varepsilon) + g^1(z_2)(1+\tanh\varepsilon)$$

and

$$h^2(z_1)(1-\tanh\varepsilon) - g^2(z_1)(1-\tanh\varepsilon) = h^2(z_2)(1-\tanh\varepsilon) - g^2(z_2)(1-\tanh\varepsilon)$$

imply that

$$h^1(z_1) + g^1(z_1) = h^1(z_2) + g^1(z_2)$$

and

$$h^2(z_1) - g^2(z_1) = h^2(z_2) - g^2(z_2)$$

since $|\tanh\varepsilon| < 1$ for all $\varepsilon \in \mathbb{R}$. Now we have that $\operatorname{Re}(f(z_1)) = \operatorname{Re}(f(z_2))$ and $\operatorname{Im}(f(z_1)) = \operatorname{Im}(f(z_2))$ which contradicts our supposition that f is univalent. Therefore if f is univalent then $\widehat{f}$ is univalent as well.

Now, since $f(0) = 0, f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$ then it can be verified that $\widehat{f}(0) = 0, \widehat{f}_z(0) = 1$ but $\widehat{f}_{\overline{z}}(0) = \tanh\varepsilon \neq 0$ when $\varepsilon \neq 0$. Therefore if $f \in S_H^O$ then $\widehat{f} \in S_H$ and furthermore, when $\varepsilon \neq 0$ then $\widehat{f} \in S_H \setminus S_H^O$. We can see that if we begin with $f \in S_H \setminus S_H^O$ then $f_{\overline{z}}(0) = b \neq 0$ and we would need to renormalize $\widehat{f}$

to give $\widehat{f} \in S_H$. We have that $\widehat{f}_z(0) = h'(0) + g'(0)\tanh\varepsilon = 1 + b\tanh\varepsilon$ giving the factor that must be divided by in order to yield $\widehat{f} \in S_H$.

We further claim that if f is locally univalent and $g'(0) = 0$ then $\widehat{f}$ will be locally univalent as well. To see this, suppose that f and is locally univalent and that $g'(0) = 0$. Then Theorem 3.4 gives us that $|g'(z)| < |h'(z)|$.

Consider that

$$\begin{aligned}
\left|\frac{\widehat{g}'(z)}{\widehat{h}'(z)}\right| &= \left|\frac{g'(z) + h'(z)\tanh\varepsilon}{h'(z) + g'(z)\tanh\varepsilon}\right| \\
&= \left|\frac{h'(z)\left(\tanh\varepsilon + \frac{g'(z)}{h'(z)}\right)}{h'(z)\left(1 + \frac{g'(z)}{h'(z)}\tanh\varepsilon\right)}\right| \\
&= \left|\frac{\tanh\varepsilon + \frac{g'(z)}{h'(z)}}{1 + \frac{g'(z)}{h'(z)}\tanh\varepsilon}\right|
\end{aligned}$$

because $h'(z) \neq 0$. Now let $\omega(z) = \dfrac{g'(z)}{h'(z)}$, $G(z) = \dfrac{\tanh\varepsilon + \omega(z)}{1 + \omega(z)\tanh\varepsilon}$ and $F(z) = \dfrac{\tanh\varepsilon + z}{1 + z\tanh\varepsilon}$. Note that $|\omega(z)| = \left|\dfrac{g'(z)}{h'(z)}\right| < 1$ by supposition and that $|F(z)| < 1$ for all $|z| < 1$ and F is analytic in $\mathbb{D}$ by a classical result from complex analysis and the fact that $|\tanh\varepsilon| < 1$ for all $\varepsilon \in \mathbb{R}$. Since h and g are analytic, h' and g' are analytic. Also since h' is nonzero on $\mathbb{D}$, ω is analytic on $\mathbb{D}$. Consider that $|1+\omega(z)\tanh\varepsilon| \geq 1-|\omega(z)||\tanh\varepsilon| > 0$ because $|\omega(z)|, |\tanh\varepsilon| < 1$ which therefore implies that $G(z)$ is analytic in $\mathbb{D}$ as well. So we have that G, F and ω are analytic in $\mathbb{D}$ and that $G(z) = F(\omega(z))$.

Since $g'(0) = 0$ then $\omega(0) = 0$ and since $|\omega(z)| < 1$, by Schwarz Lemma we have that $|\omega(z)| < |z|$ for all $|z| < 1$ since $\omega(z) \neq z$. Therefore by Definition 2.16 we have that G is subordinate to F. By Theorem 2.17 we have that $G(\mathbb{D}_r) \subset F(\mathbb{D}_r)$ for all $r < 1$ and since $|F(z)| < 1$ for all $|z| < 1$ then $F(\mathbb{D}) \subset \mathbb{D}$ implying that

$G(\mathbb{D}) \subseteq F(\mathbb{D}) \subset \mathbb{D}$; that is, $|G(z)| < 1$ for all $|z| < 1$. Therefore

$$\left|\frac{\widehat{g}'(z)}{\widehat{h}'(z)}\right| = \left|\frac{g'(z) + h'(z)\tanh\varepsilon}{h'(z) + g'(z)\tanh\varepsilon}\right| = |G(z)| < 1$$

giving us that $|\widehat{g}'(z)| < \left|\widehat{h}'(z)\right|$ for all $z \in \mathbb{D}$ meaning that $\widehat{f}$ is locally univalent.

Therefore we have shown that if f is locally univalent and $g'(0) = 0$ then $\widehat{f}$ is also locally univalent.

With $\widehat{f}(z) = h(z) + g(z)\tanh\varepsilon + \overline{g(z) + h(z)\tanh\varepsilon}$, let's consider the limiting case. If $\varepsilon \to \infty$ we have that $\tanh\varepsilon \to 1$ which would imply that $\widehat{f}(z) = h(z) + g(z) + \overline{g(z) + h(z)} = 2(h^1(z) + g^1(z)) = 2\,\mathrm{Re}(f(z))$ which is a real valued harmonic function. If we let $\varepsilon \to -\infty$ we have that $\tanh\varepsilon \to -1$ giving us that $\widehat{f}(z) = 2i(h^2(z) - g^2(z)) = 2i\,\mathrm{Im}(f(z))$ which is a purely imaginary valued harmonic function.

Let's consider some properties of convexity. Suppose $f \in S_H^O$ is convex in the direction of the real axis. Then by Theorem 3.12 we have that $h - g$ is convex in the direction of the real axis. It can be easily verified that for any real value α we have that $\alpha(h - g)$ is also convex in the direction of the real axis. Therefore

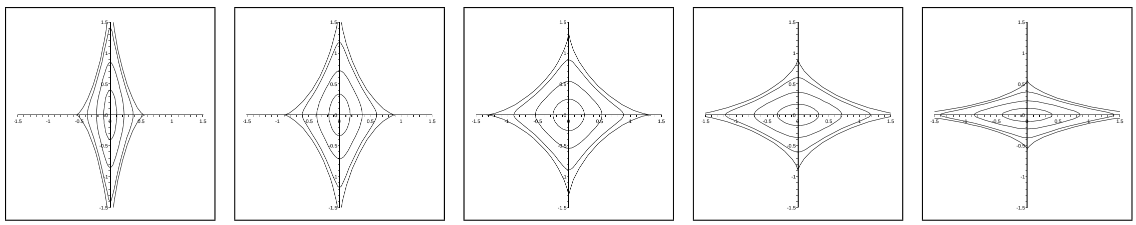
$$(1 - \tanh\varepsilon)(h - g) = (h + g\tanh\varepsilon) - (g + h\tanh\varepsilon) = \widehat{h} - \widehat{g}$$

is convex in the direction of the real axis implying that $h + g\tanh\varepsilon + \overline{g + h\tanh\varepsilon} = \widehat{h} + \overline{\widehat{g}} = \widehat{f}$ is as well by Theorem 3.12. Therefore we have proven the following two results.

Theorem 7.2. *Let $f = h + \overline{g}$ and $\widehat{f} = \widehat{h} + \overline{\widehat{g}}$ where $\widehat{h}(z) = h(z) + g(z)\tanh\varepsilon$ and $\widehat{g}(z) = g(z) + h(z)\tanh\varepsilon$. Then if $f \in S_H^O$ then $\widehat{f} \in S_H$ for any $\varepsilon \in \mathbb{R}$.*

Theorem 7.3. *Let f and $\widehat{f}$ be defined as in Theorem 7.2. Then if $f \in CD(0)$ and is univalent then $\widehat{f} \in CD(0)$ and is univalent.*

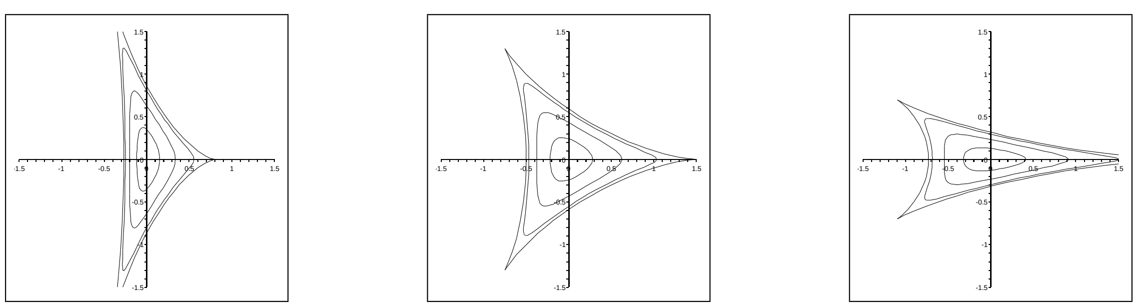
To see the effects of this Lie symmetry on the plot of the function $f(z) = z + \overline{\frac{1}{3}z^3}$, let's view several steps in the flow of $\widehat{f}$ as ε ranges from $-\frac{2}{3}$ to $\frac{2}{3}$ where $h(z) = z$ and $g(z) = \frac{1}{3}z^3$. These are given in Figure 8 as the plots of four concentric circles in $\mathbb{D}$ under $\widehat{f}$.



(a) $\varepsilon = -\frac{2}{3}$ (b) $\varepsilon = -\frac{1}{3}$ (c) $\varepsilon = 0$ (d) $\varepsilon = \frac{1}{3}$ (e) $\varepsilon = \frac{2}{3}$

Figure 8: *Images of* $\mathbb{D}$ *under the map* $\widehat{f} = h + g\tanh\varepsilon + \overline{g + h\tanh\varepsilon}$ *for several values of* ε *where* $f(z) = h(z) + \overline{g(z)} = z + \overline{\frac{1}{3}z^3}$.

If we consider the effects of this Lie symmetry on the plot of $\widehat{f}$ where $h(z) = z$ and $g(z) = \frac{1}{2}z^2$, we can see that we preserve the convexity in the direction of the real axis as we had proven earlier. This is shown for $\varepsilon = -\frac{1}{2}, 0, \frac{1}{2}$ in Figure 9. Again the plots of four concentric circles in $\mathbb{D}$ are shown.



(a) $\varepsilon = -\frac{1}{2}$ (b) $\varepsilon = 0$ (c) $\varepsilon = \frac{1}{2}$

Figure 9: *Images of* $\mathbb{D}$ *under the map* $\widehat{f} = h + g\tanh\varepsilon + \overline{g + h\tanh\varepsilon}$ *for* $\varepsilon = -\frac{1}{2}, 0, \frac{1}{2}$ *where* $f(z) = h(z) + \overline{g(z)} = z + \overline{\frac{1}{2}z^2}$.

7.4 Flow with Applications to Minimal Surfaces

Without giving any indepth background on the theory of minimal surfaces, we will give a flow that when "lifted" to a minimal surface at every stage in the flow will give all the classical associated families of surfaces to that particular surface. Begin with the infinitesimal generator $\mathbf{v} = -g^2 \frac{\partial}{\partial g^1} + g^1 \frac{\partial}{\partial g^2}$. We can see that this generator arises from $a^1 + ia^2 \equiv 0$, $a^3 + ia^4 \equiv 0$ and $(a^5 + ia^6)(g) = -g^2 + ig^1 = ig$ which is analytic in g. The generator has an exponentiation of $\widetilde{h} = h$ and

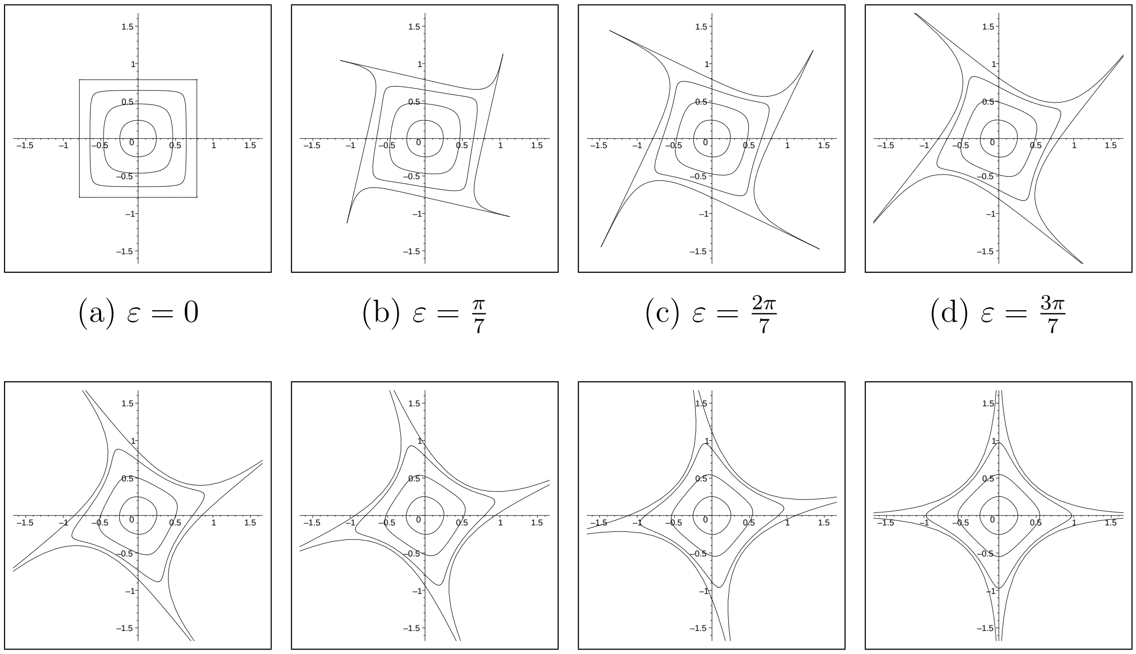
$$\begin{aligned}\widetilde{g} &= (g^1 \cos\varepsilon - g^2 \sin\varepsilon) + i(g^2 \cos\varepsilon + g^1 \cos\varepsilon) \\ &= g^1(\cos\varepsilon + i\sin\varepsilon) + ig^2(\cos\varepsilon + i\sin\varepsilon) \\ &= e^{i\varepsilon} g\end{aligned}$$

and therefore $\widetilde{f}(z) = h(z) + \overline{e^{i\varepsilon} g(z)}$. Let's look at the projection of Scherk's doubly periodic surface and the flow from this under the Lie symmetry to the classically associated surface which is Scherk's singly periodic. The projection of the doubly periodic surface comes from the harmonic function $f = h + \overline{g}$ where

$$\begin{aligned}h(z) &= \frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{1+iz}{1-iz}\right) \\ g(z) &= -\frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{1+iz}{1-iz}\right).\end{aligned}$$

Let's consider the flow of $\widetilde{f} = h + \overline{e^{i\varepsilon} g}$ as ε ranges from 0 to π as shown in Figure 10. At each step in the flow, the image may be lifted to a minimal surface.

The Lie symmetries that yield the associated surfaces were originally calculated in Robert Berry's Master's Thesis. Previosly, Necklets Bîlă [1] had considered the Lie symmetries of the minimal surface equation but the associated surfaces did not arise as a Lie symmetry in this case. In Berry's thesis, when he analyzed the Lie symmetries of complex valued harmonic functions expressed as $f = u + iv$, where



(a) $\varepsilon = 0$ (b) $\varepsilon = \frac{\pi}{7}$ (c) $\varepsilon = \frac{2\pi}{7}$ (d) $\varepsilon = \frac{3\pi}{7}$

(e) $\varepsilon = \frac{4\pi}{7}$ (f) $\varepsilon = \frac{5\pi}{7}$ (g) $\varepsilon = \frac{6\pi}{7}$ (h) $\varepsilon = \pi$

Figure 10: *Images of* $\mathbb{D}$ *under the map* $\widehat{f} = h + \overline{e^{i\varepsilon} g}$ *for* ε *ranging from* 0 *to* π *where* $f = h + \overline{g} = \frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{1+iz}{1-iz}\right) + \overline{-\frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{1+iz}{1-iz}\right)}$.

u and v are real valued harmonic, he was able to show that the associated surfaces arise as part of the symmetries of harmonic functions with a few restrictions on the transformed functions applied after their calculation such as normalization. In our analysis using harmonic functions of the form $f = h + \overline{g}$, the associated surfaces arise as a direct result of the symmetries. In order for $f = h + \overline{g}$ to lift to a minimal surface, it must be true that $\sqrt{h'g'}$ must be analytic in $\mathbb{D}$ meaning that $h'g'$ must be a perfect square. If this is true, then we can see from the Lie symmetry $\widetilde{f} = h + \overline{e^{i\varepsilon} g}$ that we will get $\widetilde{f}$ lifting to a minimal surface if f does.

8 Harmonic Area-Preserving Functions

Let's continue with $f = h + \overline{g}$ being harmonic with constant Jacobian on $\mathbb{D}$; that is, $J_f = k$ for some $k \in \mathbb{R}$ on $\mathbb{D}$. For example, if $J_f = 1$ on $\mathbb{D}$ then f is said to be an area-preserving harmonic function. Again consider the generic infinitesimal generator

$$\mathbf{v} = a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y} + a^3 \frac{\partial}{\partial h^1} + a^4 \frac{\partial}{\partial h^2} + a^5 \frac{\partial}{\partial g^1} + a^6 \frac{\partial}{\partial g^2}$$

where $a^1, \ldots, a^6$ are general functions of x, y, h^1, h^2, g^1 and g^2. In Appendix B we calculate what the a^i's must be in order for $\mathbf{v}$ to be an infinitesimal generator of the Lie symmetries of

$$\Delta = \begin{pmatrix} h^1_x - h^2_y \\ h^1_y + h^2_x \\ g^1_x - g^2_y \\ g^1_y + g^2_y \\ (h^1_x + g^1_x)(h^2_y - g^2_y) - (h^1_y + g^1_y)(h^2_x - g^2_x) - k \end{pmatrix} = 0.$$

We can see that $\Delta = 0$ gives us that h and g are analytic and $J_f = k$.

As shown in Appendix B, we get a finite dimensional Lie algebra spanned by

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial x} & \mathbf{v}_7 &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\
\mathbf{v}_2 &= \frac{\partial}{\partial y} & \mathbf{v}_8 &= -h^2\frac{\partial}{\partial h^1} + h^1\frac{\partial}{\partial h^2} \\
\mathbf{v}_3 &= \frac{\partial}{\partial h^1} & \mathbf{v}_9 &= -g^2\frac{\partial}{\partial g^1} + g^1\frac{\partial}{\partial g^2} \\
\mathbf{v}_4 &= \frac{\partial}{\partial h^2} & \mathbf{v}_{10} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + h^1\frac{\partial}{\partial h^1} + h^2\frac{\partial}{\partial h^2} + g^1\frac{\partial}{\partial g^1} + g^2\frac{\partial}{\partial g^2} \\
\mathbf{v}_5 &= \frac{\partial}{\partial g^1} & \mathbf{v}_{11} &= g^2\frac{\partial}{\partial h^1} - g^1\frac{\partial}{\partial h^2} - h^2\frac{\partial}{\partial g^1} + h^1\frac{\partial}{\partial g^2} \\
\mathbf{v}_6 &= \frac{\partial}{\partial g^2} & \mathbf{v}_{12} &= g^1\frac{\partial}{\partial h^1} + g^2\frac{\partial}{\partial h^2} + h^1\frac{\partial}{\partial g^1} + h^2\frac{\partial}{\partial g^2}
\end{aligned}$$

The exponentiations of each of the $\mathbf{v}_i$ are given in Appendix B and can be composed in any way yielding an area-preserving harmonic function. If $f = h + \overline{g}$ is area-preserving harmonic then composing the exponentiated functions from $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_7$ gives that

$$\widetilde{f}(z) = f\left(e^{i\varepsilon}z + \alpha\right) + \beta$$

is area-preserving harmonic where $\alpha, \beta \in \mathbb{C}$ and $\varepsilon \in \mathbb{R}$. This is a rotation and translation of the domain followed by a translation of the image. Composing the exponentiations from $\mathbf{v}_8, \mathbf{v}_9$ and $\mathbf{v}_{10}$ gives that

$$\widetilde{f}(z) = re^{i\varepsilon_1}h\left(\frac{z}{r}\right) + \overline{re^{i\varepsilon_2}g\left(\frac{z}{r}\right)}$$

is also area-preserving harmonic where $r > 0$ and $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$. If we let $\varepsilon_1 = -\varepsilon_2$ we get that

$$\widetilde{f}(z) = re^{i\varepsilon_1}h\left(\frac{z}{r}\right) + \overline{re^{-i\varepsilon_1}g\left(\frac{z}{r}\right)} = re^{i\varepsilon_1}f\left(\frac{z}{r}\right).$$

This is a scaling of the domain followed by a rotation and unscaling of the image. The exponentiation of $\mathbf{v}_{11}$ yields

$$\widetilde{f}(z) = f(z)\cosh\varepsilon + \overline{if(z)}\sinh\varepsilon, \quad \varepsilon \in \mathbb{R}$$

and that of $\mathbf{v}_{12}$ yields

$$\widetilde{f}(z) = f(z)\cosh\varepsilon + \overline{f(z)}\sinh\varepsilon, \quad \varepsilon \in \mathbb{R}.$$

Since the system of differential equations for the area-preserving harmonic functions gives a submanifold in the 1st order jet space of the system for harmonic functions, the Lie algebra formed by the generators for the area-preserving functions is a finite dimensional subalgebra of the infinite dimensional Lie algebra for the harmonic functions. This should be clear from a geometric and algebraic point of view, given its submanifold nature and the infinitesimal criterion.

9 Harmonic Functions with Fixed Convexity

Now we will consider our last case of symmetries. In Appendix C, we calculate the symmetries for functions $f = h + \overline{g}$ where $h - g$ is a fixed function and h and g are analytic. We want to know the symmetries of f where $h - g \in CD(0)$. Geometrically we can justify considering only the simpler case of convex in the direction of the real axis as opposed to convex in some other direction by considering all others as a rotation of this case.

By Theorem 3.12 we see that for $f = h + \overline{g}$ locally univalent then $f \in CD(0)$ and univalent if and only if $h - g \in CD(0)$ and univalent. Let $h(z) - g(z) = F(z) = F^1(x, y) + iF^2(x, y) \in CD(0)$. We will consider symmetries of $\Delta = 0$ where

$$\Delta = \begin{pmatrix} h^1_x - h^2_y \\ h^1_y + h^2_x \\ g^1_x - g^2_y \\ g^1_y + g^2_y \\ h^1 - g^1 - F^1 \\ h^2 - g^2 - F^2 \end{pmatrix}.$$

This gives that $\Delta = 0$ represents h and g being analytic and $h - g$ fixed as F. If we can show that the transformed function $\widetilde{f}$ is locally univalent then we will have that $\widetilde{f}$ will be convex in the direction of the real axis since F will be assumed to be in $CD(0)$.

We will begin again with the general infinitesimal generator

$$\mathbf{v} = a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y} + a^3\frac{\partial}{\partial h^1} + a^4\frac{\partial}{\partial h^2} + a^5\frac{\partial}{\partial g^1} + a^6\frac{\partial}{\partial g^2}$$

where $a^1, \ldots, a^6$ are general functions of x, y, h^1, h^2, g^1 and g^2. As given by the calculations in Appendix C, we have that $a^1 = a^2 = 0$, $a^3 = a^5$, $a^4 = a^6$ and

$a^3 + ia^4$ is analytic in z, h and g as defined informally in Section 7. Therefore the infinitesimal generator becomes

$$\mathbf{v} = a^3 \frac{\partial}{\partial h^1} + a^4 \frac{\partial}{\partial h^2} + a^3 \frac{\partial}{\partial g^1} + a^4 \frac{\partial}{\partial g^2},$$

where $\mathbf{v}$ generates the Lie algebra relative to the symmetries. Since a^3 and a^4 are not independent of each other, we cannot break up this generator into smaller elements of the algebra as we did for the simpler case of the symmetries of harmonic functions as shown in Section 7. We will do so when we consider some subalgebra, but in the broadest case, $\mathbf{v}$ is the only basis element of the infinite dimensional Lie algebra.

It is worth noting here that the Lie symmetries calculated in this section have broader applications than simply harmonic functions convex in the direction of the real axis because they preserve the harmonicity of f and fix $h - g$ as *any* function and not simply one that is analytic, univalent and convex in some direction. Current research is being done using these symmetries in minimal surface theory to fix the second coordinate in the Weierstrass-Enneper representation of a minimal surface. This method gives new families of associated surfaces.

9.1 A Finite Dimensional Subalgebra

One of the simplest cases of $\mathbf{v}$ is given when

$$(a^3 + ia^4)(z, h, g) = c_1 + ic_2 + (c_3 + ic_4)z + (c_5 + ic_6)h + (c_7 + ic_8)g,$$

where $c_1, \ldots, c_8$ are each arbitrary real numbers. We can verify that $a^3 + ia^4$ satisfies the infinitesimal criterion and therefore $\mathbf{v}$ as defined previously is a generator in the associated Lie algebra. As shown in Appendix C.1, this gives us the eight generators

$$\mathbf{v}_1 = \frac{\partial}{\partial h^1} + \frac{\partial}{\partial g^1}$$

$$
\begin{aligned}
\mathbf{v}_2 &= \frac{\partial}{\partial h^2} + \frac{\partial}{\partial g^2} \\
\mathbf{v}_3 &= x\frac{\partial}{\partial h^1} + y\frac{\partial}{\partial h^2} + x\frac{\partial}{\partial g^1} + y\frac{\partial}{\partial g^2} \\
\mathbf{v}_4 &= -y\frac{\partial}{\partial h^1} + x\frac{\partial}{\partial h^2} - y\frac{\partial}{\partial g^1} + x\frac{\partial}{\partial g^2} \\
\mathbf{v}_5 &= h^1\frac{\partial}{\partial h^1} + h^2\frac{\partial}{\partial h^2} + h^1\frac{\partial}{\partial g^1} + h^2\frac{\partial}{\partial g^2} \\
\mathbf{v}_6 &= -h^2\frac{\partial}{\partial h^1} + h^1\frac{\partial}{\partial h^2} - h^2\frac{\partial}{\partial g^1} + h^1\frac{\partial}{\partial g^2} \\
\mathbf{v}_7 &= g^1\frac{\partial}{\partial h^1} + g^2\frac{\partial}{\partial h^2} + g^1\frac{\partial}{\partial g^1} + g^2\frac{\partial}{\partial g^2} \\
\mathbf{v}_8 &= -g^2\frac{\partial}{\partial h^1} + g^1\frac{\partial}{\partial h^2} - g^2\frac{\partial}{\partial g^1} + g^1\frac{\partial}{\partial g^2}.
\end{aligned}
$$

Notice that none of these yield a flow in the domain space therefore no Lie symmetry will involve a reparametrization of the domain. We will simply have flows in the function space.

The calculations of the exponentiations of $\mathbf{v}_1$ through $\mathbf{v}_8$ are given in Appendix C.1. If we consider the exponentiations of $\mathbf{v}_1$,$\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ and compose them we get

$$
\begin{aligned}
\widetilde{h}(z) &= h(z) + \alpha z + \beta \\
\widetilde{g}(z) &= g(z) + \alpha z + \beta,
\end{aligned}
$$

where α and β are arbitrary complex numbers sufficiently close to 0. Exponentiating and composing $\mathbf{v}_5$ and $\mathbf{v}_6$ give us

$$
\begin{aligned}
\widetilde{h}(z) &= re^{i\varepsilon}h(z) \\
\widetilde{g}(z) &= g(z) + re^{i\varepsilon}h(z) - h(z),
\end{aligned}
$$

where $r > 0$ and is sufficiently close to 1. Similarly exponentiating and composing

$\mathbf{v}_7$ and $\mathbf{v}_8$ gives

$$\widetilde{h}(z) = h(z) + re^{i\varepsilon}g(z) - g(z)$$
$$\widetilde{g}(z) = re^{i\varepsilon}g(z)$$

again with $r > 0$ being sufficiently close to 1.

Let's consider normalizing these exponentiated functions. If we begin with $f = h + \overline{g} \in S_H$ and $h - g \in CD(0)$ univalent, can we classify $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$ in terms of schlicht functions convex in the direction of the real axis? If $f \in S_H$ then f is univalent and therefore is locally univalent and we have satisfied the conditions of Theorem 3.12 for the function f giving us that $f \in CD(0)$ if $h - g \in CD(0)$ is univalent. Unfortunately we are not able to encode the local univalence and normalization of f into our system $\Delta = 0$, which means that we must impose these conditions onto the transformed function.

9.1.1 Flows Induced by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$ on f

Let's consider the transformations given by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$. By the Lie symmetries, we are guaranteed that

$$\widetilde{f}(z) = h(z) + \alpha z + \beta + \overline{g(z) + \alpha z + \beta}$$

is harmonic and that $\widetilde{h} - \widetilde{g} = h - g$ where clearly

$$\widetilde{h}(z) = h(z) + \alpha z + \beta$$
$$\widetilde{g}(z) = g(z) + \alpha z + \beta.$$

In order to have the transformed function be in S_H, we must appropriately normalize it. Since $f \in S_H$, f is univalent with $f(0) = 0$ and $f_z(0) = 1$. We can see that $\widetilde{f}(0) = 2\,\mathrm{Re}\,\beta$ and $\widetilde{f}_z(0) = 1 + \alpha$. If we require that $\beta = 0$ then we will get that $\widetilde{f}(0) = 0$. In order to impose the normalizations of S_H on $\widetilde{f}$ let's require that $\alpha \in \mathbb{R}$ and define a new function $\widehat{f}$ by defining

$$\widehat{h}(z) = \frac{\widetilde{h}(z)}{1+\alpha} = \frac{h(z)+\alpha z}{1+\alpha}$$
$$\widehat{g}(z) = \frac{\widetilde{g}(z)}{1+\alpha} = \frac{g(z)+\alpha z}{1+\alpha}.$$

In order to have a conventional notation with our transformed functions, let $\alpha = \varepsilon$. Requiring that $\varepsilon \in \mathbb{R}$ gives us a few properties, one of which is that $\varepsilon = \overline{\varepsilon}$, yielding that

$$\widehat{f}(z) = \widehat{h}(z) + \overline{\widehat{g}(z)} = \frac{1}{1+\varepsilon}\left(h(z) + \varepsilon z + \overline{g(z)+\varepsilon z}\right) = \frac{\widetilde{f}(z)}{1+\varepsilon}.$$

Now we can see that $\widehat{f}(0) = \dfrac{\widetilde{f}(0)}{1+\varepsilon} = 0$ and $\widehat{f}_z(0) = \dfrac{\widetilde{f}_z(0)}{1+\varepsilon} = 1$, giving us that $\widehat{f} \in S_H$ if $\widehat{f}$ is univalent. We can see that if $f \in S_H^O$ then $f_{\overline{z}}(0) = 0$, implying that $\widehat{f}_{\overline{z}}(0) = \dfrac{\varepsilon}{1+\varepsilon} \neq 0$ when $\varepsilon \neq 0$, and therefore if $\widehat{f}$ is univalent then $\widehat{f} \in S_H \setminus S_H^O$ when $\varepsilon \neq 0$ and $\widehat{f} \in S_H^O$ when $\varepsilon = 0$. Since $\varepsilon = 0$ corresponds to the trivial transformation of f then we get that $\widehat{f} \in S_H \setminus S_H^O$ under any nontrivial transformation of f.

The univalence of $\widehat{f}$ follows directly from $\widetilde{f}$ being univalent since $\widehat{f}$ is simply a nonzero multiple of $\widetilde{f}$, but in general we will not be able to show that $\widehat{f}$ is univalent if f is univalent but we will be able to show that $\widehat{f}$ is univalent if $\widehat{f}$ is locally univalent. This all leads to the following result.

Theorem 9.1. *Let $f = h + \overline{g} \in S_H \cap CD(0)$ and let $\widehat{f} = \widehat{h} + \overline{\widehat{g}}$ where $\widehat{h}(z) = \dfrac{h(z)+\varepsilon z}{1+\varepsilon}$ and $\widehat{g}(z) = \dfrac{g(z)+\varepsilon z}{1+\varepsilon}$ with $\varepsilon \in (-1, \infty)$. Then if $\widehat{f}$ is locally univalent then $\widehat{f} \in S_H \cap CD(0)$.*

Proof. From what is shown above, if $f \in S_H$ then $\widehat{f}(0) = 0$ and $\widehat{f}_z(0) = 1$. Since $f \in S_H \cap CD(0)$ then f is univalent and by Theorem 3.12 we have that $h - g \in CD(0)$ and is univalent. Since $\varepsilon \in (-1, \infty)$ then $\dfrac{h-g}{1+\varepsilon} \in CD(0)$ and is univalent. This then implies that $\widehat{h} - \widehat{g} \in CD(0)$ and is univalent since

$$\begin{aligned}\frac{h(z) - g(z)}{1+\varepsilon} &= \frac{h(z) + \varepsilon z - (g(z) + \varepsilon z)}{1+\varepsilon} \\ &= \frac{h(z) + \varepsilon z}{1+\varepsilon} - \frac{g(z) + \varepsilon z}{1+\varepsilon} \\ &= \widehat{h}(z) - \widehat{g}(z).\end{aligned}$$

Since $\widehat{h} - \widehat{g}$ is univalent and convex in the direction of the real axis and $\widehat{f}$ is locally univalent by assumption, by Theorem 3.12 we get that $\widehat{f}$ is univalent and convex in the direction of the real axis. Therefore we have shown that $\widehat{f} \in S_H \cap CD(0)$. □

Note that we require $\varepsilon \in (-1, \infty)$ and not simply $\mathbb{R} \setminus \{-1\}$ because we want ε to flow over a connected subset of $\mathbb{R}$ containing 0.

Let's consider an example of the flow of a function that has its convexity preserved by the flow of ε. In Example 9.2 we will consider the projection of Enneper's minimal surface.

Example 9.2. Let $h(z) = z$ and $g(z) = \frac{1}{3}z^3$ and let $\omega(z) = \dfrac{g'(z)}{h'(z)} = z^2$. We see that ω is analytic and $|\omega(z)| = |z^2| < 1$ for all $z \in \mathbb{D}$ and therefore, by Definition 3.2, we have that f has no singular points in $\mathbb{D}$. Since f is nonzero on $\partial\mathbb{D}$, by Theorem 3.6 we have that f is univalent in $\mathbb{D}$. By checking the appropriate normalizations, we see that $f \in S_H^O$. It can be shown that f is also convex in the direction of the real axis. For a visual check of this, we can verify the convexity by inspection of Figure 11(c). Therefore $f \in S_H^O \cap CD(0)$. Now we need to find what restrictions must be made on ε to guarantee that $\widehat{f}(z) = \frac{1}{1+\varepsilon}\left(z + \varepsilon z + \overline{\frac{1}{3}z^3 + \varepsilon z}\right)$ is locally univalent.

Suppose that $\varepsilon \in [0, \infty)$ and that $z = x + iy \in \mathbb{D}$. Since $z \in \mathbb{D}$ then $|z|^2 < 1$,

$x^2 - y^2 < 1$ and since $0 \leq \varepsilon$ then $2\varepsilon(x^2 - y^2) \leq 2\varepsilon$ which yields that

$$\begin{aligned} |z^2 + \varepsilon|^2 &= (x^2 + y^2) + 2\varepsilon(x^2 - y^2) + \varepsilon^2 \\ &= |z|^2 + 2\varepsilon(x^2 - y^2) + \varepsilon^2 \\ &< 1 + 2\varepsilon + \varepsilon^2 \\ &= |1 + \varepsilon|^2. \end{aligned}$$

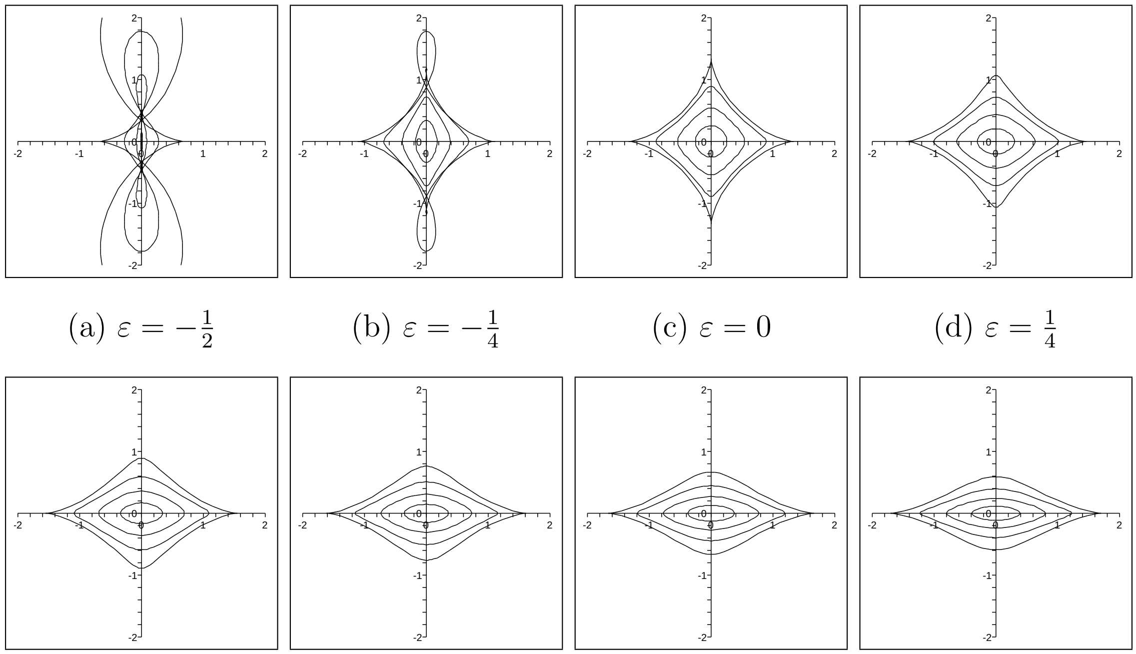
This can be simply applied to show that

$$\left| \frac{\widehat{g}'(z)}{\widehat{h}'(z)} \right| = \left| \frac{\frac{g'(z)+\varepsilon}{1+\varepsilon}}{\frac{h'(z)+\varepsilon}{1+\varepsilon}} \right| = \left| \frac{g'(z) + \varepsilon}{h'(z) + \varepsilon} \right| = \left| \frac{z^2 + \varepsilon}{1 + \varepsilon} \right| < 1.$$

This inequality yields that $\widehat{f}$ is locally univalent by Theorem 3.4 and therefore we have that $\widehat{f} \in S_H \cap CD(0)$ by Theorem 9.1 if $[0, \infty)$.

Let's analyze the function $\widehat{f}$ as ε ranges over values in $(-1, \infty)$. Consider that as ε gets large we have that $\widehat{f}(z) \approx 2x$ which gives us that the image of $\mathbb{D}$ under $\widehat{f}$ flattens out and approaches the real axis as $\varepsilon \to \infty$. A few of the images of $\mathbb{D}$ under $\widehat{f}$ as ε ranges from $-\frac{1}{2}$ to $\frac{5}{4}$ are shown in Figure 11.

Notice that as ε decreases from a positive to a negative value, we can see that the image is not univalent as we proved above thus showing necessary our original assumption that ε be nonnegative. Also notice that the convexity in the direction of the real axis is preserved as we allow $\varepsilon \geq 0$ to flow, this being a result of Theorem 9.1 since $\widehat{f}$ is locally univalent if $\varepsilon \geq 0$. Even for $\varepsilon < 0$, it appears that $\widehat{f} \in CD(0)$ as we let ε flow through negative values although this is not justified by the theorem presented in this section because $\widehat{f}$ would not be univalent.



(a) $\varepsilon = -\frac{1}{2}$ (b) $\varepsilon = -\frac{1}{4}$ (c) $\varepsilon = 0$ (d) $\varepsilon = \frac{1}{4}$

(e) $\varepsilon = \frac{1}{2}$ (f) $\varepsilon = \frac{3}{4}$ (g) $\varepsilon = 1$ (h) $\varepsilon = \frac{5}{4}$

Figure 11: *Images of* $\mathbb{D}$ *under the map* $\widehat{f} = \widehat{h} + \overline{\widehat{g}}$ *where* $\widehat{h}(z) = \dfrac{h(z) + \varepsilon z}{1+\varepsilon}$ *and* $\widehat{g}(z) = \dfrac{g(z) + \varepsilon z}{1+\varepsilon}$ *for* ε *ranging from* $-\frac{1}{2}$ *to* $\frac{5}{4}$ *where* $f(z) = h(z) + \overline{g(z)} = z + \overline{\frac{1}{3}z^3}$.

9.1.2 Flows Induced by $\mathbf{v}_5$ and $\mathbf{v}_6$ on f

Consider the exponentiations and compositions of $\mathbf{v}_5$ and $\mathbf{v}_6$ as given by

$$\widetilde{h}(z) = re^{i\varepsilon}h(z)$$

$$\widetilde{g}(z) = g(z) + re^{i\varepsilon}h(z) - h(z)$$

where $r > 0$ and is sufficiently close to 1. Define $\widetilde{f}$ by

$$\widetilde{f}(z) = \widetilde{h}(z) + \overline{\widetilde{g}(z)} = re^{i\varepsilon}h(z) + \overline{g(z) + re^{i\varepsilon}h(z) - h(z)}$$

and let's perform a similar analysis as we have previously done for $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v}_4$. First, we will consider $f = h + \overline{g} \in S_H$ and $h - g$ univalent with $h - g \in CD(0)$.

Again, in order for $\widetilde{f}$ to have the possibility of being in S_H, let's normalize $\widetilde{f}$ accordingly. We can see that $\widetilde{f}(0) = 0$ and $\widetilde{f}_z(0) = re^{i\varepsilon}$. Define $\widehat{h}$ and $\widehat{g}$ by multiplying $\widetilde{h}$ and $\widetilde{g}$, respectively, by $se^{-i\varepsilon}$ where $s = r^{-1}$ and define $\widehat{f}$ by $\widehat{f}(z) = \widehat{h}(z) + \overline{\widehat{g}(z)}$. Therefore

$$\widehat{h}(z) = se^{-i\varepsilon}\widetilde{h}(z) = h(z)$$

$$\widehat{g}(z) = se^{-i\varepsilon}\widetilde{g}(z) = h(z) - se^{-i\varepsilon}h(z) + se^{-i\varepsilon}g(z)$$

implying that

$$\widehat{f}(z) = h(z) + \overline{h(z) - se^{-i\varepsilon}h(z) + se^{-i\varepsilon}g(z)}.$$

Unlike the case for $\mathbf{v}_1, \ldots, \mathbf{v}_4$, for $\mathbf{v}_5$ and $\mathbf{v}_6$ we have that $\widehat{f}$ is not simply a complex constant multiple of $\widetilde{f}$ but with the definition of $\widehat{f}$ as given above we see that $\widehat{f}(0) = 0$ and $\widehat{f}_z(0) = 1$. Also, we see that

$$\widehat{h} - \widehat{g} = se^{-i\varepsilon}(h - g),$$

yielding that $\widehat{h} - \widehat{g}$ is univalent if and only if $h - g$ is, but this does give rise to a question of convexity. If $h - g \in CD(0)$ then $\widehat{h} - \widehat{g}$ will only be in $CD(0)$ if $se^{-i\varepsilon}$ is real. Therefore we must require that ε be an integer multiple of π. This additional requirement gives us that

$$\widehat{f}(z) = h(z) + \overline{h(z) - sh(z) + sg(z)}$$

if $\varepsilon = 2k\pi$ where $s \in \mathbb{R}$ and s is close to 1 or

$$\widehat{f}(z) = h(z) + \overline{h(z) + sh(z) - sg(z)}$$

if $\varepsilon = (2k + 1)\pi$ where s is close to -1 and in both cases $k \in \mathbb{Z}$.

For simplicity, let's consider the first case with $\varepsilon = 0$ and again to maintain uniformity in our notation, make the substitution that $s = e^{\varepsilon}$ where ε is close to 0.

Now we have that

$$\widehat{f}(z) = h(z) + \overline{h(z) - e^{\varepsilon}h(z) + e^{\varepsilon}g(z)}$$

is harmonic and we have shown that $\widehat{f}(0) = 0$, $\widehat{f}_z(0) = 1$ and that $\widehat{h} - \widehat{g}$ is univalent and in $CD(0)$.

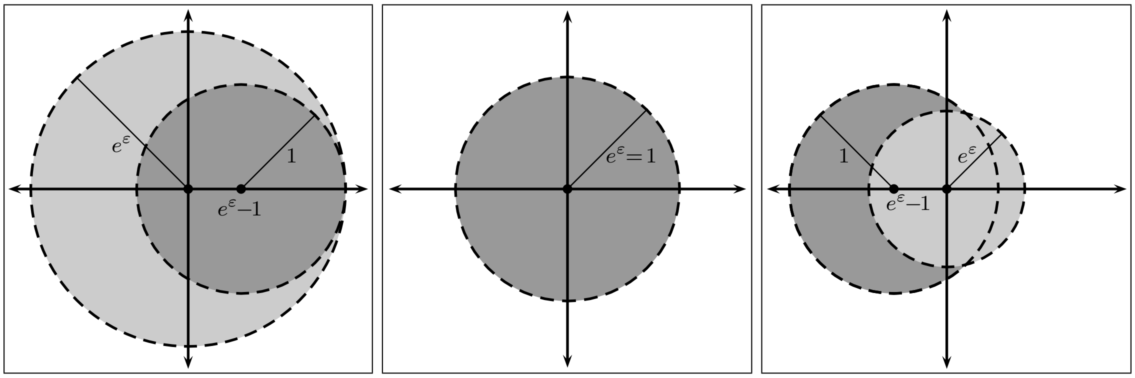
Now we want to find what restriction on ε will guarantee that $\widehat{f}$ be locally univalent. Theorem 3.4 gives us that in order to get the local univalence of $\widehat{f}$ we must find ε so that $\left|\frac{\widehat{g}'(z)}{\widehat{h}'(z)}\right| < 1$ holds true. This in turn implies that

$$\left|e^{\varepsilon}\frac{g'(z)}{h'(z)} - (e^{\varepsilon} - 1)\right| < 1.$$

If we think about this geometrically, the above inequality implies that the maximum distance between any point in the set $\{z \mid |z| < e^{\varepsilon}\}$ and the point $e^{\varepsilon} - 1$ is less than 1. This can only occur when ε is 0 which gives that $\widehat{f} = f$, the trivial transformation. This can be seen in Figure 12 where the light gray region is the set of all points where $e^{\varepsilon}\frac{g'(z)}{h'(z)}$ must lie for a given $z \in \mathbb{D}$ and the dark gray region is the set of all points whose distance from $e^{\varepsilon} - 1$ is less than 1. For this inequality to hold, the light gray region must equal the dark gray one because this is the only case where the distance between the point $e^{\varepsilon}\frac{g'(z)}{h'(z)}$ and $e^{\varepsilon} - 1$ is less than 1 implying that $\left|\left(e^{\varepsilon}\frac{g'(z)}{h'(z)}\right) - (e^{\varepsilon} - 1)\right| < 1$.

Therefore we are unable to prove anything about the local univalence of $\widehat{f}$ in general. But if we show that $\widehat{f}$ is locally univalent in a specific case, then we have Theorem 9.3.

Theorem 9.3. *Let $f = h + \overline{g} \in S_H \cap CD(0)$ and let $\widehat{f}(z) = \widehat{h}(z) + \overline{\widehat{g}(z)}$ where $\widehat{h}(z) = h(z)$ and $\widehat{g}(z) = h(z) - e^{\varepsilon}h(z) + e^{\varepsilon}g(z)$ with $\varepsilon \in \mathbb{R}$. If $\widehat{f}$ is locally univalent then $\widehat{f} \in S_H \cap CD(0)$.*



(a) For $e^{\varepsilon} > 1$ (b) For $e^{\varepsilon} = 1$ (c) For $e^{\varepsilon} < 1$

Figure 12: *Geometric interpretation of* $\left|e^{\varepsilon}\frac{g'(z)}{h'(z)} - (e^{\varepsilon} - 1)\right| < 1$

The proof of Theorem 9.3 is nearly identical to the proof of Theorem 9.1 where we can see that if $h - g \in CD(0)$ and univalent then $\widehat{h} - \widehat{g} = e^{\varepsilon}(h - g) \in CD(0)$ and is univalent.

Note that if we begin with $f \in S_H^O$ then we get that $f_{\overline{z}}(0) = 0$, but $\widehat{f}_{\overline{z}}(0) = 1 - e^{\varepsilon} \neq 0$ for $\varepsilon \neq 0$ and therefore when we transform f nontrivially we get that $\widehat{f} \in S_H \setminus S_H^O$.

9.1.3 Flows Induced by $\mathbf{v}_7$ and $\mathbf{v}_8$ on f

Consider the exponentiation and composition that arises from $\mathbf{v}_7$ and $\mathbf{v}_8$. If we let $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$ be the transform of $f = h + \overline{g}$ then we have that

$$\widetilde{h}(z) = h(z) + re^{i\varepsilon}g(z) - g(z)$$

$$\widetilde{g}(z) = re^{i\varepsilon}g(z)$$

implying that

$$\widetilde{f}(z) = h(z) + re^{i\varepsilon}g(z) - g(z) + \overline{re^{i\varepsilon}g(z)}.$$

Notice that if we begin with $f \in S_H^O$ then we have that $f(0) = 0$, $f_z(0) = 1$ and $f_{\overline{z}}(0) = 0$, which implies that $\widetilde{f}(0) = 0$, $\widetilde{f}_z(0) = 1$ and $\widetilde{f}_{\overline{z}}(0) = 0$, giving us that if $\widetilde{f}$ is univalent then $\widetilde{f} \in S_H^O$ without any further normalization. If we begin with $f \in S_H$ and let $g'(0) = b_2$ then we get that $\widetilde{f}(0) = 0$ and $f_z(0) = 1 + re^{i\varepsilon}b_2 - b_2$ which would need further normalization in order to possibly be in S_H.

Let's consider $f \in S_H^O$ with $h - g \in CD(0)$ and analyze $\widetilde{f}(z) = h(z) + re^{i\varepsilon}g(z) - g(z) + \overline{re^{i\varepsilon}g(z)}$. If we expand g as $g = g^1 + ig^2$ we get the simplification of $\widetilde{f}$ as

$$\begin{aligned}\widetilde{f}(z) &= h(z) + re^{i\varepsilon}g(z) - g(z) + \overline{re^{i\varepsilon}g(z)} \\ &= h(z) - g(z) + 2rg^1(z)\left(\frac{e^{i\varepsilon} + e^{-i\varepsilon}}{2}\right) - 2rg^2(z)\left(\frac{e^{i\varepsilon} - e^{-i\varepsilon}}{2i}\right) \\ &= h(z) - g(z) + 2r\left(g^1(z)\cos\varepsilon - g^2(z)\sin\varepsilon\right)\end{aligned}$$

showing that $\widetilde{f}$ is $h - g$ plus a continuous real-valued function.

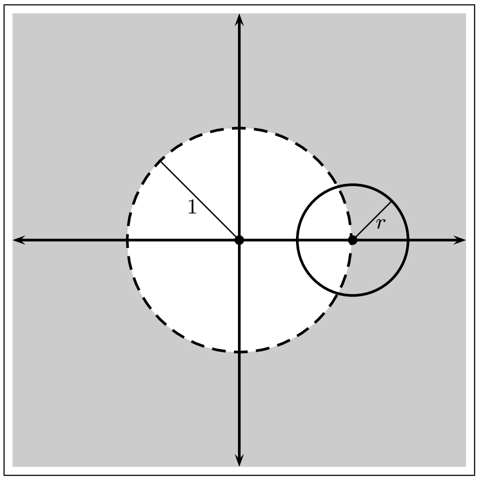
As before, we want to know what conditions must be placed on r and ε so that $\widetilde{f}$ is locally univalent when f is. The local univalence of $\widetilde{f}$ is equivalent to requiring that $\left|\frac{\widetilde{h}'(z)}{\widetilde{g}'(z)}\right| > 1$ on $\mathbb{D}$ which in turn implies that

$$\left|\frac{h'(z)}{g'(z)} - \left(1 - re^{i\varepsilon}\right)\right| > |r|.$$

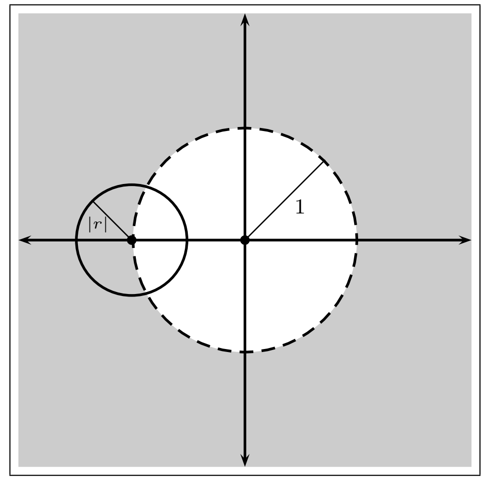
Geometrically we may consider this as requiring that the minimum distance from outside the unit disk to somewhere on the circle centered at 1 of radius $|r|$ be at least $|r|$. This can be seen in Figure 13 where the light gray region is the set of all possible values of $\frac{h'(z)}{g'(z)}$ and the solid circle is the set of all possible values for $1 - re^{i\varepsilon}$.

We can see that this inequality is satisfied for $-1 < r < 1$ and $\varepsilon = \pi k$ with $k \in \mathbb{Z}$. We may thus assume that $\varepsilon = 0$. Since $r = 0$ would imply that $h + \overline{g} = h - g$ yielding that $g \equiv 0$, we may additionally assume that $0 < |r| < 1$. We can now state the following two results.

Theorem 9.4. *Let $f = h + \overline{g} \in S_H^O \cap CD(0)$ and let $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$ where $\widetilde{h}(z) = h(z) + re^{i\varepsilon}g(z) - g(z)$ and $\widetilde{g}(z) = re^{i\varepsilon}g(z)$. If $\widetilde{f}$ is locally univalent then $\widetilde{f} \in S_H^O \cap CD(0)$*



(a) For $r > 0$



(b) For $r < 0$

Figure 13: *Geometric interpretation of* $\left|\frac{h'(z)}{g'(z)} - (1 - re^{i\varepsilon})\right| > |r|$

for any $\varepsilon, r \in \mathbb{R}$. *If* $\varepsilon = 0$ *and* $-1 < r < 1$ *then* $\widehat{f}$ *is locally univalent and* $\widetilde{f} \in S_H^O \cap CD(0)$.

Proof. We showed previously that if $f \in S_H^O$ then $\widetilde{f}(0) = 0$, $\widetilde{f}_z(0) = 1$ and $\widetilde{f}_{\overline{z}}(0) = 0$. Now since $f \in CD(0)$ and is univalent then $h - g \in CD(0)$ and is univalent by Theorem 3.12 which gives us that $\widetilde{h} - \widetilde{g}$ is as well. Therefore $\widetilde{f} \in CD(0)$ and is univalent implying that $\widetilde{f} \in S_H^O$ because of the normalizations shown implying that $\widetilde{f} \in S_H^O \cap CD(0)$. The second statement follows directly from the first and by the argument given in association with Figure 13. □

We give both parts of this theorem because there may be functions f where the geometric interpretation as shown in Figure 13 is too broad; that is, $\left|\frac{h'(z)}{g'(z)}\right|$ may be greater than some $\delta > 1$ on $\mathbb{D}$, which would allow for ε to flow away from 0 and not be required to remain fixed.

9.2 An Infinite Dimensional Subalgebra

Let's begin again with $f = h + \overline{g} \in S_H$ and $h - g \in CD(0)$ univalent and consider a specific infinitesimal generator. In Appendix C.2 we show the calculations for the infinite dimensional subalgebra relative to the Lie symmetries of the harmonic functions where $h - g$ is a fixed function and where we let $(a^3 + ia^4)(z, g) = \phi^1(z) + i\phi^2(z) + g^1 + ig^2 = \phi(z) + g$ with ϕ being an analytic function of only z. Recall that the general infinitesimal generator is

$$\mathbf{v} = a^3 \frac{\partial}{\partial h^1} + a^4 \frac{\partial}{\partial h^2} + a^3 \frac{\partial}{\partial g^1} + a^4 \frac{\partial}{\partial g^2},$$

where $a^3 + ia^4$ is analytic in z, h and g. In this section we will consider the generator

$$\mathbf{v} = \left(\phi^1(z) + g^1\right) \frac{\partial}{\partial h^1} + \left(\phi^2(z) + g^2\right) \frac{\partial}{\partial h^2} + \left(\phi^1(z) + g^1\right) \frac{\partial}{\partial g^1} + \left(\phi^2(z) + g^2\right) \frac{\partial}{\partial g^2}.$$

The exponentiation of $\mathbf{v}$ as shown in Appendix C.2 gives us that

$$\widetilde{h}(z) = h(z) - g(z) + e^{\varepsilon} g(z) + \phi(z) \left(e^{\varepsilon} - 1\right)$$
$$\widetilde{g}(z) = e^{\varepsilon} g(z) + \phi(z) \left(e^{\varepsilon} - 1\right)$$

and making the substitutions $\alpha = e^{\varepsilon}$ and $\varphi = \left(e^{\varepsilon} - 1\right) \phi$ yields

$$\widetilde{h}(z) = h(z) - g(z) + \alpha g(z) + \varphi(z)$$
$$\widetilde{g}(z) = \alpha g(z) + \varphi(z)$$

where $\alpha > 0$. This in turn gives us that the transformed function $\widetilde{f}$ is given by

$$\widetilde{f}(z) = h(z) - g(z) + \alpha g(z) + \varphi(z) + \overline{\alpha g(z) + \varphi(z)}.$$

We can see that φ is analytic if and only if ϕ is analytic. We will be concerned with the nontrivial transformations of f which are those where $\alpha \neq 1$; that is, $\varphi \neq 0$ if $\phi \not\equiv 0$. We can simplify $\widetilde{f}$ to

$$\widetilde{f}(z) = h(z) - g(z) + 2\alpha g^1(z) + \varphi^1(z)$$

which shows that $\widetilde{f}$ is $h-g$ plus a continuous real-valued function as we did above in the finite dimensional case.

The technique of Lie symmetries was essential in leading us to the function

$$\widetilde{f}(z) = h(z) - g(z) + \alpha g(z) + \varphi(z) + \overline{\alpha g(z) + \varphi(z)}$$

but we can see that $\widetilde{f}$ is harmonic regardless of what we let α be and we need not force φ to be 0 when $\alpha = 1$. By relaxing these conditions we can see that we preserve $h - g = \widetilde{h} - \widetilde{g}$ and therefore we will continue the results from this point with φ being analytic and $\alpha \in \mathbb{C}$.

We can form a similar result as those given above by using Theorem 3.12 in the same way as before. This gives us:

Theorem 9.5. *Let $h-g \in CD(0)$ with $h-g$ univalent. Then if $\widetilde{f}(z) = h(z)-g(z)+\alpha g(z) + \varphi(z) + \overline{\alpha g(z) + \varphi(z)}$ is locally univalent then $\widetilde{f} \in CD(0)$ and is univalent.*

Proof. Since $h - g = \widetilde{h} - \widetilde{g}$ and $\widetilde{f}$ is locally univalent then by Theorem 3.12 we have that $\widetilde{f}$ is globally univalent and $\widetilde{f} \in CD(0)$. □

For $\widetilde{f}$ to be locally univalent we must have that

$$\left|\frac{\widetilde{g}'(z)}{\widetilde{h}'(z)}\right| < 1$$

on $\mathbb{D}$. If we let $\dfrac{\widetilde{g}'(z)}{\widetilde{h}'(z)} = \dfrac{N(z)}{D(z)}$ and force $\dfrac{N(z)}{D(z)} \in \mathbb{D}$ for all $z \in \mathbb{D}$ then we will have that $\left|\dfrac{\widetilde{g}'(z)}{\widetilde{h}'(z)}\right| < 1$ implying that $\widetilde{f}$ is locally univalent.

We can see that since $\dfrac{\widetilde{g}'}{\widetilde{h}'} = \dfrac{N}{D}$ then $\dfrac{N}{D} = \dfrac{\varphi' + \alpha g'}{h' - g' + \varphi' + \alpha g'}$ allowing us to solve for φ in terms of N and D. Since $\dfrac{N}{D} = \dfrac{\varphi' + \alpha g'}{h' - g' + \varphi' + \alpha g'}$ then $Nh' - Ng' + N\varphi' + N\alpha g' = D\varphi' + D\alpha g'$ implying that $\varphi'(D-N) = N(h'-g') - \alpha g'(D-N)$. Therefore

$\varphi' = (h' - g')\dfrac{N}{D-N} - \alpha g'$ and integrating this in closed form gives us that

$$\varphi(z) = \int \left[(h'(z) - g'(z)) \frac{N(z)}{D(z) - N(z)} \right] dz - \alpha g(z).$$

Therefore if $\dfrac{N}{D}$ maps $\mathbb{D}$ into $\mathbb{D}$ then we will have that $\widetilde{f}$ will be locally univalent and therefore univalent and convex in the direction of the real axis by Theorem 9.5.

Once we have the function φ we can simplify $\widetilde{h}$ and $\widetilde{g}$ to

$$\widetilde{h}(z) = h(z) - g(z) + \int \left[(h'(z) - g'(z)) \frac{N(z)}{D(z) - N(z)} \right] dz \tag{6}$$

$$\widetilde{g}(z) = \int \left[(h'(z) - g'(z)) \frac{N(z)}{D(z) - N(z)} \right] dz. \tag{7}$$

With the above analysis we can restate Theorem 9.5 in the following way.

Theorem 9.6. *Let $h - g \in CD(0)$ with $h - g$ univalent and let $\widetilde{h}$ and $\widetilde{g}$ be defined by (6) and (7), respectively. If $\left|\dfrac{N(z)}{D(z)}\right| < 1$ for all $z \in \mathbb{D}$ then $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$ is univalent and is in $CD(0)$.*

The proof of Theorem 9.6 follows directly from what is shown previously and from Theorem 9.5.

Let's consider a few examples of functions N and D that satisfy the above conditions.

Example 9.7. Consider the projection of Enneper's surface as shown in Figure 11(c) and given by $h(z) = z$ and $g(z) = \frac{1}{3}z^3$ and suppose that $N(z) = (z + \alpha)^2$ and $D(z) = (1 + \alpha z)^2$, where $-1 < \alpha < 1$. This gives us that $\dfrac{N}{D}$ is the square of a fractional linear transformation that maps $\mathbb{D}$ onto $\mathbb{D}$ and therefore $\left|\dfrac{N(z)}{D(z)}\right| < 1$ for all $z \in \mathbb{D}$. Now we have that

$$\begin{aligned}
\widetilde{h}(z) &= z - \frac{1}{3}z^3 + \int \left[\left(1 - z^2\right) \frac{(z+\alpha)^2}{(1-z^2)(1-\alpha^2)} \right] dz \\
&= z - \frac{1}{3}z^3 + \frac{1}{1-\alpha^2} \int \left(z^2 + 2\alpha z + \alpha^2\right) dz \\
&= z - \frac{1}{3}z^3 + \frac{\frac{1}{3}z^3 + \alpha z^2 + \alpha^2 z}{1-\alpha^2} \\
&= \left(\frac{1}{1-\alpha^2}\right) \left(z + \alpha z^2 + \frac{\alpha^2}{3} z^3\right)
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{g}(z) &= \int \left[\left(1 - z^2\right) \frac{(z+\alpha)^2}{(1-z^2)(1-\alpha^2)} \right] dz \\
&= \frac{1}{1-\alpha^2} \int \left(z^2 + 2\alpha z + \alpha^2\right) dz \\
&= \frac{\frac{1}{3}z^3 + \alpha z^2 + \alpha^2 z}{1-\alpha^2} \\
&= \left(\frac{1}{1-\alpha^2}\right) \left(\alpha^2 z + \alpha z^2 + \frac{1}{3} z^3\right).
\end{aligned}$$

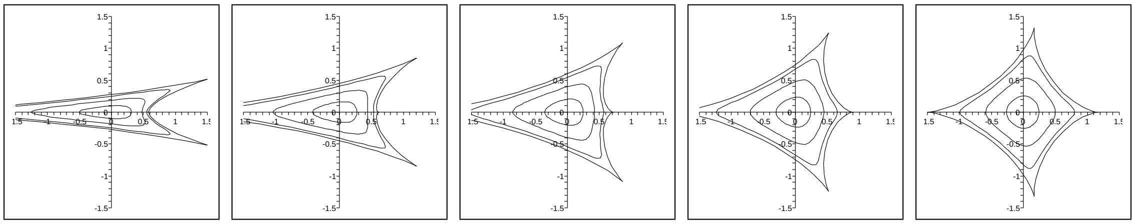
Let's normalize $\widetilde{f}$. If we let $\widehat{h} = (1-\alpha^2)\widetilde{h}$ and $\widehat{g} = (1-\alpha^2)\widetilde{g}$ then we get that $\widehat{f} = (1-\alpha^2)\widetilde{f}$; that is

$$\begin{aligned}
\widehat{h}(z) &= z + \alpha z^2 + \frac{\alpha^2}{3} z^3 \\
\widehat{g}(z) &= \alpha^2 z + \alpha z^2 + \frac{1}{3} z^3
\end{aligned}$$

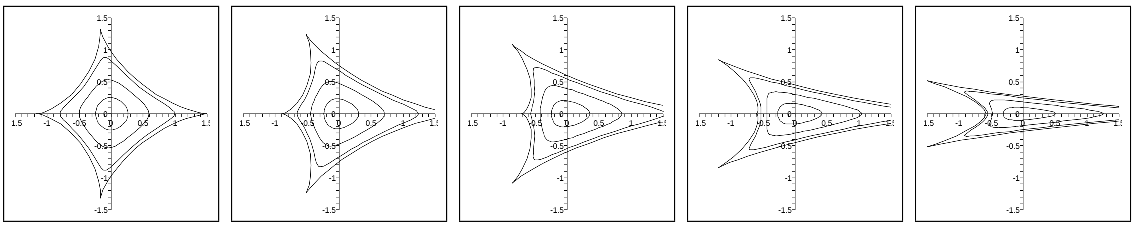
giving that

$$\widehat{f}(z) = \left(z + \alpha z^2 + \frac{\alpha^2}{3} z^3\right) + \overline{\left(\alpha^2 z + \alpha z^2 + \frac{1}{3} z^3\right)}.$$

We see that this gives that $\widehat{f}(0) = 0$ and $f_z(0) = 1$. Theorem 9.6 gives that $\widetilde{f} \in CD(0)$ is univalent and since $\widehat{f}$ is a real valued non-zero multiple of $\widetilde{f}$ then $\widehat{f} \in CD(0)$ and is univalent. Therefore $\widehat{f} \in S_H \cap CD(0)$ for all $-1 < \alpha < 1$. In Figure 14, several images of $\mathbb{D}$ under $\widehat{f}$ as α ranges from -0.777 to 0.777 are shown and we see that the convexity and univalence of $\widehat{f}$ are preserved as we flow between the functions.



(a) $\alpha = -0.777$ (b) $\alpha = -0.605$ (c) $\alpha = -0.432$ (d) $\alpha = -0.259$ (e) $\alpha = -0.086$



(f) $\alpha = 0.086$ (g) $\alpha = 0.259$ (h) $\alpha = 0.432$ (i) $\alpha = 0.605$ (j) $\alpha = 0.777$

Figure 14: *Images of $\mathbb{D}$ under the map $\widehat{f} = \widehat{h} + \overline{\widehat{g}}$ where $\widehat{h}(z) = z + \alpha z^2 + \dfrac{\alpha^2}{3} z^3$ and $\widehat{g}(z) = \alpha^2 z + \alpha z^2 + \dfrac{1}{3} z^3$ for α ranging from -0.777 to 0.777.*

Let's consider the limiting cases of $\widehat{f}$ as α approaches -1 and 1. It will be helpful to consider the real and imaginary parts of $\widehat{f}$ separately. We see that

$$\begin{aligned} \operatorname{Re}\left\{\widehat{f}(z)\right\} &= \operatorname{Re}\left\{\widehat{h}(z) + \widehat{g}(z)\right\} \\ &= \operatorname{Re}\left\{(1+\alpha^2)z + 2\alpha z^2 + \frac{1+\alpha^2}{3} z^3\right\} \end{aligned}$$

and

$$\begin{aligned} \operatorname{Im}\left\{\widehat{f}(z)\right\} &= \operatorname{Im}\left\{\widehat{h}(z) - \widehat{g}(z)\right\} \\ &= \operatorname{Re}\left\{(1-\alpha^2)\left(z - \frac{1}{3}z^3\right)\right\}. \end{aligned}$$

We can see that

$$\lim_{\alpha \to 1^-} \operatorname{Re}\left\{\widehat{f}(z)\right\} = \operatorname{Re}\left\{2z + 2z^2 + \frac{2}{3}z^3\right\}$$

and

$$\lim_{\alpha\to -1^+} \operatorname{Re}\left\{\widehat{f}(z)\right\} = \operatorname{Re}\left\{2z - 2z^2 + \frac{2}{3}z^3\right\}.$$

For the imaginary part of $\widehat{f}$ we get

$$\lim_{\alpha\to 1^-} \operatorname{Im}\left\{\widehat{f}(z)\right\} = \lim_{\alpha\to 1^-} \operatorname{Re}\left\{(1-\alpha^2)\left(z - \frac{1}{3}z^3\right)\right\} = 0$$

and similarly for α approaching -1 from the right. In the limiting case we see that $\widehat{f}$ becomes purely real.

In essence, Example 9.7 shows taking a harmonic univalent polynomial in S_H and generating a new class of harmonic univalent polynomials in S_H. With the exception of Suffridge's paper [15], the field of harmonic univalent polynomials has very few known results. This approach with Lie symmetries may prove useful in solving some results in this area.

Example 9.8. Consider the projection of Scherk's doubly periodic minimal surface which is given by

$$\begin{aligned} h(z) &= \frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{1+iz}{1-iz}\right) \\ g(z) &= -\frac{1}{4}\ln\left(\frac{1+z}{1-z}\right) - \frac{i}{4}\ln\left(\frac{1+iz}{1-iz}\right). \end{aligned}$$

If we let $N(z) = \alpha^2 z^2$ and $D(z) = 1$ then we have that $\left|\frac{N(z)}{D(z)}\right| < 1$ on $\mathbb{D}$ if $|\alpha| \leq 1$ with $\alpha \neq -1, 1$.

Solving for $\widetilde{h}(z)$ and $\widetilde{g}(z)$ gives us that

$$\begin{aligned}
\widetilde{h}(z) &= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + \int\left[\left(\frac{1}{1-z^2}\right)\left(\frac{\alpha^2 z^2}{1-\alpha^2 z^2}\right)\right]dz \\
&= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + \int \frac{\alpha^2 z^2}{(1-\alpha^2 z^2)(1-z^2)}dz \\
&= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + \frac{\alpha^2}{2(1-\alpha^2)}\int\left(\frac{1}{1+z} + \frac{1}{1-z} - \frac{1}{1+\alpha z} - \frac{1}{1-\alpha z}\right)dz \\
&= \frac{1}{2}\ln\left(\frac{1+z}{1-z}\right) + \frac{\alpha^2}{2(1-\alpha^2)}\left(\ln\left(\frac{1+z}{1-z}\right) - \frac{1}{\alpha}\ln\left(\frac{1+\alpha z}{1-\alpha z}\right)\right) \\
&= \frac{1}{2(1-\alpha^2)}\ln\left(\frac{1+z}{1-z}\right) - \frac{\alpha}{2(1-\alpha^2)}\ln\left(\frac{1+\alpha z}{1-\alpha z}\right)
\end{aligned}$$

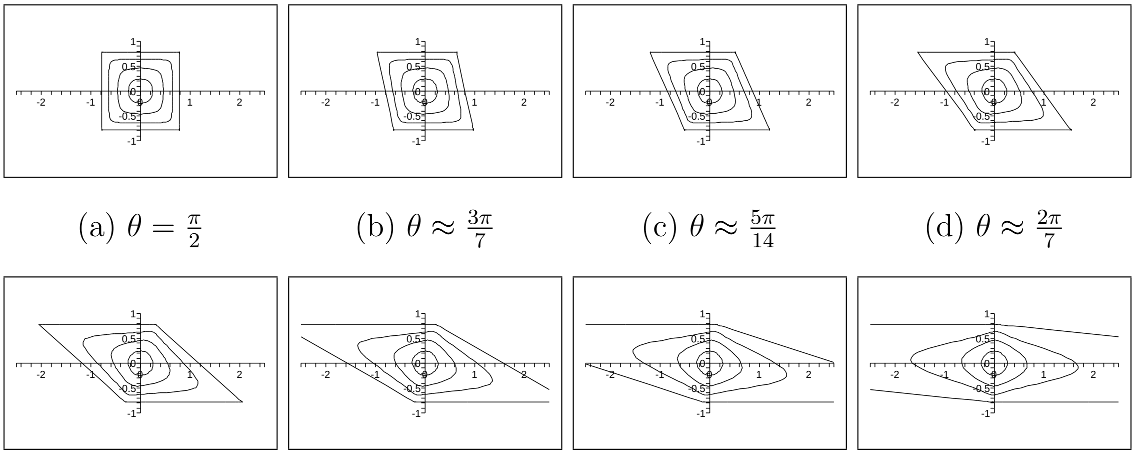
and

$$\begin{aligned}
\widetilde{g}(z) &= \int\left[\left(\frac{1}{1-z^2}\right)\left(\frac{\alpha^2 z^2}{1-\alpha^2 z^2}\right)\right]dz \\
&= \int \frac{\alpha^2 z^2}{(1-\alpha^2 z^2)(1-z^2)}dz \\
&= \int\left(\frac{1}{1+z} + \frac{1}{1-z} - \frac{1}{1+\alpha z} - \frac{1}{1-\alpha z}\right)dz \\
&= \frac{\alpha^2}{2(1-\alpha^2)}\left(\ln\left(\frac{1+z}{1-z}\right) - \frac{1}{\alpha}\ln\left(\frac{1+\alpha z}{1-\alpha z}\right)\right) \\
&= \frac{\alpha^2}{2(1-\alpha^2)}\ln\left(\frac{1+z}{1-z}\right) - \frac{\alpha}{2(1-\alpha^2)}\ln\left(\frac{1+\alpha z}{1-\alpha z}\right)
\end{aligned}$$

thus giving that $\widetilde{f}(z) = \widetilde{h}(z) + \overline{\widetilde{g}(z)}$ is univalent and is convex in the direction of the real axis by Theorem 9.6.

If we let $\alpha = i$ then $\widetilde{h}(z) = h(z)$ and $\widetilde{g}(z) = g(z)$ giving that $\widetilde{f}(z)$ is the projection of Scherk's doubly periodic surface. If we let $\alpha = e^{i\theta}$ and allow θ to flow from $\frac{\pi}{2}$ to 0 then $\widetilde{f}$ will become the projection of the helicoid in the limiting case. This was proven in [6]. Figure 15 shows images of $\widetilde{f}$ as θ flows from $\frac{\pi}{2}$ to 0.

Example 9.9. This next example is a family of slit mappings that were studied previously by Dorff and Suffridge [5]. It is interesting that the manner in which we devised the family of slit mappings in this example is different from how Dorff and



(a) $\theta = \frac{\pi}{2}$ (b) $\theta \approx \frac{3\pi}{7}$ (c) $\theta \approx \frac{5\pi}{14}$ (d) $\theta \approx \frac{2\pi}{7}$

(e) $\theta \approx \frac{3\pi}{14}$ (f) $\theta \approx \frac{\pi}{7}$ (g) $\theta \approx \frac{\pi}{14}$ (h) $\theta \approx 0$

Figure 15: *Images of* $\mathbb{D}$ *under the map* $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$ *where* $\widetilde{h}(z) = \dfrac{1}{2(1-\alpha^2)} \ln\left(\dfrac{1+z}{1-z}\right) - \dfrac{\alpha}{2(1-\alpha^2)} \ln\left(\dfrac{1+\alpha z}{1-\alpha z}\right)$ *and* $\widetilde{g}(z) = \dfrac{\alpha^2}{2(1-\alpha^2)} \ln\left(\dfrac{1+z}{1-z}\right) - \dfrac{\alpha}{2(1-\alpha^2)} \ln\left(\dfrac{1+\alpha z}{1-\alpha z}\right)$ *for* $\alpha = e^{i\theta}$ *with* θ *ranging from* $\frac{\pi}{2}$ *to* 0.

Suffridge approached the problem. To understand the significance of this family of mappings, we first must give some background.

For $f \in S_H^O$, the inner mapping radius, $\rho_O(f)$, of the domain $f(\mathbb{D})$ is the real number $F'(0)$, where $F(z)$ is the analytic function that maps $\mathbb{D}$ onto $f(\mathbb{D})$ and satisfies the conditions $F(0) = 0$, $F'(0) > 0$. By the Riemann Mapping Theorem, there always exists a unique such F. If $f \in S_H$, the inner mapping radius is denoted by $\rho(f)$. The lower bound for $\rho(f)$ is 0. It was conjectured by Sheil-Small ([3], [13]) that the lower bound for $\rho_O(f)$ is $\frac{2}{3}$. The upper bound for $\rho(f)$ cannot be larger than 2π, because of the Koebe $\frac{1}{4}$-theorem and Hall's result [11] showing that $f(\mathbb{D})$ omits some point on any circle of radius R, where $R \geq r = \frac{\pi}{2}$. Similarly, $\rho_O(f)$ is bounded above by $\dfrac{8\pi\sqrt{3}}{9} < 4.837$. Sheil-Small also conjectured that $\rho(f) \leq \frac{\pi}{2}$ ([3], [13]).

In [5] Dorff and Suffridge presented a collection of univalent, harmonic 1-slit mappings, $f = h + \overline{g}$ with $g'(z) = \omega(z)h'(z)$, whose slit is on the negative real axis. By changing $\omega(z)$, they were able to slide the slit away from the origin. For $f \in S_H^O$, the tip of the slit can be brought as close as $-\frac{1}{6}$ and as far as $-\frac{1}{2}$. The inner mapping radius for this last function is 2. When they enlarged the class so that $f \in S_H$, the slit point moved from 0 to -1, and thus the inner mapping radius was brought arbitrarily close to 4. Hence, these functions disproved the conjectures of Sheil-Small and provided the largest known values for the inner mapping radius. In this example, we get these same results but by a different approach than the one used by Dorff and Suffridge.

We will show the derivation of this class of slit mappings using Theorem 9.6, which was not the method used in [5]. Let h and g be defined by

$$h(z) = \frac{z - \frac{1}{2}z^2 - \frac{1}{6}z^3}{(1-z)^3}$$

and

$$g(z) = \frac{\frac{1}{2}z^2 - \frac{1}{6}z^3}{(1-z)^3}.$$

Here we have that

$$h(z) - g(z) = \frac{z}{(1-z)^2}$$

yielding that $h - g$ is the slit mapping with slit contained in the negative real axis and therefore $h - g \in CD(0)$. Suppose that $h(z_1) - g(z_1) = h(z_2) - g(z_2)$ for $z_1 \neq z_2$, with $z_1, z_2 \in \mathbb{D}$. This implies that $\dfrac{z_1}{(1-z_1)^2} = \dfrac{z_2}{(1-z_2)^2} \Rightarrow z_1 - 2z_1z_2 + z_1z_2^2 = z_2 - 2z_1z_2 + z_2z_1^2$ and hence $z_1 - z_2 = z_1z_2(z_1 - z_2)$. Since $z_1 \neq z_2$, we may divide by $z_1 - z_2$ to get that $z_1z_2 = 1$ but this implies that $|z_1| \geq 1$ or $|z_2| \geq 1$ which contradicts the fact that $z_1, z_2 \in \mathbb{D}$ and thus our supposition was false. Therefore $h - g$ is univalent and convex in the direction of the real axis.

Let $\dfrac{N(z)}{D(z)} = \dfrac{z(z+\alpha)}{1+\alpha z}$ and we can see that $\left|\dfrac{N}{D}\right| < 1$ for all $z \in \mathbb{D}$ and $-1 \leq \alpha \leq 1$ since $\dfrac{N}{D}$ is the product of a fractional linear transformation with norm no greater than 1 and the identity map on $\mathbb{D}$. From this definition and (6) and (7) we get $\widetilde{h}$ and $\widetilde{g}$ defined by

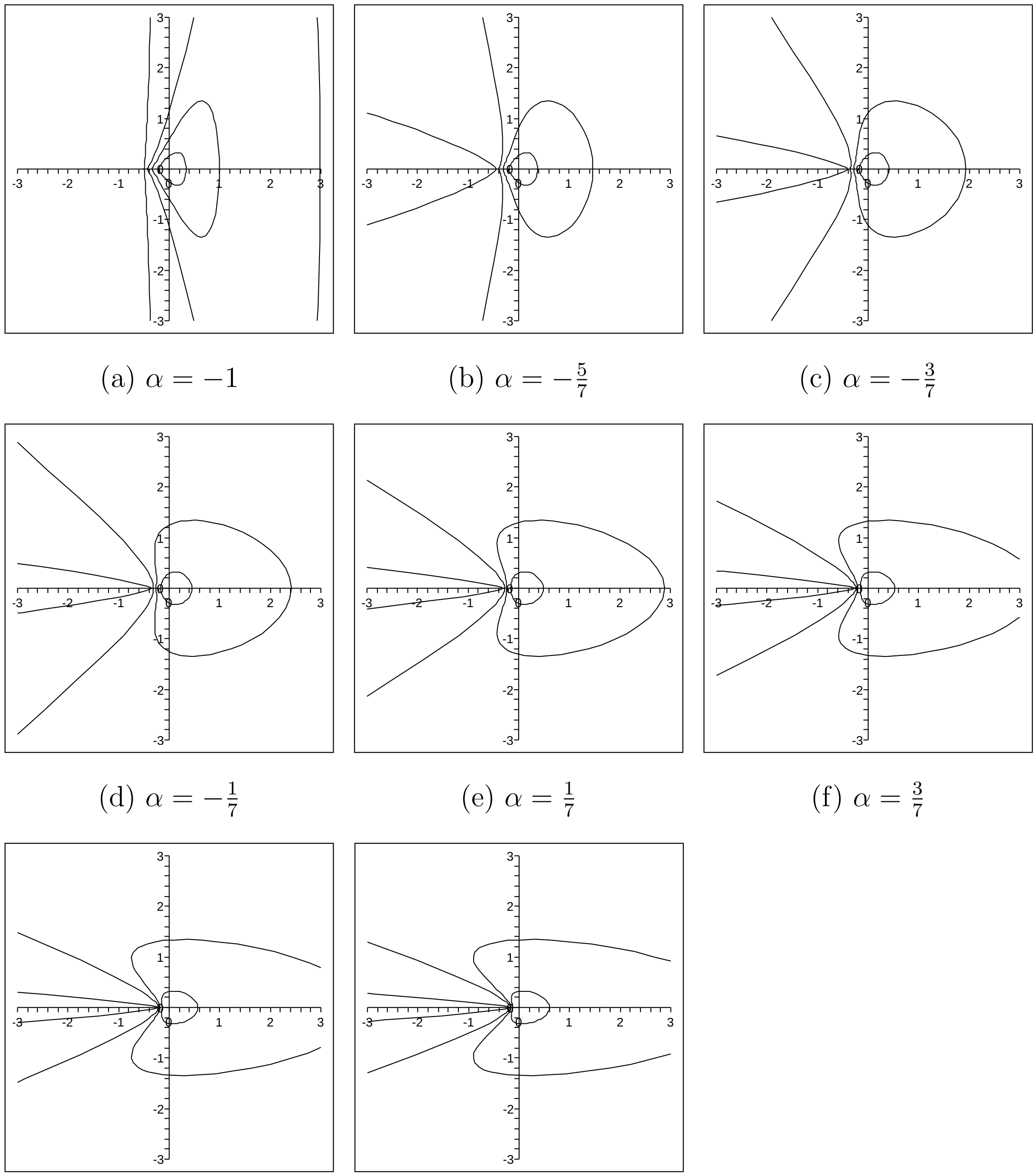
$$\begin{aligned}
\widetilde{h}(z) &= \frac{z}{(1-z)^2} + \int \left[\frac{z(\alpha+z)(1+z)}{(1-z)^4(1+z)}\right] dz \\
&= \frac{z}{(1-z)^2} + \int \left[\frac{z(\alpha+z)}{(1-z)^4}\right] dz \\
&= \frac{z + \left(\frac{\alpha}{2} - 1\right) z^2 + \left(\frac{1}{3} - \frac{\alpha}{6}\right) z^3}{(1-z)^3}
\end{aligned}$$

and

$$\begin{aligned}
\widetilde{g}(z) &= \int \left[\frac{z(\alpha+z)(1+z)}{(1-z)^4(1+z)}\right] dz \\
&= \int \left[\frac{z(\alpha+z)}{(1-z)^4}\right] dz \\
&= \frac{\frac{\alpha}{2} z^2 + \left(\frac{1}{3} - \frac{\alpha}{6}\right) z^3}{(1-z)^3}.
\end{aligned}$$

If we let $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$ then since $\widetilde{h} - \widetilde{g} = h - g \in CD(0)$ is univalent, we get that $\widetilde{f} \in CD(0)$ and is univalent by Theorem 9.6. These two properties can be seen partially in Figures 16 and 17.

Figure 16 shows several images of four concentric circles in $\mathbb{D}$ under the map $\widetilde{f}$. We see that as α approaches -1 we get a half-plane map and as α flows from -1 to 1 we see that the slit moves in towards $-\frac{1}{6}$. Figure 17 gives a closer view of the movement of the slit.

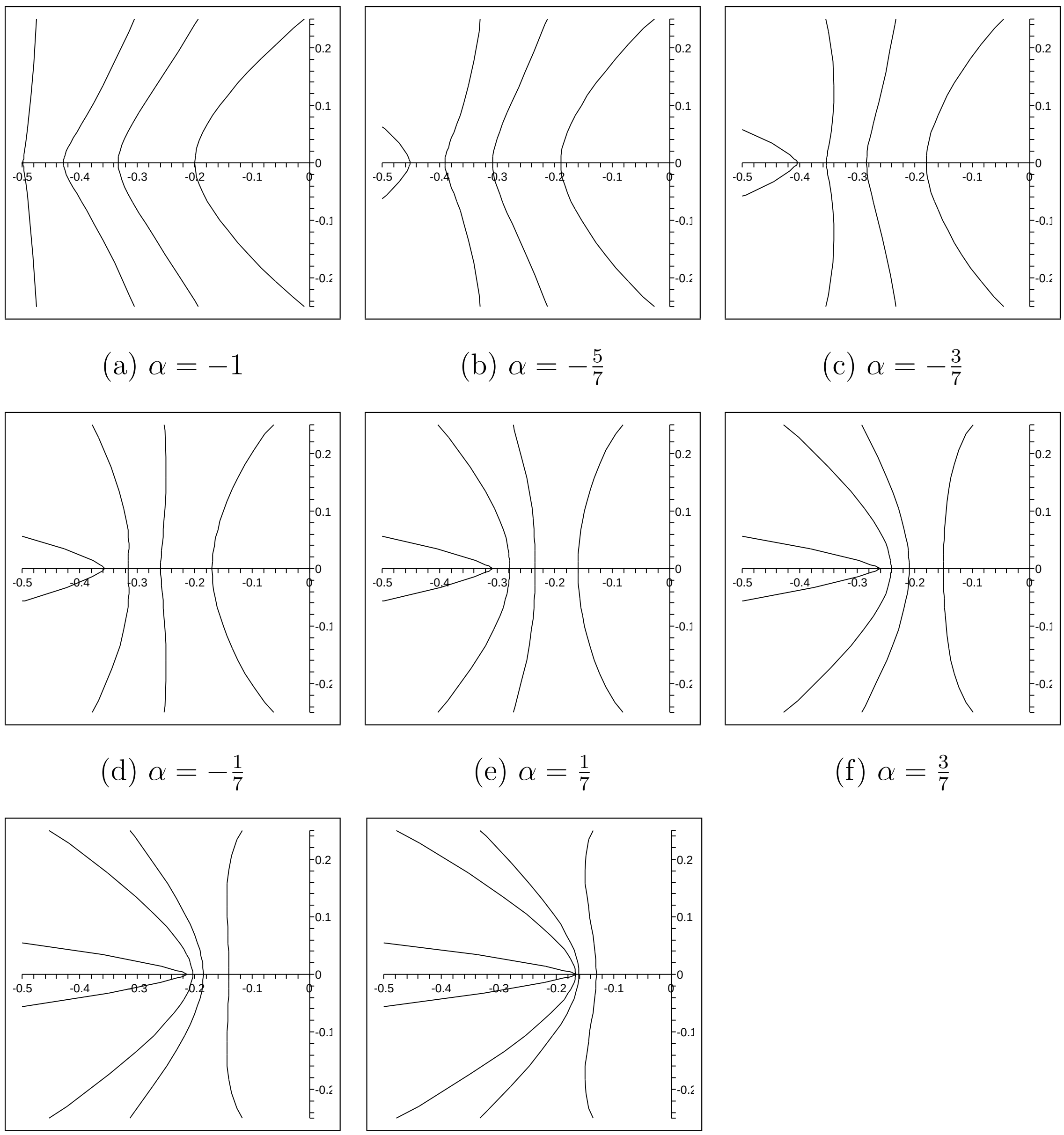


(a) $\alpha = -1$ (b) $\alpha = -\frac{5}{7}$ (c) $\alpha = -\frac{3}{7}$

(d) $\alpha = -\frac{1}{7}$ (e) $\alpha = \frac{1}{7}$ (f) $\alpha = \frac{3}{7}$

(g) $\alpha = \frac{5}{7}$ (h) $\alpha = 1$

Figure 16: *Images of* $\mathbb{D}$ *under the map* $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$ *where* $\widetilde{h}(z) = \dfrac{z + \left(\frac{\alpha}{2} - 1\right) z^2 + \left(\frac{1}{3} - \frac{\alpha}{6}\right) z^3}{(1-z)^3}$ *and* $\widetilde{g}(z) = \dfrac{\frac{\alpha}{2} z^2 + \left(\frac{1}{3} - \frac{\alpha}{6}\right) z^3}{(1-z)^3}$ *for* α *ranging from* -1 *to* 1.



(a) $\alpha = -1$ (b) $\alpha = -\frac{5}{7}$ (c) $\alpha = -\frac{3}{7}$

(d) $\alpha = -\frac{1}{7}$ (e) $\alpha = \frac{1}{7}$ (f) $\alpha = \frac{3}{7}$

(g) $\alpha = \frac{5}{7}$ (h) $\alpha = 1$

Figure 17: *Images of* $\mathbb{D}$ *under the map* $\widetilde{f} = \widetilde{h} + \overline{\widetilde{g}}$ *as in Figure 16 for* α *ranging from* -1 *to* 1 *with a viewing region of* $[-0.5, 0] \times [-0.25, 0.25]$.

A Calculation for Harmonic Functions

Let $h = h^1 + ih^2$ and $g = g^1 + ig^2$ be analytic functions of $z = x + iy$. The Cauchy-Riemann equations give us that h and g are analytic if and only if

$$h^1_x - h^2_y = 0$$

$$h^1_y + h^2_x = 0$$

$$g^1_x - g^2_y = 0$$

$$g^1_y + g^2_x = 0.$$

Therefore, if let

$$\Delta = \begin{pmatrix} h^1_x - h^2_y \\ h^1_y + h^2_x \\ g^1_x - g^2_y \\ g^1_y + g^2_y \end{pmatrix}$$

then $\Delta = 0$ represents h and g being analytic.

Now consider the generic infinitesimal generator $\mathbf{v}$ for the Lie symmetries of $\Delta = 0$. The generator $\mathbf{v}$ has the form

$$\mathbf{v} = a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y} + a^3 \frac{\partial}{\partial h^1} + a^4 \frac{\partial}{\partial h^2} + a^5 \frac{\partial}{\partial g^1} + a^6 \frac{\partial}{\partial g^2}$$

where $a^1, \ldots, a^6$ are arbitrary functions of x, y, h^1, h^2, g^1 and g^2. Theorem 5.35 gives us that $\mathbf{v}$ is the infinitesimal generator of the Lie symmetries if

$$\mathsf{pr}^{(1)}\mathbf{v}[\Delta_v(x, u^{(1)})] = 0, \quad v = 1, \ldots, 4 \quad \text{whenever} \quad \Delta(x, u^{(1)}) = 0.$$

Therefore, we want to find what each of the a^i must be so that the infinitesimal criterion holds true.

First we must prolong $\mathbf{v}$ to $\mathsf{pr}^{(1)}\mathbf{v}$. Using the prolongation formula as given in

Theorem 5.33 we get that

$$\begin{aligned}\mathsf{pr}^{(1)}\mathbf{v} = a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y} + a^3\frac{\partial}{\partial h^1} + a^4\frac{\partial}{\partial h^2} + a^5\frac{\partial}{\partial g^1} + a^6\frac{\partial}{\partial g^2}\\ + \Phi^1\frac{\partial}{\partial h^1_x} + \Phi^2\frac{\partial}{\partial h^1_y} + \Phi^3\frac{\partial}{\partial h^2_x} + \Phi^4\frac{\partial}{\partial h^2_y}\\ + \Phi^5\frac{\partial}{\partial g^1_x} + \Phi^6\frac{\partial}{\partial g^1_y} + \Phi^7\frac{\partial}{\partial g^2_x} + \Phi^8\frac{\partial}{\partial g^2_y}\end{aligned}$$

where

$$\begin{aligned}
\Phi^1 &= a^3_x + a^3_{h^1}h^1_x + a^3_{h^2}h^2_x + a^3_{g^1}g^1_x + a^3_{g^2}g^2_x - h^1_x a^1_x - a^1_{h^1}{h^1_x}^2 - h^1_x a^1_{h^2}h^2_x - h^1_x a^1_{g^1}g^1_x - \\
&\quad h^1_x a^1_{g^2}g^2_x - h^1_y a^2_x - h^1_y a^2_{h^1}h^1_x - h^1_y a^2_{h^2}h^2_x - h^1_y a^2_{g^1}g^1_x - h^1_y a^2_{g^2}g^2_x \\
\Phi^2 &= a^3_y + a^3_{h^1}h^1_y + a^3_{h^2}h^2_y + a^3_{g^1}g^1_y + a^3_{g^2}g^2_y - h^1_x a^1_y - h^1_x a^1_{h^1}h^1_y - h^1_x a^1_{h^2}h^2_y - h^1_x a^1_{g^1}g^1_y - \\
&\quad h^1_x a^1_{g^2}g^2_y - h^1_y a^2_y - a^2_{h^1}{h^1_y}^2 - h^1_y a^2_{h^2}h^2_y - h^1_y a^2_{g^1}g^1_y - h^1_y a^2_{g^2}g^2_y \\
\Phi^3 &= a^4_x + a^4_{h^1}h^1_x + a^4_{h^2}h^2_x + a^4_{g^1}g^1_x + a^4_{g^2}g^2_x - h^2_x a^1_x - h^2_x a^1_{h^1}h^1_x - a^1_{h^2}{h^2_x}^2 - h^2_x a^1_{g^1}g^1_x - \\
&\quad h^2_x a^1_{g^2}g^2_x - h^2_y a^2_x - h^2_y a^2_{h^1}h^1_x - h^2_y a^2_{h^2}h^2_x - h^2_y a^2_{g^1}g^1_x - h^2_y a^2_{g^2}g^2_x \\
\Phi^4 &= a^4_y + a^4_{h^1}h^1_y + a^4_{h^2}h^2_y + a^4_{g^1}g^1_y + a^4_{g^2}g^2_y - h^2_x a^1_y - h^2_x a^1_{h^1}h^1_y - h^2_x a^1_{h^2}h^2_y - h^2_x a^1_{g^1}g^1_y - \\
&\quad h^2_x a^1_{g^2}g^2_y - h^2_y a^2_y - h^2_y a^2_{h^1}h^1_y - a^2_{h^2}{h^2_y}^2 - h^2_y a^2_{g^1}g^1_y - h^2_y a^2_{g^2}g^2_y \\
\Phi^5 &= a^5_x + a^5_{h^1}h^1_x + a^5_{h^2}h^2_x + a^5_{g^1}g^1_x + a^5_{g^2}g^2_x - g^1_x a^1_x - g^1_x a^1_{h^1}h^1_x - g^1_x a^1_{h^2}h^2_x - a^1_{g^1}{g^1_x}^2 - \\
&\quad g^1_x a^1_{g^2}g^2_x - g^1_y a^2_x - g^1_y a^2_{h^1}h^1_x - g^1_y a^2_{h^2}h^2_x - g^1_y a^2_{g^1}g^1_x - g^1_y a^2_{g^2}g^2_x \\
\Phi^6 &= a^5_y + a^5_{h^1}h^1_y + a^5_{h^2}h^2_y + a^5_{g^1}g^1_y + a^5_{g^2}g^2_y - g^1_x a^1_y - g^1_x a^1_{h^1}h^1_y - g^1_x a^1_{h^2}h^2_y - g^1_x a^1_{g^1}g^1_y - \\
&\quad g^1_x a^1_{g^2}g^2_y - g^1_y a^2_y - g^1_y a^2_{h^1}h^1_y - g^1_y a^2_{h^2}h^2_y - a^2_{g^1}{g^1_y}^2 - g^1_y a^2_{g^2}g^2_y \\
\Phi^7 &= a^6_x + a^6_{h^1}h^1_x + a^6_{h^2}h^2_x + a^6_{g^1}g^1_x + a^6_{g^2}g^2_x - g^2_x a^1_x - g^2_x a^1_{h^1}h^1_x - g^2_x a^1_{h^2}h^2_x - g^2_x a^1_{g^1}g^1_x - \\
&\quad a^1_{g^2}{g^2_x}^2 - g^2_y a^2_x - g^2_y a^2_{h^1}h^1_x - g^2_y a^2_{h^2}h^2_x - g^2_y a^2_{g^1}g^1_x - g^2_y a^2_{g^2}g^2_x \\
\Phi^8 &= a^6_y + a^6_{h^1}h^1_y + a^6_{h^2}h^2_y + a^6_{g^1}g^1_y + a^6_{g^2}g^2_y - g^2_x a^1_y - g^2_x a^1_{h^1}h^1_y - g^2_x a^1_{h^2}h^2_y - g^2_x a^1_{g^1}g^1_y - \\
&\quad g^2_x a^1_{g^2}g^2_y - g^2_y a^2_y - g^2_y a^2_{h^1}h^1_y - g^2_y a^2_{h^2}h^2_y - g^2_y a^2_{g^1}g^1_y - a^2_{g^2}{g^2_y}^2.
\end{aligned}$$

Now we need to find $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_i]$ for $i = 1, 2, 3, 4$. By applying Definition 5.24 we get that

$$
\begin{aligned}
\mathsf{pr}^{(1)}\mathbf{v}[\Delta_1] \;=\;& -h^1_x a^1_{g^2} g^2_x - h^1_x a^1_{h^2} h^2_x + h^2_y a^2_{h^1} h^1_y + a^3_x - h^1_y a^2_x - h^1_y a^2_{g^1} g^1_x - h^1_y a^2_{g^2} g^2_x + \\
& h^2_x a^1_{h^2} h^2_y + h^2_x a^1_{g^1} g^1_y + h^2_x a^1_{g^2} g^2_y + h^2_y a^2_{g^1} g^1_y + h^2_y a^2_{g^2} g^2_y - a^4_y + h^2_x a^1_{h^1} h^1_y - \\
& h^1_y a^2_{h^2} h^2_x - h^1_x a^1_{g^1} g^1_x - h^1_y a^2_{h^1} h^1_x + a^3_{h^1} h^1_x + a^3_{h^2} h^2_x + a^3_{g^1} g^1_x + a^3_{g^2} g^2_x - \\
& h^1_x a^1_x - a^1_{h^1} {h^1_x}^2 - a^4_{h^1} h^1_y - a^4_{h^2} h^2_y - a^4_{g^1} g^1_y - a^4_{g^2} g^2_y + h^2_x a^1_y + h^2_y a^2_y + a^2_{h^2} {h^2_y}^2 \\
\mathsf{pr}^{(1)}\mathbf{v}[\Delta_2] \;=\;& -h^1_y a^2_{g^1} g^1_y - h^2_x a^1_{g^1} g^1_x - h^1_y a^2_{g^2} g^2_y + a^4_{g^2} g^2_x - a^2_{h^1} {h^1_y}^2 + a^4_{h^1} h^1_x + a^4_x - h^1_y a^2_y - \\
& h^1_x a^1_{h^1} h^1_y - h^1_x a^1_{h^2} h^2_y - h^1_x a^1_{g^1} g^1_y - h^1_x a^1_{g^2} g^2_y - h^1_y a^2_{h^2} h^2_y - h^2_x a^1_{h^1} h^1_x - \\
& h^2_x a^1_{g^2} g^2_x - h^2_y a^2_{h^1} h^1_x - h^2_y a^2_{h^2} h^2_x - h^2_y a^2_{g^1} g^1_x - h^2_y a^2_{g^2} g^2_x + a^3_y + a^3_{h^1} h^1_y + \\
& a^3_{h^2} h^2_y + a^3_{g^1} g^1_y + a^3_{g^2} g^2_y - h^1_x a^1_y + a^4_{h^2} h^2_x + a^4_{g^1} g^1_x - h^2_x a^1_x - a^1_{h^2} {h^2_x}^2 - h^2_y a^2_x \\
\mathsf{pr}^{(1)}\mathbf{v}[\Delta_3] \;=\;& -g^1_y a^2_{h^1}\; h^1_x + g^2_x a^1_{g^2} g^2_y - g^1_x a^1_{g^2} g^2_x + g^2_x a^1_{h^1} h^1_y - g^1_x a^1_{h^2} h^2_x + g^2_x a^1_{h^2} h^2_y - \\
& g^1_x a^1_{h^1} h^1_x - g^1_y a^2_{h^2} h^2_x - g^1_y a^2_{g^1} g^1_x - g^1_y a^2_{g^2} g^2_x + a^5_x + g^2_x a^1_{g^1} g^1_y + g^2_y a^2_{h^1} h^1_y + \\
& g^2_y a^2_{h^2} h^2_y + g^2_y a^2_{g^1} g^1_y - a^6_y - a^6_{h^1} h^1_y - a^6_{h^2} h^2_y - a^6_{g^1} g^1_y - a^6_{g^2} g^2_y + g^2_x a^1_y + \\
& g^2_y a^2_y + a^2_{g^2} {g^2_y}^2 + a^5_{h^1} h^1_x + a^5_{h^2} h^2_x + a^5_{g^1} g^1_x + a^5_{g^2} g^2_x - g^1_x a^1_x - a^1_{g^1} {g^1_x}^2 - g^1_y a^2_x \\
\mathsf{pr}^{(1)}\mathbf{v}[\Delta_4] \;=\;& -g^1_y a^2_{h^2} h^2_y - g^1_x a^1_{g^2} g^2_y - g^2_x a^1_{g^1} g^1_x - g^1_x a^1_{h^1} h^1_y - g^2_y a^2_{h^1} h^1_x - g^2_x a^1_{h^2} h^2_x - \\
& g^2_x a^1_{h^1} h^1_x - g^1_x a^1_{h^2} h^2_y - g^1_x a^1_{g^1} g^1_y - g^1_y a^2_{h^1} h^1_y - g^1_y a^2_{g^2} g^2_y - g^2_y a^2_{h^2} h^2_x - \\
& g^2_y a^2_{g^1} g^1_x - g^2_y a^2_{g^2} g^2_x + a^6_{h^2} h^2_x + a^6_{g^1} g^1_x + a^6_{g^2} g^2_x - g^2_x a^1_x - a^1_{g^2} {g^2_x}^2 - g^2_y a^2_x + \\
& a^5_y + a^6_x + a^5_{h^1} h^1_y + a^5_{h^2} h^2_y + a^5_{g^1} g^1_y + a^5_{g^2} g^2_y - g^1_x a^1_y - g^1_y a^2_y - a^2_{g^1} {g^1_y}^2 + a^6_{h^1} h^1_x.
\end{aligned}
$$

The infinitesimal criterion gives that $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_i]$ should be 0 for each $i = 1, 2, 3, 4$ when $\Delta = 0$. Making the substitutions of $\Delta = 0$ into each of the above expressions and setting them equal to 0 gives us a system of polynomials in x, y, h^1, h^2, g^1, g^2, h^1_x, h^2_x, g^1_x, g^2_x, h^1_y, h^2_y, g^1_y and g^2_y. Collecting in these variables and equating coefficients gives us the following system of partial differential equations.

$$
\begin{aligned}
a^2_x + a^6_{g^1} + a^1_y + a^5_{g^2} &= 0 & a^5_{g^2} + a^6_{g^1} - a^1_y - a^2_x &= 0\\
-a^5_{g^1} - a^2_y + a^6_{g^2} + a^1_x &= 0 & a^6_{h^1} + a^5_{h^2} &= 0\\
a^5_{h^1} - a^6_{h^2} &= 0 & a^3_{g^1} - a^4_{g^2} &= 0\\
a^1_{h^1} - a^2_{h^2} &= 0 & -a^2_{g^2} + a^1_{g^1} &= 0\\
a^1_{g^2} + a^2_{g^1} &= 0 & a^3_y + a^4_x &= 0\\
-a^4_y + a^3_x &= 0 & a^3_{h^2} + a^1_y + a^4_{h^1} + a^2_x &= 0\\
a^3_{h^1} - a^1_x + a^2_y - a^4_{h^2} &= 0 & a^4_{g^1} + a^3_{g^2} &= 0\\
-a^5_{g^1} - a^1_x + a^6_{g^2} + a^2_y &= 0 & a^5_x - a^6_y &= 0\\
-a^4_{h^2} + a^3_{h^1} - a^2_y + a^1_x &= 0 & a^3_{h^2} - a^1_y - a^2_x + a^4_{h^1} &= 0\\
a^1_{h^2} + a^2_{h^1} &= 0 & a^5_y + a^6_x &= 0
\end{aligned}
$$

This system can be simplified to

$$
\begin{aligned}
a^i_x &= a^{i+1}_y & a^i_{h^1} &= a^{i+1}_{h^2} & a^i_{g^1} &= a^{i+1}_{g^2}\\
a^i_y &= -a^{i+1}_x & a^i_{h^2} &= -a^{i+1}_{h^1} & a^i_{g^2} &= -a^{i+1}_{g^1}
\end{aligned}
$$

for $i = 1, 3, 5$ and gives us that $a^1 + ia^2$, $a^3 + ia^4$ and $a^5 + ia^6$ are each analytic in the complex variables z, h and g as defined in Section 7. These calculations were mainly done using the software system Maple 9.5 with the Vessiot add-in package.

Since each of $a^1 + ia^2$, $a^3 + ia^4$ and $a^5 + ia^6$ are independent of each other then we can consider $\mathbf{v}$ as the sum of

$$\mathbf{v}_1 = a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y}$$
$$\mathbf{v}_2 = a^3 \frac{\partial}{\partial h^1} + a^4 \frac{\partial}{\partial h^2}$$
$$\mathbf{v}_3 = a^5 \frac{\partial}{\partial g^1} + a^6 \frac{\partial}{\partial g^2}.$$

With this we get that the infinite dimensional Lie algebra associated with the Lie symmetries of the harmonic functions is spanned by the set of infinitesimal generators, $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

B Calculation for Area-Preserving Harmonic Functions

Instead of simply an area-preserving harmonic function, let's consider the broader case of harmonic with constant Jacobian. For a function $f = h + \overline{g}$ we have that h and g are analytic and $J_f = k$ for some constant k. These properties can be represented by the system

$$\Delta = \begin{pmatrix} h^1_x - h^2_y \\ h^1_y + h^2_x \\ g^1_x - g^2_y \\ g^1_y + g^2_y \\ (h^1_x + g^1_x)(h^2_y - g^2_y) - (h^1_y + g^1_y)(h^2_x - g^2_x) - k \end{pmatrix} = 0.$$

If we let $\Delta_H = 0$ be the system as given in Appendix A then we can see that $\mathscr{S}_\Delta \subset \mathscr{S}_{\Delta_H}$. That is that the subvariety of Δ is contained in the subvariety of Δ_H where the subvariety of Δ, $\mathscr{S}_\Delta$, is defined by (5) on page 46. This means that the manifold represented by the set of all points in the first jet space $X \times U \times U_1$

where $\Delta = 0$ as given by $\mathscr{S}_\Delta$ is a submanifold of $\mathscr{S}_{\Delta_H}$. Therefore if we find the Lie symmetries that have flows contained in $\mathscr{S}_\Delta$ then we will have found a set of Lie symmetries that have flows contained in $\mathscr{S}_{\Delta_H}$ as well. Simply put, the infinitesimal generators of the Lie symmetries of $\Delta = 0$ will form a Lie subalgebra of the generators for Δ_H.

Now let's calculate the infinitesimal generators for $\Delta = 0$. If we begin again with the generic generator

$$\mathbf{v} = a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y} + a^3 \frac{\partial}{\partial h^1} + a^4 \frac{\partial}{\partial h^2} + a^5 \frac{\partial}{\partial g^1} + a^6 \frac{\partial}{\partial g^2}$$

we will have the same first prolongation as given in Appendix A. That is

$$\begin{aligned} \mathsf{pr}^{(1)}\mathbf{v} = a^1 \frac{\partial}{\partial x} + a^2 \frac{\partial}{\partial y} + a^3 \frac{\partial}{\partial h^1} + a^4 \frac{\partial}{\partial h^2} + a^5 \frac{\partial}{\partial g^1} + a^6 \frac{\partial}{\partial g^2} \\ + \Phi^1 \frac{\partial}{\partial h^1_x} + \Phi^2 \frac{\partial}{\partial h^1_y} + \Phi^3 \frac{\partial}{\partial h^2_x} + \Phi^4 \frac{\partial}{\partial h^2_y} \\ + \Phi^5 \frac{\partial}{\partial g^1_x} + \Phi^6 \frac{\partial}{\partial g^1_y} + \Phi^7 \frac{\partial}{\partial g^2_x} + \Phi^8 \frac{\partial}{\partial g^2_y} \end{aligned}$$

where $\Phi^i, i = 1, \ldots, 8$ are defined in Appendix A.

To satisfy the infinitesimal criterion we must calculate each of $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_i]$ for $i = 1, 2, 3, 4, 5$. For $i = 1, \ldots, 4$, we have $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_i]$ as given in Appendix A and using the definition of Lie derivative as given in Definition 5.24 we get that $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_5]$ is given by

$$a^4_{g^2} g^2_y g^1_x - a^5_{g^1} g^1_y h^2_x + a^4_{g^1} g^1_y h^1_x - a^5_{h^1} h^1_x g^2_y - a^3_{h^2} h^2_x g^2_y - h^2_y a^2_y h^1_x + a^5_{h^1} h^1_x h^2_y - a^4_{g^1} g^1_x h^1_y -$$
$$a^4_{h^1} h^1_x g^1_y + a^2_{h^1} {h^1_y}^2 h^2_x + a^3_{g^2} g^2_x h^2_y + h^2_x a^1_x g^1_y + h^2_x a^1_{g^1} g^1_x h^1_y + a^4_{h^2} h^2_y g^1_x + h^1_y a^2_y h^2_x + a^4_y g^1_x -$$
$$a^2_{h^2} {h^2_y}^2 h^1_x + a^1_{h^2} {h^2_x}^2 g^1_y - g^2_x a^1_x g^1_y - a^3_y h^2_x - a^5_y h^2_x + a^5_{h^2} h^2_y g^2_x + h^2_x a^1_{h^1} h^1_x g^1_y - h^1_x a^1_{g^2} g^2_x h^2_y - a^4_x g^1_y -$$
$$a^3_x g^2_y + a^5_x h^2_y - a^4_x h^1_y + a^5_y g^2_x + a^3_y g^2_x - a^2_{g^1} {g^1_y}^2 g^2_x + h^2_y a^2_{h^2} h^2_x g^1_y + a^4_y h^1_x - h^2_y a^2_{h^1} h^1_y g^1_x - a^6_y h^1_x -$$
$$a^6_y g^1_x - a^5_x g^2_y + a^6_x h^1_y + a^6_x g^1_y + a^3_x h^2_y + h^1_y a^2_{g^1} g^1_y h^2_x - h^1_y a^2_{g^1} g^1_y g^2_x + h^1_y a^2_{g^2} g^2_y h^2_x + h^2_x a^1_{g^1} g^1_x g^1_y +$$

$$h^2_x a^1_{g^2} g^2_x h^1_y + h^2_x a^1_{g^2} g^2_x g^1_y - h^2_y a^2_{g^1} g^1_x g^1_y - h^2_x a^1_{h^2} h^2_y g^1_x - h^2_y a^2_{g^1} g^1_y h^1_x - h^2_y a^2_{g^2} g^2_y h^1_x + g^1_x a^1_{h^1} h^1_x g^2_y +$$

$$g^1_x a^1_{h^2} h^2_x g^2_y - g^1_x a^1_{g^2} g^2_x h^2_y + g^1_x a^1_{g^2} g^2_x g^2_y + g^1_y a^2_{g^1} g^1_x g^2_y - g^1_y a^2_{g^2} g^2_x g^2_y - g^1_x a^1_{g^1} g^1_y g^2_x + g^1_y a^2_{h^1} h^1_y h^2_x -$$

$$g^1_y a^2_{h^1} h^1_y g^2_x - g^1_y a^2_{h^2} h^2_y g^2_x + g^1_y a^2_{g^2} g^2_y h^2_x - g^2_x a^1_{h^1} h^1_x g^1_y - g^2_x a^1_{h^2} h^2_x h^1_y - g^2_x a^1_{h^2} h^2_x g^1_y - g^2_x a^1_{g^1} g^1_x h^1_y +$$

$$g^2_y a^2_{h^1} h^1_y g^1_x + g^2_y a^2_{h^2} h^2_y h^1_x + g^2_y a^2_{h^2} h^2_y g^1_x + g^2_y a^2_{g^1} g^1_y h^1_x - h^1_x a^1_{h^2} h^2_x h^2_y + h^1_x a^1_{g^1} g^1_x g^2_y - h^1_x a^1_{h^1} h^1_y g^2_x +$$

$$h^1_x a^1_{h^2} h^2_x g^2_y - h^1_x a^1_{g^1} g^1_x h^2_y - g^1_x a^1_{h^1} h^1_x h^2_y - h^1_y a^2_{h^2} h^2_y g^2_x - h^2_y a^2_{g^2} g^2_y g^1_x + h^1_x a^1_{g^2} g^2_x g^2_y - h^1_y a^2_{h^1} h^1_x h^2_y +$$

$$h^1_y a^2_{h^1} h^1_x g^2_y + h^1_y a^2_{h^2} h^2_x h^2_y - h^1_y a^2_{g^2} g^2_x g^2_y + h^1_x a^1_{h^1} h^1_y h^2_x - g^1_y a^2_y g^2_x + a^5_{g^1} g^1_y g^2_x + g^1_y a^2_y h^2_x - h^1_x a^1_x h^2_y -$$

$$a^5_{g^2} g^2_y h^2_x + a^2_{g^1} {g^1_y}^2 h^2_x + a^3_{h^1} h^1_x h^2_y - a^3_{h^1} h^1_x g^2_y + a^3_{g^1} g^1_x h^2_y - a^3_{g^1} g^1_x g^2_y + h^1_x a^1_x g^2_y - a^1_{h^1} {h^1_x}^2 h^2_y +$$

$$a^1_{h^1} {h^1_x}^2 g^2_y - a^3_{h^1} h^1_y h^2_x + a^3_{h^1} h^1_y g^2_x + a^3_{h^2} h^2_y g^2_x - a^3_{g^1} g^1_y h^2_x + a^3_{g^1} g^1_y g^2_x - a^3_{g^2} g^2_y h^2_x - h^1_y a^2_y g^2_x -$$

$$a^2_{h^1} {h^1_y}^2 g^2_x - a^4_{h^2} h^2_x h^1_y - a^4_{h^2} h^2_x g^1_y - a^4_{g^2} g^2_x h^1_y - a^4_{g^2} g^2_x g^1_y + h^2_x a^1_x h^1_y + a^1_{h^2} {h^2_x}^2 h^1_y - a^5_{h^1} h^1_y h^2_x +$$

$$a^4_{h^2} h^2_y h^1_x + a^4_{h^1} h^1_y g^1_x + a^4_{g^2} g^2_y h^1_x - h^2_y a^2_y g^1_x - a^2_{h^2} {h^2_y}^2 g^1_x - a^5_{h^2} h^2_x g^2_y + a^5_{g^1} g^1_x h^2_y - a^5_{g^1} g^1_x g^2_y +$$

$$a^5_{g^2} g^2_x h^2_y - g^1_x a^1_x h^2_y + g^1_x a^1_x g^2_y - a^1_{g^1} {g^1_x}^2 h^2_y + a^1_{g^1} {g^1_x}^2 g^2_y + a^5_{h^1} h^1_y g^2_x + a^6_{h^2} h^2_x h^1_y + a^6_{h^2} h^2_x g^1_y +$$

$$a^6_{h^1} h^1_x g^1_y + a^6_{g^1} g^1_x h^1_y + a^6_{g^2} g^2_x h^1_y + a^6_{g^2} g^2_x g^1_y - g^2_x a^1_x h^1_y - a^1_{g^2} {g^2_x}^2 h^1_y - a^1_{g^2} {g^2_x}^2 g^1_y - a^6_{h^1} h^1_y g^1_x -$$

$$a^6_{h^2} h^2_y h^1_x - a^6_{h^2} h^2_y g^1_x - a^6_{g^1} g^1_y h^1_x - a^6_{g^2} g^2_y h^1_x - a^6_{g^2} g^2_y g^1_x + g^2_y a^2_y h^1_x + g^2_y a^2_y g^1_x + a^2_{g^2} {g^2_y}^2 h^1_x + a^2_{g^2} {g^2_y}^2 g^1_x.$$

Again, while their sizes may be intimidating, each of the $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_i]$ is simply a polynomial in the variables x, y, h^1, h^2, g^1, g^2, h^1_x, h^2_x, g^1_x, g^2_x, h^1_y, h^2_y, g^1_y and g^2_y. Now to satisfy the infinitesimal criterion, we must again make the substitutions of $\Delta = 0$ into each of the $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_i]$, group the polynomials by their terms and equate each of them with 0. Doing this will yield the following system of partial differential equations.

$$\begin{aligned}
-a^1_x + a^5_{h^1} - a^6_{h^2} + a^3_{h^1} + a^4_{h^2} - a^2_y &= 0 & a^2_{g^1} + a^1_{g^2} &= 0\\
-a^6_{g^2} - a^5_{h^1} + a^4_{h^2} - a^6_{h^2} - a^3_{h^1} + a^3_{g^1} + a^5_{g^1} + a^4_{g^2} &= 0 & -a^1_{g^1} + a^2_{g^2} &= 0\\
a^3_{h^1} - a^2_y - a^4_{h^2} + a^1_x &= 0 & a^1_{h^2} + a^2_{h^1} &= 0\\
-a^1_x - a^2_y + a^3_{g^1} + a^5_{g^1} - a^4_{g^2} + a^6_{g^2} &= 0 & a^4_x + a^3_y &= 0\\
-a^1_y - a^2_x + a^3_{h^2} + a^4_{h^1} &= 0 & a^2_{h^2} &= 0
\end{aligned}$$

$$a^4_{h^1} + a^6_{g^1} + a^3_{g^2} - a^4_{g^1} - a^6_{h^1} + a^3_{h^2} + a^5_{g^2} + a^5_{h^2} = 0 \qquad a^2_{h^1} - a^1_{h^2} = 0$$

$$a^2_x + a^6_{g^1} + a^1_y + a^5_{g^2} = 0 \qquad a^1_{g^2} - a^2_{g^1} = 0$$

$$-a^3_x - a^5_x - a^4_y + a^1_{h^1} + a^6_y = 0 \qquad a^5_{h^1} - a^6_{h^2} = 0$$

$$a^1_{g^1} + a^1_{h^1} + a^2_{g^2} = 0 \qquad a^3_{g^2} + a^4_{g^1} = 0$$

$$a^4_x + a^3_y + a^5_y - a^6_x = 0 \qquad a^3_{g^1} - a^4_{g^2} = 0$$

$$-a^5_x + a^1_{h^1} - a^3_x + a^4_y - a^6_y = 0 \qquad a^1_{h^1} = 0$$

$$-a^2_y + a^6_{g^2} + a^1_x - a^5_{g^1} = 0 \qquad a^6_x + a^5_y = 0$$

$$-a^6_{g^2} + a^1_x + a^5_{g^1} - a^2_y = 0 \qquad -a^6_y + a^5_x = 0$$

$$a^4_{h^1} + a^3_{h^2} + a^1_y + a^2_x = 0 \qquad -a^2_{h^2} + a^1_{h^1} = 0$$

$$-a^5_{g^2} + a^2_x - a^6_{g^1} + a^1_y = 0 \qquad a^4_y - a^3_x + a^1_{h^1} = 0$$

$$a^4_{h^2} + a^1_x - a^3_{h^1} - a^2_y = 0 \qquad a^5_{h^2} + a^6_{h^1} = 0$$

$$-a^4_x + a^5_y + a^6_x + a^3_y = 0$$

Solving the above system of partial differential equations gives us that

$$a^1(z,h,g) = c_{10}x - c_7 y + c_1$$

$$a^2(z,h,g) = c_{10}y + c_7 x + c_2$$

$$a^3(z,h,g) = c_{12}g^1 + c_{11}g^2 + c_{10}h^1 - c_8 h^2 + c_3$$

$$a^4(z,h,g) = c_{12}g^2 + c_{10}h^2 - c_{11}g^1 + c_8 h^1 + c_4$$

$$a^5(z,h,g) = c_{10}g^1 + c_{12}h^1 - c_9 g^2 - c_{11}h^2 + c_5$$

$$a^6(z,h,g) = c_9 g^1 + c_{11}h^1 + c_{12}h^2 + c_{10}g^2 + c_6$$

where each c_i is an arbitrary real number.

Substituting each of the a^i into $\mathbf{v}$ and then factoring with respect to the c_i will give us

$$\begin{aligned}
\mathbf{v} = c_1\left(\frac{\partial}{\partial x}\right) + c_2\left(\frac{\partial}{\partial y}\right) + c_3\left(\frac{\partial}{\partial h^1}\right) + c_4\left(\frac{\partial}{\partial h^2}\right) + c_5\left(\frac{\partial}{\partial g^1}\right) + c_6\left(\frac{\partial}{\partial g^2}\right) \\
+ c_7\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) + c_8\left(-h^2\frac{\partial}{\partial h^1} + h^1\frac{\partial}{\partial h^2}\right) + c_9\left(-g^2\frac{\partial}{\partial g^1} + g^1\frac{\partial}{\partial g^2}\right) \\
+ c_{10}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + h^1\frac{\partial}{\partial h^1} + h^2\frac{\partial}{\partial h^2} + g^1\frac{\partial}{\partial g^1} + g^2\frac{\partial}{\partial g^2}\right) \\
+ c_{11}\left(g^2\frac{\partial}{\partial h^1} - g^1\frac{\partial}{\partial h^2} - h^2\frac{\partial}{\partial g^1} + h^1\frac{\partial}{\partial g^2}\right) \\
+ c_{12}\left(g^1\frac{\partial}{\partial h^1} + g^2\frac{\partial}{\partial h^2} + h^1\frac{\partial}{\partial g^1} + h^2\frac{\partial}{\partial g^2}\right).
\end{aligned}$$

Successively setting all c_i to 0 except for one of them gives us the following infinitesimal generators

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial x} & \mathbf{v}_7 &= -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \\
\mathbf{v}_2 &= \frac{\partial}{\partial y} & \mathbf{v}_8 &= -h^2\frac{\partial}{\partial h^1} + h^1\frac{\partial}{\partial h^2} \\
\mathbf{v}_3 &= \frac{\partial}{\partial h^1} & \mathbf{v}_9 &= -g^2\frac{\partial}{\partial g^1} + g^1\frac{\partial}{\partial g^2} \\
\mathbf{v}_4 &= \frac{\partial}{\partial h^2} & \mathbf{v}_{10} &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + h^1\frac{\partial}{\partial h^1} + h^2\frac{\partial}{\partial h^2} + g^1\frac{\partial}{\partial g^1} + g^2\frac{\partial}{\partial g^2} \\
\mathbf{v}_5 &= \frac{\partial}{\partial g^1} & \mathbf{v}_{11} &= g^2\frac{\partial}{\partial h^1} - g^1\frac{\partial}{\partial h^2} - h^2\frac{\partial}{\partial g^1} + h^1\frac{\partial}{\partial g^2} \\
\mathbf{v}_6 &= \frac{\partial}{\partial g^2} & \mathbf{v}_{12} &= g^1\frac{\partial}{\partial h^1} + g^2\frac{\partial}{\partial h^2} + h^1\frac{\partial}{\partial g^1} + h^2\frac{\partial}{\partial g^2}
\end{aligned}$$

where $\mathbf{v}_i$ corresponds to setting all but c_i to 0. Since the set $\{\mathbf{v}_1, \ldots, \mathbf{v}_{12}\}$ spans a Lie algebra over $\mathbb{R}$ corresponding to the Lie symmetries of $\Delta = 0$, then we need not consider $\mathbf{v}_1$ as $\mathbf{v}_1 = c_1\frac{\partial}{\partial x}$ since we can multiply by c_1^{-1} for $c_1 \neq 0$ giving us $\mathbf{v}_1 = \frac{\partial}{\partial x}$. This is true in general thus justifying the 12 infinitesimal generators that we have stated.

Let's consider the exponentiations of each of these. First consider $\mathbf{v}_1, \ldots, \mathbf{v}_6$ and their respective exponentiations. For each of these, the boundary valued differential equation to solve for each of the exponentiations is essentially the same. Consider that $\mathbf{v}_1 = \dfrac{\partial}{\partial x}$. This leads to the equation

$$\frac{d\widetilde{x}}{d\varepsilon} = 1$$

with $\widetilde{x}|_0 = x$. We can see that this has solution $\widetilde{x} = x + \varepsilon$. Each of $\widetilde{y}, \widetilde{h}^1, \ldots, \widetilde{g}^2$ remains unchanged yielding that $\widetilde{z} = z + \varepsilon$, $\widetilde{h}(z) = h(z)$ and $\widetilde{g}(z) = g(z)$. The exponentiations for $\mathbf{v}_2, \ldots, \mathbf{v}_6$ are derived in the same way.

For $\mathbf{v}_7, \mathbf{v}_8$ and $\mathbf{v}_9$ we must solve basically the same system of differential equations as we have for $\mathbf{v}_7$ which is the system

$$\frac{d\widetilde{x}}{d\varepsilon} = -\widetilde{y}, \quad \frac{d\widetilde{y}}{d\varepsilon} = \widetilde{x}$$

subject to $\widetilde{x}|_0 = x$ and $\widetilde{y}|_0 = y$. This has solution $\widetilde{x} = x\cos\varepsilon - y\sin\varepsilon$ and $\widetilde{y} = y\cos\varepsilon + x\sin\varepsilon$. This gives us that

$$\begin{aligned}
\widetilde{z} &= \widetilde{x} + i\widetilde{y} \\
&= (x\cos\varepsilon - y\sin\varepsilon) + i(y\cos\varepsilon + x\sin\varepsilon) \\
&= x(\cos\varepsilon + i\sin\varepsilon) + iy(\cos\varepsilon + i\sin\varepsilon) \\
&= e^{i\varepsilon}(x + iy) \\
&= e^{i\varepsilon} z.
\end{aligned}$$

For $\mathbf{v}_7$ we get $\widetilde{h}(z) = e^{i\varepsilon} h(z)$ and for $\mathbf{v}_8$ we get $\widetilde{g}(z) = e^{i\varepsilon} g(z)$.

For $\mathbf{v}_{10}$ we must solve

$$\frac{d\widetilde{x}}{d\varepsilon} = \widetilde{x}, \quad \frac{d\widetilde{y}}{d\varepsilon} = \widetilde{y}, \quad \frac{d\widetilde{h}^1}{d\varepsilon} = \widetilde{h}^1, \quad \frac{d\widetilde{h}^2}{d\varepsilon} = \widetilde{h}^2, \quad \frac{d\widetilde{g}^1}{d\varepsilon} = \widetilde{g}^1, \quad \frac{d\widetilde{g}^2}{d\varepsilon} = \widetilde{g}^2$$

subject to $\widetilde{x}|_0 = x, \ldots, \widetilde{g}^2|_0 = g^2$. The system has solution $\widetilde{x} = e^{\varepsilon} x, \ldots, \widetilde{g}^2 = e^{\varepsilon} g^2$ thus giving us that $\widetilde{z} = e^{\varepsilon} z$, $\widetilde{h}(z) = e^{\varepsilon} h(z)$ and $\widetilde{g}(z) = e^{\varepsilon} g(z)$.

At this point notice that our notation is slightly misleading in that each of the exponentiated functions $\widetilde{h}$ and $\widetilde{g}$ technically should have independent variable $\widetilde{z}$ instead of z, that is that $\widetilde{h}(z)$ should really be $\widetilde{h}(\widetilde{z})$. The adoption of the notation as given in this paper is for simplicity and it will always be clear from context what is meant. We must therefore solve for z in terms of $\widetilde{z}$ in order to find the transformed functions $\widetilde{h}$ and $\widetilde{g}$. Doing so gives us the inverse transform of z and since ε acting on z is a Lie group action then the inverse action is simply the action of the inverse of ε on z, that is $z = e^{-\varepsilon}\widetilde{z}$. We can see this from Definition 5.22. Therefore $\widetilde{h}(z)$ is given by $\widetilde{h}(z) = e^{\varepsilon}h\left(e^{-\varepsilon}z\right)$ and $\widetilde{g}(z)$ by $\widetilde{g}(z) = e^{\varepsilon}g\left(e^{-\varepsilon}z\right)$.

For the exponentiation of $\mathbf{v}_{11}$ we must solve

$$\frac{d\widetilde{h}^1}{d\varepsilon} = \widetilde{g}^2, \quad \frac{d\widetilde{h}^2}{d\varepsilon} = -\widetilde{g}^1, \quad \frac{d\widetilde{g}^1}{d\varepsilon} = -\widetilde{h}^2, \quad \frac{d\widetilde{g}^2}{d\varepsilon} = \widetilde{h}^1$$

subject to the same boundary conditions as above. This system has solution

$$\begin{aligned}
\widetilde{h}^1 &= h^1\cosh\varepsilon + g^2\sinh\varepsilon \\
\widetilde{h}^2 &= h^2\cosh\varepsilon - g^1\sinh\varepsilon \\
\widetilde{g}^1 &= g^1\cosh\varepsilon - h^2\sinh\varepsilon \\
\widetilde{g}^2 &= g^2\cosh\varepsilon + h^1\sinh\varepsilon
\end{aligned}$$

which gives us that

$$\begin{aligned}
\widetilde{h} &= (h^1\cosh\varepsilon + g^2\sinh\varepsilon) + i(h^2\cosh\varepsilon - g^1\sinh\varepsilon) \\
&= h\cosh\varepsilon - ig\sinh\varepsilon \\
\widetilde{g} &= (g^1\cosh\varepsilon - h^2\sinh\varepsilon) + i(g^2\cosh\varepsilon + h^1\sinh\varepsilon) \\
&= g\cosh\varepsilon + ih\sinh\varepsilon.
\end{aligned}$$

Since the exponentiation of $\mathbf{v}_{12}$ is very similar to that of $\mathbf{v}_{11}$, we will not show its calculation. The exponentiations of each of the $\mathbf{v}_i$ are given by

$$
\begin{aligned}
\mathbf{v}_1:\quad & \widetilde{h}(z) = h(z-\varepsilon), && \widetilde{g}(z) = g(z-\varepsilon) \\
\mathbf{v}_2:\quad & \widetilde{h}(z) = h(z-i\varepsilon), && \widetilde{g}(z) = g(z-i\varepsilon) \\
\mathbf{v}_3:\quad & \widetilde{h}(z) = h(z)+\varepsilon, && \widetilde{g}(z) = g(z) \\
\mathbf{v}_4:\quad & \widetilde{h}(z) = h(z)+i\varepsilon, && \widetilde{g}(z) = g(z) \\
\mathbf{v}_5:\quad & \widetilde{h}(z) = h(z), && \widetilde{g}(z) = g(z)+\varepsilon \\
\mathbf{v}_6:\quad & \widetilde{h}(z) = h(z), && \widetilde{g}(z) = g(z)+i\varepsilon \\
\mathbf{v}_7:\quad & \widetilde{h}(z) = h\left(e^{-i\varepsilon}z\right), && \widetilde{g}(z) = g\left(e^{-i\varepsilon}z\right) \\
\mathbf{v}_8:\quad & \widetilde{h}(z) = e^{i\varepsilon}h(z), && \widetilde{g}(z) = g(z) \\
\mathbf{v}_9:\quad & \widetilde{h}(z) = h(z), && \widetilde{g}(z) = e^{i\varepsilon}g(z) \\
\mathbf{v}_{10}:\quad & \widetilde{h}(z) = e^{\varepsilon}h(e^{-\varepsilon}z), && \widetilde{g}(z) = e^{\varepsilon}g(e^{-\varepsilon}z) \\
\mathbf{v}_{11}:\quad & \widetilde{h}(z) = h(z)\cosh\varepsilon - ig(z)\sinh\varepsilon, && \widetilde{g}(z) = g(z)\cosh\varepsilon + ih(z)\sinh\varepsilon \\
\mathbf{v}_{12}:\quad & \widetilde{h}(z) = h(z)\cosh\varepsilon + ig(z)\sinh\varepsilon, && \widetilde{g}(z) = g(z)\cosh\varepsilon + ih(z)\sinh\varepsilon
\end{aligned}
$$

where we can see that $\widetilde{h}(z) = h(z)$ and $\widetilde{g}(z) = g(z)$ for each of the above exponentiations when $\varepsilon = 0$, thus verifying the trivial action of ε on h and g.

These exponentiations can be composed in any way to give another transform. This is the composition of the action of ε on z, h and g and not simply the composition of the transformed functions. Since each ε is independent of the others, we will notate the differences by subscripts. For an example of composing consider $h(z) \mapsto e^{i\varepsilon_1}h(z)$ and $h(z) \mapsto h(z) + \varepsilon_2$, which compose to give that

$$h(z) \mapsto e^{i\varepsilon_1}h(z) \mapsto \left(e^{i\varepsilon_1}h(z)\right) + \varepsilon_2 = e^{i\varepsilon_1}h(z) + \varepsilon_2$$

or composition in the other direction gives

$$h(z) \mapsto h(z) + \varepsilon_2 \mapsto e^{i\varepsilon_1}\left(h(z) + \varepsilon_2\right) = e^{i\varepsilon_1}h(z) + \varepsilon_2 e^{i\varepsilon_1}$$

where either composition is valid and are not necessarily equal.

C Calculation for Harmonic Functions with Prescribed Convexity

The title for this appendix and for Section 9 is a little misleading in that it relies on Theorem 3.12 to give us convexity by fixing $h - g$ when $\widetilde{f}$ is locally univalent, which cannot be encoded in the system $\Delta = 0$. We will calculate the Lie symmetries for h and g analytic with $h - g$ being a fixed function and we will let $h - g$ be an analytic univalent function convex in the direction of the real axis and apply Theorem 3.12 to $\widetilde{f}$ to get its convexity in the direction of the real axis. If $\widetilde{f}$ is locally univalent then Theorem 3.12 gives us that $\widetilde{f}$ is convex in the direction of the real axis.

Let's begin with our system of differential equations,

$$\Delta = \begin{pmatrix} h^1_x - h^2_y \\ h^1_y + h^2_x \\ g^1_x - g^2_y \\ g^1_y + g^2_y \\ h^1 - g^1 - F^1 \\ h^2 - g^2 - F^2 \end{pmatrix} = 0$$

where $F^1(z) + iF^2(z)$ is analytic and fixed. We can see that $\Delta = 0$ represents h and g being analytic and $h - g$ being a fixed function. Let the general infinitesimal generator be

$$\mathbf{v} = a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y} + a^3\frac{\partial}{\partial h^1} + a^4\frac{\partial}{\partial h^2} + a^5\frac{\partial}{\partial g^1} + a^6\frac{\partial}{\partial g^2}$$

as given in Appendix A and Appendix B and note that the first prolongation is the same as well. Therefore

$$\begin{aligned}\mathsf{pr}^{(1)}\mathbf{v} = {} & a^1\frac{\partial}{\partial x} + a^2\frac{\partial}{\partial y} + a^3\frac{\partial}{\partial h^1} + a^4\frac{\partial}{\partial h^2} + a^5\frac{\partial}{\partial g^1} + a^6\frac{\partial}{\partial g^2}\\ & + \Phi^1\frac{\partial}{\partial h^1_x} + \Phi^2\frac{\partial}{\partial h^1_y} + \Phi^3\frac{\partial}{\partial h^2_x} + \Phi^4\frac{\partial}{\partial h^2_y}\\ & + \Phi^5\frac{\partial}{\partial g^1_x} + \Phi^6\frac{\partial}{\partial g^1_y} + \Phi^7\frac{\partial}{\partial g^2_x} + \Phi^8\frac{\partial}{\partial g^2_y}\end{aligned}$$

where $\Phi^i, i = 1, \ldots, 8$ are defined in Appendix A.

Now to satisfy the infinitesimal criterion we must find the Lie derivatives of the prolongation of $\mathbf{v}$ for each of Δ_i. For $\Delta_1, \ldots, \Delta_4$, these are the same as in Appendix A so we need only calculate $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_5]$ and $\mathsf{pr}^{(1)}\mathbf{v}[\Delta_6]$. Definition 5.24 gives us that

$$\begin{aligned}\mathsf{pr}^{(1)}\mathbf{v}[\Delta_5] &= a^3 - a^5 - a^1F^1_x - a^2F^1_y\\ \mathsf{pr}^{(1)}\mathbf{v}[\Delta_6] &= a^4 - a^6 - a^1F^2_x - a^2F^2_y.\end{aligned}$$

Making the substitution of $\Delta = 0$ into each of the prolongations, setting them to 0 and comparing the coefficients of these polynomials gives us the following system of differential equations.

$$\begin{aligned} a^3_x - a^4_y &= 0 & a^2_{h^1} + a^1_{h^2} &= 0\\ a^2 &= 0 & a^3 - a^5 &= 0\\ -a^2_{g^2} + a^1_{g^1} &= 0 & a^2_{g^1} + a^1_{g^2} &= 0\\ a^4_{g^1} + a^3_{g^2} &= 0 & a^4_{h^2} - a^3_{h^1} + a^1_x - a^2_y &= 0\end{aligned}$$

$$a^4_{h^1} + a^3_{h^2} + a^1_y + a^2_x = 0 \qquad -a^2_{h^2} + a^1_{h^1} = 0$$

$$-a^4_{g^2} + a^3_{g^1} = 0 \qquad a^3_y + a^4_x = 0$$

$$-a^3_{h^2} + a^1_y - a^4_{h^1} + a^2_x = 0 \qquad -a^4_{h^2} + a^3_{h^1} + a^1_x - a^2_y = 0$$

$$a^6_y - a^5_x = 0 \qquad a^5_{g^2} + a^2_x + a^6_{g^1} + a^1_y = 0$$

$$-a^6_{h^2} + a^5_{h^1} = 0 \qquad a^6_{h^1} + a^5_{h^2} = 0$$

$$-a^2_y - a^5_{g^1} + a^1_x + a^6_{g^2} = 0 \qquad a^5_y + a^6_x = 0$$

$$a^1_x - a^2_y + a^5_{g^1} - a^6_{g^2} = 0 \qquad -a^5_{g^2} + a^1_y - a^6_{g^1} + a^2_x = 0$$

$$a^1 = 0 \qquad a^4 - a^6 = 0$$

Solving this system yields $a^1, a^2 \equiv 0$, $a^3 = a^5$, $a^4 = a^6$ and that $a^3 + ia^4$ is analytic in each of z, h and g. Therefore we have that

$$\mathbf{v} = a^3\frac{\partial}{\partial h^1} + a^4\frac{\partial}{\partial h^2} + a^3\frac{\partial}{\partial g^1} + a^4\frac{\partial}{\partial g^2}$$

is the infinitesimal generator for the Lie symmetries. Since the coefficients of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are 0 then we can see that a reparametrization of the domain in any way is not given by the Lie symmetries. We should note that since a^3 and a^4 are not independent of each other then we cannot split the infinitesimal generator into smaller elements. Therefore $\mathbf{v}$ alone spans the Lie algebra corresponding to the Lie symmetries of harmonic functions with $h - g$ being fixed.

C.1 A Finite Dimensional Lie Subalgebra

We want to consider a subalgebra of the algebra spanned by the generator as given in Appendix C. If we let

$$a^3(z, h, g) = c_1 + c_3x - c_4y + c_5h^1 - c_6h^2 + c_7g^1 - c_8g^2$$

and

$$a^4(z,h,g) = c_2 + c_3y + c_4x + c_5h^2 + c_6h^1 + c_7g^2 + c_8g^1$$

where each c_i is an arbitrary real number then it can be verified that

$$(a^3 + ia^4)(z,h,g) = (c_1 + ic_2) + (c_3 + ic_4)z + (c_5 + ic_6)h + (c_7 + ic_8)g$$

is analytic in z, h and g. If we substitute a^3 and a^4 into $\mathbf{v}$ and factor in each c_i we get that

$$\begin{aligned}
\mathbf{v} &= a^3\frac{\partial}{\partial h^1} + a^4\frac{\partial}{\partial h^2} + a^3\frac{\partial}{\partial g^1} + a^4\frac{\partial}{\partial g^2} \\
&= c_1\left(\frac{\partial}{\partial h^1} + \frac{\partial}{\partial h^2}\right) + c_2\left(\frac{\partial}{\partial h^2} + \frac{\partial}{\partial g^2}\right) \\
&\quad + c_3\left(x\frac{\partial}{\partial h^1} + y\frac{\partial}{\partial h^2} + x\frac{\partial}{\partial g^1} + y\frac{\partial}{\partial g^2}\right) \\
&\quad + c_4\left(-y\frac{\partial}{\partial h^1} + x\frac{\partial}{\partial h^2} - y\frac{\partial}{\partial g^1} + x\frac{\partial}{\partial g^2}\right) \\
&\quad + c_5\left(h^1\frac{\partial}{\partial h^1} + h^2\frac{\partial}{\partial h^2} + h^1\frac{\partial}{\partial g^1} + h^2\frac{\partial}{\partial g^2}\right) \\
&\quad + c_6\left(-h^2\frac{\partial}{\partial h^1} + h^1\frac{\partial}{\partial h^2} - h^2\frac{\partial}{\partial g^1} + h^1\frac{\partial}{\partial g^2}\right) \\
&\quad + c_7\left(g^1\frac{\partial}{\partial h^1} + g^2\frac{\partial}{\partial h^2} + g^1\frac{\partial}{\partial g^1} + g^2\frac{\partial}{\partial g^2}\right) \\
&\quad + c_8\left(-g^2\frac{\partial}{\partial h^1} + g^1\frac{\partial}{\partial h^2} - g^2\frac{\partial}{\partial g^1} + g^1\frac{\partial}{\partial g^2}\right).
\end{aligned}$$

By successively setting the c_i to 0 we get the following eight infinitesimal generators:

$$\begin{aligned}
\mathbf{v}_1 &= \frac{\partial}{\partial h^1} + \frac{\partial}{\partial g^1} \\
\mathbf{v}_2 &= \frac{\partial}{\partial h^2} + \frac{\partial}{\partial g^2} \\
\mathbf{v}_3 &= x\frac{\partial}{\partial h^1} + y\frac{\partial}{\partial h^2} + x\frac{\partial}{\partial g^1} + y\frac{\partial}{\partial g^2} \\
\mathbf{v}_4 &= -y\frac{\partial}{\partial h^1} + x\frac{\partial}{\partial h^2} - y\frac{\partial}{\partial g^1} + x\frac{\partial}{\partial g^2} \\
\mathbf{v}_5 &= h^1\frac{\partial}{\partial h^1} + h^2\frac{\partial}{\partial h^2} + h^1\frac{\partial}{\partial g^1} + h^2\frac{\partial}{\partial g^2}
\end{aligned}$$

$$\mathbf{v}_6 = -h^2\frac{\partial}{\partial h^1} + h^1\frac{\partial}{\partial h^2} - h^2\frac{\partial}{\partial g^1} + h^1\frac{\partial}{\partial g^2}$$
$$\mathbf{v}_7 = g^1\frac{\partial}{\partial h^1} + g^2\frac{\partial}{\partial h^2} + g^1\frac{\partial}{\partial g^1} + g^2\frac{\partial}{\partial g^2}$$
$$\mathbf{v}_8 = -g^2\frac{\partial}{\partial h^1} + g^1\frac{\partial}{\partial h^2} - g^2\frac{\partial}{\partial g^1} + g^1\frac{\partial}{\partial g^2}$$

where $\mathbf{v}_i$ corresponds to setting all but c_i to 0. These eight generators span a finite dimensional Lie subalgebra of the infinite dimensional one calculated in Appendix C.

Let's consider the exponentiations of each of the $\mathbf{v}_i$. The exponentiation for $\mathbf{v}_1$ comes from solving the system

$$\frac{d\widetilde{h}^1}{d\varepsilon} = 1, \qquad \frac{d\widetilde{g}^1}{d\varepsilon} = 1$$

subject to the aforementioned boundary conditions. This system has solution $\widetilde{h}^1 = h^1 + \varepsilon$ and $\widetilde{g}^1 = g^1 + \varepsilon$ which yields the exponentiation of $\widetilde{h}(z) = h(z) + \varepsilon$ and $\widetilde{g}(z) = g(z) + \varepsilon$. The exponentiation of $\mathbf{v}_2$ is similar and yields $\widetilde{h}(z) = h(z) + i\varepsilon$ and $\widetilde{g}(z) = g(z) + i\varepsilon$.

To exponentiate $\mathbf{v}_3$ we solve

$$\frac{d\widetilde{h}^1}{d\varepsilon} = \widetilde{x}, \qquad \frac{d\widetilde{h}^2}{d\varepsilon} = \widetilde{y}, \qquad \frac{d\widetilde{g}^1}{d\varepsilon} = \widetilde{x}, \qquad \frac{d\widetilde{g}^2}{d\varepsilon} = \widetilde{y}$$

which has solution $\widetilde{h}^1 = h^1 + \varepsilon x$, $\widetilde{h}^2 = h^2 + \varepsilon y$, $\widetilde{g}^1 = g^1 + \varepsilon x$ and $\widetilde{g}^2 = g^2 + \varepsilon y$. This solution gives us the exponentiation of $\widetilde{h}(z) = h(z) + \varepsilon z$ and $\widetilde{g}(z) = g(z) + \varepsilon z$. The exponentiation of $\mathbf{v}_4$ is similar and yields $\widetilde{h}(z) = h(z) + i\varepsilon z$ and $\widetilde{g}(z) = g(z) + i\varepsilon z$.

The exponentiation of $\mathbf{v}_5$ and $\mathbf{v}_7$ are also similar. The exponentiation of $\mathbf{v}_5$ arises from the system

$$\frac{d\widetilde{h}^1}{d\varepsilon} = \widetilde{h}^1, \qquad \frac{d\widetilde{h}^2}{d\varepsilon} = \widetilde{h}^2, \qquad \frac{d\widetilde{g}^1}{d\varepsilon} = \widetilde{h}^1, \qquad \frac{d\widetilde{g}^2}{d\varepsilon} = \widetilde{h}^2$$

which has solution $\widetilde{h}^1 = e^\varepsilon h^1$, $\widetilde{h}^2 = e^\varepsilon h^2$, $\widetilde{g}^1 = g^1 + e^\varepsilon h^1 - h^1$ and $\widetilde{g}^2 = g^2 + e^\varepsilon h^2 - h^2$. This gives us that $\widetilde{h}(z) = e^\varepsilon h(z)$ and $\widetilde{g}(z) = g(z) + e^\varepsilon h(z) - h(z)$. For $\mathbf{v}_7$ we have that $\widetilde{h}(z) = h(z) + e^\varepsilon g(z) - g(z)$ and $\widetilde{g}(z) = e^\varepsilon g(z)$.

In order to exponentiate $\mathbf{v}_6$ we find that the solution of

$$\frac{d\widetilde{h}^1}{d\varepsilon} = -\widetilde{h}^2, \qquad \frac{d\widetilde{h}^2}{d\varepsilon} = \widetilde{h}^1, \qquad \frac{d\widetilde{g}^1}{d\varepsilon} = -\widetilde{h}^2, \qquad \frac{d\widetilde{g}^2}{d\varepsilon} = \widetilde{h}^1$$

is $\widetilde{h}^1 = h^1 \cos\varepsilon - h^2 \sin\varepsilon$, $\widetilde{h}^2 = h^2 \cos\varepsilon + h^1 \sin\varepsilon$, $\widetilde{g}^1 = g^1 + h^1 \cos\varepsilon - h^2 \sin\varepsilon - h^1$ and $\widetilde{g}^2 = g^2 + h^2 \cos\varepsilon + h^1 \sin\varepsilon - h^2$. This gives us that $\widetilde{h}(z) = e^{i\varepsilon} h(z)$ and $\widetilde{g}(z) = g(z) + e^{i\varepsilon} h(z) - h(z)$. This is similar to the exponentiation of $\mathbf{v}_8$ which gives that $\widetilde{h}(z) = h(z) + e^{i\varepsilon} g(z) - g(z)$ and $\widetilde{g}(z) = e^{i\varepsilon} g(z)$.

The exponentiations of each of the $\mathbf{v}_i$ are given by

$$\begin{aligned}
\mathbf{v}_1 :&\quad \widetilde{h}(z) = h(z) + \varepsilon, && \widetilde{g}(z) = g(z) + \varepsilon \\
\mathbf{v}_2 :&\quad \widetilde{h}(z) = h(z) + i\varepsilon, && \widetilde{g}(z) = g(z) + i\varepsilon \\
\mathbf{v}_3 :&\quad \widetilde{h}(z) = h(z) + \varepsilon z, && \widetilde{g}(z) = g(z) + \varepsilon z \\
\mathbf{v}_4 :&\quad \widetilde{h}(z) = h(z) + i\varepsilon z, && \widetilde{g}(z) = g(z) + i\varepsilon z \\
\mathbf{v}_5 :&\quad \widetilde{h}(z) = e^{\varepsilon} h(z), && \widetilde{g}(z) = g(z) + e^{\varepsilon} h(z) - h(z) \\
\mathbf{v}_6 :&\quad \widetilde{h}(z) = e^{i\varepsilon} h(z), && \widetilde{g}(z) = g(z) + e^{i\varepsilon} h(z) - h(z) \\
\mathbf{v}_7 :&\quad \widetilde{h}(z) = h(z) + e^{\varepsilon} g(z) - g(z), && \widetilde{g}(z) = e^{\varepsilon} g(z) \\
\mathbf{v}_8 :&\quad \widetilde{h}(z) = h(z) + e^{i\varepsilon} g(z) - g(z), && \widetilde{g}(z) = e^{i\varepsilon} g(z)
\end{aligned}$$

where we can see that $\widetilde{h}(z) = h(z)$ and $\widetilde{g}(z) = g(z)$ for each of the above exponentiations when $\varepsilon = 0$ verifying the trivial action of ε on h and g.

C.2 An Infinite Dimensional Lie Subalgebra

We will begin with an arbitrary analytic function $\phi(z) = \phi^1(z) + i\phi^2(z)$ of strictly the variable z. If we let $a^3(z, g) = \phi^1(z) + g^1$ and $a^4(z, g) = \phi^2(z) + g^2$ we have satisfied the necessary conditions for $a^3 + ia^4$ as given in Appendix C, that is that

$(a^3 + ia^4)(z, g) = \phi(z) + g$ is analytic in z and in g. Therefore

$$\mathbf{v} = (\phi^1(z) + g^1)\frac{\partial}{\partial h^1} + (\phi^2(z) + g^2)\frac{\partial}{\partial h^2} + (\phi^1(z) + g^1)\frac{\partial}{\partial g^1} + (\phi^2(z) + g^2)\frac{\partial}{\partial g^2}$$

will be an infinitesimal generator of Lie symmetries for harmonic functions with $h-g$ fixed and will span an infinite dimensional subalgebra of the algebra presented in Appendix C.

To exponentiate this we must solve the system

$$\frac{d\widetilde{h}^1}{d\varepsilon} = \phi(\widetilde{z}) + \widetilde{g}^1, \quad \frac{d\widetilde{h}^2}{d\varepsilon} = \phi^2(\widetilde{z}) + \widetilde{g}^2, \quad \frac{d\widetilde{g}^1}{d\varepsilon} = \phi^1(\widetilde{z}) + \widetilde{g}^1, \quad \frac{d\widetilde{g}^2}{d\varepsilon} = \phi^2(\widetilde{z}) + \widetilde{g}^2$$

where $\phi(\widetilde{z})$ is not a function of ε because we have no flow in the $\dfrac{\partial}{\partial x}$ or $\dfrac{\partial}{\partial y}$ coordinates. Therefore we could consider

$$\frac{d\widetilde{x}}{d\varepsilon} = 0 \text{ and } \frac{d\widetilde{y}}{d\varepsilon} = 0$$

as part of the system as well but this gives us the trivial solution of $\widetilde{z} = z$ and therefore the entire system has solution

$$\begin{aligned}
\widetilde{h}^1 &= h^1 + e^{\varepsilon} g^1 - g^1 + \phi^1(z)\,(e^{\varepsilon} - 1) \\
h^2 &= \widetilde{h}^2 + e^{\varepsilon} g^2 - g^2 + \phi^2(z)\,(e^{\varepsilon} - 1) \\
\widetilde{g}^1 &= e^{\varepsilon} g^1 + \phi^1(z)\,(e^{\varepsilon} - 1) \\
\widetilde{g}^2 &= e^{\varepsilon} g^2 + \phi^2(z)\,(e^{\varepsilon} - 1)
\end{aligned}$$

and if we make the substitution that $\varphi = \phi\,(e^{\varepsilon} - 1)$ then we get the exponentiation of

$$\widetilde{h}(z) = h(z) + e^{\varepsilon} g(z) - g(z) + \varphi(z)$$

and

$$\widetilde{g}(z) = e^{\varepsilon} g(z) + \varphi(z).$$

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